

# Preface

This book focuses primarily on applications of mathematical analysis to stock price models with stochastic volatility and more general stochastic asset price models. The central objective of the book is to characterize limiting behavior of several important functions associated with such models, e.g., stock price densities, call and put pricing functions, and implied volatilities.

Stock price models with stochastic volatility have been developed in the last decades to improve pricing and hedging performance of the classical Black–Scholes model and to account for certain imperfections in it. The main shortcoming of the Black–Scholes model is its constant volatility assumption. Statistical analysis of stock market data shows that the volatility of a stock is a time-dependent quantity. Moreover, it exhibits various random features. Stochastic volatility models address this randomness by assuming that both the stock price and the volatility are stochastic processes affected by different sources of risk. Unlike the Black–Scholes model, stock price models with stochastic volatility explain such stylized facts as the implied volatility smile and skew. They can also incorporate the leverage effect, that is, the tendency of the volatility of the stock to increase when the stock price decreases. Stochastic volatility models reflect the leverage effect by imposing the restriction that the stock price and the volatility are negatively correlated.

An important problem in mathematical finance is to describe the asymptotic behavior of the stock price density in a stochastic volatility model. Once we have a good understanding of how this density changes, we can estimate many other characteristics of the model, for example, left and right tails of stock return distributions, option pricing functions, and implied volatilities. Asymptotic formulas for distribution tails of stock returns in a stochastic volatility model can be used to analyze how well the model addresses the tail risk. In financial practice, the tail risk is defined as the probability that stock returns will move more than three standard deviations beyond the mean. The Black–Scholes model underestimates the tail risk, since the probability of extreme variations of stock returns in this model is negligible. This follows from the fact that distribution tails of stock returns in the Black–Scholes model decay like Gaussian density functions. In the present book, we obtain sharp asymptotic formulas with relative error estimates for stock price densities in three

popular stock price models with stochastic volatility: the Hull–White model, the Stein–Stein model, and the Heston model. These formulas show that for the above-mentioned models, the stock price distributions have Pareto type tails, that is to say, the tails decay like regularly varying functions. As a consequence, the Hull–White, Stein–Stein, and Heston models estimate the probability of abrupt downward movements of stock prices (disastrous scenarios) better than the Black–Scholes model.

The implied volatility associated with the call pricing function in a stochastic asset price model may be poetically described as the reflection of this function in the Black–Scholes mirror. One can obtain the implied volatility by inverting the Black–Scholes call pricing function and composing the inverse function with the call pricing function of our interest. A substantial part of the present book discusses model free asymptotic formulas for the implied volatility at extreme strikes in general asset price models. Some of the reasons why such asymptotic formulas are important are the following. On the one hand, these formulas help to check whether the given stochastic asset price model produces a skewed volatility pattern often observed in real markets. On the other hand, since the implied volatility at extreme strikes is associated with out-of-the-money and in-the-money put and call options, the analysis of the implied volatility for large and small strikes quantifies the expectations and fears of investors of possible large upward or downward movements in asset prices. Note that buying out-of-the money put options has been a popular hedging strategy against negative tail risk.

The text is organized as follows. The main emphasis in Chaps. 1–7 is on special stochastic volatility models (the Hull–White, Stein–Stein, and Heston models). In Chap. 1, we consider stochastic processes, which play the role of volatility in these models, i.e., geometric Brownian motion, Ornstein–Uhlenbeck process, and Cox–Ingersoll–Ross process (Feller process). Chapter 2 introduces general correlated stock price models with stochastic volatility. It also discusses risk-neutral measures in such models. Chapter 3 is concerned with realized volatility and mixing distributions. For an uncorrelated stochastic volatility model, the mixing distribution is the law of the realized volatility, while for correlated models, mixing distributions are defined as joint distributions of various combinations of the variance of the stock price, the integrated volatility, and the integrated variance. Chapter 4 considers integral transforms of mixing distribution densities, and provides explicit formulas for the stock price density in terms of mixing distributions. In Chap. 5 we prove a Tauberian theorem for the two-sided Laplace transform, and also Abelian theorems for fractional integrals and for integral operators with log-normal kernels. The Tauberian theorem is used in Chap. 5 to characterize the asymptotics of mixing distributions by inverting their Laplace transforms approximately, while the Abelian theorem for fractional integrals is a helpful tool in the study of mixing distributions in the Hull–White model. In Chap. 6 we provide asymptotic formulas with error estimates for the stock price distribution densities in the Hull–White, Stein–Stein, and Heston models. For the correlated Heston model the proof of the asymptotic formula is based on affine principles, while in the absence of correlation an alternative proof of the asymptotic formula is given. In the latter proof the Abelian theorem for integral operators with log-normal kernels plays an important role. Finally, in

Chap. 7 we include a short exposition of the theory of regularly varying functions. This chapter also considers Pareto type distributions and their applications.

The second part of the book (Chaps. 8–11) is devoted to general call and put pricing functions in no-arbitrage setting and to the Black–Scholes implied volatility. In the beginning of Chap. 8 we prove a characterization theorem for call pricing functions, and at the end of this chapter we establish sharp asymptotic formulas with error estimates for the call pricing functions in the Hull–White, Stein–Stein, and Heston models. Chapter 8 also presents an analytical proof of the Black–Scholes call option pricing formula, which is arguably the most famous formula of mathematical finance. Chapter 9 introduces the notion of implied volatility (or “smile”) and provides model free asymptotic formulas for the implied volatility at extreme strikes. One more topic discussed in Chap. 9 concerns certain symmetries hidden in option pricing models. The contents of Chap. 10 can be guessed from its title “More Formulas for Implied Volatility”. It is shown in this chapter that R. Lee’s moment formulas for the implied volatility and the tail-wing formulas due to S. Benaim and P. Friz can be derived from more general results established in Chap. 9. Chapter 10 also presents an important result obtained by E. Renault and N. Touzi, which can be shortly presented as follows: “The absence of correlation between the stock price and the volatility implies smile”. The last section of Chap. 10 deals with J. Gatheral’s SVI parameterization of implied variance. SVI parameterization provides a good approximation to implied variance observed in the markets and also to implied variance used in stochastic volatility models. Finally, in Chap. 11 we study implied volatility in models without moment explosions. Here we show that V.V. Piterbarg’s conjecture concerning the limiting behavior of implied volatility in models without moment explosions is true in a modified form. Chapter 11 also studies smile asymptotics in various special models, e.g., the displaced diffusion model, the constant elasticity of variance model, SV1 and SV2 models introduced by L.C.G. Rogers and L.A.M. Veraart, and the finite moment log-stable models developed by P. Carr and L. Wu.

We will next make a brief comparison between the present book and the following related books: [Lew00, Gat06, H-L09], and [FPSS11]. It is easy to check that although the books on the previous list and the present book have the same main heroes (stochastic volatility models, stock price densities, option pricing functions, and implied volatilities), they differ substantially from each other with respect to the choice of special topics. For example, the book by A. Lewis [Lew00] deals with various methods of option pricing under stochastic volatility, and the topics covered in [Lew00] include the volatility of volatility series expansions, volatility explosions, and related corrections in option pricing formulas. The book by J. Gatheral [Gat06] is a rich source of information on implied and local volatilities in stochastic stock price models and on the asymptotic and dynamic behavior of volatility surfaces. In particular, the book [Gat06] discusses early results on smile asymptotics for large and small strikes. The book by P. Henry-Labordère [H-L09] uses powerful methods of differential geometry and mathematical physics to study the asymptotic behavior of implied volatility in local and stochastic volatility models. For instance, heat kernel expansions in Riemannian manifolds and Schrödinger semigroups with

Kato class potentials play an important role in [H-L09]. The book by J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Sølna [FPSS11] is devoted to pricing and hedging of financial derivatives in stochastic volatility models. In [FPSS11], regular and singular perturbation techniques are used to study small parameter asymptotics of option pricing functions and implied volatilities. The authors of [FPSS11] obtain first and second order approximations to implied volatility in single-factor and multifactor stochastic volatility models, and explain how to use these approximations to calibrate stochastic volatility models and price more complex derivative contracts. A more detailed comparison shows that a large part of the material appearing in the present book is not covered by the books on the list. Moreover, to the best of the author's knowledge, many of the results discussed in the present book have never been published before in book form. These results include sharp asymptotic formulas with error estimates for stock price densities, option pricing functions, and implied volatilities in special stochastic volatility models, and sharp model free asymptotic formulas for implied volatilities.

This book is aimed at a variety of people: researchers in the field of financial mathematics, professional mathematicians interested in applications of mathematical analysis to finance, and advanced graduate students thinking of a career in applied analysis or financial mathematics. It is assumed that the reader is familiar with basic definitions and facts from probability theory, stochastic differential equations, asymptotic analysis, and complex analysis. The book does not aspire to completeness, several important topics related to its contents have been omitted. For example, small and large maturity asymptotics of implied volatility, affine models, local martingale option pricing models, or applications of geometric methods to the study of implied volatility are not discussed in the book. The reader can find selected references to publications on the missing subjects in the sections "Notes and References" that conclude each chapter, or search the bibliography at the end of the book for additional reading.

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