Preface

This work is ultimately directed at understanding the diffeomorphism groups of elliptic three-manifolds—those closed three-manifolds that admit a Riemannian metric of constant positive curvature. The main results concern the Smale Conjecture. The original Smale Conjecture, proven by A. Hatcher [24], asserts that if M is the 3-sphere with the standard constant curvature metric, the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ from the isometry group to the diffeomorphism group is a homotopy equivalence. The *Generalized Smale Conjecture* (henceforth just called the Smale Conjecture) asserts this whenever M is an elliptic three-manifold.

Here are our main results:

1. The Smale Conjecture holds for elliptic three-manifolds containing geometrically incompressible Klein bottles (Theorem 1.2). These include all quaternionic and prism manifolds.
2. The Smale Conjecture holds for all lens spaces $L(m, q)$ with $m \geq 3$ (Theorem 1.3).

Many of the cases in Theorem 1.2 were proven a number of years ago by N. Ivanov [33–36] (see Sect. 1.2).

Some of our other results concern the groups of diffeomorphisms $\mathrm{Diff}(\Sigma)$ and fiber-preserving diffeomorphisms $\mathrm{Diff}_f(\Sigma)$ of a Seifert-fibered Haken three-manifold Σ and the coset space $\mathrm{Diff}(\Sigma)/\mathrm{Diff}_f(\Sigma)$, which is called the space of Seifert fiberings (equivalent to the given fibering) of Σ.

3. Apart from a small list of known exceptions, $\mathrm{Diff}_f(\Sigma) \to \mathrm{Diff}(\Sigma)$ is a homotopy equivalence (Theorem 3.15).
4. The space of Seifert fiberings of Σ has contractible components (Theorem 3.14) and apart from a small list of known exceptions, it is contractible (Theorem 3.15).

These may be already accepted as part of the overall three-dimensional landscape, but we are unable to find any serious treatment of them. And we have found that the development of the necessary tools and their application to the three-dimensional context goes well beyond a routine exercise.

Table 1 Status of the Smale conjecture

Case	SC proven?
S^3	Hatcher [24]
$\mathbb{RP}^3$	
Lens spaces	Chapter 5
Prism and quaternionic manifolds	Ivanov [33–36] and Chap. 4
Tetrahedral manifolds	
Octahedral manifolds	
Icosahedral manifolds	

This manuscript includes work done more than 20 years ago, as well as work recently completed. In the mid-1980s, two of the authors (DM and JHR) sketched an argument proving the Smale Conjecture for the three-manifolds that contain one-sided Klein bottles (other than the lens space $L(4, 1)$). That method, which ultimately became Chap. 4, underwent a long evolution as various additions were made to fill in technical details.

The case of one-sided Klein bottles includes some lens spaces—those of the form $L(4n, 2n - 1)$ for $n \geq 2$. But for the general lens space case, a different approach using Heegaard tori was developed by SH and DM starting around 2000. It is based on a powerful methodology developed by JHR and M. Scharlemann [58]. It turned out that JHR was working on the Smale Conjecture for lens spaces along exactly the same lines as SH and DM, so the efforts were combined in the work that became Chap. 5.

One more case of the Smale Conjecture may be accessible to existing techniques. It seems likely that A. Hatcher's approach to the S^3 case in [24] would also serve for $\mathbb{RP}^3$, but this has yet to be carried out.

In summary, this is where the Smale Conjecture now stands (Table 1).

Our work on the Smale Conjecture requires some basic theory about spaces of mappings of smooth manifolds, such as the fact that diffeomorphism groups of compact manifolds and spaces of embeddings of submanifolds have the homotopy type of CW-complexes, a result originally proven by R. Palais. This theory is well known to global analysts and others, but not to many low-dimensional topologists. Also, most sources do not discuss the case of manifolds with boundary, and we know of no existing treatment of the case of fiber-preserving diffeomorphisms and embeddings, which is the context of much of our technical work. For this reason, we have included a fair dose of foundational material on diffeomorphism groups in Chap. 2, which includes the case of manifolds with boundary, with the additional boundary control that we will need.

A more serious gap in the literature is the absence of versions of the fundamental restriction fibration theorems of Palais and Cerf in the context of fibered (and Seifert-fibered) manifolds. These extensions of the well-known theory require some new ideas, which were developed by JK and DM and form most of Chap. 3. We work in a class of singular fiberings large enough to include all Seifert fiberings of three-manifolds, except some fiberings of lens spaces. These results are heavily used in our work in Chaps. 4 and 5. Our results on fiber-preserving diffeomorphisms and the space of fibered structures of a Seifert-fibered Haken three-manifold are applications of this work, and they also appear in Chap. 3.

Much of our work in this text is unusually detailed and technical. In considerable part, this not only arises from its inherent complication, but it also reflects the fact that over the years we have filled in many arguments in response to recommendations from various readers. Unfortunately, one reader's "too sketchy" can be another's "too much elaboration of well-known facts," and personally we find some of the current exposition to be somewhat too long and too detailed. To provide an alternative, we have included Sects. 4.2 and 5.1, which are overviews of the proofs of the main results. In the actual proofs, we trust that each reader will simply accept the "obvious" parts and focus on the "nontrivial" parts, whichever they may be.

We have made the text self-contained, when possible, and sought useful references when not. We do assume that the reader is comfortable with basic topology and differential topology of manifolds, group actions on manifolds, Riemannian metrics, fibrations, and so on. We freely use classical three-manifold topology, such as I-bundles and Seifert-fibered structures and two-dimensional orbifolds, the Jaco–Shalen–Johannson decomposition, the results of Waldhausen, and hyperbolic three-manifolds, as well as major developments such as the results (but not the methods) of Perelman. We rather freely use facts about spaces of isometries and diffeomorphisms of surfaces and commonly encountered three-manifolds. Here, familiarity with papers of A. Hatcher such as [22, 23] would be very helpful. In the realm of infinite-dimensional topology, we use some basics about Fréchet manifolds and some standard theorems, see Sect. 2.1 for a discussion. Additionally we draw on the theory of singularities, modestly in Chap. 4 and in quite a bit more depth in Chap. 5. Both chapters make heavy use of the parameterized methods in the aforementioned papers of Hatcher, and in the latter chapter, familiarity with the Rubinstein–Scharlemann graphic [58] will be very helpful.

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