Chapter 1
Elliptic Three-Manifolds and the Smale Conjecture

As noted in the Preface, the Smale Conjecture is the assertion that the inclusion $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence whenever M is an elliptic three-manifold, that is, a three-manifold with a Riemannian metric of constant positive curvature (which may be assumed to be 1). The Geometrization Conjecture, now proven by Perelman, shows that all closed three-manifolds with finite fundamental group are elliptic.

In this chapter, we will first review elliptic three-manifolds and their isometry groups. In the second section, we will state our main results on the Smale Conjecture, and provide some historical context. In the final two sections, we discuss isometries of nonelliptic three-manifolds, and address the possibility of applying Perelman's methods to the Smale Conjecture.

1.1 Elliptic Three-Manifolds and Their Isometries

The elliptic three-manifolds were completely classified long ago. They are exactly the three-manifolds whose universal cover can be uniformized as the unit sphere S^3 in $\mathbb{R}^4$ so that $\pi_1(M)$ acts freely as a subgroup of $\mathrm{Isom}_+(S^3) = \mathrm{SO}(4)$. The subgroups of SO(4) that act freely were first determined by Hopf and Seifert–Threlfall, and reformulated using quaternions by Hattori. References include [74] (pp. 226–227), [49] (pp. 103–113), [60] (pp. 449–457), [46, 59].

The isometry groups of elliptic three-manifolds have also been known for a long time, and are topologically rather simple: they are compact Lie groups of dimension at most 6. A detailed calculation of the isometry groups of elliptic three-manifolds was given in [46], and in this section we will recall the resulting groups.

To set notation, recall that there is a well-known twofold covering $S^3 \to \mathrm{SO}(3)$, which is a homomorphism when S^3 is regarded as the group of unit quaternions (see Sect. 4.3 for a fuller discussion). The elements of SO(3) that preserve a given axis, say the z-axis, form the orthogonal subgroup O(2). We will denote

S. Hong et al., *Diffeomorphisms of Elliptic 3-Manifolds*, Lecture Notes in Mathematics 2055, DOI 10.1007/978-3-642-31564-0_1,

Table 1.1 Isometry groups of $M = S^3/G$ ($m > 2, n > 1$)

G	M	$\mathrm{Isom}(M)$	$\mathscr{I}(M)$
Q_8	Quaternionic	$\mathrm{SO}(3) \times S_3$	S_3
$Q_8 \times C_n$	Quaternionic	$\mathrm{O}(2) \times S_3$	$C_2 \times S_3$
D^*_{4m}	Prism	$\mathrm{SO}(3) \times C_2$	C_2
$D^*_{4m} \times C_n$	Prism	$\mathrm{O}(2) \times C_2$	$C_2 \times C_2$
Index 2 diagonal	Prism	$\mathrm{O}(2) \times C_2$	$C_2 \times C_2$
T^*_{24}	Tetrahedral	$\mathrm{SO}(3) \times C_2$	C_2
$T^*_{24} \times C_n$	Tetrahedral	$\mathrm{O}(2) \times C_2$	$C_2 \times C_2$
Index 3 diagonal	Tetrahedral	$\mathrm{O}(2)$	C_2
O^*_{48}	Octahedral	$\mathrm{SO}(3)$	$\{1\}$
$O^*_{48} \times C_n$	Octahedral	$\mathrm{O}(2)$	C_2
I^*_{120}	Icosahedral	$\mathrm{SO}(3)$	$\{1\}$
$I^*_{120} \times C_n$	Icosahedral	$\mathrm{O}(2)$	C_2

by $\mathrm{O}(2)^*$ the inverse image in S^3 of $\mathrm{O}(2)$. When H_1 and H_2 are groups, each containing -1 as a central involution, the quotient $(H_1 \times H_2)/\langle(-1,-1)\rangle$ is denoted by $H_1 \mathbin{\widetilde{\times}} H_2$. In particular, $\mathrm{SO}(4)$ itself is $S^3 \mathbin{\widetilde{\times}} S^3$, and contains the subgroups $S^1 \mathbin{\widetilde{\times}} S^3$, $\mathrm{O}(2)^* \mathbin{\widetilde{\times}} \mathrm{O}(2)^*$, and $S^1 \mathbin{\widetilde{\times}} S^1$. The latter is isomorphic to $S^1 \times S^1$, but it is sometimes useful to distinguish between them. Finally, $\mathrm{Dih}(S^1 \times S^1)$ is the semidirect product $(S^1 \times S^1) \circ C_2$, where C_2 acts by complex conjugation in both factors.

There are twofold covering homomorphisms

$$\mathrm{O}(2)^* \times \mathrm{O}(2)^* \to \mathrm{O}(2)^* \mathbin{\widetilde{\times}} \mathrm{O}(2)^* \to \mathrm{O}(2) \times \mathrm{O}(2) \to \mathrm{O}(2) \mathbin{\widetilde{\times}} \mathrm{O}(2)\,.$$

Each of these groups is diffeomorphic to four disjoint copies of the torus, but they are pairwise nonisomorphic. Indeed, they are easily distinguished by examining their subsets of order two elements. Similarly, $S^1 \times S^3$ and $S^1 \mathbin{\widetilde{\times}} S^3$ are diffeomorphic, but nonisomorphic.

Table 1.1 gives the isometry groups of the elliptic three-manifolds with noncyclic fundamental group. The first column, G, indicates the fundamental group of M, where C_m denotes a cyclic group of order m, and D^*_{4m}, T^*_{24}, O^*_{48}, and I^*_{120} are the binary dihedral, tetrahedral, octahedral, and icosahedral groups of the indicated orders. The groups called index 2 and index 3 diagonal are certain subgroups of $D^*_{4m} \times C_{4m}$ and $T^*_{24} \times C_{6n}$ respectively. The last two columns give the full isometry group $\mathrm{Isom}(M)$, and the group $\mathscr{I}(M)$ of path components of $\mathrm{Isom}(M)$.

Table 1.2 gives the isometry groups of the elliptic three-manifolds with cyclic fundamental group. These are the 3-sphere $L(1,0)$, real projective space $L(2,1)$, and the lens spaces $L(m,q)$ with $m \geq 3$.

Section 4.3 contains the detailed calculation of $\mathrm{isom}(M)$, the connected component of id_M in $\mathrm{Isom}(M)$, for the elliptic three-manifolds that contain one-sided incompressible Klein bottles (the quaternionic and prism manifolds, and the lens spaces of the form $L(4n, 2n-1)$), since the notation and some of the mechanics of this calculation are needed for the arguments in Chap. 4.

Table 1.2 Isometry groups of $L(m,q)$

m, q	$\mathrm{Isom}(L(m,q))$	$\mathscr{I}(L(m,q))$
$m = 1$ ($L(1,0) = S^3$)	$\mathrm{O}(4)$	C_2
$m = 2$ ($L(2,1) = \mathbb{RP}^3$)	$(\mathrm{SO}(3) \times \mathrm{SO}(3)) \circ C_2$	C_2
$m > 2$, m odd, $q = 1$	$\mathrm{O}(2)^* \mathbin{\widetilde{\times}} S^3$	C_2
$m > 2$, m even, $q = 1$	$\mathrm{O}(2) \times \mathrm{SO}(3)$	C_2
$m > 2$, $1 < q < m/2$, $q^2 \not\equiv \pm 1 \bmod m$	$\mathrm{Dih}(S^1 \times S^1)$	C_2
$m > 2$, $1 < q < m/2$, $q^2 \equiv -1 \bmod m$	$(S^1 \mathbin{\widetilde{\times}} S^1) \circ C_4$	C_4
$m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \bmod m$, $\gcd(m, q+1)\gcd(m, q-1) = m$	$\mathrm{O}(2) \mathbin{\widetilde{\times}} \mathrm{O}(2)$	$C_2 \times C_2$
$m > 2$, $1 < q < m/2$, $q^2 \equiv 1 \bmod m$, $\gcd(m, q+1)\gcd(m, q-1) = 2m$	$\mathrm{O}(2) \times \mathrm{O}(2)$	$C_2 \times C_2$

1.2 The Smale Conjecture

Smale [64] proved that for the standard round 2-sphere S^2, the inclusion of the isometry group $\mathrm{O}(3)$ into the diffeomorphism group $\mathrm{Diff}(S^2)$ is a homotopy equivalence. He conjectured that the analogous result holds true for the 3-sphere, that is, that $\mathrm{O}(4) \to \mathrm{Diff}(S^3)$ is a homotopy equivalence. Cerf [11] proved that the inclusion induces a bijection on path components, and the full conjecture was proven by Hatcher [24].

A weak form of the (generalized) Smale Conjecture is known. In [46], the calculations of $\mathrm{Isom}(M)$ for elliptic three-manifolds are combined with results on mapping class groups of many authors, including [2, 5, 6, 56, 57], to obtain the following statement:

Theorem 1.1. *Let M be an elliptic three-manifold. Then the inclusion of* $\mathrm{Isom}(M)$ *into* $\mathrm{Diff}(M)$ *is a bijection on path components.*

This can be called the "π_0-part" of the Smale Conjecture. By virtue of this result, to prove the Smale Conjecture for any elliptic three-manifold, it is sufficient to prove that the inclusion $\mathrm{isom}(M) \to \mathrm{diff}(M)$ of the connected components of the identity map in $\mathrm{Isom}(M)$ and $\mathrm{Diff}(M)$ is a homotopy equivalence.

The earliest work on the Smale Conjecture was by N. Ivanov. Certain elliptic three-manifolds contain one-sided geometrically incompressible Klein bottles. Fixing such a Klein bottle K_0, called the base Klein bottle, the remainder of the three-manifold is an open solid torus, and (up to isotopy) there are two Seifert fiberings, one for which the Klein bottle is fibered by nonsingular fibers (the "meridional" fibering), and one for which it contains two exceptional fibers of type $(2,1)$ (the "longitudinal" fibering). As will be detailed in Sect. 4.1 below, the manifolds then fall into four types:

(I) Those for which neither the meridional nor the longitudinal fibering is nonsingular on the complement of K_0.
(II) Those for which only the longitudinal fibering is nonsingular on the complement of K_0. These are the lens spaces $L(4n, 2n-1)$, $n \geq 2$.

(III) Those for which only the meridional fibering is nonsingular on the complement of K_0.

(IV) The lens space $L(4,1)$, for which both the meridional and longitudinal fiberings are nonsingular on the complement of K_0.

Cases I and III are the quaternionic and prism manifolds.

Ivanov announced the Smale Conjecture for Cases I and II in [33,34], and gave a detailed proof for Case I in [35,36]. One of our main theorems extends those results to all cases:

Theorem 1.2 (Smale Conjecture for elliptic three-manifolds containing incompressible Klein bottles). *Let M be an elliptic three-manifold containing a geometrically incompressible Klein bottle. Then* $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ *is a homotopy equivalence.*

Theorem 1.2 is proven in Chap. 4, except for the case of $L(4,1)$, which is proven in Chap. 5.

Our second main result concerns lens spaces, which for us refers only to the lens spaces $L(m,q)$ with $m \geq 3$:

Theorem 1.3 (Smale Conjecture for lens spaces). *For any lens space L, the inclusion* $\mathrm{Isom}(L) \to \mathrm{Diff}(L)$ *is a homotopy equivalence.*

One consequence of the Smale Conjecture is the determination of the homeomorphism type of $\mathrm{Diff}(M)$. Recall that a Fréchet space is a locally convex complete metrizable linear space. In Sect. 2.1, we will review the fact that if M is a closed smooth manifold, then with the C^∞-topology, $\mathrm{Diff}(M)$ is a separable infinite-dimensional manifold locally modeled on the Fréchet space of smooth vector fields on M. By the Anderson–Kadec Theorem [4, Corollary VI.5.2], every infinite-dimensional separable Fréchet space is homeomorphic to $\mathbb{R}^\infty$, the countable product of lines. A theorem of Henderson and Schori ([4, Theorem IX.7.3], originally announced in [28]) shows that if Y is any locally convex space with Y homeomorphic to Y^∞, then manifolds locally modeled on Y are homeomorphic whenever they have the same homotopy type. Applying this with $Y = \mathbb{R}^\infty$, our main theorems give immediately the homeomorphism type of $\mathrm{Diff}(M)$:

Corollary 1.1. *Let M be an elliptic three-manifold which either contains an incompressible Klein bottle or is a lens space $L(m,q)$ with $m \geq 3$. Then* $\mathrm{Diff}(M)$ *is homeomorphic to* $\mathrm{Isom}(M) \times \mathbb{R}^\infty$.

Combining this with the calculations of $\mathrm{Isom}(M)$ in Table 1.1 gives the following homeomorphism classification of $\mathrm{Diff}(M)$, in which P_n denotes the discrete space with n points:

Corollary 1.2. *Let M be an elliptic three-manifold, not a lens space, containing an incompressible Klein bottle.*

1. *If M is the quaternionic manifold with fundamental group $Q_8 = D_8^*$, then* $\mathrm{Diff}(M) \approx P_6 \times \mathrm{SO}(3) \times \mathbb{R}^\infty$.

2. *If M is a quaternionic manifold with fundamental group $Q_8 \times C_n$, $n > 2$, then* $\mathrm{Diff}(M) \approx P_{12} \times S^1 \times \mathbb{R}^\infty$.
3. *If M is a prism manifold with fundamental group D^*_{4m}, $m \geq 3$, then* $\mathrm{Diff}(M) \approx P_2 \times \mathrm{SO}(3) \times \mathbb{R}^\infty$.
4. *If M is any other prism manifold, then* $\mathrm{Diff}(M) \approx P_4 \times S^1 \times \mathbb{R}^\infty$.

As above, using Table 1.2, we obtain a complete classification of $\mathrm{Diff}(L)$ for lens spaces into four homeomorphism types:

Corollary 1.3. *For a lens space $L(m,q)$ with $m \geq 3$, the homeomorphism type of* $\mathrm{Diff}(L)$ *is as follows:*

1. *For m odd,* $\mathrm{Diff}(L(m,1)) \approx P_2 \times S^1 \times S^3 \times \mathbb{R}^\infty$.
2. *For m even,* $\mathrm{Diff}(L(m,1)) \approx P_2 \times S^1 \times \mathrm{SO}(3) \times \mathbb{R}^\infty$.
3. *For $q > 1$ and $q^2 \not\equiv \pm 1 \pmod m$,* $\mathrm{Diff}(L(m,q)) \approx P_2 \times S^1 \times S^1 \times \mathbb{R}^\infty$.
4. *For $q > 1$ and $q^2 \equiv \pm 1 \pmod m$,* $\mathrm{Diff}(L(m,q)) \approx P_4 \times S^1 \times S^1 \times \mathbb{R}^\infty$.

We remark that the homeomorphism classification is quite different from the isomorphism classification. In fact, for *any* smooth manifold, the *isomorphism type* of $\mathrm{Diff}(M)$ determines M. That is, an abstract isomorphism between the diffeomorphism groups of two differentiable manifolds must be induced by a diffeomorphism between the manifolds [3, 13, 66].

The Smale Conjecture has some other applications, beyond the problem of understanding $\mathrm{Diff}(M)$. Ivanov's results were used in [43] to construct examples of homeomorphisms of reducible three-manifolds that are homotopic but not isotopic. Our results show that the construction applies to a larger class of three-manifolds. In [55], Theorem 1.2 was applied to the classification problem for three-manifolds which have metrics of positive Ricci curvature and universal cover S^3.

The Smale Conjecture has attracted the interest of physicists studying the theory of quantum gravity. Certain physical configuration spaces can be realized as the quotient space of a principal $\mathrm{Diff}_1(M, x_0)$-bundle with contractible total space, where $\mathrm{Diff}_1(M, x_0)$ denotes the subgroup of $\mathrm{Diff}(M, x_0)$ that induce the identity on the tangent space to M at x_0. (This group is homotopy equivalent to $\mathrm{Diff}(M \# D^3 \ \mathrm{rel}\ \partial D^3)$.) Consequently the loop space of the configuration space is weakly homotopy equivalent to $\mathrm{Diff}_1(M, x_0)$. Physical significance of $\pi_0(\mathrm{Diff}(M))$ for quantum gravity was first pointed out in [14]. See also [1, 18, 30, 65, 73]. The physical significance of some higher homotopy groups of $\mathrm{Diff}(M)$ was examined by Giulini [17].

1.3 The Weak Smale Conjecture

For an arbitrary three-manifold M, we may say that M satisfies the Smale Conjecture if $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a homotopy equivalence for a Riemannian metric on M of maximal symmetry (that is, one for which the Lie group $\mathrm{Isom}(M)$ has maximal dimension and maximal number of components). In general, however, the SC does not extend beyond the elliptic case. The three-torus T^3 provides a simple example: $\mathrm{Diff}(T^3)$ has infinitely many components (since taking the induced outer automorphism on $\pi_1(T^3)$ defines a continuous surjection from $\mathrm{Diff}(T^3)$ onto $\mathrm{GL}(3,\mathbb{Z})$), but $\mathrm{Isom}(T^3)$ is a compact Lie group so has only finitely many components. In this example, however, the inclusion $\mathrm{isom}(M) \to \mathrm{diff}(M)$ of the connected components of the identity map in $\mathrm{Isom}(M)$ and $\mathrm{Diff}(M)$ is a homotopy equivalence. This and other examples motivate us to define the *Weak Smale Conjecture* (WSC) for M to be the assertion that the inclusion $\mathrm{isom}(M) \to \mathrm{diff}(M)$ is a homotopy equivalence. Note that the SC for M is equivalent to the assertion that $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ is a bijection on path components (the "π_0-part" of the conjecture) and the WSC, a fact used in the previous section to reduce the SC for elliptic three-manifolds to the WSC.

The WSC holds in some important cases, such as T^3, and the SC even extends for some classes of nonelliptic three-manifolds. In the remainder of this section, we will survey what is currently known for the nonelliptic closed orientable cases.

For closed Haken three-manifolds, $\mathrm{isom}(M)$ is $(S^1)^k$, where k is the rank of the center of $\pi_1(M)$. Explicitly, k is 3 when $M = T^3$, 1 for Seifert-fibered Haken three-manifolds with orientable quotient orbifold, and 0 otherwise. Work of Hatcher [22] and Ivanov [31, 32] shows that $\mathrm{isom}(M) \to \mathrm{diff}(M)$ is a homotopy equivalence, that is, the WSC (in [22], the results are stated for PL homeomorphisms, but the Smale Conjecture for S^3 extends the results to the smooth category).

Using his "insulator" methodology, 'Gabai [15] proved that the components of $\mathrm{Diff}(M)$ are contractible for all hyperbolic three-manifolds. He deduced the SC for these manifolds, showing in fact that both $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ and $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1(M))$ are homotopy equivalences for finite-volume hyperbolic three-manifolds (for hyperbolic three-manifolds that are also Haken, this was already known by Mostow Rigidity, Waldhausen's Theorem, and the work of Hatcher and Ivanov already discussed). The same statements have now been proven by Soma and the third author [47] when M has an $\mathbb{H}^2 \times \mathbb{R}$ or $\widetilde{\mathrm{SL}}_2(\mathbb{R})$ geometry and its (unique, up to isotopy) Seifert-fibered structure has base orbifold the 2-sphere with three cone points. This is expected to hold for the Nil geometry as well.

As for the non-irreducible case, Hatcher [23, 25] proved that $\mathrm{Diff}(S^2 \times S^1)$ is homotopy equivalent to $\mathrm{O}(2) \times \mathrm{O}(3) \times \Omega\,\mathrm{O}(3)$, where $\Omega\,\mathrm{O}(3)$ is the space of loops in $\mathrm{O}(3)$. In this case, the product metric is maximally symmetric and $\mathrm{Isom}(S^2 \times S^1)$ is diffeomorphic to $\mathrm{O}(2) \times \mathrm{O}(3)$, but the "rotation" components involving nontrivial elements of $\Omega\,\mathrm{O}(3)$ are not isotopic to isometries. The latter is geometrically obvious since no such element can preserve the geodesics of the form $\{x\} \times S^1$.

For the remaining non-irreducible three-manifolds, the WSC is known to fail in most cases. The second and third authors [39] proved that when M has at least three nonsimply connected prime summands, or one $S^2 \times S^1$-summand and one other prime summand with infinite fundamental group, $\pi_1(\mathrm{diff}(M))$ is not finitely generated, so the WSC fails drastically. When M is a connected sum $(S^2 \times S^1)\#P$ with $\pi_1(P)$ finite, the WSC fails at least when $\pi_1(P)$ has order more than 2. For by [39], $\pi_1(\mathrm{diff}(M))$ has a free abelian summand of rank $n-1$, where n is the order of $\pi_1(P)$. On the other hand, $\pi_1(\mathrm{Isom}(M))$ has rank at most 1. This can be seen using the fibration $\mathrm{Isom}(M) \to M$ with fiber $\mathrm{Isom}(M, x_0)$ the isometries preserving a basepoint of M. In the associated exact sequence, $\pi_1(\mathrm{Isom}(M)) \to \pi_1(M)$ is the trivial homomorphism, since the trace of any isotopy from the identity to the identity is a central element of the fundamental group, and $\pi_1(M)$ is a nontrivial free product so is centerless. Therefore $\pi_1(\mathrm{Isom}(M, x_0)) \to \pi_1(\mathrm{Isom}(M))$ is surjective. Now $\mathrm{Isom}(M, x_0)$ is a Lie subgroup of the isometries of the unit tangent 2-sphere of M at x_0, and so the connected component of the identity is a connected subgroup of SO(3) and can only be either trivial, S^1, or SO(3) itself (actually, the latter case cannot occur, since the action of $\mathrm{Isom}(M, x_0)$ on M lifts to an action with fixed point on the Freudenthal endpoint compactification of the universal cover of M, which is S^3. The fixed point set of this action contains the Cantor set of endpoints, so has dimension at least 1).

1.4 Perelman's Methods

It is natural to ask whether the Smale Conjecture can be proven using the methodology that G. Perelman developed to prove the Geometrization Conjecture. The Smale Conjecture would follow if there were a flow retracting the space $\mathscr{R}$ of all Riemannian metrics on an elliptic three-manifold M to the subspace $\mathscr{R}_c$ of metrics of constant positive curvature. Here is why this is so. First, note that by rescaling, $\mathscr{R}_c$ deformation retracts to the subspace $\mathscr{R}_1$ of metrics of constant curvature 1. Now, $\mathrm{Diff}(M)$ acts by pullback on $\mathscr{R}_1$; this action is transitive (given two constant curvature metrics on M, the developing map gives a diffeomorphism which is an isometry between the lifted metrics on the universal cover, and since the action of $\pi_1(M)$ is known to be unique up to conjugation by an isometry, this diffeomorphism can be composed with some isometry to make it equivariant) and the stabilizer of each point is a subgroup conjugate to $\mathrm{Isom}(M)$, so $\mathscr{R}_1$ may be identified with the coset space $\mathrm{Isom}(M)\backslash \mathrm{Diff}(M)$. On the other hand, $\mathscr{R}$ is contractible (M is parallelizable and one can use a Gram–Schmidt orthonormalization process). So the existence of a flow retracting $\mathscr{R}$ to $\mathscr{R}_c$ would imply that $\mathrm{Isom}(M)\backslash \mathrm{Diff}(M)$ is contractible, which is equivalent to the Smale Conjecture. Finding a flow that retracts $\mathscr{R}$ to $\mathscr{R}_c$ is, of course, the rough idea of the Hamilton–Perelman program. At the present time, however, we do not see any way to carry this out, due to the formation of singularities and the requisite surgery of necks, and we are unaware of any progress in this direction.

http://www.springer.com/978-3-642-31563-3

Diffeomorphisms of Elliptic 3-Manifolds
Hong, S.; Kalliongis, J.; McCullough, D.; Rubinstein, J.H.
2012, X, 155 p. 22 illus., Softcover
ISBN: 978-3-642-31563-3