Chapter 2
Approximating Casimir–Polder Potentials

As seen in the previous chapter, dispersion forces can be expressed in terms of the classical Green's tensor for the electromagnetic field and the polarisabilities and magnetisabilities of the atoms. In order to study the position-dependence of a dispersion force for a particular arrangement of bodies, one needs to calculate the respective Green's tensor by solving the inhomogeneous Helmholtz equation (2.149). For many arrangements displaying a high degree of symmetry, e.g., free space, planar, spherical, or cylindrical multilayer systems, the Green's tensor is available in closed form [1]. Exploiting this fact, one can find exact and explicit expressions for, e.g., the Casimir force between two plates (Sect. 3.3 of Vol. I); the CP potential of an atom in various planar multilayer systems (Sect. 4.6 of Vol. I) or next to a sphere (Sect. 4.7 of Vol. I); and the vdW potential of two atoms in free space (Sect. 5.4 of Vol. I), in front of a plate (Sect. 5.5.1 of Vol. I) or next to a sphere (Sect. 5.5.2 of Vol. I). A brief summary of most of these results can be found in Table 3.1 of Sect. 3.1 in this volume.

For configurations displaying less symmetry, approximative methods are required. In this chapter, we consider arrangements which deviate only slightly from a highly symmetrical one. We begin by showing how the Green's tensor can be approximated in this case. We use the approximate Green's tensor to express the CP potential in terms of multiple volume integrals or as a sum over bodies. These two alternative forms are illustrated by considering the CP potential of an atom interacting with a weakly dielectric ring and an inhomogeneous half space. In addition, we discuss the convergence of the Born expansion by studying an atom next to a metal plate or sphere.

2.1 Born Expansions of the Green's Tensor

A powerful tool for obtaining approximate solutions to the Helmholtz equation is the Dyson equation. As shown in the following, it can be used to obtain the Born expansion of the Green's tensor as a systematic power-series expansion. We begin

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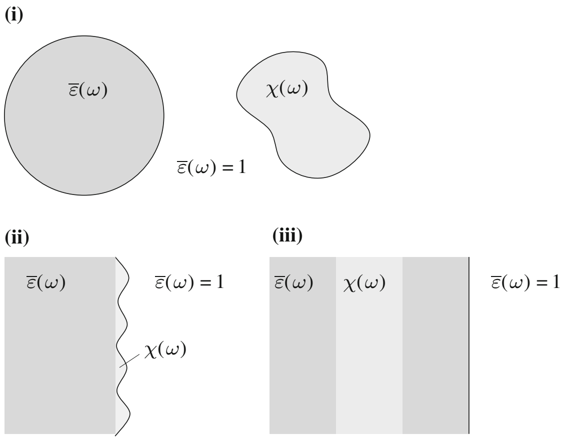

Fig. 2.1 Examples for permittivity decompositions: (**i**) weakly dielectric body next to a sphere; (**ii**) surface roughness of a plate; (**iii**) inhomogeneous half space

with the case of purely electric bodies and then proceed to the purely magnetic and fully magnetoelectric cases.

2.1.1 Electric Bodies

Let us consider an arrangement of purely electric bodies for which the permittivity can be decomposed as

$$\varepsilon(\boldsymbol{r},\omega)=\overline{\varepsilon}(\boldsymbol{r},\omega)+\chi(\boldsymbol{r},\omega)\ . \tag{2.1}$$

Here, $\overline{\varepsilon}(\boldsymbol{r},\omega)$ describes some background bodies with the corresponding Green's tensor being the known solution to

$$\left[\nabla\times\nabla\times-\frac{\omega^2}{c^2}\,\overline{\varepsilon}(\boldsymbol{r},\omega)\right]\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega)=\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}') \tag{2.2}$$

and $\chi(\boldsymbol{r},\omega)$ describes small corrections to this background. The decomposition (2.1) applies to a large variety of cases. As illustrated in Fig. 2.1, it can be used to study a weakly dielectric body of unusual shape in the possible presence of highly symmetric background bodies (i), surface roughness (ii) or inhomogeneities of a body's permittivity (iii).

Using the decomposition of the permittivity, the differential equation (1.14) for the full Green's tensor can be written as

$$\left[\nabla\times\nabla\times - \frac{\omega^2}{c^2}\,\bar{\varepsilon}(\boldsymbol{r},\omega)\right]\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = \delta(\boldsymbol{r}-\boldsymbol{r}') + \frac{\omega^2}{c^2}\,\chi(\boldsymbol{r},\omega)\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\;. \tag{2.3}$$

Its solution can be written in the form of a Dyson equation [2]

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\;, \tag{2.4}$$

as can easily be verified by direct substitution upon exploiting the fact that $\overline{\boldsymbol{G}}$ is a solution to (2.3).

By repeated use of the Dyson equation, one can obtain an expansion of $\boldsymbol{G}$ in powers of χ, which is known as the Born expansion. We start the series by using the zero-order approximation $\boldsymbol{G} = \overline{\boldsymbol{G}}$ in the Dyson equation to obtain the solution to linear order in χ

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}',\omega)\;. \tag{2.5}$$

Substituting this solution back into the Dyson equation, we obtain a better approximation which is correct to quadratic order χ,

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = {}&\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}',\omega)\\ &+\frac{\omega^4}{c^4}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\int \mathrm{d}^3s'\,\chi(\boldsymbol{s}',\omega)\\ &\times\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{s}',\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s}',\boldsymbol{r}',\omega)\;.\end{aligned} \tag{2.6}$$

Iterating in this way, the full Born expansion of the Green's tensor is found to be [3]

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \sum_{K=1}^{\infty}\Delta_K\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) \tag{2.7}$$

with

$$\begin{aligned}\Delta_K\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = {}&\frac{\omega^{2K}}{c^{2K}}\int \mathrm{d}^3s_1\,\chi(\boldsymbol{s}_1,\omega)\cdots\int \mathrm{d}^3s_K\,\chi(\boldsymbol{s}_K,\omega)\\ &\times\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s}_1,\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega)\cdots\overline{\boldsymbol{G}}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\end{aligned} \tag{2.8}$$

denoting corrections of order K in χ.

An alternative expansion can be obtained by isolating the singular part of the Green's tensor [4]: According to (A.18), the Green's tensor in an infinite bulk medium can be written as

$$\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) = -\frac{c^2}{3\omega^2\varepsilon(\boldsymbol{r},\omega)}\,\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}') + \boldsymbol{H}(\boldsymbol{r},\boldsymbol{r}',\omega) \tag{2.9}$$

where $\boldsymbol{H}$ is free of delta-function singularities. This decomposition remains true in the general case of an arbitrary arrangement of bodies. Applying it to $\overline{\boldsymbol{G}}$ and substituting it into the Dyson equation, we find

$$\begin{aligned}&\frac{\overline{\varepsilon}(\boldsymbol{r},\omega)+\frac{1}{3}\,\chi(\boldsymbol{r},\omega)}{\overline{\varepsilon}(\boldsymbol{r},\omega)}\,\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\\ &\qquad = \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\,.\end{aligned} \tag{2.10}$$

Introducing the auxiliary tensor

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) = \frac{\overline{\varepsilon}(\boldsymbol{r},\omega)+\frac{1}{3}\,\chi(\boldsymbol{r},\omega)}{\overline{\varepsilon}(\boldsymbol{r},\omega)}\,\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\,, \tag{2.11}$$

the new Dyson equation takes the more explicit form

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) &= \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega)\\ &\quad + \frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\frac{\chi(\boldsymbol{s},\omega)\overline{\varepsilon}(\boldsymbol{s},\omega)}{\overline{\varepsilon}(\boldsymbol{s},\omega)+\frac{1}{3}\,\chi(\boldsymbol{s},\omega)}\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\boldsymbol{F}(\boldsymbol{s},\boldsymbol{r}',\omega)\,.\end{aligned} \tag{2.12}$$

It can easily be solved by repeated iteration, leading to a Born series

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \sum_{K=1}^{\infty}\Delta_K\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) \tag{2.13}$$

with

$$\begin{aligned}&\Delta_K\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega)\\ &= \frac{\omega^{2K}}{c^{2K}}\int \mathrm{d}^3s_1\,\frac{\chi(\boldsymbol{s}_1,\omega)\overline{\varepsilon}(\boldsymbol{s}_1,\omega)}{\overline{\varepsilon}(\boldsymbol{s}_1,\omega)+\frac{1}{3}\,\chi(\boldsymbol{s}_1,\omega)}\cdots\int \mathrm{d}^3s_K\,\frac{\chi(\boldsymbol{s}_K,\omega)\overline{\varepsilon}(\boldsymbol{s}_K,\omega)}{\overline{\varepsilon}(\boldsymbol{s}_K,\omega)+\frac{1}{3}\,\chi(\boldsymbol{s}_K,\omega)}\\ &\quad\times\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s}_1,\omega)\cdot\overline{\boldsymbol{H}}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega)\cdots\overline{\boldsymbol{G}}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\,.\end{aligned} \tag{2.14}$$

When both field point $\boldsymbol{r}$ and source point $\boldsymbol{r}'$ are situated in free space, we have $\chi(\boldsymbol{r},\omega)=0$ and $\overline{\varepsilon}(\boldsymbol{r},\omega)=1$. The Green's tensor then coincides with the auxiliary tensor, $\boldsymbol{F}(\boldsymbol{r}, \boldsymbol{r}', \omega) = \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega)$, and its alternative Born series is given by (2.7) with

$$\begin{aligned}\Delta_K \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega) &= \frac{\omega^{2K}}{c^{2K}} \int \mathrm{d}^3 s_1 \, \frac{\chi(\boldsymbol{s}_1, \omega)\overline{\varepsilon}(\boldsymbol{s}_1, \omega)}{\overline{\varepsilon}(\boldsymbol{s}_1, \omega) + \frac{1}{3}\chi(\boldsymbol{s}_1, \omega)} \cdots \int \mathrm{d}^3 s_K \, \frac{\chi(\boldsymbol{s}_K, \omega)\overline{\varepsilon}(\boldsymbol{s}_K, \omega)}{\overline{\varepsilon}(\boldsymbol{s}_K, \omega) + \frac{1}{3}\chi(\boldsymbol{s}_K, \omega)} \\ &\quad \times \overline{\boldsymbol{H}}(\boldsymbol{r}, \boldsymbol{s}_1, \omega)\cdot\overline{\boldsymbol{H}}(\boldsymbol{s}_1, \boldsymbol{s}_2, \omega) \cdots \overline{\boldsymbol{H}}(\boldsymbol{s}_K, \boldsymbol{r}', \omega) \end{aligned} \tag{2.15}$$

denoting corrections of order K in $\chi\overline{\varepsilon}/(\overline{\varepsilon} + \frac{1}{3}\chi)$.

The two Born series differ in their expansion parameters. The expansion (2.7) with (2.8) is more intuitive, because the perturbative parameter $\chi = \varepsilon - 1$ on a free-space background is simply the electric susceptibility (1.109). The alternative Born expansion with (2.15) is based on the perturbative parameter $\chi\overline{\varepsilon}/(\overline{\varepsilon} + \frac{1}{3}\chi)$. It is favourable for metals with large χ, ensuring better convergence in this case. In particular, in the perfect conductor limit $\chi \to \infty$, each of the terms in the series (2.8) obviously diverges, whereas the terms (2.15) remain finite with a perturbative parameter $\chi\overline{\varepsilon}/\left(\overline{\varepsilon} + \frac{1}{3}\chi\right) \to 3$.

2.1.2 Magnetic Bodies

In the case of purely magnetic bodies, we decompose the inverse permeability according to

$$\frac{1}{\mu(\boldsymbol{r}, \omega)} = \frac{1}{\overline{\mu}(\boldsymbol{r}, \omega)} - \zeta(\boldsymbol{r}, \omega) \,. \tag{2.16}$$

Note that the correction ζ coincides with the magnetic susceptibility (1.09) in the case of a free-space background, $\zeta = 1 - 1/\mu$. The unperturbed Green's tensor is now given by

$$\left[\nabla \times \frac{1}{\overline{\mu}(\boldsymbol{r}, \omega)} \nabla \times - \frac{\omega^2}{c^2}\right] \overline{\boldsymbol{G}}(\boldsymbol{r}, \boldsymbol{r}', \omega) = \boldsymbol{\delta}(\boldsymbol{r} - \boldsymbol{r}') \tag{2.17}$$

and the Helmholtz equation (1.14) for the full Green's tensor reads

$$\begin{aligned}\left[\nabla \times \frac{1}{\overline{\mu}(\boldsymbol{r}, \omega)} \nabla \times - \frac{\omega^2}{c^2}\right] &\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega) \\ &= \boldsymbol{\delta}(\boldsymbol{r} - \boldsymbol{r}') + \nabla \times \zeta(\boldsymbol{r}, \omega) \nabla \times \boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega) \,. \end{aligned} \tag{2.18}$$

Using the background solution (2.17) and employing partial integration, we can easily verify that the Helmholtz equation is solved by the Dyson equation

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) &= \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) \\ &\quad - \int \mathrm{d}^3 s\, \zeta(\boldsymbol{s},\omega)\Big[\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\Big]\cdot\Big[\nabla_{\boldsymbol{s}}\times\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\Big]. \end{aligned} \tag{2.19}$$

Starting from the unperturbed solution $\boldsymbol{G} = \overline{\boldsymbol{G}}$, the Dyson equation yields the linear Born expansion

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) &= \overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) \\ &\quad - \int \mathrm{d}^3 s\, \zeta(\boldsymbol{s},\omega)\Big[\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\Big]\cdot\Big[\nabla_{\boldsymbol{s}}\times\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}',\omega)\Big]. \end{aligned} \tag{2.20}$$

Iterative use of the Dyson equation leads to a full Born series (2.7) with terms

$$\begin{aligned} \Delta_K \boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) &= (-1)^K \int \mathrm{d}^3 s_1\, \zeta(\boldsymbol{s}_1,\omega)\cdots\int \mathrm{d}^3 s_K\, \zeta(\boldsymbol{s}_K,\omega) \\ &\quad \times\Big[\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s}_1,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_1}\Big]\cdot\Big[\nabla_{\boldsymbol{s}_1}\times\overline{\boldsymbol{G}}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_2}\Big] \\ &\quad \cdots\Big[\nabla_{\boldsymbol{s}_K}\times\overline{\boldsymbol{G}}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\Big]. \end{aligned} \tag{2.21}$$

An alternative expansion can again be obtained by isolating the singular part of the Green's tensor. Applying the duality transformation (A.14) to the separation (2.9), we find

$$\begin{aligned} \nabla\times\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\times\overleftarrow{\nabla}' &= -\frac{\omega^2}{c^2}\,\mu(\boldsymbol{r},\omega)\boldsymbol{G}^{\circledast}(\boldsymbol{r},\boldsymbol{r}',\omega)\mu(\boldsymbol{r}',\omega) - \mu(\boldsymbol{r},\omega)\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}') \\ &= -\tfrac{2}{3}\,\mu(\boldsymbol{r},\omega)\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}') + \nabla\times\boldsymbol{H}(\boldsymbol{r},\boldsymbol{r}',\omega)\times\overleftarrow{\nabla}'. \end{aligned} \tag{2.22}$$

By contrast, the tensors $\nabla\times\boldsymbol{G} = \nabla\times\boldsymbol{H}$ and $\boldsymbol{G}\times\overleftarrow{\nabla}' = \boldsymbol{H}\times\overleftarrow{\nabla}'$ do not exhibit any delta-function part for the assumed independent electric and magnetic medium properties. As an intermediate step, we take the curl of the Dyson equation (2.19). Substituting the above separation into the result, we arrive at

$$\begin{aligned} &\big[1-\tfrac{2}{3}\zeta(\boldsymbol{r},\omega)\overline{\mu}(\boldsymbol{r},\omega)\big]\nabla\times\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) \\ &\quad = \nabla\times\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) - \int \mathrm{d}^3 s\, \zeta(\boldsymbol{s},\omega)\Big[\nabla\times\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\Big]\cdot\Big[\nabla_{\boldsymbol{s}}\times\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\Big]. \end{aligned} \tag{2.23}$$

Introducing an auxiliary tensor

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) = \left[1 - \tfrac{2}{3}\zeta(\boldsymbol{r},\omega)\overline{\mu}(\boldsymbol{r},\omega)\right]\nabla\times\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\ , \tag{2.24}$$

this equation takes the form

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) &= \nabla\times\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) - \int \mathrm{d}^3 s\, \frac{\zeta(\boldsymbol{s},\omega)}{1-\frac{2}{3}\zeta(\boldsymbol{s},\omega)\overline{\mu}(\boldsymbol{s},\omega)} \\ &\quad\times \left[\nabla\times\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\right]\cdot\left[\nabla_{\boldsymbol{s}}\times\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\right].\end{aligned} \tag{2.25}$$

It can easily be solved by successive iterations, leading to

$$\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) = \nabla\times\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega) + \sum_{K=1}^{\infty}\Delta_K\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) \tag{2.26}$$

with

$$\begin{aligned}\Delta_K\boldsymbol{F}(\boldsymbol{r},\boldsymbol{r}',\omega) &= (-1)^K \int \mathrm{d}^3 s_1\, \frac{\zeta(\boldsymbol{s}_1,\omega)}{1-\frac{2}{3}\,\zeta(\boldsymbol{s}_1,\omega)\overline{\mu}(\boldsymbol{s}_1,\omega)} \\ &\quad\cdots\int \mathrm{d}^3 s_K\, \frac{\zeta(\boldsymbol{s}_K,\omega)}{1-\frac{2}{3}\,\zeta(\boldsymbol{s}_K,\omega)\overline{\mu}(\boldsymbol{s}_K,\omega)}\left[\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s}_1,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_1}\right] \\ &\quad\cdot\left[\nabla_{\boldsymbol{s}_1}\times\overline{\boldsymbol{H}}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_2}\right]\cdots\left[\nabla_{\boldsymbol{s}_K}\times\overline{\boldsymbol{H}}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\right].\end{aligned} \tag{2.27}$$

Substituting this result together with (2.24) into the original Dyson equation (2.19), we obtain the alternative Born expansion for $\boldsymbol{G}$. For $\boldsymbol{r}$ and $\boldsymbol{r}'$ in free space, it has the form (2.7) with

$$\begin{aligned}\Delta_K\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega) &= (-1)^K \int \mathrm{d}^3 s_1\, \frac{\zeta(\boldsymbol{s}_1,\omega)}{1-\frac{2}{3}\,\zeta(\boldsymbol{s}_1,\omega)\overline{\mu}(\boldsymbol{s}_1,\omega)} \\ &\quad\cdots\int \mathrm{d}^3 s_K\, \frac{\zeta(\boldsymbol{s}_K,\omega)}{1-\frac{2}{3}\,\zeta(\boldsymbol{s}_K,\omega)\overline{\mu}(\boldsymbol{s}_K,\omega)}\left[\overline{\boldsymbol{H}}(\boldsymbol{r},\boldsymbol{s}_1,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_1}\right] \\ &\quad\cdot\left[\nabla_{\boldsymbol{s}_1}\times\overline{\boldsymbol{H}}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}_2}\right]\cdots\left[\nabla_{\boldsymbol{s}_K}\times\overline{\boldsymbol{H}}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\right].\end{aligned} \tag{2.28}$$

2.1.3 Electromagnetic Bodies

Finally, let us consider the most general case of a magnetoelectric correction

$$\varepsilon(\boldsymbol{r},\omega)=\overline{\varepsilon}(\boldsymbol{r},\omega)+\chi(\boldsymbol{r},\omega)\,,\qquad \frac{1}{\mu(\boldsymbol{r},\omega)}=\frac{1}{\overline{\mu}(\boldsymbol{r},\omega)}-\zeta(\boldsymbol{r},\omega)\,. \tag{2.29}$$

The unperturbed Green's tensor is then given by

$$\left[\nabla\times\frac{1}{\overline{\mu}(\boldsymbol{r},\omega)}\nabla\times-\frac{\omega^2}{c^2}\,\overline{\varepsilon}(\boldsymbol{r},\omega)\right]\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega)=\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}') \tag{2.30}$$

and the differential equation (1.14) reads

$$\begin{aligned}&\left[\nabla\times\frac{1}{\overline{\mu}(\boldsymbol{r},\omega)}\nabla\times-\frac{\omega^2}{c^2}\,\overline{\varepsilon}(\boldsymbol{r},\omega)\right]\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\\&=\boldsymbol{\delta}(\boldsymbol{r}-\boldsymbol{r}')+\frac{\omega^2}{c^2}\,\chi(\boldsymbol{r},\omega)\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)+\nabla\times\zeta(\boldsymbol{r},\omega)\nabla\times\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)\,.\end{aligned} \tag{2.31}$$

The corresponding Dyson equation is simply a combination of those for purely electric (2.4) or purely magnetic bodies (2.19):

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)=\;&\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega)+\frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\\&-\int \mathrm{d}^3s\,\zeta(\boldsymbol{s},\omega)\left[\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\right]\cdot\left[\nabla_{\boldsymbol{s}}\times\boldsymbol{G}(\boldsymbol{s},\boldsymbol{r}',\omega)\right].\end{aligned} \tag{2.32}$$

Within linear order in χ and ζ, the Green's tensor can hence be approximated as

$$\begin{aligned}\boldsymbol{G}(\boldsymbol{r},\boldsymbol{r}',\omega)=\;&\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{r}',\omega)+\frac{\omega^2}{c^2}\int \mathrm{d}^3s\,\chi(\boldsymbol{s},\omega)\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}',\omega)\\&-\int \mathrm{d}^3s\,\zeta(\boldsymbol{s},\omega)\left[\overline{\boldsymbol{G}}(\boldsymbol{r},\boldsymbol{s},\omega)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\right]\cdot\left[\nabla_{\boldsymbol{s}}\times\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}',\omega)\right].\end{aligned} \tag{2.33}$$

Obtaining the full Born series is greatly facilitated by introducing the electric–magnetic tensors $\boldsymbol{G}_{\lambda\lambda'}$ ($\lambda,\lambda'=e,m$) according to (1.172)–(1.175). In terms of these quantities, the Dyson equation takes the simple form

$$\boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) - \sum_{\lambda''=e,m} \int \mathrm{d}^3 s\, \chi_{\lambda''}(\boldsymbol{s},\omega) \overline{\boldsymbol{G}}_{\lambda\lambda''}(\boldsymbol{r},\boldsymbol{s},\omega) \cdot \boldsymbol{G}_{\lambda''\lambda'}(\boldsymbol{s},\boldsymbol{r}',\omega) \tag{2.34}$$

where we have defined $\chi_e = \chi$, $\chi_m = \zeta$. By iterating the Dyson equation, we obtain the Born expansion

$$\boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) + \sum_{K=1}^{\infty} \Delta_K \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) \tag{2.35}$$

with

$$\begin{aligned} &\Delta_K \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) \\ &= (-1)^K \sum_{\lambda_1=e,m} \int \mathrm{d}^3 s_1\, \chi_{\lambda_1}(\boldsymbol{s}_1,\omega) \cdots \sum_{\lambda_K=e,m} \int \mathrm{d}^3 s_K\, \chi_{\lambda_K}(\boldsymbol{s}_K,\omega) \\ &\quad \times \overline{\boldsymbol{G}}_{\lambda\lambda_1}(\boldsymbol{r},\boldsymbol{s}_1,\omega) \cdot \overline{\boldsymbol{G}}_{\lambda_1\lambda_2}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega) \cdots \overline{\boldsymbol{G}}_{\lambda_K\lambda'}(\boldsymbol{s}_K,\boldsymbol{r}',\omega) \end{aligned} \tag{2.36}$$

denoting contributions of order K in χ and ζ.

To obtain the alternative Born series, we substitute the decompositions (2.9) and (2.22) into the Dyson equation (2.34) to find

$$f_\lambda(\boldsymbol{r},\omega) \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) - \sum_{\lambda''=e,m} \int \mathrm{d}^3 s\, \chi_{\lambda''}(\boldsymbol{s},\omega) \overline{\boldsymbol{H}}_{\lambda\lambda''}(\boldsymbol{r},\boldsymbol{s},\omega) \cdot \boldsymbol{G}_{\lambda''\lambda'}(\boldsymbol{s},\boldsymbol{r}',\omega) \tag{2.37}$$

with

$$f_e(\boldsymbol{r},\omega) = \frac{\overline{\varepsilon}(\boldsymbol{r},\omega) + \frac{1}{3}\chi(\boldsymbol{r},\omega)}{\overline{\varepsilon}(\boldsymbol{r},\omega)} \,, \tag{2.38}$$

$$f_m(\boldsymbol{r},\omega) = 1 - \tfrac{2}{3}\zeta(\boldsymbol{r},\omega)\overline{\mu}(\boldsymbol{r},\omega) \,. \tag{2.39}$$

Introducing the auxiliary tensors

$$\boldsymbol{F}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = f_\lambda(\boldsymbol{r},\omega) \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) \,, \tag{2.40}$$

these equations take the form

$$\boldsymbol{F}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = \overline{\boldsymbol{G}}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) - \sum_{\lambda''=e,m} \int \mathrm{d}^3 s\, g_{\lambda''}(\boldsymbol{s},\omega) \overline{\boldsymbol{H}}_{\lambda\lambda''}(\boldsymbol{r},\boldsymbol{s},\omega) \cdot \boldsymbol{F}_{\lambda''\lambda'}(\boldsymbol{s},\boldsymbol{r}',\omega) \tag{2.41}$$

with

$$g_e(\boldsymbol{r},\omega) = \frac{\overline{\varepsilon}(\boldsymbol{r},\omega)\chi(\boldsymbol{r},\omega)}{\overline{\varepsilon}(\boldsymbol{r},\omega) + \frac{1}{3}\chi(\boldsymbol{r},\omega)}\,, \tag{2.42}$$

$$g_m(\boldsymbol{r},\omega) = \frac{\zeta(\boldsymbol{r},\omega)}{1 + \frac{2}{3}\zeta(\boldsymbol{r},\omega)\overline{\mu}(\boldsymbol{r},\omega)}\,. \tag{2.43}$$

We solve these equations for the auxiliary tensors iteratively. When source and field points are situated in free space, this solution coincides with the required solution for the Green's tensor, $\boldsymbol{F}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) = \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega)$. The alternative Born expansion is then given by (2.35) with

$$\begin{aligned} &\Delta_K \boldsymbol{G}_{\lambda\lambda'}(\boldsymbol{r},\boldsymbol{r}',\omega) \\ &= (-1)^K \sum_{\lambda_1=e,m} \int \mathrm{d}^3 s_1\, g_{\lambda_1}(\boldsymbol{s}_1,\omega) \cdots \sum_{\lambda_K=e,m} \int \mathrm{d}^3 s_K\, g_{\lambda_K}(\boldsymbol{s}_K,\omega) \\ &\quad \times \overline{\boldsymbol{H}}_{\lambda\lambda_1}(\boldsymbol{r},\boldsymbol{s}_1,\omega) \cdot \overline{\boldsymbol{H}}_{\lambda_1\lambda_2}(\boldsymbol{s}_1,\boldsymbol{s}_2,\omega) \cdots \overline{\boldsymbol{H}}_{\lambda_K\lambda'}(\boldsymbol{s}_K,\boldsymbol{r}',\omega)\,. \end{aligned} \tag{2.44}$$

2.2 Casimir–Polder Potential via Volume Integrals

The Born expansions for the Green's tensor in their various forms can be used to approximate dispersion forces involving weakly magnetoelectric, rough or inhomogeneous bodies to arbitrary order. We will restrict our attention to the CP potential of a single atom, bearing in mind that approximations of the Casimir force between bodies or body-assisted vdW potentials of two atoms can be developed in a completely analogous way.

2.2.1 Arbitrary Background

We start with an electric ground-state atom in an environment of purely electric bodies. Substituting the linear Born expansion (2.5), we find that to linear order in χ, the CP potential (1.126) or (1.132) can be approximated as [2, 3]

$$U(\boldsymbol{r}_A) = \overline{U}(\boldsymbol{r}_A) + \Delta U(\boldsymbol{r}_A)\,. \tag{2.45}$$

Here,

$$\overline{U}(\boldsymbol{r}_A) = \frac{\hbar\mu_0}{2\pi} \int_0^\infty \mathrm{d}\xi\, \xi^2 \alpha(\mathrm{i}\xi) \mathrm{tr}\, \overline{\boldsymbol{G}}^{(1)}(\boldsymbol{r}_A, \boldsymbol{r}_A, \mathrm{i}\xi) \tag{2.46}$$

is the potential associated with the background bodies and

$$\Delta U(\boldsymbol{r}_A) = -\frac{\hbar\mu_0}{2\pi c^2} \int_0^\infty \mathrm{d}\xi\, \xi^4 \alpha(\mathrm{i}\xi) \int \mathrm{d}^3 s\, \chi(\boldsymbol{s}, \mathrm{i}\xi) \mathrm{tr}\Big[\overline{\boldsymbol{G}}(\boldsymbol{r}_A, \boldsymbol{s}, \mathrm{i}\xi) \cdot \overline{\boldsymbol{G}}(\boldsymbol{s}, \boldsymbol{r}_A, \mathrm{i}\xi)\Big] \tag{2.47}$$

is the first-order correction due to $\chi(\boldsymbol{r}, \omega)$. Using (2.8), the full Born expansion of the CP potential for purely electric bodies reads

$$U(\boldsymbol{r}_A) = \overline{U}(\boldsymbol{r}_A) + \sum_{K=1}^{\infty} \Delta_K U(\boldsymbol{r}_A) \tag{2.48}$$

with

$$\begin{aligned} \Delta_K U(\boldsymbol{r}_A) = \frac{(-1)^K \hbar\mu_0}{2\pi c^{2K}} & \int_0^\infty \mathrm{d}\xi\, \xi^{2K+2} \alpha(\mathrm{i}\xi) \\ & \times \int \mathrm{d}^3 s_1\, \chi(\boldsymbol{s}_1, \mathrm{i}\xi) \cdots \int \mathrm{d}^3 s_K\, \chi(\boldsymbol{s}_K, \mathrm{i}\xi) \\ & \times \mathrm{tr}\Big[\overline{\boldsymbol{G}}(\boldsymbol{r}_A, \boldsymbol{s}_1, \mathrm{i}\xi) \cdot \overline{\boldsymbol{G}}(\boldsymbol{s}_1, \boldsymbol{s}_2, \mathrm{i}\xi) \cdots \overline{\boldsymbol{G}}(\boldsymbol{s}_K, \boldsymbol{r}_A, \mathrm{i}\xi)\Big]. \end{aligned} \tag{2.49}$$

Isolating the singular part of the Green's tensor via (2.9) and noting that the atom is always assumed to be situated in a small free-space region, we can employ the alternative expansion (2.15). It leads to an alternative Born series for the CP potential with terms

$$\begin{aligned} \Delta_K U(\boldsymbol{r}_A) = \frac{(-1)^K \hbar\mu_0}{2\pi c^{2K}} & \int_0^\infty \mathrm{d}\xi\, \xi^{2K+2} \alpha(\mathrm{i}\xi) \\ & \times \int \mathrm{d}^3 s_1\, \frac{\chi(\boldsymbol{s}_1, \mathrm{i}\xi)\overline{\varepsilon}(\boldsymbol{s}_1, \mathrm{i}\xi)}{\overline{\varepsilon}(\boldsymbol{s}_1, \mathrm{i}\xi) + \frac{1}{3}\chi(\boldsymbol{s}_1, \mathrm{i}\xi)} \cdots \int \mathrm{d}^3 s_K\, \frac{\chi(\boldsymbol{s}_K, \omega)\overline{\varepsilon}(\boldsymbol{s}_K, \mathrm{i}\xi)}{\overline{\varepsilon}(\boldsymbol{s}_K, \mathrm{i}\xi) + \frac{1}{3}\chi(\boldsymbol{s}_K, \mathrm{i}\xi)} \\ & \times \mathrm{tr}\Big[\overline{\boldsymbol{H}}(\boldsymbol{r}_A, \boldsymbol{s}_1, \mathrm{i}\xi) \cdot \overline{\boldsymbol{H}}(\boldsymbol{s}_1, \boldsymbol{s}_2, \mathrm{i}\xi) \cdots \overline{\boldsymbol{H}}(\boldsymbol{s}_K, \boldsymbol{r}_A, \mathrm{i}\xi)\Big] \end{aligned} \tag{2.50}$$

which is favourable for metals.

The linear correction for magnetic bodies can be found by substituting (2.20) for the Green's tensor into the CP potential (1.126):

$$\Delta U(\boldsymbol{r}_A) = -\frac{\hbar\mu_0}{2\pi}\int_0^\infty \mathrm{d}\xi\,\xi^2\alpha(\mathrm{i}\xi)\int \mathrm{d}^3s\,\zeta(\boldsymbol{s},\mathrm{i}\xi)$$
$$\times\,\mathrm{tr}\Big[\overline{\boldsymbol{G}}(\boldsymbol{r}_A,\boldsymbol{s},\mathrm{i}\xi)\times\overleftarrow{\nabla}_{\boldsymbol{s}}\cdot\nabla_{\boldsymbol{s}}\times\overline{\boldsymbol{G}}(\boldsymbol{s},\boldsymbol{r}_A,\mathrm{i}\xi)\Big]\,. \tag{2.51}$$

Note that within linear order, electric and magnetic corrections to the CP potential decouple, so the correction due to weakly magnetoelectric bodies is simply the sum of (2.47) and (2.51).

Finally, the CP potential (1.146) with (1.177) of an electromagnetic atom in the presence of magnetoelectric bodies can be approximated by making use of the Born expansion (2.36). We find

$$U_\lambda(\boldsymbol{r}_A) = \overline{U}_\lambda(\boldsymbol{r}_A) + \sum_{K=1}^{\infty}\Delta_K U_\lambda(\boldsymbol{r}_A) \tag{2.52}$$

with

$$\overline{U}_\lambda(\boldsymbol{r}_A) = \frac{\hbar}{2\pi\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha_\lambda(\mathrm{i}\xi)\mathrm{tr}\,\overline{\boldsymbol{G}}^{(1)}_{\lambda\lambda}(\boldsymbol{r}_A,\boldsymbol{r}_A,\mathrm{i}\xi) \tag{2.53}$$

and

$$\Delta_K U_\lambda(\boldsymbol{r}_A) = \frac{(-1)^K\hbar}{2\pi\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha_\lambda(\mathrm{i}\xi)$$
$$\times\sum_{\lambda_1=e,m}\int \mathrm{d}^3s_1\,\chi_{\lambda_1}(\boldsymbol{s}_1,\mathrm{i}\xi)\cdots\sum_{\lambda_K=e,m}\int \mathrm{d}^3s_K\,\chi_{\lambda_K}(\boldsymbol{s}_K,\mathrm{i}\xi)$$
$$\times\,\mathrm{tr}\Big[\overline{\boldsymbol{G}}_{\lambda\lambda_1}(\boldsymbol{r}_A,\boldsymbol{s}_1,\mathrm{i}\xi)\cdot\overline{\boldsymbol{G}}_{\lambda_1\lambda_2}(\boldsymbol{s}_1,\boldsymbol{s}_2,\mathrm{i}\xi)\cdots\overline{\boldsymbol{G}}_{\lambda_K\lambda}(\boldsymbol{s}_K,\boldsymbol{r}_A,\mathrm{i}\xi)\Big]\,. \tag{2.54}$$

With the atom being situated in a free-space region, we can make use of (2.44) to find the alternative series

$$\Delta_K U_\lambda(\boldsymbol{r}_A) = \frac{(-1)^K\hbar}{2\pi\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha_\lambda(\mathrm{i}\xi)$$
$$\times\sum_{\lambda_1=e,m}\int \mathrm{d}^3s_1\,g_{\lambda_1}(\boldsymbol{s}_1,\mathrm{i}\xi)\cdots\sum_{\lambda_K=e,m}\int \mathrm{d}^3s_K\,g_{\lambda_K}(\boldsymbol{s}_K,\mathrm{i}\xi)$$
$$\times\,\mathrm{tr}\Big[\overline{\boldsymbol{H}}_{\lambda\lambda_1}(\boldsymbol{r}_A,\boldsymbol{s}_1,\mathrm{i}\xi)\cdot\overline{\boldsymbol{H}}_{\lambda_1\lambda_2}(\boldsymbol{s}_1,\boldsymbol{s}_2,\mathrm{i}\xi)\cdots\overline{\boldsymbol{H}}_{\lambda_K\lambda}(\boldsymbol{s}_K,\boldsymbol{r}_A,\mathrm{i}\xi)\Big]\,, \tag{2.55}$$

recall (2.42) and (2.43).

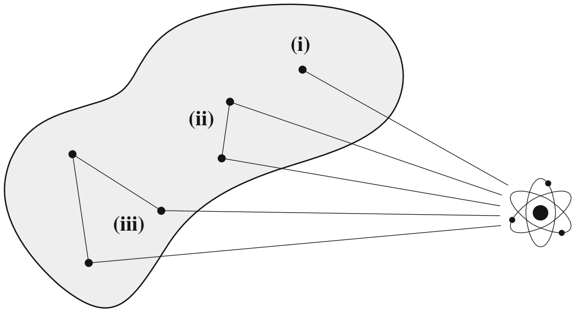

Fig. 2.2 Born expansion: (**i**) First, (**ii**) second and (**iii**) third-order contributions to CP potential of a ground-state atom near a weakly magnetoelectric body

2.2.2 Weakly Magnetodielectric Bodies in Free Space

With the aid of the various Born expansions presented above, the CP potential can be approximated as a series of multiple volume integrals over products of the respective background Green's tensors. Let us now consider the simplest and most important special case in a little more detail: When only weakly magnetodielectric bodies are present, we can choose the background to be free space, $\overline{\varepsilon}(\boldsymbol{r},\omega)\equiv 1$, $\overline{\mu}(\boldsymbol{r},\omega)\equiv 1$, so that all present bodies are characterised by $\chi(\boldsymbol{r},\omega)=\varepsilon(\boldsymbol{r},\omega)-1$ and $\zeta(\boldsymbol{r},\omega)=1-1/\mu(\boldsymbol{r},\omega)$. The background Green's tensor is then identical to the free-space one and the background potential $\overline{U}$ vanishes. The first few terms of the Born expansion are schematically represented in Fig. 2.2.

Using the explicit form (A.21) of the free-space Green's tensor and noting that the atom is well-separated from the bodies so that the delta function does not contribute, the leading-order potential (2.47) for weakly dielectric bodies reads [3]

$$U(\boldsymbol{r}_A)=-\frac{\hbar}{16\pi^3\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\int \mathrm{d}^3s\,\frac{\chi(\boldsymbol{s},\mathrm{i}\xi)}{|\boldsymbol{r}_A-\boldsymbol{s}|^6}\,g(\xi|\boldsymbol{r}_A-\boldsymbol{s}|/c)\,, \tag{2.56}$$

where $g(x)=\mathrm{e}^{-2x}(3+6x+5x^2+2x^3+x^4)$. Within this approximation, the CP force between an electric ground-state atom and purely dielectric bodies of arbitrary shapes is thus given by a single volume integral over attractive central forces, since

$$\boldsymbol{\nabla}\left[\frac{g(\xi r/c)}{r^6}\right]=-\frac{2\boldsymbol{r}}{r^8}\left[\mathrm{e}^{-2x}(9+18x+16x^2+8x^3+3x^4+x^5)\right]_{x=\xi r/c}\,. \tag{2.57}$$

These forces closely resemble the vdW force between two electric atoms as given by (5.97) in Vol. I. In Chap. 3 of this volume, we will discuss in more detail in how CP forces and vdW forces are related in general.

The CP potential can be further simplified by considering the retarded and non-retarded limits of large and small atom–body separations. In the retarded limit $r_-\gg c/\omega_-$ (r_-: minimum atom–body distance, ω_-: minimum of all relevant atomic and medium resonance frequencies), the exponential contained in g effectively limits the ξ-integral to a range $0 \le \xi \lesssim c/(2r_-) \ll \omega_-$, cf. Fig. 3.7(ii) of Vol. I for details. We may hence replace the atom and body response functions by their static values $\alpha(\mathrm{i}\xi) \simeq \alpha(0) \equiv \alpha$ and $\chi(\boldsymbol{r}, \mathrm{i}\xi) \simeq \chi(\boldsymbol{r}, 0) \equiv \chi(\boldsymbol{r})$ and perform the ξ-integral by means of

$$\int_0^\infty \mathrm{d}x\, g(x) = \int_0^\infty \mathrm{d}x\, (3 + 6x + 5x^2 + 2x^3 + x^4)\mathrm{e}^{-2x} = \frac{23}{4} \tag{2.58}$$

to find

$$U(\boldsymbol{r}_A) = -\frac{23\hbar c\alpha}{64\pi^3\varepsilon_0} \int \mathrm{d}^3 s\, \frac{\chi(\boldsymbol{s})}{|\boldsymbol{r}_A - \boldsymbol{s}|^7}\,. \tag{2.59}$$

In the opposite nonretarded limit $r_+ \ll c/\omega_+$ (r_+: maximum atom–body distance, ω_+: maximum of all relevant atomic and medium resonance frequencies), the atom and body response functions restrict the ξ-integral to a range where $\xi|\boldsymbol{r}_A - \boldsymbol{s}|/c \le \xi r_+/c \le \omega_+ r_+/c \ll 1$, cf. Fig. 3.7(iii) of Vol. I. We may hence make the approximation $g(\xi|\boldsymbol{r}_A - \boldsymbol{s}|/c) \simeq g(0) = 3$, leading to

$$U(\boldsymbol{r}_A) = -\frac{3\hbar}{16\pi^3\varepsilon_0} \int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi) \int \mathrm{d}^3 s\, \frac{\chi(\boldsymbol{s}, \mathrm{i}\xi)}{|\boldsymbol{r}_A - \boldsymbol{s}|^6}\,. \tag{2.60}$$

In order to be able to judge the reliability of the approximation, it is useful to also consider the the second-order correction $\Delta_2 U(\boldsymbol{r}_A)$, which according to (2.49) consists of a double volume integral over a product of three Green's tensors. Using the decomposition of the free-space Green's tensor (2.9) and noting that the atom is well separated from all present bodies, only the middle Green's tensor contains a delta-function contribution in addition to its regular part. The second-order contribution can thus be separated into a single-point term, which results from the delta function and a two-point correlation term containing three regular Green's tensors:

$$\Delta_2 U(\boldsymbol{r}_A) = \Delta_2^1 U(\boldsymbol{r}_A) + \Delta_2^2 U(\boldsymbol{r}_A)\,. \tag{2.61}$$

With the free-space Green's tensor (A.21), the single-point term reads [3]

$$\Delta_2^1 U(\boldsymbol{r}_A) = \frac{\hbar}{48\pi^3\varepsilon_0} \int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi) \int \mathrm{d}^3 s\, \frac{\chi^2(\boldsymbol{s}, \mathrm{i}\xi)}{|\boldsymbol{r}_A - \boldsymbol{s}|^6}\, g(\xi|\boldsymbol{r}_A - \boldsymbol{s}|/c)\,. \tag{2.62}$$

It differs from the linear contribution only via the replacement $\chi \mapsto -\frac{1}{3}\chi^2$. The single-point part of the second-order correction thus leads to a reduction of the leading-order linear result. Its retarded and nonretarded limits obviously read

$$\Delta_2^1 U(\boldsymbol{r}_A) = \frac{23\hbar c\alpha}{192\pi^3\varepsilon_0}\int \mathrm{d}^3 s\, \frac{\chi^2(\boldsymbol{s})}{|\boldsymbol{r}_A - \boldsymbol{s}|^7} \tag{2.63}$$

and

$$\Delta_2^1 U(\boldsymbol{r}_A) = \frac{\hbar}{16\pi^3\varepsilon_0}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\int \mathrm{d}^3 s\, \frac{\chi^2(\boldsymbol{s},\mathrm{i}\xi)}{|\boldsymbol{r}_A - \boldsymbol{s}|^6}\,, \tag{2.64}$$

respectively.

The two-point contribution is much more complex. Performing the trace over the product of three regular free-space Green's tensors (A.21), we find [3]

$$\begin{aligned}\Delta_2^2 U(\boldsymbol{r}_A) = {} & \frac{\hbar}{128\pi^4\varepsilon_0}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi) \\ & \times \int \mathrm{d}^3 s_1 \int \mathrm{d}^3 s_2\, \frac{\chi(\boldsymbol{s}_1,\mathrm{i}\xi)\chi(\boldsymbol{s}_2,\mathrm{i}\xi)}{r_1^3 r_2^3 r_3^3}\, g(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\xi)\,. \end{aligned} \tag{2.65}$$

Here, we have defined the function

$$\begin{aligned} g(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,\xi) = {} & \mathrm{e}^{-\xi(r_1+r_2+r_3)/c}\big[3a(\xi r_1/c)a(\xi r_2/c)a(\xi r_3/c) \\ & - b(\xi r_1/c)a(\xi r_2/c)a(\xi r_3/c) - a(\xi r_1/c)b(\xi r_2/c)a(\xi r_3/c) \\ & - a(\xi r_1/c)a(\xi r_2/c)b(\xi r_3/c) + b(\xi r_1/c)b(\xi r_2/c)a(\xi r_3/c)(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)^2 \\ & + a(\xi r_1/c)b(\xi r_2/c)b(\xi r_3/c)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)^2 \\ & + b(\xi r_1/c)a(\xi r_2/c)b(\xi r_3/c)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)^2 \\ & - b(\xi r_1/c)b(\xi r_2/c)b(\xi r_3/c)(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)\big] \end{aligned} \tag{2.66}$$

with $a(x) = 1 + x + x^2$ and $b(x) = 3 + 3x + x^2$ and introduced the abbreviating notation

$$r_1 = r_A - s_1\,, \qquad r_2 = s_1 - s_2\,, \qquad r_3 = s_2 - r_A\,, \tag{2.67}$$

with $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ being the associated unit vectors.

In the retarded limit, we may replace α and χ by their static values, so that

$$\Delta_2^2 U(\boldsymbol{r}_A) = \frac{\hbar\alpha}{128\pi^4\varepsilon_0} \int \mathrm{d}^3 s_1 \int \mathrm{d}^3 s_2 \, \frac{\chi(\boldsymbol{s}_1)\chi(\boldsymbol{s}_2)}{r_1^3 r_2^3 r_3^3} \int_0^\infty \mathrm{d}\xi \, g(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \xi) \,. \tag{2.68}$$

The integral over ξ can then be performed by expanding the products of polynomials in (2.66) and using

$$\int_0^\infty \mathrm{d}\xi \left(\frac{\xi r_1}{c}\right)^i \left(\frac{\xi r_2}{c}\right)^j \left(\frac{\xi r_3}{c}\right)^k \mathrm{e}^{-\xi(r_1+r_2+r_3)/c} = \frac{(i+j+k)!\, r_1^i r_2^j r_3^k c}{(r_1+r_2+r_3)^{i+j+k+1}} \,. \tag{2.69}$$

The result of this rather tedious calculation can be written in a relatively compact form with the aid of the triangle formula

$$A_\triangle \equiv 1 - (\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 - (\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)^2 - (\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)^2 + 2(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1) = 0 \tag{2.70}$$

which is an immediate consequence of (2.67). Adding the expression

$$\begin{aligned} 0 = 6A_\triangle + \frac{6A_\triangle}{(r_1+r_2+r_3)^6} \Big\{ & \big[r_1^5(r_2+r_3) + r_2^5(r_3+r_1) + r_3^5(r_1+r_2)\big] \\ & + 7\big[r_1^4(r_2^2+r_3^2) + r_2^4(r_3^2+r_1^2) + r_3^4(r_1^2+r_2^2)\big] + 12\big(r_1^3 r_2^3 + r_2^3 r_3^3 + r_3^3 r_1^3\big) \\ & + 12 r_1 r_2 r_3 \big(r_1^3 + r_2^3 + r_3^3\big) + 138 r_1^2 r_2^2 r_3^2 \\ & + 52 r_1 r_2 r_3 \big[r_1 r_2 (r_1+r_2) + r_2 r_3 (r_2+r_3) + r_3 r_1 (r_3+r_1)\big]\Big\} \end{aligned} \tag{2.71}$$

to our intermediate result for (2.68), we obtain

$$\begin{aligned} \Delta_2^2 U(\boldsymbol{r}_A) = \frac{\hbar c\alpha}{32\pi^4\varepsilon_0} \int \mathrm{d}^3 s_1 \int \mathrm{d}^3 s_2 \, & \frac{\chi(\boldsymbol{s}_1)\chi(\boldsymbol{s}_2)}{r_1^3 r_2^3 r_3^3 (r_1+r_2+r_3)} \\ & \times \big[f_1(r_1,r_2,r_3) + f_2(r_3,r_1,r_2)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 + f_2(r_1,r_2,r_3)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)^2 \\ & + f_2(r_2,r_3,r_1)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)^2 + f_3(r_1,r_2,r_3)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)\big] \end{aligned} \tag{2.72}$$

with

$$f_1(r_1,r_2,r_3) = 9 - 39\,\frac{\sigma_2}{\sigma_1^2} + 22\,\frac{\sigma_3}{\sigma_1^3} + 54\,\frac{\sigma_2^2}{\sigma_1^4} - 65\,\frac{\sigma_2\sigma_3}{\sigma_1^5} + 20\,\frac{\sigma_3^2}{\sigma_1^6} \,, \tag{2.73}$$

$$f_2(r_1, r_2, r_3) = 3\left[\frac{r_1^2}{\sigma_1^2} + \frac{3r_1^2(r_2+r_3)}{\sigma_1^3} + \frac{4r_2r_3(3r_1^2 - r_2r_3)}{\sigma_1^4} - \frac{20r_1r_2^2r_3^2}{\sigma_1^5}\right], \tag{2.74}$$

$$f_3(r_1, r_2, r_3) = -1 - 39\,\frac{\sigma_2}{\sigma_1^2} + 17\,\frac{\sigma_3}{\sigma_1^3} + 72\,\frac{\sigma_2^2}{\sigma_1^4} - 75\,\frac{\sigma_2\sigma_3}{\sigma_1^5} + 20\,\frac{\sigma_3^2}{\sigma_1^6} \tag{2.75}$$

and $\sigma_i = r_1^i + r_2^i + r_3^i$.

In the opposite, nonretarded limit, we approximate

$$\begin{aligned} &g(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, \xi) \simeq g(\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3, 0) \\ &= 3\big\{-2 + 3\big[(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)^2 + (\boldsymbol{e}_2\cdot\boldsymbol{e}_3)^2 + (\boldsymbol{e}_3\cdot\boldsymbol{e}_1)^2\big] - 9(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)\big\} \\ &= 3\big[1 - 3(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)\big], \end{aligned} \tag{2.76}$$

where the triangle formula (2.70) has again been used. This leads to

$$\begin{aligned} \Delta_2^2 U(\boldsymbol{r}_A) = \frac{3\hbar}{128\pi^4\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\int \mathrm{d}^3 s_1 \int \mathrm{d}^3 s_2\,\chi(\boldsymbol{s}_1, \mathrm{i}\xi)\chi(\boldsymbol{s}_2, \mathrm{i}\xi) \\ \times\,\frac{1 - 3(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)}{r_1^3 r_2^3 r_3^3}\,. \end{aligned} \tag{2.77}$$

The integrand of the two-point contribution $\Delta_2^2 U$ can be positive or negative, depending on the relative positions of the atom and the two integration points inside the body. The magnitude of the second-order correction resulting from the double integral therefore sensitively depends on the shape of the body.

Let us next consider the leading, linear CP potential (2.51) due to weakly magnetic bodies. Calculating the left and right curls of the free-space Green's tensor (A.21),

$$\nabla\times\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega) = -\frac{\mathrm{e}^{\mathrm{i}\omega\rho/c}}{4\pi\rho^2}\left(1 - \frac{\mathrm{i}\omega\rho}{c}\right)\boldsymbol{e}_\rho\times\boldsymbol{I}\,, \tag{2.78}$$

$$\boldsymbol{G}(\boldsymbol{r}, \boldsymbol{r}', \omega)\times\overleftarrow{\nabla}' = \frac{\mathrm{e}^{\mathrm{i}\omega\rho/c}}{4\pi\rho^2}\left(1 - \frac{\mathrm{i}\omega\rho}{c}\right)\boldsymbol{I}\times\boldsymbol{e}_\rho\,, \tag{2.79}$$

and evaluating the trace via $\mathrm{tr}[\boldsymbol{e}\times\boldsymbol{I}\times\boldsymbol{e}] = -2$, we find [3]

$$\begin{aligned} U(\boldsymbol{r}_A) &= \Delta U(\boldsymbol{r}_A) \\ &= \frac{\hbar\mu_0}{16\pi^3}\int_0^\infty \mathrm{d}\xi\,\xi^2\alpha(\mathrm{i}\xi)\int \mathrm{d}^3 s\,\frac{\zeta(\boldsymbol{s}, \mathrm{i}\xi)}{|\boldsymbol{r}_A - \boldsymbol{s}|^4}\,h(\xi|\boldsymbol{r}_A - \boldsymbol{s}|/c) \end{aligned} \tag{2.80}$$

with $h(x) = \mathrm{e}^{-2x}(1 + 2x + x^2)$. Forming the gradient

$$\nabla\left[\frac{h(\xi r/c)}{r^4}\right] = -\frac{2\boldsymbol{r}}{r^6}\left[e^{-2x}(2 + 4x + 3x^2 + x^3)\right]_{x=\xi r/c}, \tag{2.81}$$

we note that the CP force on an electric ground-state atom near weakly magnetic bodies is a volume integral over repulsive central forces. The integrand in (2.80) is very similar to the respective vdW force between an electric and a paramagnetic atom, cf. (5.124) in Vol I. In the retarded limit, we replace α and ζ by their static values and carry out the ξ-integral by means of

$$\int\limits_0^\infty \mathrm{d}x\, x^2 h(x) = \int\limits_0^\infty \mathrm{d}x\, x^2(1 + 2x + x^2)\mathrm{e}^{-2x} = \frac{7}{4} \tag{2.82}$$

to find

$$U(\boldsymbol{r}_A) = \frac{7\hbar c\alpha}{64\pi^3\varepsilon_0}\int \mathrm{d}^3s\,\frac{\zeta(\boldsymbol{s})}{|\boldsymbol{r}_A - \boldsymbol{s}|^7}\,. \tag{2.83}$$

In the nonretarded limit, the approximation $h(x) \simeq h(0) = 1$ leads to

$$U(\boldsymbol{r}_A) = \frac{\hbar\mu_0}{16\pi^3}\int\limits_0^\infty \mathrm{d}\xi\,\xi^2\alpha(\mathrm{i}\xi)\int \mathrm{d}^3s\,\frac{\zeta(\boldsymbol{s},\mathrm{i}\xi)}{|\boldsymbol{r}_A - \boldsymbol{s}|^4}\,. \tag{2.84}$$

2.2.3 *Atom Next to a Ring*

Let us apply the general results of the previous section to a specific body. We consider the CP potential of an atom placed on the symmetry axis of a thin homogeneous ring of radius R, circular cross section πa^2 ($a \ll R$) and volume $V = 2\pi^2 Ra^2$, the atom being separated from the centre of the ring by a distance z_A (Fig. 2.3). For a thin ring, we have $|\boldsymbol{r}_A - \boldsymbol{s}| \simeq \sqrt{z_A^2 + R^2} = \rho_A$, so the volume integral in (2.56) becomes trivial, resulting in the attractive first-order CP potential

$$\Delta_1 U(\rho_A) = -\frac{\hbar V}{16\pi^3\varepsilon_0\rho_A^6}\int\limits_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)g(\xi\rho_A/c) \tag{2.85}$$

of a weakly dielectric ring of permittivity $\chi(\mathrm{i}\xi)$. Its retarded and nonretarded limits (2.59) and (2.60) are given by

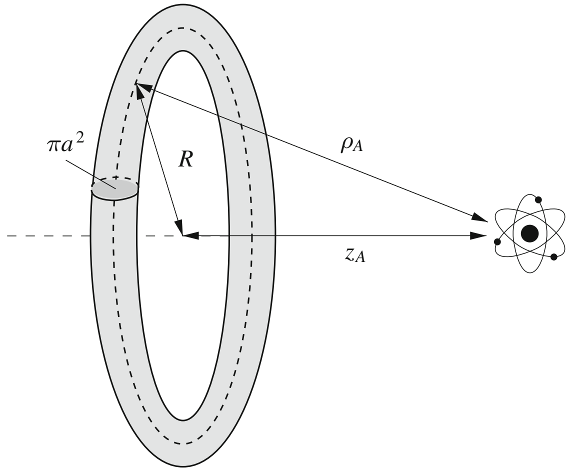

Fig. 2.3 Atom next to a ring

$$\Delta_1 U(\rho_A) = -\frac{23\hbar c V \alpha\chi}{64\pi^3\varepsilon_0\rho_A^7} \tag{2.86}$$

and

$$\Delta_1 U_A(\rho_A) = -\frac{3\hbar V}{16\pi^3\varepsilon_0\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)\ , \tag{2.87}$$

respectively.

To assess the reliability of these first-order results, we also consider the second-order correction (2.61). The value of the single-point term $\Delta_2^1 U$ can be obtained by applying the replacement $\chi \mapsto -\frac{1}{3}\chi^2$ to the linear results, cf. the remark below (2.62). We hence have

$$\Delta_2^1 U(\rho_A) = \frac{\hbar V}{48\pi^3\varepsilon_0\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi) g(\xi\rho_A/c) \tag{2.88}$$

with retarded and nonretarded limits

$$\Delta_2^1 U(\rho_A) = \frac{23\hbar c V \alpha\chi}{192\pi^3\varepsilon_0\rho_A^7}\ , \tag{2.89}$$

$$\Delta_2^1 U_A(\rho_A) = \frac{\hbar V}{16\pi^3\varepsilon_0\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)\ . \tag{2.90}$$

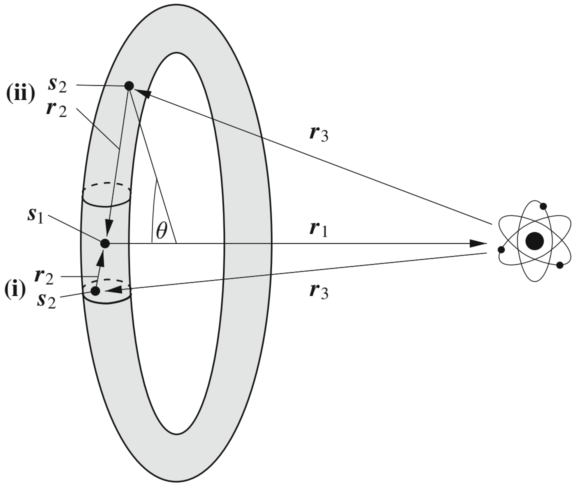

Fig. 2.4 Calculation of the two-point contributions (**i**) and (**ii**) to the CP potential of a ring

The two-point correlation term $\Delta_2^2 U$ is a lot more difficult to evaluate. In particular, it contains an apparent singularity at $\boldsymbol{s}_1 = \boldsymbol{s}_2$ that has to be treated with care. Starting with the retarded limit (2.72), we replace the variable $\boldsymbol{s}_1$ by its average across the cross section of the ring ($|\boldsymbol{r}_A - \boldsymbol{s}_1| \simeq \rho_A$) and carry out the $\boldsymbol{s}_1$-integral. In order to perform the $\boldsymbol{s}_2$-integral, we split the integration volume into two regions (i) and (ii) as illustrated in Fig. 2.4: Region (i) is an approximately cylindrical volume of cross section πa^2 and length $2l$ centred around $\boldsymbol{s}_1$; it contains the apparent singularity. In this region, the separation vector $\boldsymbol{r}_2 = \boldsymbol{s}_1 - \boldsymbol{s}_2$ may be parametrised by local cylindrical coordinates (ρ, ϕ, z). Region (ii) is the remaining open ring where $\boldsymbol{r}_2$ is adequately described by a separation angle θ. In the limit of a thin ring ($a \ll R$), we can choose the length of the cylindrical region such that $a \ll l \ll R$. With this separation of the $\boldsymbol{s}_2$-integral, the two-point term takes the form

$$
\begin{aligned}
\Delta_2^2 U(\boldsymbol{r}_A) &\equiv \Delta_2^{2(\mathrm{i})} U(\boldsymbol{r}_A) + \Delta_2^{2(\mathrm{ii})} U(\boldsymbol{r}_A) \\
&= \frac{\hbar c V \alpha \chi^2}{32\pi^4 \varepsilon_0} \left\{ \int_{-l}^{l} \mathrm{d}z \int_0^a \mathrm{d}\rho\, \rho \int_0^{2\pi} \mathrm{d}\phi + \pi a^2 \int_{l/R}^{2\pi - l/R} R\,\mathrm{d}\theta \right\} \\
&\quad \times \frac{1}{r_1^3 r_2^3 r_3^3 (r_1 + r_2 + r_3)} \big[f_1(r_1, r_2, r_3) + f_2(r_3, r_1, r_2)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)^2 \\
&\quad + f_2(r_1, r_2, r_3)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)^2 + f_2(r_2, r_3, r_1)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1)^2 \\
&\quad + f_3(r_1, r_2, r_3)(\boldsymbol{e}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{e}_2 \cdot \boldsymbol{e}_3)(\boldsymbol{e}_3 \cdot \boldsymbol{e}_1) \big] ,
\end{aligned}
\tag{2.91}
$$

For the integral over the cylindrical region (i), we may use the approximations $r_3 \simeq r_1 = \rho_A$, $r_2 = \sqrt{z^2 + \rho^2} \ll \rho_A$, $-\boldsymbol{e}_2 \cdot \boldsymbol{e}_3 \simeq \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = \rho \cos(\phi)/\sqrt{z^2 + \rho^2}$, $\boldsymbol{e}_3 \cdot \boldsymbol{e}_1 \simeq -1$ for $a \ll R$ (Fig. 2.4); and the functions (2.73)–(2.75) simplify to

$f_1(r_1,r_2,r_3) \simeq \frac{13}{8}$, $f_2(r_3,r_1,r_2) \simeq f_2(r_1,r_2,r_3) \simeq \frac{15}{8}$, $f_2(r_2,r_3,r_1) \simeq -\frac{3}{4}$ and $f_3(r_1,r_2,r_3) \simeq -\frac{51}{8}$. With these estimates, we have

$$\begin{aligned}
\Delta_2^{2(\mathrm{i})}U(\boldsymbol{r}_A) &= \frac{7\hbar c V\alpha\chi^2}{512\pi^4\varepsilon_0\rho_A^7}\int_{-l}^{l}\mathrm{d}z\int_0^a \mathrm{d}\rho\,\rho\int_0^{2\pi}\mathrm{d}\phi\,\frac{\rho^2+z^2-3\rho^2\cos^2\phi}{\sqrt{z^2+\rho^2}^{\,5}}\\
&= \frac{7\hbar c V\alpha\chi^2}{512\pi^3\varepsilon_0\rho_A^7}\int_{-l}^{l}\mathrm{d}z\int_0^a \mathrm{d}\rho\,\rho\,\frac{2z^2-\rho^2}{\sqrt{z^2+\rho^2}^{\,5}} = \frac{7\hbar c V\alpha\chi^2}{512\pi^3\varepsilon_0\rho_A^7}\int_{-l}^{l}\mathrm{d}z\,\frac{a^2}{\sqrt{z^2+a^2}^{\,3}}\\
&= \frac{7\hbar c V\alpha\chi^2}{256\pi^3\varepsilon_0\rho_A^7}\,\frac{(l/a)}{\sqrt{1+(l/a)^2}} \simeq \frac{7\hbar c V\alpha\chi^2}{256\pi^3\varepsilon_0\rho_A^7}
\end{aligned} \tag{2.92}$$

for $a \ll l$. The integral thus remains finite although the integration region (i) contains the point $\boldsymbol{s}_2 \simeq \boldsymbol{s}_1$ where the denominator of the integrand vanishes.

In the open-ring region (ii), the estimates $r_3 \simeq r_1 = \rho_A$, $r_2 \simeq 2R|\sin(\theta/2)|$, $\boldsymbol{e}_1\cdot\boldsymbol{e}_2 \simeq \boldsymbol{e}_2\cdot\boldsymbol{e}_3 \simeq -(R/\rho_A)|\sin(\theta/2)|$, $\boldsymbol{e}_3\cdot\boldsymbol{e}_1 \simeq 2(R^2/\rho_A^2)|\sin(\theta/2)|-1$ hold for $a \ll R$. Due to the denominator $r_2^3 \propto \sin^3(\theta/2)$, the main contribution to the θ-integral in (2.91) comes from regions where $\sin(\theta/2) \ll 1$. We may hence apply a Taylor expansion in powers of $\sin(\theta/2)$ and retain only the terms proportional to $f_1(r_1,r_2,r_3) \simeq \frac{13}{8}$ and $f_2(r_2,r_3,r_1) \simeq -\frac{3}{4}$. With these approximations, (2.91) leads to

$$\begin{aligned}
\Delta_2^{2(\mathrm{ii})}U(\boldsymbol{r}_A) &= \frac{7\hbar c V\alpha\chi^2 a^2}{4096\pi^3\varepsilon_0\rho_A^7 R^2}\int_{l/R}^{2\pi-l/R}\frac{\mathrm{d}\theta}{|\sin^3(\theta/2)|}\\
&\simeq \frac{7\hbar c V\alpha\chi^2 a^2}{4096\pi^3\varepsilon_0\rho_A^7 R^2}\times 2\int_{l/R}^{\infty}\frac{\mathrm{d}\theta}{(\theta/2)^3} = \frac{7\hbar c V\alpha\chi^2}{512\pi^3\varepsilon_0\rho_A^7}\,\frac{a^2}{l^2} \simeq 0
\end{aligned} \tag{2.93}$$

when $l \ll R$ and $a \ll l$. The contribution from the open-ring region hence becomes negligible for a thin ring. In this limit, the two-point term is dominated by the contribution from the cylindrical region and we have

$$\Delta_2^2 U(\rho_A) = \frac{7\hbar c V\alpha\chi^2}{256\pi^3\varepsilon_0\rho_A^7}\,. \tag{2.94}$$

The nonretarded limit (2.77) of the two-point term can be calculated in a completely analogous way by again using the above splitting into two regions:

$$\Delta_2^2 U(\boldsymbol{r}_A) \equiv \Delta_2^{2(\mathrm{i})}U(\boldsymbol{r}_A) + \Delta_2^{2(\mathrm{ii})}U(\boldsymbol{r}_A)$$

$$= \frac{3\hbar V}{128\pi^4\varepsilon_0}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi)$$
$$\times\left\{\int_{-l}^{l} dz\int_0^a d\rho\,\rho\int_0^{2\pi} d\phi+\pi a^2\int_{l/R}^{2\pi-l/R} R d\theta\right\}$$
$$\times\frac{1-3(\boldsymbol{e}_1\cdot\boldsymbol{e}_2)(\boldsymbol{e}_2\cdot\boldsymbol{e}_3)(\boldsymbol{e}_3\cdot\boldsymbol{e}_1)}{r_1^3r_2^3r_3^3}\,. \tag{2.95}$$

Using the approximations above (2.92), the integral over the cylindrical region is found to be

$$\Delta_2^{2(\mathrm{i})}U(\boldsymbol{r}_A)=\frac{3\hbar V}{128\pi^4\varepsilon_0\rho_A^6}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi)\int_{-l}^{l} dz\int_0^a d\rho\,\rho\int_0^{2\pi} d\phi$$
$$\times\frac{\rho^2+z^2-3\rho^2\cos^2\phi}{\sqrt{z^2+\rho^2}^{\,5}}=\frac{3\hbar V}{64\pi^3\varepsilon_0\rho_A^6}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi)\,\frac{(l/a)}{\sqrt{1+(l/a)^2}}$$
$$\simeq\frac{3\hbar V}{64\pi^3\varepsilon_0\rho_A^6}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi) \tag{2.96}$$

for $a \ll l$. With estimates above (2.93), the open-ring integral becomes

$$\Delta_2^{2(\mathrm{ii})}U(\boldsymbol{r}_A)=\frac{3\hbar V a^2}{1024\pi^3\varepsilon_0\rho_A^6R^2}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi)\int_{l/R}^{2\pi-l/R}\frac{d\theta}{|\sin^3(\theta/2)|}$$
$$=\frac{3\hbar V}{128\pi^3\varepsilon_0\rho_A^6}\,\frac{a^2}{l^2}\simeq 0 \tag{2.97}$$

when $a \ll l \ll R$, so that the total two-point term reads

$$\Delta_2^2U(\rho_A)=\frac{3\hbar V}{64\pi^3\varepsilon_0\rho_A^6}\int_0^\infty d\xi\,\alpha(i\xi)\chi^2(i\xi) \tag{2.98}$$

for a thin ring in the nonretarded limit.

Combining our retarded results (2.86), (2.89) and (2.94), the quadratic Born expansion of the retarded CP potential of an atom near a thin ring is given by [3]

$$\begin{aligned} U(\rho_A) &= \Delta_1 U(\rho_A) + \Delta_2^1 U(\rho_A) + \Delta_2^2 U(\rho_A) \\ &= -\frac{23\hbar c V\alpha}{64\pi^3\varepsilon_0\rho_A^7}\left(\chi - \tfrac{1}{3}\chi^2 - \tfrac{7}{92}\chi^2\right) \\ &= -\frac{23\hbar c V\alpha}{64\pi^3\varepsilon_0\rho_A^7}\left(\chi - 0.333\chi^2 - 0.076\chi^2\right). \end{aligned} \tag{2.99}$$

The respective Born expansion in the nonretarded limit follows from (2.87), (2.90) and (2.98):

$$\begin{aligned} U(\rho_A) &= \Delta_1 U(\rho_A) + \Delta_2^1 U(\rho_A) + \Delta_2^2 U(\rho_A) \\ &= -\frac{3\hbar V}{16\pi^3\varepsilon_0\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\left[\chi(\mathrm{i}\xi) - \tfrac{1}{3}\chi^2(\mathrm{i}\xi) - \tfrac{1}{4}\chi^2(\mathrm{i}\xi)\right] \\ &= -\frac{3\hbar V}{16\pi^3\varepsilon_0\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\left[\chi(\mathrm{i}\xi) - 0.333\chi^2(\mathrm{i}\xi) - 0.250\chi^2(\mathrm{i}\xi)\right]. \end{aligned} \tag{2.100}$$

The CP potential is seen to be attractive and proportional to $1/\rho_A^7$ and $1/\rho_A^6$ in the retarded and nonretarded limits, respectively. In both cases, the two quadratic contributions each reduce the potential in comparison to its linear approximation, but they do not change its sign. We further note that the two-point term has its strongest influence in the nonretarded limit where it leads to a very slow convergence of the Born series.

Let us next consider a weakly magnetic ring, restricting our attention to the leading-order contribution (2.80). Using $|\boldsymbol{r}_A - \boldsymbol{s}| \simeq \sqrt{z_A^2 + R^2} = \rho_A$ and carrying out the trivial volume integral, we find

$$U(\rho_A) = \frac{\hbar\mu_0 V}{16\pi^3\rho_A^4}\int_0^\infty \mathrm{d}\xi\, \xi^2\alpha(\mathrm{i}\xi)\zeta(\mathrm{i}\xi)h(\xi\rho_A/c) \tag{2.101}$$

with retarded and nonretarded limits

$$U(\rho_A) = \frac{7\hbar c\alpha\zeta V}{64\pi^3\varepsilon_0\rho_A^7} \tag{2.102}$$

and

$$U(\rho_A) = \frac{\hbar\mu_0 V}{16\pi^3\rho_A^4}\int_0^\infty \mathrm{d}\xi\, \xi^2\alpha(\mathrm{i}\xi)\zeta(\mathrm{i}\xi)\,, \tag{2.103}$$

recall (2.83) and (2.84). In contrast to the potential of a weakly dielectric ring, that of a weakly magnetic ring is repulsive. It is governed by $1/\rho_A^7$ and $1/\rho_A^4$ power laws in the retarded and nonretarded limits, respectively.

So far, we have concentrated on the CP potential of a purely electric atom. A magnetic atom could be treated by starting from the respective general Born series (2.52)–(2.54). A much simpler route is based on duality as discussed in Sect. 1.3. We have seen that CP potentials in free space are invariant under a simultaneous replacement $\alpha \mapsto \beta/c^2$, $\varepsilon \leftrightarrow \mu$. With $\varepsilon = 1+\chi$ and $1/\mu = 1-\zeta$, the latter replacement amounts to

$$\chi \mapsto \frac{1}{1-\zeta} - 1 = \zeta + \zeta^2 + \cdots\,, \qquad \zeta \mapsto 1 - \frac{1}{1+\chi} = \chi - \chi^2 + \cdots \tag{2.104}$$

Applying this duality transformation to (2.99) and (2.100), we find that within quadratic order in ζ, the retarded and nonretarded CP interaction of a magnetic atom with a weakly magnetic ring is given by

$$U_m(\rho_A) = -\frac{23\hbar c\mu_0 V\beta}{64\pi^3\varepsilon_0\rho_A^7}\left(\zeta + \tfrac{163}{276}\,\zeta^2\right) \tag{2.105}$$

and

$$U_m(\rho_A) = -\frac{3\hbar\mu_0 V}{16\pi^3\rho_A^6}\int_0^\infty \mathrm{d}\xi\, \beta(\mathrm{i}\xi)\Big[\zeta(\mathrm{i}\xi) + \tfrac{5}{12}\,\zeta^2(\mathrm{i}\xi)\Big]\,, \tag{2.106}$$

respectively. Similarly, (2.102) and (2.103) imply that the linear CP potential of a magnetic atom interacting with a weakly dielectric ring reads

$$U_m(\rho_A) = \frac{7\hbar c\mu_0\beta\chi V}{64\pi^3\rho_A^7} \tag{2.107}$$

and

$$U_m(\rho_A) = \frac{\hbar\mu_0 V}{16\pi^3 c^2\rho_A^4}\int_0^\infty \mathrm{d}\xi\, \xi^2\beta(\mathrm{i}\xi)\chi(\mathrm{i}\xi) \tag{2.108}$$

in the retarded and nonretarded limits.

2.2.4 Atom Next to a Metal Plate or Sphere

The Born expansion of the CP potential is a rapidly converging series for weakly magnetodielectric bodies, provided that $\chi, \zeta \ll 1$. In this case, the total potential can be well approximated by calculating just the first few terms of the series, as we have done for the ring. For metals, on the contrary, we typically have $\chi \gg 1$. Even the perturbative parameter $\chi/(1+\frac{1}{3}\chi)$ of the alternative Born series (2.50) can be very close to its limiting value 3 which is realised for perfect conductors.

In order to assess the reliability of the Born expansion for metals, let us consider two geometries where the exact results are known, namely the nonretarded CP potentials of an atom next to a perfectly metal plate and sphere. Recall from (4.137) and (4.138) of Vol. I that the nonretarded potential of an atom at distance z_A from an semi-infinite metal half space of permittivity $\varepsilon(\omega)$ is given by

$$U(z_A) = -\frac{\hbar}{16\pi^2\varepsilon_0 z_A^3}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\,\frac{\varepsilon(\mathrm{i}\xi)-1}{\varepsilon(\mathrm{i}\xi)+1}\,. \tag{2.109}$$

We invert ($\chi = \varepsilon - 1$)

$$\frac{\chi}{1+\frac{1}{3}\chi} = \frac{3(\varepsilon-1)}{\varepsilon+2} \tag{2.110}$$

to find

$$\varepsilon = \frac{3+2\chi/\left(1+\frac{1}{3}\chi\right)}{3-\chi/\left(1+\frac{1}{3}\chi\right)}\,. \tag{2.111}$$

With this relation, we can express the exact nonretarded potential in terms of the perturbative parameter $\chi/(1+\frac{1}{3}\chi)$:

$$U(z_A) = -\frac{\hbar}{16\pi^2\varepsilon_0 z_A^3}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\,\frac{3\chi(\mathrm{i}\xi)/\left[1+\frac{1}{3}\chi(\mathrm{i}\xi)\right]}{6+\chi(\mathrm{i}\xi)/\left[1+\frac{1}{3}\chi(\mathrm{i}\xi)\right]}\,. \tag{2.112}$$

The terms (2.50) of the Born series are the unique terms of a Taylor expansion in powers of $\chi/(1+\frac{1}{3}\chi)$. With the exact solution for the total potential being known, we can thus deduce these terms without explicitly performing the spatial integrations that occur in (2.50). Instead, we simply expand (2.112) in powers of $\chi/(1+\frac{1}{3}\chi)$:

$$
\begin{aligned}
U(z_A) &= \varDelta_1 U(z_A) + \varDelta_2 U(z_A) + \varDelta_3 U(z_A) + \varDelta_4 U(z_A) + \cdots \\
&= -\frac{\hbar}{16\pi^2\varepsilon_0 z_A^3}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\left\{\frac{3}{6}\,\frac{\chi(\mathrm{i}\xi)}{1+\frac{1}{3}\chi(\mathrm{i}\xi)} - \frac{3}{6^2}\left[\frac{\chi(\mathrm{i}\xi)}{1+\frac{1}{3}\chi(\mathrm{i}\xi)}\right]^2\right. \\
&\qquad \left. + \frac{3}{6^3}\left[\frac{\chi(\mathrm{i}\xi)}{1+\frac{1}{3}\chi(\mathrm{i}\xi)}\right]^3 - \frac{3}{6^4}\left[\frac{\chi(\mathrm{i}\xi)}{1+\frac{1}{3}\chi(\mathrm{i}\xi)}\right]^4 + \cdots\right\}.
\end{aligned} \tag{2.113}
$$

The convergence of this series becomes slowest in the limit of a perfect conductor $\chi/(1+\frac{1}{3}\chi) \to 3$ where

$$
\begin{aligned}
U(z_A) &= \varDelta_1 U(z_A) + \varDelta_2 U(z_A) + \varDelta_3 U(z_A) + \varDelta_4 U(z_A) + \cdots \\
&= (\tfrac{3}{2} - \tfrac{3}{4} + \tfrac{3}{8} - \tfrac{3}{16} + \tfrac{3}{32} - \tfrac{3}{64} + \tfrac{3}{128} - \tfrac{3}{512} + \cdots)U(z_A) \\
&= (1.5 - 0.75 + 0.38 - 0.19 + 0.09 \\
&\quad - 0.05 + 0.02 - 0.01 + \cdots)U(z_A)\,.
\end{aligned} \tag{2.114}
$$

We see that for a perfectly conducting plate in the nonretarded limit, the Born series converges very slowly. As many as eight terms have to be included in order to reduce the error to about 1 %. Recall that for a weakly dielectric ring we had found that the series converges faster in the retarded limit, so we may expect a better convergence for larger distances for the metal plate as well. Nevertheless, this study shows that the Born expansion leads to a very poor approximation for metals. Retaining only the first one or two terms of the series, we can expect qualitatively correct results at best.

The convergence of the Born series strongly depends on the shape of the bodies under consideration. To show this, let us next consider the nonretarded potential of an atom at distance r_A from the centre of a small metal sphere of radius R, as given by (4.237) together with (4.231) in Vol. I:

$$
U(r_A) = -\frac{3\hbar R^3}{4\pi^2\varepsilon_0 r_A^6}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\,\frac{\varepsilon(\mathrm{i}\xi)-1}{\varepsilon(\mathrm{i}\xi)+2}\,. \tag{2.115}
$$

Using (2.110), we can write it as

$$
U(r_A) = -\frac{\hbar R^3}{4\pi^2\varepsilon_0 r_A^6}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\,\frac{\chi(\mathrm{i}\xi)}{1+\frac{1}{3}\chi(\mathrm{i}\xi)}\,. \tag{2.116}
$$

When using $\chi/(1+\frac{1}{3}\chi)$ as perturbative parameter, the sphere potential hence agrees exactly with its first-order expansion:

$$U(z_A) = \Delta_1 U(z_A)\ . \tag{2.117}$$

This perfect convergence is a result of the small-sphere geometry, where many-atom effects cancel due to the symmetry of the problem [5].

Our two examples mark the two extremes of an unbounded geometry on the one hand and a compact one on the other. For other body shapes, we may therefore expect the convergence speed to lie in between the slow convergence of the plate and the extremely rapid one found for the sphere. The nonretarded CP potential of an atom in front of a perfectly conducting plate with its slow convergence (2.114) may be viewed as a worst-case scenario with respect to distance as well as body material and shape.

2.3 Casimir–Polder Potential via Body Decomposition

As we have seen when studying the ring in the previous section, the evaluation of higher-order terms in the Born expansion can be very difficult for concrete examples. As an alternative, the Born series can be used to establish an expansion of the CP potential based on body decomposition.

2.3.1 Summation Formulae

Beginning with purely dielectric bodies, we assume that the arrangement described by $\chi(\boldsymbol{r},\omega)$ can be decomposed into a set of smaller, homogeneous bodies n with constant permittivities $\chi_n(\omega)$ and volumes V_n, so that

$$\chi(\boldsymbol{r},\omega) = \sum_n \chi_n(\omega) 1_{V_n}(\boldsymbol{r}) \tag{2.118}$$

with

$$1_V(\boldsymbol{r}) = \begin{cases} 1 \text{ if } \boldsymbol{r} \in V, \\ 0 \text{ else} \end{cases} \tag{2.119}$$

being the characteristic function of a volume V. This body decomposition applies to a number of cases as illustrated in Fig. 2.5: For instance, it can used to separate cavities into its mirror components (i) or to decompose planar (ii) or spherical (iii) stratified bodies with inhomogeneous permittivities into homogeneous layers.

Using the body decomposition, the terms (2.49) of the Born expansion take the form

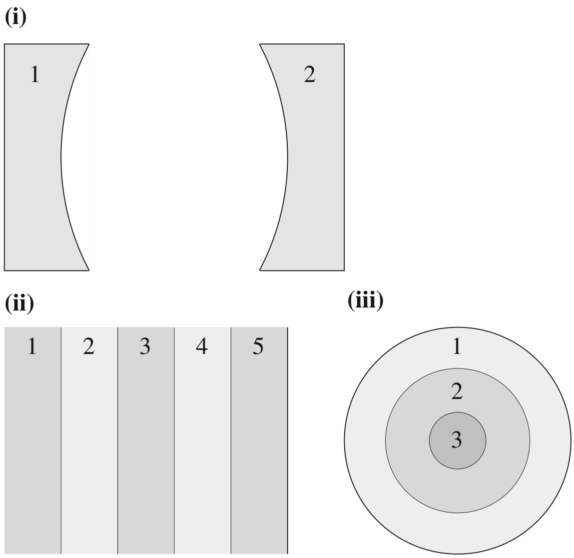

Fig. 2.5 Examples for body decompositions: (**i**) cavity, (**ii**) stratified half space, (**iii**) stratified sphere

$$\Delta_K U(\boldsymbol{r}_A) = \frac{(-1)^K \hbar\mu_0}{2\pi c^{2K}} \int_0^\infty \mathrm{d}\xi\, \xi^{2K+2} \alpha(\mathrm{i}\xi) \\ \times \sum_{n_1} \chi_{n_1}(\mathrm{i}\xi) \int_{V_{n_1}} \mathrm{d}^3 s_1 \cdots \sum_{n_K} \chi_{n_K}(\mathrm{i}\xi) \int_{V_{n_K}} \mathrm{d}^3 s_K \\ \times \mathrm{tr}\Big[\overline{\boldsymbol{G}}(\boldsymbol{r}_A, \boldsymbol{s}_1, \mathrm{i}\xi)\cdot \overline{\boldsymbol{G}}(\boldsymbol{s}_1, \boldsymbol{s}_2, \mathrm{i}\xi) \cdots \overline{\boldsymbol{G}}(\boldsymbol{s}_K, \boldsymbol{r}_A, \mathrm{i}\xi)\Big]. \qquad (2.120)$$

We rearrange the multiple sums over the bodies: First, we identify how many bodies contribute by writing

$$\Delta_K U(\boldsymbol{r}_A) = \sum_{L=1}^{K} \Delta_K^L U(\boldsymbol{r}_A)\,, \qquad (2.121)$$

with $\Delta_K^L U$ being the sum of all L-body contributions to the CP potential. Next, we distinguish which body contributes to which order by specifying

$$\Delta_K^L U(\boldsymbol{r}_A) = \sum_{\substack{n_1 < \cdots < n_L \\ k_1 + \cdots + k_L = K}} \Delta U_{n_1 \cdots n_L}^{k_1 \ldots k_L}(\boldsymbol{r}_A)\,, \qquad (2.122)$$

where the terms $\Delta U_{n_1 \ldots n_L}^{k_1 \ldots k_L}$ contain contributions from the susceptibilities χ_{n_j} of each of the bodies n_j to a specific order k_j. Explicitly, they are given by

$$
\begin{aligned}
\Delta U_{n_1\ldots n_L}^{k_1\ldots k_L}(\boldsymbol{r}_A) = {} & \frac{(-1)^K \hbar\mu_0}{2\pi c^{2K}} \int\limits_0^\infty \mathrm{d}\xi\, \xi^{2K+2} \alpha(\mathrm{i}\xi) \\
& \times \sum_{m_1,\ldots,m_L \in \{n_1,\ldots,n_L\}} \chi_{m_1}(\mathrm{i}\xi) \int\limits_{V_{m_1}} \mathrm{d}^3 s_1 \cdots \chi_{m_K}(\mathrm{i}\xi) \int\limits_{V_{m_K}} \mathrm{d}^3 s_K \\
& \times \mathrm{tr}\Big[\overline{\boldsymbol{G}}(\boldsymbol{r}_A, \boldsymbol{s}_1, \mathrm{i}\xi)\cdot\overline{\boldsymbol{G}}(\boldsymbol{s}_1, \boldsymbol{s}_2, \mathrm{i}\xi)\cdots\overline{\boldsymbol{G}}(\boldsymbol{s}_K, \boldsymbol{r}_A, \mathrm{i}\xi)\Big]
\end{aligned} \tag{2.123}
$$

where the sum runs only over those terms fulfilling the additional constraint that each n_j occurs exactly k_j times.

As an example, note that the total linear contribution in χ reads

$$
\Delta_1 U(\boldsymbol{r}_A) = \Delta_1^1 U(\boldsymbol{r}_A) = \sum_n \Delta U_n^1(\boldsymbol{r}_A)\,. \tag{2.124}
$$

Within this leading order, the CP potential is additive: The total potential for a set of bodies n is simply the sum over the individual potentials ΔU_n^1 associated with these bodies.

Additivity breaks down already in the second order

$$
\Delta_2 U(\boldsymbol{r}_A) = \Delta_2^1 U(\boldsymbol{r}_A) + \Delta_2^2 U(\boldsymbol{r}_A) = \sum_n \Delta U_n^2(\boldsymbol{r}_A) + \sum_{m<n} \Delta U_{mn}^{11}(\boldsymbol{r}_A) \tag{2.125}
$$

due to the presence of the two-body potentials ΔU_{mn}^{11}. In general, L-body potentials start to appear when considering the term $\Delta_L U$ of the Born expansion, provided that enough bodies are present in a given arrangement.

Our decomposition (2.121) together with (2.122) is useful in cases where the exact potentials associated with the individual bodies of the decomposition are known. In this case, the terms $\Delta U_{n_1\ldots n_L}^{k_1\ldots k_L}$ can be uniquely identified by performing a Taylor expansion in the bodies' susceptibilities. The explicit knowledge of (2.123) let alone the tedious evaluation of the integrals appearing therein is then not required. Recall our study of the metal half space in the previous section where we were also able to deduce the terms of the Born expansion without explicitly calculating them, by expanding the known exact solution instead.

The developed body decomposition can also be used for metal bodies. To that end, we apply exactly the same steps as before to the respective Born expansion (2.50). The result is again of the form (2.121) with (2.122), $\Delta U_{n_1\ldots n_L}^{k_1\ldots k_L}$ now denotes contributions from the alternative expansion parameters $\chi_{n_j}/(1+\frac{1}{3}\chi_{n_j})$ of bodies n_j with powers k_j. Finally, the body decomposition can be generalised to electromagnetic atoms in magnetoelectric environments. Starting from (2.118) together with

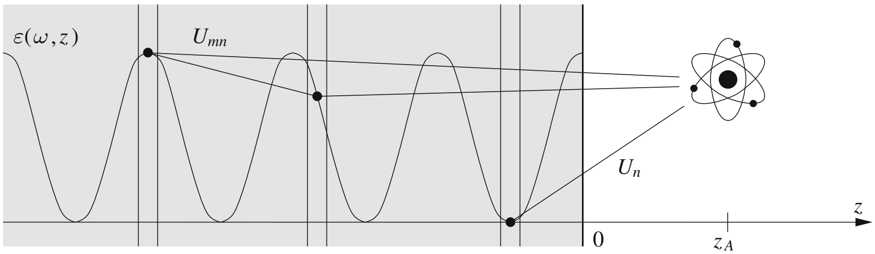

Fig. 2.6 Atom in front of an inhomogeneous half space

$$\zeta(\boldsymbol{r}, \omega) = \sum_n \zeta_n(\omega) 1_{V_n}(\boldsymbol{r}) \tag{2.126}$$

the respective Born expansion (2.54), we again arrive at (2.121) and (2.122) where $\Delta U^{k_1 \ldots k_L}_{n_1 \ldots n_L}$ represents contributions containing χ_{n_j}, ζ_{n_j} with total powers k_j.

2.3.2 *Atom in Front of an Inhomogeneous Half Space*

As an example for the use of body decomposition, let us consider an atom at a distance z_A from an inhomogeneous half space as sketched in Fig. 2.6 whose susceptibility only depends on z,

$$\varepsilon(\boldsymbol{r}, \omega) = 1 + \chi(\omega) p(-z) \ . \tag{2.127}$$

Note that such a body whose material properties only change in one direction is commonly known as a stratified body. The profile function is normalised such that $0 \leq p(-z) \leq 1$ and we have $p(-z) = 0$ for $z < 0$.

We decompose the half space into a number of plates of asymptotically small thickness d such that the susceptibility is approximately constant for each plate. The CP potential associated with such plate is known: As seen from (4.101) in Vol. I, the potential for a plate of permittivity $\varepsilon(\omega)$ and thickness d at separation s from an atom reads

$$U(s) = \frac{\hbar\mu_0}{8\pi^2} \int\limits_0^\infty \mathrm{d}\xi\, \xi^2 \alpha(\mathrm{i}\xi) \int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\, \mathrm{e}^{-2\kappa^\perp s} \left[r_s + \left(1 - 2\,\frac{\kappa^{\perp 2} c^2}{\xi^2} \right) r_p \right] \tag{2.128}$$

where

$$r_s = \frac{\left[\kappa^{\perp 2} - \kappa_1^{\perp 2}\right] \tanh\left(\kappa_1^{\perp} d\right)}{2\kappa^{\perp}\kappa_1^{\perp} + \left[\kappa^{\perp 2} + \kappa_1^{\perp 2}\right] \tanh\left(\kappa_1^{\perp} d\right)} , \tag{2.129}$$

$$r_p = \frac{\left[\varepsilon^2(\mathrm{i}\xi)\kappa^{\perp 2} - \kappa_1^{\perp 2}\right] \tanh\left(\kappa_1^{\perp} d\right)}{2\varepsilon(\mathrm{i}\xi)\kappa^{\perp}\kappa_1^{\perp} + \left[\varepsilon^2(\mathrm{i}\xi)\kappa^{\perp 2} + \kappa_1^{\perp 2}\right] \tanh\left(\kappa_1^{\perp} d\right)} \tag{2.130}$$

with $\kappa_1^{\perp} = \sqrt{[\varepsilon(\mathrm{i}\xi) - 1]\xi^2/c^2 + \kappa^{\perp 2}}$ being the reflection coefficients of the plate for s- and p-polarised waves. In the limit of an asymptotically thin plate, $\sqrt{\varepsilon} d \ll s$, we may replace the reflection coefficients by their leading-order Taylor expansion in $\kappa_1^{\perp} d$,

$$r_s \simeq \frac{[\kappa^{\perp 2} - \kappa_1^{\perp 2}]d}{2\kappa^{\perp}} = -\frac{[\varepsilon(\mathrm{i}\xi) - 1]\kappa^{\perp} d}{2} \frac{\xi^2}{\kappa^{\perp 2} c^2} , \tag{2.131}$$

$$r_p \simeq \frac{[\varepsilon^2(\mathrm{i}\xi)\kappa^{\perp 2} - \kappa_1^{\perp 2}]d}{2\varepsilon(\mathrm{i}\xi)\kappa^{\perp}} = \frac{[\varepsilon^2(\mathrm{i}\xi) - 1]\kappa^{\perp} d}{2\varepsilon(\mathrm{i}\xi)} - \frac{[\varepsilon(\mathrm{i}\xi) - 1]\kappa^{\perp} d}{2\varepsilon(\mathrm{i}\xi)} \frac{\xi^2}{\kappa^{\perp 2} c^2} , \tag{2.132}$$

so that

$$U(s) = -\frac{\hbar d}{16\pi^2\varepsilon_0} \int_0^{\infty} \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)[\varepsilon(\mathrm{i}\xi) - 1] \int_{\xi/c}^{\infty} \mathrm{d}\kappa^{\perp}\, \kappa^{\perp 3} \mathrm{e}^{-2\kappa^{\perp} s}$$
$$\times \left[\frac{2\varepsilon(\mathrm{i}\xi) + 2}{\varepsilon(\mathrm{i}\xi)} - \frac{\varepsilon(\mathrm{i}\xi) + 3}{\varepsilon(\mathrm{i}\xi)} \frac{\xi^2}{\kappa^{\perp 2} c^2} + \frac{\varepsilon(\mathrm{i}\xi) + 1}{\varepsilon(\mathrm{i}\xi)} \frac{\xi^4}{\kappa^{\perp 4} c^4}\right] . \tag{2.133}$$

Labelling the plates by n such that each plate is at position $z = -nd$ and hence at a distance $s = z_A + nd$ from the atom, the potential of plate n reads

$$U_n(z_A) = -\frac{\hbar d}{16\pi^2\varepsilon_0} \int_0^{\infty} \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)[\varepsilon_n(\mathrm{i}\xi) - 1] \int_{\xi/c}^{\infty} \mathrm{d}\kappa\, \kappa^{\perp 3} \mathrm{e}^{-2\kappa(z_A + nd)}$$
$$\times \left[\frac{2\varepsilon_n(\mathrm{i}\xi) + 2}{\varepsilon_n(\mathrm{i}\xi)} - \frac{\varepsilon_n(\mathrm{i}\xi) + 3}{\varepsilon_n(\mathrm{i}\xi)} \frac{\xi^2}{\kappa^{\perp 2} c^2} + \frac{\varepsilon_n(\mathrm{i}\xi) + 1}{\varepsilon_n(\mathrm{i}\xi)} \frac{\xi^4}{\kappa^{\perp 4} c^4}\right] \tag{2.134}$$

where $\varepsilon_n(\omega) = 1 + \chi_n(\omega)$ and $\chi_n(\omega) = \chi(\omega) p(nd)$. Applying a leading-order Taylor expansion in χ_n, we find

$$\Delta U_n^1(z_A) = -\frac{\hbar d}{8\pi^2\varepsilon_0}\int\limits_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi_n(\mathrm{i}\xi)\int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 3}\mathrm{e}^{-2\kappa^\perp(z_A+nd)}$$
$$\times\left(2-2\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right). \tag{2.135}$$

According to (2.124), the leading-order potential $\Delta_1 U(z_A)$ of the inhomogeneous half space can be obtained by summing over these thin-plate potentials. In the limit of asymptotically thin plates, the sum becomes an integral ($d\sum_n = \int_{-\infty}^0 \mathrm{d}z$ with $z=-nd$), so we have

$$\Delta_1 U(z_A) = -\frac{\hbar}{8\pi^2\varepsilon_0}\int\limits_{-\infty}^0 \mathrm{d}z\int\limits_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)p(-z)\int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 3}$$
$$\times\,\mathrm{e}^{-2\kappa^\perp(z_A-z)}\left(2-2\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right). \tag{2.136}$$

After making the substitution $z\mapsto -z$, we have

$$\Delta_1 U(z_A) = -\frac{\hbar}{8\pi^2\varepsilon_0}\int\limits_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)\int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 3}\mathrm{e}^{-2\kappa^\perp z_A}$$
$$\times\left(2-2\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right)\int\limits_0^\infty \mathrm{d}z\,\mathrm{e}^{-2\kappa^\perp z}p(z)\,. \tag{2.137}$$

The quadratic correction $\Delta_2 U$ to this potential consists of a single-plate term $\Delta_2^1 U$ and a two-plate contribution $\Delta_2^2 U$, recall (2.124). The single-plate term can be easily found by performing a second-order Taylor expansion in χ_n of the single-plate potential U_n given above,

$$\Delta U_n^2(z_A) = \frac{\hbar d}{16\pi^2\varepsilon_0}\int\limits_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi_n(\mathrm{i}\xi)\int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 3}\mathrm{e}^{-2\kappa^\perp(z_A+nd)}$$
$$\times\left(2-3\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right), \tag{2.138}$$

followed by an integration over all plates:

$$\Delta_2^1 U(z_A) = \frac{\hbar}{16\pi^2\varepsilon_0} \int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi) \int_{\xi/c}^\infty \mathrm{d}\kappa^\perp\, \kappa^{\perp 3} \mathrm{e}^{-2\kappa^\perp z_A} \times \left(2 - 3\,\frac{\xi^2}{\kappa^{\perp 2}c^2} + \frac{\xi^4}{\kappa^{\perp 4}c^4}\right) \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-2\kappa^\perp z} p(z)\,. \tag{2.139}$$

For the two-plate contribution, we require the CP potential of an atom at a distance s from two plates of permittivities $\varepsilon(\omega)$, $\varepsilon'(\omega)$, thicknesses d, d' and separation l. This potential is again of the form (2.128) where the reflection coefficients have to be replaced by those of the two-plate system, $r_\sigma \mapsto \bar{r}_\sigma$. These coefficients can be obtained by repeated use of the recursion relations (A.39) and (A.40) in App. A.3.2. To leading order in $\kappa_1^\perp d$, $\kappa_1^\perp d'$, one finds

$$\begin{aligned}
\bar{r}_s &\simeq r_s + \left(1 - \frac{\kappa^{\perp 2} + \kappa_1^{\perp 2}}{\kappa^{\perp 2}}\,\kappa^\perp d\right) \mathrm{e}^{-2\kappa l} r_s' \\
&= r_s + \left\{1 - 2\kappa^\perp d - [\varepsilon(\mathrm{i}\xi) - 1]\kappa^\perp d\, \frac{\xi^2}{\kappa^{\perp 2}c^2}\right\} \mathrm{e}^{-2\kappa^\perp l} r_s'\,,
\end{aligned} \tag{2.140}$$

$$\begin{aligned}
\bar{r}_p &\simeq r_p + \left[1 - \frac{\varepsilon^2(\mathrm{i}\xi)\kappa^{\perp 2} + \kappa_1^{\perp 2}}{\varepsilon(\mathrm{i}\xi)\kappa^{\perp 2}}\,\kappa^\perp d\right] \mathrm{e}^{-2\kappa^\perp l} r_p' \\
&= r_p + \left\{1 - 2\kappa^\perp d - \frac{[\varepsilon(\mathrm{i}\xi) - 1]^2 \kappa^\perp d}{\varepsilon(\mathrm{i}\xi)} \right. \\
&\qquad \left. - \frac{[\varepsilon(\mathrm{i}\xi) - 1]\kappa^\perp d}{\varepsilon(\mathrm{i}\xi)}\, \frac{\xi^2}{\kappa^{\perp 2}c^2}\right\} \mathrm{e}^{-2\kappa^\perp l} r_p'\,,
\end{aligned} \tag{2.141}$$

with r_σ, r_σ' being the single-plate reflection coefficients as given by (2.131) and (2.132). The first term in (2.140), (2.141) describes reflection at the front plate, while the second term is associated with transmission through the front plate, propagation to the rear plate, reflection at the rear plate, propagation back to and transmission through the front plate. Substituting the reflection coefficients into (2.128), we obtain single-plate terms which depend on ε or ε' only and a two-plate term that depends on both ε and ε'. The single-plate potential has already been treated. Noting that the plates are at positions $z = -md$ and $z' = -nd$ so that $s = z_A + md$ and $l = nd - md$, the two-plate potential reads

$$U_{mn}(z_A) = \frac{\hbar d d'}{16\pi^2\varepsilon_0} \int\limits_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)[\varepsilon_m(\mathrm{i}\xi)-1][\varepsilon_n(\mathrm{i}\xi)-1] \int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\, \kappa^{\perp 4} \mathrm{e}^{-2\kappa^\perp(z_A+nd)}$$
$$\times \left\{2\,\frac{[\varepsilon_m(\mathrm{i}\xi)-1][\varepsilon_m(\mathrm{i}\xi)+1]}{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)} - \frac{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)+3\varepsilon_m(\mathrm{i}\xi)-3\varepsilon_n(\mathrm{i}\xi)-1}{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)}\,\frac{\xi^2}{\kappa^{\perp 2}c^2}\right.$$
$$\left. - \frac{\varepsilon_m(\mathrm{i}\xi)+\varepsilon_n(\mathrm{i}\xi)+2}{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)}\,\frac{\xi^4}{\kappa^{\perp 4}c^4} + \frac{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)+1}{\varepsilon_m(\mathrm{i}\xi)\varepsilon_n(\mathrm{i}\xi)}\,\frac{\xi^6}{\kappa^{\perp 6}c^6}\right\}. \tag{2.142}$$

The required leading-order, linear term in χ_m, χ_n reads

$$\Delta U^{11}_{mn}(z_A) = \frac{\hbar\mu_0 d d'}{8\pi^2} \int\limits_0^\infty \mathrm{d}\xi\, \xi^2 \alpha(\mathrm{i}\xi)\chi_m(\mathrm{i}\xi)\chi_n(\mathrm{i}\xi) \int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\, \kappa^{\perp 2}$$
$$\times\, \mathrm{e}^{-2\kappa^\perp(z_A+nd)}\left(2 - 2\,\frac{\xi^2}{\kappa^{\perp 2}c^2} + \frac{\xi^4}{\kappa^{\perp 4}c^4}\right). \tag{2.143}$$

Summing over all plates in accordance with (2.125), the total two-plate contribution is given by ($dd'\sum_{m<n} = \int_{-\infty}^0 \mathrm{d}z \int_{-\infty}^z \mathrm{d}z'$ with $z = -md$, $z' = -nd$)

$$\Delta_2^2 U(z_A) = \frac{\hbar\mu_0}{8\pi^2} \int\limits_0^\infty \mathrm{d}\xi\, \xi^2 \alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi) \int\limits_{\xi/c}^\infty \mathrm{d}\kappa^\perp\, \kappa^{\perp 2} \mathrm{e}^{-2\kappa^\perp z_A}$$
$$\times \left(2 - 2\,\frac{\xi^2}{\kappa^{\perp 2}c^2} + \frac{\xi^4}{\kappa^{\perp 4}c^4}\right) \int\limits_0^\infty \mathrm{d}z\, p(z) \int\limits_0^\infty \mathrm{d}z'\, \mathrm{e}^{-2\kappa^\perp z'} p(z') \tag{2.144}$$

where we have made the substitutions $z \mapsto -z$, $z' \mapsto -z'$.

As an example, let us consider a dielectric medium whose permittivity oscillates as a function of z, with the profile function being given by

$$p(z) = \cos^2(kz)\Theta(-z) \tag{2.145}$$

The parameter k determines the period λ of the permittivity oscillations according to $\lambda = \pi/k$.

With this choice of profile function, the z- and z'-integrals in (2.137), (2.139) and (2.144) can be evaluated explicitly,

$$\int_0^\infty \mathrm{d}z\, \mathrm{e}^{-2\kappa^\perp z}\cos^2(kz) = \frac{2\kappa^{\perp 2}+k^2}{4\kappa^\perp(\kappa^{\perp 2}+k^2)}\,, \tag{2.146}$$

$$\int_0^\infty \mathrm{d}z\, \cos^2(kz)\int_z^\infty \mathrm{d}z'\, \mathrm{e}^{-2\kappa^\perp z'}\cos^2(kz') = \frac{2\kappa^{\perp 6}+8\kappa^{\perp 4}k^2+5\kappa^{\perp 2}k^4+2k^6}{8\kappa^{\perp 2}(\kappa^{\perp 2}+k^2)^2(\kappa^{\perp 2}+4k^2)}\,, \tag{2.147}$$

and we find

$$\Delta_1 U(z_A) = -\frac{\hbar}{32\pi^2\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi)\int_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 2}\mathrm{e}^{-2\kappa^\perp z_A}\,\frac{2\kappa^{\perp 2}+k^2}{\kappa^{\perp 2}+k^2}$$
$$\times\left(2-2\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right), \tag{2.148}$$

$$\Delta_2^1 U(z_A) = \frac{\hbar}{64\pi^2\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi)\int_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\kappa^{\perp 2}\mathrm{e}^{-2\kappa^\perp z_A}\,\frac{2\kappa^{\perp 2}+k^2}{\kappa^{\perp 2}+k^2}$$
$$\times\left(2-3\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right), \tag{2.149}$$

$$\Delta_2^2 U(z_A) = \frac{\hbar\mu_0}{64\pi^2}\int_0^\infty \mathrm{d}\xi\,\xi^2\alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi)\int_{\xi/c}^\infty \mathrm{d}\kappa^\perp\,\mathrm{e}^{-2\kappa^\perp z_A}$$
$$\times\frac{2\kappa^{\perp 6}+8\kappa^{\perp 4}k^2+5\kappa^{\perp 2}k^4+2k^6}{(\kappa^{\perp 2}+k^2)^2(\kappa^{\perp 2}+4k^2)}\left(2-2\,\frac{\xi^2}{\kappa^{\perp 2}c^2}+\frac{\xi^4}{\kappa^{\perp 4}c^4}\right). \tag{2.150}$$

The potential simplifies considerably in the retarded and nonretarded limits. In the retarded limit $z_A \gg c/\omega_-$, the exponential $\exp^{-2\kappa^\perp z_A}$ restricts the ξ-integral to a range where $0 \le \xi \lesssim c/(2z_A) \ll \omega_-$, so that we may make the approximations $\alpha(\mathrm{i}\xi) \simeq \alpha$ and $\chi(\mathrm{i}\xi) \simeq \chi$. Introducing the new integration variables $v = \kappa^\perp c/\xi$ and $s = \kappa^\perp z_A = \xi z_A v/c$, we subsequently transform the integrals according to $\int_0^\infty \mathrm{d}\xi \int_{\xi/c}^\infty \mathrm{d}\kappa^\perp = \int_0^\infty \mathrm{d}\xi\,\xi\int_1^\infty \mathrm{d}v/c = (c/z_A^2)\int_0^\infty \mathrm{d}s\,s\int_1^\infty \mathrm{d}v/v^2$. The integrals can then be carried out to give

$$\Delta_1 U(z_A) = -\frac{23\hbar c\alpha\chi}{480\pi^2\varepsilon_0 z_A^4}\int_0^\infty \mathrm{d}s\, s^3 \mathrm{e}^{-2s}\,\frac{2s^2+(kz_A)^2}{s^2+(kz_A)^2}\,, \tag{2.151}$$

$$\Delta_2^1 U(z_A) = \frac{3\hbar c\alpha\chi^2}{160\pi^2\varepsilon_0 z_A^4}\int_0^\infty \mathrm{d}s\, s^3 \mathrm{e}^{-2s}\,\frac{2s^2+(kz_A)^2}{s^2+(kz_A)^2}\,, \tag{2.152}$$

$$\Delta_2^2 U(z_A) = \frac{43\hbar c\alpha\chi^2}{6720\pi^2\varepsilon_0 z_A^4}\int_0^\infty \mathrm{d}s\, s^3 \mathrm{e}^{-2s} \times \frac{2s^6+8s^4(kz_A)^2+5s^2(kz_A)^4+2(kz_A)^6}{[s^2+(kz_A)^2]^2[s^2+4(kz_A)^2]}\,. \tag{2.153}$$

In the nonretarded limit $z_A \ll c/\omega_+$, the atom and medium response functions restrict the ξ-integral to values such that $\xi/\kappa^\perp c \lesssim \xi z_A/c \le \omega_+ z_A/c \ll 1$. We may hence set the lower limit of the $\kappa^\perp$-integral to zero and discard higher-order terms in $\xi/\kappa^\perp c$ in its integrand. After again using $s = \kappa^\perp z_A$, we find

$$\Delta_1 U(z_A) = -\frac{\hbar}{16\pi^2\varepsilon_0 z_A^3}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi(\mathrm{i}\xi) \times \int_0^\infty \mathrm{d}s\, s^2 \mathrm{e}^{-sx}\,\frac{2s^2+(kz_A)^2}{s^2+(kz_A)^2}\,, \tag{2.154}$$

$$\Delta_2^1 U(z_A) = \frac{\hbar}{32\pi^2\varepsilon_0 z_A^3}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi) \times \int_0^\infty \mathrm{d}s\, s^2 \mathrm{e}^{-2s}\,\frac{2s^2+(kz_A)^2}{s^2+(kz_A)^2}\,, \tag{2.155}$$

$$\Delta_2^2 U(z_A) = \frac{\hbar\mu_0}{32\pi^2 z_A}\int_0^\infty \mathrm{d}\xi\, \xi^2\alpha(\mathrm{i}\xi)\chi^2(\mathrm{i}\xi)\int_0^\infty \mathrm{d}s\, \mathrm{e}^{-2s} \times \frac{2s^6+8s^4(kz_A)^2+5s^2(kz_A)^4+2(kz_A)^6}{[s^2+(kz_A)^2]^2[s^2+4(kz_A)^2]}\,. \tag{2.156}$$

It is instructive to consider limits of small- and large-scale oscillations of the half-space susceptibility. When the period of the oscillations is much larger than the atom–surface distance, $\lambda \gg z_A$, i.e., $kz_A \ll 1$, then the s-integrals in (2.151)–(2.153) can be performed easily. Adding the results, we obtain the total retarded CP potential

$$U(z_A) = \frac{C_4}{z_A^4}\,. \tag{2.157}$$

It agrees with that of a homogeneous semi-infinite half space of susceptibility χ, where

$$C_4 = -\frac{\hbar c\alpha}{\pi^2\varepsilon_0}\left(\tfrac{23}{640}\,\chi - \tfrac{9}{640}\,\chi^2 - \tfrac{43}{8960}\,\chi^2\right) \tag{2.158}$$

is simply the second-order approximation in χ to the exact half-space coefficient

$$C_4 = -\frac{3\hbar c\alpha}{64\pi^2\varepsilon_0}\int_1^\infty \mathrm{d}v\,\left[\left(\frac{2}{v^2}-\frac{1}{v^4}\right)\frac{\varepsilon v-\sqrt{\varepsilon\mu-1+v^2}}{\varepsilon v+\sqrt{\varepsilon\mu-1+v^2}} - \frac{1}{v^4}\,\frac{\mu v-\sqrt{\varepsilon\mu-1+v^2}}{\mu v+\sqrt{\varepsilon\mu-1+v^2}}\right], \tag{2.159}$$

cf. (4.133), (4.134) and (4.143) in Vol. I. In the opposite limit of the oscillation period being much smaller than the atom-surface distance $\lambda \ll z_A$, i.e., $kz_A \gg 1$, the integrals in (2.151)–(2.153) can again be performed and we find a total potential (2.157) with coefficient

$$C_4 = -\frac{\hbar c\alpha}{\pi^2\varepsilon_0}\left(\tfrac{23}{1280}\,\chi - \tfrac{9}{1280}\,\chi^2 - \tfrac{43}{35840}\,\chi^2\right). \tag{2.160}$$

We see that the linear contribution in χ is simply given by one half its value for the homogeneous half space. This is in accordance with the simple intuition that the potential for a half space with rapid permittivity oscillations should be determined by the average permittivity. However, the two-plate contribution is reduced to one quarter of its homogeneous-case value for the rapidly oscillating half space, leading to a failure of this simple intuition. Due to many-body correlations, the potential of a half space with rapidly oscillating permittivity is hence slightly more than one half the value for a corresponding homogeneous half space. The value of the potential for intermediate values of z_A/λ can be given as ($k = \pi/\lambda$)

$$U(z_A) = \frac{C_4 f(z_A/\lambda)}{z_A^4} \tag{2.161}$$

where C_4 is the homogeneous-case coefficient (2.158) and the normalised potential $f(z_A/\lambda) = U(z_A)/U(z_A)|_{\lambda\to\infty}$ with

$$f(x) = \frac{6720}{322-169\chi}\left\{\frac{23-9\chi}{360}\int_0^\infty \mathrm{d}s\, s^2 \mathrm{e}^{-2s}\,\frac{2s^2+\pi^2x^2}{s^2+\pi^2x^2} - \frac{43\chi}{5040}\int_0^\infty \mathrm{d}s\,\mathrm{e}^{-2s}\,\frac{2s^6+8s^4\pi^2x^2+5s^2\pi^4x^4+2\pi^6x^6}{[s^2+\pi^2x^2]^2[s^2+4\pi^2x^2]}\right\} \tag{2.162}$$

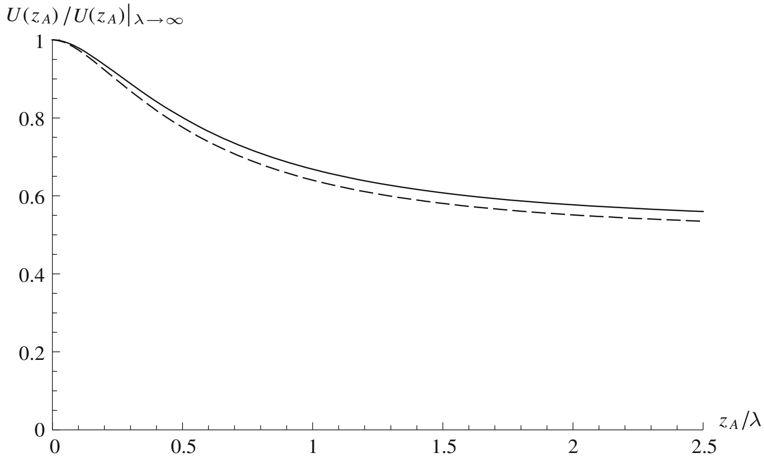

Fig. 2.7 Normalised retarded (*solid line*) and nonretarded (*dashed line*) CP potentials of an atom in front of a half space with spatially oscillating permittivity (where $\chi = 0.5$ for the retarded potential)

depends on the dimensionless parameter z_A/λ. As we will see in Sect. 3.2 below, $f(x)$ is an example of a scaling function. It is depicted in Fig. 2.7 and describes the effect of the permittivity-oscillations. As seen, the CP potential is gradually reduced from its homogeneous-case value as the atom-surface distance increases. As a consequence, it decreases more strongly with distance than $1/z_A^4$ in the transition region $z_A \simeq \lambda$.

Asymptotes for the nonretarded potential can be found in a similar way. Note that the two-plate contribution (2.156) becomes negligible in comparison to (2.154) and (2.155) in the nonretarded limit, because it increases less strongly with decreasing z_A. For large-scale oscillations $\lambda \gg z_A$, we may perform the s-integrals to find

$$U(z_A) = \frac{C_3}{z_A^3} \tag{2.163}$$

where

$$C_3 = -\frac{\hbar}{16\pi^2\varepsilon_0}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\left[\chi(\mathrm{i}\xi) - \tfrac{1}{2}\chi^2(\mathrm{i}\xi)\right] \tag{2.164}$$

is simply the second-order approximation to the coefficient

$$C_3 = -\frac{\hbar}{16\pi^2\varepsilon_0}\int_0^\infty \mathrm{d}\xi\, \alpha(\mathrm{i}\xi)\,\frac{\varepsilon(\mathrm{i}\xi)-1}{\varepsilon(\mathrm{i}\xi)+1} \tag{2.165}$$

for a homogeneous half space. In the opposite limit of small-scale oscillations $\lambda \ll z_A$, the potential is governed by

$$C_3 = -\frac{\hbar}{32\pi^2\varepsilon_0}\int_0^\infty \mathrm{d}\xi\,\alpha(\mathrm{i}\xi)\left[\chi(\mathrm{i}\xi) - \tfrac{1}{2}\chi^2(\mathrm{i}\xi)\right] \tag{2.166}$$

and hence equal to one half the homogeneous-case result. This is a consequence of the fact that the two-plate term does not contribute in the nonretarded limit. The behaviour of the potential for between the two extremes reads

$$U(z_A) = \frac{C_3 f(z_A/\lambda)}{z_A^3}\,. \tag{2.167}$$

The nonretarded coefficient C_3 is given according to (2.164) and we find a normalised potential $f(z_A/\lambda) = U(z_A)/U(z_A)|_{\lambda\to\infty}$ as given by the function

$$f(x) = \frac{4}{3}\int_0^\infty \mathrm{d}s\, s^2 \mathrm{e}^{-2s}\,\frac{2s^2+\pi^2x^2}{s^2+\pi^2x^2}\,. \tag{2.168}$$

It is displayed in Fig. 2.7. Again, we see that the CP potential is reduced from its homogeneous-case value as the distance increases in comparison to the oscillation period. In contrast to the retarded case, the normalised potential is reduced to exactly one half in the limit of small-scale oscillations; this is due to the absence of a two-plate contribution in the nonretarded limit.

References

1. W.C. Chew, *Waves and Fields in Inhomogeneous Media* (IEEE, New York, 1995)
2. S.Y. Buhmann, D.G. Welsch, Prog. Quantum Electron. **31**(2), 51 (2007)
3. S.Y. Buhmann, D.G. Welsch, Appl. Phys. B **82**(2), 189 (2006)
4. R. Golestantian, Phys. Rev. A **80**(1), 012519 (2009)
5. H.Y. Kim, J. Sofo, D. Velegol, M.W. Cole, A.A. Lucas, Langmuir **23**(4), 1735 (2007)

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