Chapter 2
Harmonic Oscillator as an Effective Theory

Abstract The concepts of Effective Theories are illustrated allegorically within the context of one of the most ubiquitous models of oscillating physical phenomena—the harmonic oscillator.

2.1 Basics of the Harmonic Oscillator

The concepts and issues related to effective theories can be illustrated quite nicely by the harmonic oscillator problem. The harmonic oscillator is one of the most ubiquitous mathematical models of physics phenomena. It is present in almost every system with a restoring force, which includes the galaxy, solar system, springs, atoms, molecules, and innumerable other configurations.

The main point I would like to illustrate is that the lowest order effective potential for the harmonic oscillator is an excellent approximation to the motion of a system over a wide range of amplitudes. However, at some point it breaks down when the amplitude is large enough, and then control over the system is lost unless a deeper theory is understood. We shall not go into the construction of deeper theories in this chapter, but rather focus on the domain of applicability of the harmonic oscillator effective theory, and show how small corrections can be anticipated and then measured by precise experiments to start building a more complete picture of the potential governing the system.

To keep the illustration simple, we will restrict ourselves to one-dimensional harmonic motion of a particle subject to the restoring potential $V(x) = kx^2/2$. The Lagrangian of the system is then

$$L = \int dt \left(m\frac{\dot{x}^2}{2} - k\frac{x^2}{2} \right). \tag{2.1}$$

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From the principle of least action the equation of motion gives Newton's second law of motion $F = ma$ the form

$$m\ddot{x} = -kx \Longrightarrow m\ddot{x} + kx = 0. \tag{2.2}$$

Defining $\omega^2 = k/m$, we can rewrite this as

$$\ddot{x} + \omega^2 x = 0 \tag{2.3}$$

which has the solution

$$x(t) = A\sin(\omega t) \tag{2.4}$$

where A is the amplitude, and the boundary condition of $x = 0$ at $t = 0$ is enforced.

Let us review a few basic facts about the harmonic oscillator solution. The period is

$$T_{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}. \tag{2.5}$$

The amplitude A of motion is related to the initial velocity by equating full potential energy at maximum amplitude to the full kinetic energy at maximum velocity:

$$\frac{1}{2}mv_{max}^2 = \frac{1}{2}kA^2 \Longrightarrow A = v_{max}\sqrt{\frac{m}{k}} = \frac{v_{max}}{\omega} = \frac{v_{max}T_{period}}{2\pi}. \tag{2.6}$$

It should also be noted that the period of the harmonic motion is not dependent on the amplitude of the motion. This is clear from Eq. 2.5 where it is shown that the period only depends on the input parameters m and k. The amplitude and maximum velocity conspire with each other such that v_{max}/A is always equal to $\sqrt{k/m}$.

2.2 Ubiquity of the Harmonic Oscillator

The harmonic oscillator problem is ubiquitous in physics, describing small motions of an object attached to a string, molecules vibrating in crystals, electrical circuit response, etc. There is a straightforward reason why there are so many examples that follow simple harmonic behavior. Let us suppose that the equilibrium point (i.e., the minimum of the potential) is about the origin. Then, the potential for motion is a power series of the form

$$V(x) = a_2x^2 + a_3x^3 + a_4x^4 + \cdots. \tag{2.7}$$

We do not write down a constant term or a term linear in x because the first is irrelevant and the second term cannot be present if $x = 0$ is a local minimum. If it is present, one shifts x to cancel it, which is the place of the new extremum.[1] There are an infinite number of potentials that can be written down, with various relative weightings of x^4, x^{12}, etc. The motions of a particle or entity about the equilibrium can be very different depending on the potential.

Nevertheless, the universal quality of harmonic motion is ubiquitous because at values of x below some critical value x_{crit} the potential is always dominated by the x^2 term. For example, in comparing the a_2x^2 term to the a_3x^3 term, the ratio is

$$\frac{a_2x^2}{a_3x^3} = \frac{a_2}{a_3}\frac{1}{x} \Longrightarrow a_2x^2 \text{ term dominates over } a_3x^3 \text{ when } x < x_{crit} = \frac{a_2}{a_3}. \quad (2.8)$$

In other words, small enough amplitudes are always very well described by simple harmonic motion in a x^2 potential.

In the following we will investigate an abstract system that has harmonic oscillation in the "low-energy limit", when the amplitude is small. We shall see that through a combination of precision measurements and venturing into the high-energy unknown we can learn more about the system. In the course of these investigations I wish to give a sense of the usefulness of thinking in terms of effective theories, as well as seeing the limitations of it.

2.3 First Theory

Let us suppose that there exists a System[2] that appears to be undergoing harmonic oscillation. For simplicity, the System will be chosen to have lengths of amplitude and times for the period of motion to be measured most conveniently in meters and seconds; however, this is only for intuitive concreteness, and one can multiply these units by orders of magnitude in any direction as appropriate for different systems.

In the earliest stages of investigation of the System we see that it is undergoing oscillatory behavior with a period of about 10 s. The resolution of the instrumentation is not good enough to resolve any deviations from pure harmonic motion, and so we posit that the motion is governed by the potential

$$V(x) = \frac{x^2}{2} \Longrightarrow \ddot{x} + \omega^2 x = 0 \quad \text{(Theory 1)}. \quad (2.9)$$

Let us now suppose that we try to test this theory by precision measurements. Again, at the early stages of experimenting on a system, the resolution may not be so good. Let us suppose that is the case for our simple System, and assume that the period is measured to be

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[1] If for some reason $a_2 = 0$, then a_3 will need to be zero also, otherwise $x = 0$ is not a local minimum, and the first term to worry about is x^4. This is a complication that we need not worry about for now.

[2] We capitalize System to give it a reference name for rest of the discussion.

$$T_{period} = 10\,\mathrm{s} \pm 0.3\,\mathrm{s} \quad \text{(Measurement 1)}. \tag{2.10}$$

This period of motion can be accommodated by our theory as long as

$$\omega = 0.63 \pm 0.02\,\mathrm{s}^{-2} \quad \text{(Parameter Fit 1)}. \tag{2.11}$$

It is no mystery that we could find a value of ω that fit the period. No matter how well we measure the period, it is only one observable and the theory has one parameter that can always be adjusted to match it. We need more observables to test the validity of the theory more fully.

2.4 Second Theory

Another drawback of having just one observable is that there are an infinite number of theories that we could write down trivially whose parameters could be adjusted in an infinite continuum of values to accommodate the measurement. One such theory has the same potential as Theory 1 except for now we add an x^3 correction term to the potential,

$$V(x) = k\frac{x^2}{2}\left(1 + \frac{2x}{3\Lambda_A}\right) \Longrightarrow \ddot{x} + \omega_A^2 x\left(1 + \frac{x}{\Lambda_A}\right) = 0 \quad \text{(Theory 2)} \tag{2.12}$$

where ω_A and Λ_A, a new length scale, are two parameters that can have a relation between them that give the same period. Here are two values:

$$\omega_A = 0.63\,\mathrm{s}^{-2} \quad \text{and} \quad \Lambda_\mathrm{A} = \infty \tag{2.13}$$

$$\omega_A = 0.631\,\mathrm{s}^{-2} \quad \text{and} \quad \Lambda_\mathrm{A} = 250\,\mathrm{m} \quad \text{(Parameter Fit 2)} \tag{2.14}$$

where the first line is equivalent to Theory 1 and the second line is just one parameter fit out of an infinite number of possibilities.

Upon close inspection of Theory 2 we notice that the correction term always generates a force of the same direction no matter what the value of x: it pushes the particle away from the origin when x is negative and pulls it back to the origin when $x > 0$, whereas the first term always is restoring. This should create an asymmetry in the time it takes for the Particle to cross $x = 0$ half-way through its full periodic motion compared to the time it takes to cross $x = 0$ again on its second half of the motion. We can compute this difference in time. Even though the total period $T_{period} = 10\,\mathrm{s}$ stays the same, the first and second halves of the distance covered by the motion would be asymmetric if x/Λ_A is not too suppressed:

$$T_{period}^{+1/2} \neq T_{period}^{-1/2} \quad \text{but} \quad T_{period} = T_{period}^{+1/2} + T_{period}^{-1/2} = 10\,\mathrm{s}. \tag{2.15}$$

Therefore, an important additional observable to measure are these "half periods" to see if they are antisymmetric as Theory 2 predicts.

Let us now suppose that there are improvements in the experimental instrumentation such that we can measure each "half period", $T^{+1/2}_{period}$ and $T^{-1/2}_{period}$, and it can be done to accuracies of 0.01 s. And let us suppose that after some time of measurement it is determined that

$$\begin{aligned} T^{+1/2}_{period} &= 5.05\,\text{s}, \quad T^{-1/2}_{period} = 5.06\,\text{s}, \quad \text{and} \\ T_{period} &= 10.11 \pm 0.01\,\text{s} \quad \text{(Measurements 2)}. \end{aligned} \tag{2.16}$$

To within the error bars of 0.01 s the two period halves are equal.

The usual scientific approach to the present situation is to say that the simpler model wins out if it accommodates the data as well as the more complicated theory. Thus, the community of scholars faced with the measurements above may well conclude that Theory 1 is correct, or conclude that even if the x/Λ_A term is present it is so suppressed that it is immaterial to the physics.

As we shall discuss later, this is the kind of statement that one might find in particle physics when considering higher dimensional operators of Standard Model particles. As in particle physics we may hold firm to the idea that there is no reason why these extra terms should not exist. Indeed, in an effective theory the full series expansion of additional terms should exist. But we must acknowledge that their coefficients may be too small to discern from our experiments.

2.5 Fancy Explanations

Not seeing the effects of the asymmetric x/Λ_A term after greatly improving the experimental situation to look for it would likely get the community thinking hard for the reasons of that failure. As we already mentioned, the diehard believers would just say that Λ_A has a value just higher than the experimental sensitivities would see. Others would invent reasons for why x/Λ_A should never have been there in the first place. These reasons need to be based on some kind of symmetry argument.

There are two straight-forward symmetry arguments that would banish the x/Λ_A correction to the potential. The first argument is to presume that the potential has an $x \to -x$ discrete symmetry. This would banish all odd corrections that could give rise to asymmetric half periods. Our next correction would then be x^2/Λ^2. We will investigate the experimental consequences of that potential shortly.

Another symmetry argument that says the harmonic oscillator lagrangian is exact with a conformal symmetry, $x \to \lambda x$ where λ is some arbitrary scaling parameter. Although the Lagrangian is not invariant under this, the equations of motion are. It is this scaling symmetry that tells us that time observables are independent of the spatial scaling. In other words, the (time) period is independent of the (spatial) amplitude.

There is a temptation of smart people to promote the most sophisticated and fancy arguments to explain the phenomena. It is not very sophisticated to say "the additional terms are too small to see". But it is fancy to say things like "conformal symmetry" and "discrete symmetry." And if the experimental situation languishes long enough theorists can become even more sophisticated with their description of why these terms must be banished, and look down upon people who do not catch the fever of fancy explanations. And if it goes on even longer it will be so entrenched in the highest schools of the land, that few will want to challenge it by proposing ways to find evidence for non-fancy corrections to the spatial scale-invariant theory.

2.6 Third Theory

Nevertheless, let us suppose that we take courage and wish to press forward in testing Theory 1 yet again. Odd corrections may exist, but we may need orders of magnitude more precision to see evidence for $T^{+1/2}_{period} \neq T^{-1/2}_{period}$. We may have more luck introducing only even power corrections to the potential. So we shall do this by introducing

$$V(x) = k\frac{x^2}{2}\left(1 - \frac{x^2}{2\Lambda_B^2}\right) \Longrightarrow \ddot{x} + \omega_B^2 x\left(1 - \frac{x^2}{\Lambda_B^2}\right) = 0 \quad \text{(Theory 3)} \tag{2.17}$$

What can we do to test and try to strain the theory? We know that measuring the half-periods does no good. However, being excellent students of the prevailing scale-invariant idea, we know that the period should not change depending on the amplitude. We need to find a way to perturb the system to increase the amplitude and see if the period changes.[3]

Let us suppose in our system that the particle passes through the origin with velocity of 10 m/s. Changing it requires significant technical skill, but we find a way to do it. We increase the energy into the system and obtain a new initial velocity of 15 m/s, which increases the amplitude by approximately 50 %. Upon measuring the period we get

$$T_{period} = 10.25\,\text{s} \pm 0.01\,\text{s} \quad \text{(Measurement 3)} \tag{2.18}$$

which differs by many standard deviations from the 10.11 s value obtained when $v_{initial} = 10\,\text{m/s}$, and is a clear signal for breaking of the spatial scale invariance of

[3] It is here I would like to remind the reader again that this is a fanciful allegory to how experiment and theory interplay on the effective theory stage, and although a simple macroscopic harmonic motion system can be manipulated and measured in all sorts of ways with ease, sometimes other systems are significantly more challenging to do the analogy of measuring half periods or of increasing the amplitudes.

the equations of motion. This is the first firm proof that the exact harmonic motion law of $V(x) \propto x^2$ is not fully respected.

We are likely to be quite excited about this, because we posited a theory that said there should be violations of scale invariance when the amplitude grows. And now that we see it we want to fit the parameters. Here is one such choice that works well

$$\omega_B = 0.63\,\mathrm{s}^{-2} \quad \text{and} \quad \Lambda_B = 95\,\mathrm{m} \quad \text{(Parameter Fit 3)}. \tag{2.19}$$

The two measurements at two different velocities are accommodated by these two choices of parameters.

Theory 3 is “better” than the old simple harmonic oscillator law of Theory 1, because it accounts for all the data. It accounts for equal half periods, and accounts for the measurements when the initial velocity is at $v = 10\,\mathrm{m/s}$ and at $v = 15\,\mathrm{m/s}$. However, Theory 3 is not the only theory that could do this. We could have had an x^6 correction, for example, that would have fit just as well this limited amount of data. Dissatisfaction may set in that we cannot be confident of any precise formulation of the theory to describe the system. If arbitrary corrections are allowed now, then anything goes.

This is both the beauty and the frustration of effective theories. Being committed to the notion that all terms should be allowed in a potential consistent with the symmetries we believe to be sacrosanct, and then test them with ever increasing experimental sophistication, has given us insight that deviations from the pure harmonic oscillator potential are possible. However, these ideas of effective theory appear to have muddied the waters rather than have led to “the theory.” We come to the realization that this is one of the limitations of effective theories. By itself it cannot raise you to a deeper physical insight. It is merely a statement that all operators (i.e., all corrections) should be added to your theory and then experiment can measure or put limitations on the couplings. However, if you do happen onto a deeper theoretical insight, that can put order to all the operators that may arise.

2.7 Deep Theory Conjecture

Now let us suppose that we let our success get to our heads, and we become supremely confident that we know of a deeper theory to explain the data. Nevermind how we came to it—that is not important here—but suppose the deep theory we become convinced of is

$$V = \omega_T^2 L_T \left[1 - \cos(x/L_T)\right] \Longrightarrow m\ddot{x} + \omega_T^2 L_T \sin(x/L_T) = 0 \quad \text{(Theory 4)}. \tag{2.20}$$

The data that has been taken to date suggests that

$$\omega_T = 0.63\,\mathrm{s}^{-2} \quad \text{and} \quad L_T = 38.8\,\mathrm{m} \quad \text{(Parameter Fit 4)}. \tag{2.21}$$

We note that there is no difference between Theory 3 predictions and Theory 4 predictions as long as the initial speed stays below 20 m/s and the timing resolution is not better than 0.01 s.

However, we can make a bold prediction based on our new deep and fundamental theory conjecture: if the initial velocity is doubled to 30 m/s the period jumps to 11.23 s, whereas for Theory 3 the prediction is 11.36 s. Experimentalists may puzzle over how to double the initial velocity for many years, but finally are able to do it. When they collect the data, they find $T_{period} = 11.35\,\mathrm{s} \pm 0.01$ s, which is a dramatic confirmation of Theory 3, and the hubris of the conjecturing Theory 4 is defeated.

2.8 Ultimate Test?

After the extreme test of Theory 3, which was years in the making and passed so decisively and impressively, the smart people figure out lots of fancy language to explain why it had to be true and what symmetry properties it has. It is written in every textbook. However, there was one more experiment that people wished to do. For years it has been suggested that if you are able to reach initial speeds greater than 42 m/s the Particle will never come back. In other words, the initial energy will be so great that it will exceed the confining potential barrier of Theory 3. However, getting to 42 m/s is a technological nightmare, and it will take decades to do it.

But let us suppose that after decades of R&D, it has been figured out how to launch the particle to speeds of 50 m/s from $x = 0$. When the experiment is conducted the particle flies off into the unknown. Twenty seconds go by, one minute goes by, an hour goes by, days and months go by, and the particle has never returned. Scientists are not surprised, but a little disappointed. It would be so much fun for a new anomaly to happen, but the theory looks solid and inviolate.

The scientists may move on, and study other things like sandpiles and solar flares. But one day, many years later, the particle returns! And nobody knows why, except a bright young student who realizes that the next term in the effective potential may have been what returned it.

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