Chapter 2
Premiums and Reserves in Multiple Decrement Model

2.1 Introduction

A guiding principle in the determination of premiums for a variety of life insurance products is:

$$\text{Expected present value of inflow} = \text{Expected present value of outflow.}$$

All the books listed in Sect. 1.1 of Chap. 1 thoroughly discuss the computation of premium for a variety of standard insurance products, for a single life and for a group in single decrement models. In this chapter we discuss its extension for multiple decrement models, when the benefit depends upon the mode of exit from the group of active insureds. Section 2.2 discusses how the multiple decrement model studied in Chap. 1 is useful to find the actuarial present value of benefit when it depends on the mode of decrement. Actuarial present value of the inflow to the insurance company, via premiums, does not depend on the mode of decrement. Hence, this part of the premium computations remains the same as for the single decrement model. When the two components of premiums are determined, premiums are calculated using the equivalence principle. Section 2.3 discusses the premium computations. In many life insurance products there is a provision of riders. For example, in whole life insurance the base policy specifies the benefit to be payable at the moment of death or at the end of year of death. Extra benefit will be payable if the death is due to a specific cause, such as an accident. The premium is then specified in two parts, one corresponding to the base policy and the extra premium corresponding to extra benefit. We will discuss computations of premiums in the presence of rider. Another important actuarial calculation is the reserve, that is, valuation of an insurance product at certain time points when the policy is in force. In Sect. 2.4 we illustrate the computation of reserve in the setup of multiple decrements.

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2.2 Actuarial Present Value of Benefit

Actuarial applications of multiple decrement models arise when the amount of benefit payment depends on the mode of exit from the group of active insureds. Our aim is to find the actuarial present value of the benefits in multiple decrement models, when the benefit is payable either at the moment of death or at the end of year of death. In these two approaches it depends upon the joint distribution of $T(x)$ and $J(x)$ or on the joint distribution of $K(x)$ and $J(x)$, respectively, and on the effective rate of interest. We assume that the rate of interest is deterministic and remains constant throughout the period of the policy. In practice the rate of interest fluctuates. If it is assumed to be deterministic but varying over certain time periods, then the actuarial present values of benefit or the annuity of premiums can be obtained on similar lines as that for constant rate, with different values of v or δ for the different time periods. The rate of interest is sometimes modeled as a random variable. We explore the modifications needed in Chap. 6.

Suppose that the underlying mortality model is the multiple decrement model with m causes of decrement. We consider the general setup in which benefit depends on the cause of decrement. This approach will be useful in the theory of pension funding in the next chapter. Suppose that $B_{x+t}^{(j)}$ denotes the value of a benefit at age $x+t$ for a decrement at that age by cause j. Then the actuarial present value of the benefit to be payable at the moment of death of (x), denoted in general by $\bar{A}$, is defined as $\bar{A} = E(B_{x+T(x)}^{(J(x))} v^{T(x)})$. We derive its expression in terms of basic functions as follows:

$$\begin{aligned}
\bar{A} &= E\big(B_{x+T(x)}^{(J(x))} v^{T(x)}\big) \\
&= E_{J(x)}\big[E_{T(x)|J(x)}\big(B_{x+T(x)}^{(J(x))} v^{T(x)}\big)|J(x)\big] \\
&= \sum_{j=1}^{m}\left[\int_0^\infty \big(B_{x+t}^{(j)} v^t f(t,j)/h_j\big)\, dt\right] h_j \\
&= \sum_{j=1}^{m} \int_0^\infty B_{x+t}^{(j)} v^t \, {}_tp_x^{(\tau)} \mu_{x+t}^{(j)}\, dt. \qquad (2.1)
\end{aligned}$$

If $m = 1$ and $B_{x+t}^{(j)} = 1$, $\bar{A}$ reduces to $\bar{A}_x$, the net single premium for whole life insurance with benefit payable at the moment of death. In general it is not easy to find these integrals. Some simplification can be obtained under certain assumptions. The most frequently made assumption is the assumption of uniformity for fractional ages. Suppose that we apply the uniform distribution assumption for each unit age interval in the jth integral, in (2.1). Then we have $T(x) = K(x) + U(x)$, where $U(x)$ has the uniform distribution on (0, 1), and, further, $U(x)$ and $K(x)$ are independent random variables. Under this assumption, ${}_sp_{x+k}^{(\tau)} \mu_{x+k+s}^{(j)} = q_{x+k}^{(j)}$. With this, the jth integral in (2.1) reduces as follows:

$$\begin{aligned}\int_0^\infty B_{x+t}^{(j)} v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} \, dt &= \sum_{k=0}^{\infty} \int_0^1 v^{k+s} B_{x+k+s}^{(j)} \, {}_{k+s} p_x^{(\tau)} \mu_{x+k+s}^{(j)} \, ds \\ &= \sum_{k=0}^{\infty} v^{k+1} \, {}_k p_x^{(\tau)} q_{x+k}^{(j)} \int_0^1 B_{x+k+s}^{(j)} (1+i)^{1-s} \, ds \\ &= \sum_{k=0}^{\infty} v^{k+1/2} \, {}_k p_x^{(\tau)} q_{x+k}^{(j)} B_{x+k+1/2}^{(j)},\end{aligned}$$

where last step is obtained by the midpoint rule for the integral. The ${}_k p_x^{(\tau)}$ and $q_{x+k}^{(j)}$ values are available from the underlying multiple decrement table. Thus,

$$\bar{A} = \sum_{j=1}^{m} \sum_{k=0}^{\infty} v^{k+1/2} \, {}_k p_x^{(\tau)} q_{x+k}^{(j)} B_{x+k+1/2}^{(j)}$$

gives a practical formula for the evaluation of the integral.

We illustrate the computation for the n-year term insurance in the setup of a double indemnity provision in which the death benefit is doubled when death is caused by an accident. Let $J = 1$ for death by nonaccidental means, and $J = 2$ for death by accident and suppose that $B_{x+t}^{(1)} = 1$ and $B_{x+t}^{(2)} = 2$. We denote the net single premium for an n-year term insurance by $\bar{A}T$, and it is given by

$$\bar{A}T = \int_0^n v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(1)} \, dt + 2 \int_0^n v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(2)} \, dt.$$

We now assume that each decrement in the multiple decrement context has a uniform distribution in each year of age, and the first step is to break the expression into separate integrals for each of the years involved. The first integral can be expressed as

$$\int_0^n v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(1)} \, dt = \sum_{k=0}^{n-1} v^k \, {}_k p_x^{(\tau)} \int_0^1 v^s \, {}_s p_{x+k}^{(\tau)} \mu_{x+k+s}^{(1)} \, ds.$$

Under the assumption of uniformity, we get

$$\int_0^n v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(1)} \, dt = \sum_{k=0}^{n-1} v^{k+1} \, {}_k p_x^{(\tau)} q_{x+k}^{(1)} \int_0^1 (1+i)^{1-s} \, ds = \frac{i}{\delta} \sum_{k=0}^{n-1} v^{k+1} \, {}_k p_x^{(\tau)} q_{x+k}^{(1)}.$$

Applying a similar argument for the second integral and combining, we get

$$\bar{A}T = \frac{i}{\delta} \left[\sum_{k=0}^{n-1} v^{k+1} \, {}_k p_x^{(\tau)} \left(q_{x+k}^{(1)} + 2 q_{x+k}^{(2)} \right) \right]$$

$$= \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}q_{x+k}^{(2)} + \frac{i}{\delta}\sum_{k=0}^{n-1} v^{k+1}\,{}_kp_x^{(\tau)}q_{x+k}^{(\tau)}$$

$$= \bar{A}_{x:\bar{n}|}^{1(2)} + \bar{A}_{x:\bar{n}|}^{1},$$

where $\bar{A}_{x:\bar{n}|}^{1(2)}$ is the net single premium for a term insurance of 1 covering death from accidental means, and $\bar{A}_{x:\bar{n}|}^{1}$ is the net single premium for a term insurance of 1 covering death from all causes. The next example illustrates the computation for specified mortality pattern and for specified benefit values.

Example 2.2.1 A whole life insurance of 10000 payable at the moment of death of (x) includes a double-indemnity provision. This provision pays an additional death benefit of 10000 during the first 20 years if death is by accidental means. It is given that $\delta = 0.05$, $\mu_{x+t}^{(\tau)} = 0.005$ for $t \geq 0$, and $\mu_{x+t}^{(1)} = 0.001$ for $t \geq 0$, where $\mu_{x+t}^{(1)}$ is the force of decrement due to death by accidental means. Calculate the net single premium for this insurance.

Solution From the given information, the net single premium is the actuarial present value of the benefit of 10000 in whole life insurance plus the actuarial present value of the benefit of 10000 in 20-year term insurance if death is due to accident. Thus it is given by the expression

$$10000\left[\int_0^\infty e^{-\delta t}\,{}_tp_x^{(\tau)}\mu_{x+t}^{(\tau)}\,dt + \int_0^{20} e^{-\delta t}\,{}_tp_x^{(\tau)}\mu_{x+t}^{(1)}\,dt\right]$$

$$= 10000\left[\int_0^\infty e^{-0.05t}e^{-0.005t}0.005\,dt + \int_0^{20} e^{-0.05t}e^{-0.005t}0.001\,dt\right]$$

$$= 10000\left[\frac{0.005}{0.055} + \frac{0.001}{0.055}\left(1 - e^{-1.1}\right)\right] = 1030.$$

Example 2.2.2 For a 20-year term insurance issued to (30), the following information is given.

(i) $\mu_{30+t}^{(1)} = 0.0005t$, where (1) represents death by accidental means.
(ii) $\mu_{30+t}^{(2)} = 0.0025t$, where (2) represents death by other means.
(iii) The benefit is 2000 units if death occurs by accidental means and 1000 units if death occurs by other means.
(iv) Benefits are payable at the moment of death.

Taking $\delta = 0.06$, find the purchasing price of this insurance.

Solution From the given information we have

$$\mu_{x+t}^{(\tau)} = 0.003t \quad \Rightarrow \quad {}_tp_x^{(\tau)} = \exp\left[-\int_0^t \mu_{x+s}^{(\tau)}\,ds\right] = e^{-0.0015t^2}.$$

The actuarial present value of accidental death benefit is

$$\int_0^{20} B_{30+t}^{(1)} e^{-\delta t}\, {}_tp_{30}^{(\tau)}\, \mu_{30+t}^{(1)}\, dt = 2000 \int_0^{20} e^{-0.06t} e^{-0.0015t^2} (0.0005t)\, dt.$$

To workout this integral, we complete the square in the exponent of e and substitute $0.0015(t+20)^2 = u$. Further we use the incomplete gamma function to find the value of the integral. Thus, we have

$$\begin{aligned}
&2000 \int_0^{20} e^{-0.06t} e^{-0.0015t^2} (0.0005t)\, dt \\
&= \int_0^{20} (t + 20 - 20) e^{-0.0015(t^2+40t+400-400)}\, dt \\
&= \frac{e^{0.6}}{0.003} \Bigg\{ \int_{0.6}^{2.4} e^{-u}\, du - 20(0.0015)^{0.5} \Gamma(0.5) \\
&\quad \times \big[pgamma(2.4, 0.5, 1) - pgamma(0.6, 0.5, 1)\big] \Bigg\} \\
&= 74.05.
\end{aligned}$$

The actuarial present value of other death benefit is calculated using the similar procedure adopted for the actuarial present value of accidental death benefit and is given by

$$\begin{aligned}
1000 \int_0^{20} B_{30+t}^{(2)} e^{-\delta t}\, {}_tp_{30}^{(\tau)}\, \mu_{30+t}^{(2)}\, dt &= 1000 \int_0^{20} e^{-0.06t} e^{-0.0015t^2} (0.0025t)\, dt \\
&= 185.12.
\end{aligned}$$

The actuarial present value of death benefit, when death may be due to any cause, is 259.17, which is the purchasing price of the insurance product.

We have seen how to find the actuarial present value of the benefits in multiple decrement models, when the benefit is payable at the moment of death. When the benefit is payable at the end of year of death, we use the joint distribution of $K(x)$ and $J(x)$ instead of the joint distribution of $T(x)$ and $J(x)$. Let $\{p(k, j), k = 0, 1, \ldots, j = 1, 2, \ldots, m\}$ denote the joint probability mass function of $K(x)$ and $J(x)$. The actuarial present value of the benefit to be payable at the end of the year of death in whole life insurance for multiple decrement model, to be in general denoted by A, is given by

$$A = E\big(B_{x+K(x)}^{(J(x))} v^{K(x)+1}\big) = \sum_{j=1}^{m} \sum_{k=0}^{\infty} v^{k+1} B_{x+k}^{(j)} p(k, j).$$

Similarly, the actuarial present value of the benefit to be payable at the end of the year of death in n-year term insurance for multiple decrement model is given by

$$AT = \sum_{j=1}^{m} \sum_{k=0}^{n-1} v^{k+1} B_{x+k}^{(j)} p(k, j).$$

Actuarial present values for other insurance products are defined analogously.

In the next section we discuss the procedure for premium computation and illustrate it with examples.

2.3 Computation of Premiums

The actuarial present value of the benefit to be payable at the moment of death or at the end of year of death, is one of the two components in premium calculation. The other component in premium calculation is the expected present value of inflow to the insurance company via premiums. These computations remain exactly the same as in the setup of single decrement model and are not changed in view of various modes of decrement. Suppose that the premiums are paid as continuous whole life annuity at the rate of P per annum. Then the actuarial present value of the premiums is $P\bar{a}_x$, and $\bar{a}_x$ is given by

$$\bar{a}_x = \int_0^\infty v^t \, {}_t p_x^{(\tau)} \, dt.$$

Suppose that the premiums are paid as continuous n-year temporary life annuity at the rate of P per annum. Then the actuarial present value of the premiums is $P\bar{a}_{x:\bar{n}|}$, and $\bar{a}_{x:\bar{n}|}$ is given by

$$\bar{a}_{x:\bar{n}|} = \int_0^n v^t \, {}_t p_x^{(\tau)} \, dt.$$

If the premiums are paid as discrete whole life annuity due at the rate of P per annum, then the actuarial present value of the premiums is $P\ddot{a}_x$, and $\ddot{a}_x$ is given by

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_k p_x^{(\tau)}.$$

Suppose that the premiums are payable as a discrete n-year temporary life annuity due at the rate of P per annum. Then the actuarial present value of the premiums is $P\ddot{a}_{x:\bar{n}|}$, and $\ddot{a}_{x:\bar{n}|}$ is given by

$$\ddot{a}_{x:\bar{n}|} = \sum_{k=0}^{n-1} v^k \, {}_k p_x^{(\tau)}.$$

Thus, once we have knowledge about the survival function ${}_t p_x^{(\tau)}$, we can find out the actuarial present value of the premiums for various modes of premium payments, such as continuous premium and discrete premium. With the equivalence principle, premium is then obtained as

$$\text{Premium} = \frac{\text{Actuarial Present Value of Benefits}}{\text{Actuarial Present Value of Premium Annuity}}.$$

The following examples illustrate how the premium computations can be done once the multiple decrement model is specified in terms of forces of mortality.

Example 2.3.1 For the insurance product, mortality pattern, and force of interest as specified in Example 2.2.1, find the premium payable as

(i) the whole life continuous annuity,
(ii) whole life annuity due, and
(iii) 10-year temporary continuous life annuity.

Solution For the given insurance product, mortality pattern, and force of interest, we have obtained the actuarial present value of the benefit payable at the moment of death. It is Rs 1030. To find the premium payable as (i) the whole life continuous annuity, (ii) whole life annuity due, and (iii) 10-year temporary continuous life annuity, we compute $\bar{a}_x$, $\ddot{a}_x$, and $\bar{a}_{x:\overline{10|}}$, respectively, for the given mortality and interest pattern. By definition,

$$\bar{a}_x = \int_0^\infty v^t\, {}_t p_x^{(\tau)}\, dt = \int_0^\infty e^{-0.05t} e^{-0.005t}\, dt = 18.18182,$$

$$\ddot{a}_x = \sum_{k=0}^\infty v^k\, {}_k p_x^{(\tau)} = \sum_{k=0}^\infty e^{-0.05k} e^{-0.005k} = \left(1 - e^{-0.055}\right)^{-1} = 18.6864 > \bar{a}_x,$$

$$\bar{a}_{x:\overline{10|}} = \int_0^{10} v^t\, {}_t p_x^{(\tau)}\, dt = \int_0^{10} e^{-0.05t} e^{-0.005t} = 7.6918.$$

Hence, (i) the premium payable as the whole life continuous annuity is $1030/18.18182 = 56.65$, (ii) the premium payable as the whole life annuity due is $1030/18.6864 = 55.12$, and (iii) the premium payable as the 10-year temporary continuous life annuity is $1030/7.6918 = 133.91$. It is to be noted that the premium payable as the 10-year temporary continuous life annuity is highest among these three modes as the premium paying period is limited. Further, the premium payable as the whole life continuous annuity is slightly higher than that payable as the whole life annuity due.

Example 2.3.2 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu_{x+t}^{(1)} = BC^{x+t}, \quad t \geq 0, \qquad \mu_{x+t}^{(2)} = A, \quad t \geq 0,\ A \geq 0,\ B \geq 0,\ C \geq 1.$$

Suppose $A = 0.0008$, $B = 0.00011$, and $C = 1.095$. Further, the force of interest is $\delta = 0.05$. The benefit to be payable at the moment of death is specified as 1000 units if death is due to cause 1 and 2000 units if the death is due to cause 2.

(i) Find the actuarial present value of the benefit payable to (x) for $x = 30, 40, 50,$ and 60 for the whole life insurance.
(ii) Find the premium payable as the continuous whole life annuity by (x) for $x = 30, 40, 50,$ and 60 for the whole life insurance. Decompose the total premium according to two causes of decrement.
(iii) Find the premium payable as the continuous n-year temporary life annuity by (30) for the whole life insurance, for $n = 1, 2, \ldots, 10$. Decompose the total premium according to two causes of decrement.
(iv) Find the premium payable as the continuous n-year temporary life annuity by (30), for n-year term insurance, for $n = 1, 2, \ldots, 10$.

Solution For the given model, $\mu_{x+s}^{(\tau)} = \mu_{x+s}^{(1)} + \mu_{x+s}^{(2)} = BC^{x+s} + A$. We have derived in Example 1.2.2 the probability density function of T as

$$g(t) = {}_t p_x^{(\tau)} \mu_{x+t}^{(\tau)} = \exp\bigl[-\bigl(At + mC^x\bigl(C^t - 1\bigr)\bigr)\bigr]\bigl(A + BC^{x+t}\bigr) \quad \text{if } t \geq 0$$

and the joint distribution of T and J as

$$f(t, j) = \begin{cases} \exp[-(At + mC^x(C^t - 1))](BC^{x+t}) & \text{if } t \geq 0,\ j = 1, \\ \exp[-(At + mC^x(C^t - 1))](A) & \text{if } t \geq 0,\ j = 2. \end{cases}$$

Let mC^x denote by α_x. The actuarial present value of the benefit to be payable at the moment of death of (x), denoted by $\bar{A}$, is given by

$$\begin{aligned} \bar{A} &= \sum_{j=1}^{2} \int_0^\infty B_{x+t}^{(j)} v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} \, dt \\ &= 1000BC^x e^{\alpha_x} \int_0^\infty e^{-\delta t - At - \alpha_x C^t} C^t \, dt + 2000Ae^{\alpha_x} \int_0^\infty e^{-\delta t - At - \alpha_x C^t} \, dt \\ &= 1000BC^x e^{\alpha_x} \int_0^\infty e^{-\delta_1 t - \alpha_x C^t} C^t \, dt + 2000Ae^{\alpha_x} \int_0^\infty e^{-\delta_1 t - \alpha_x C^t} \, dt, \end{aligned}$$

where $\delta_1 = \delta + A$. Then, substituting $C^t = y$, we get $e^{-\delta_1 t} = y^{-\delta_1/\log C}$, $C^t \log C \, dt = dy$, and the range of integration is from 1 to ∞. Suppose $(-\delta_1/\log C) + 1 = \lambda_1$. Then, as in Example 1.2.2, the first integral in $\bar{A}$ simplifies to $1000e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1} P[W_1 \geq 1]$, where W_1 follows the gamma distribution with shape parameter λ_1 and scale parameter α_x, provided that $\lambda_1 > 0$. Now, using the fact that $g(t)$ is a density function, we get

$$1 = \int_0^\infty g(t) \, dt = \int_0^\infty \exp\bigl[-\bigl(At + mC^x\bigl(C^t - 1\bigr)\bigr)\bigr]\bigl(A + BC^{x+t}\bigr) \, dt.$$

Hence,

$$Ae^{\alpha_x}\int_0^\infty \exp\bigl[-\bigl(At+\alpha_x C^t\bigr)\,dt\bigr] = 1-e^{\alpha_x}\Gamma(\lambda)\alpha_x^{1-\lambda}P[W\geq 1],$$

where $\lambda = (-A/\log C)+1$, and W has the gamma distribution with shape parameter λ and scale parameter α_x. Using this approach, the second term in $\bar{A}$ can be expressed as

$$\frac{2000A[1-e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P[W_1\geq 1]]}{\delta_1}.$$

Hence, adding the two components of $\bar{A}$, we get

$$\bar{A} = \frac{2000A + 1000(\delta - A)[e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P[W_1\geq 1]]}{A+\delta}.$$

Now the premium is payable as the continuous whole life annuity. So we need to find the actuarial present value of the premiums. It is given by $P\bar{a}_x$, and $\bar{a}_x$ is given by

$$\begin{aligned}
\bar{a}_x &= \int_0^\infty v^t\,{}_tp_x^{(\tau)}\,dt \\
&= e^{\alpha_x}\int_0^\infty e^{-\delta t - At - \alpha_x C^t}\,dt = e^{\alpha_x}\int_0^\infty e^{-\delta_1 t-\alpha_x C^t}\,dt \\
&= \frac{1-e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P[W_1\geq 1]}{\delta_1}.
\end{aligned}$$

The last equality follows using similar arguments as in the second integral of $\bar{A}$.

Suppose that the premium is payable as the continuous n-year temporary life annuity. The actuarial present value of the premiums is given by $P\bar{a}_{x:\overline{n}|}$, and $\bar{a}_{x:\overline{n}|}$ is given by

$$\begin{aligned}
\bar{a}_{x:\overline{n}|} &= \int_0^n v^t\,{}_tp_x^{(\tau)}\,dt \\
&= e^{\alpha_x}\int_0^n e^{-\delta t - At - \alpha_x C^t}\,dt = e^{\alpha_x}\int_0^n e^{-\delta_1 t-\alpha_x C^t}\,dt,
\end{aligned}$$

where $\delta_1 = A+\delta$. To find this integral, we proceed as follows. We know that, for the two decrement-model in this example,

$$\int_0^n g(t)\,dt = P\bigl[T(x)\leq n\bigr] = {}_nq_x^{(\tau)} = 1-{}_np_x^{(\tau)} = 1-\exp\bigl[-An-\alpha_x C^n+\alpha_x\bigr].$$

On the other hand, $\int_0^n g(t)\,dt$ can also be expressed as follows.:

$$\begin{aligned}\int_0^n g(t)\,dt &= \int_0^n {}_tp_x^{(\tau)}\big(A+BC^{x+t}\big)\,dt\\&= A\int_0^n {}_tp_x^{(\tau)}\,dt + \int_0^n {}_tp_x^{(\tau)}\big(BC^{x+t}\big)\,dt\\&= Ae^{\alpha_x}\int_0^n \exp\big(-At-\alpha_x C^t\big)\,dt\\&\quad + BC^x e^{\alpha_x}\int_0^n \exp\big(-At-\alpha_x C^t\big)C^t\,dt.\end{aligned}$$

The second integral in the above equation can be evaluated as $e^{\alpha_x}\Gamma(\lambda)\alpha_x^{1-\lambda}P[1\le W\le C^n]$, where $(-\delta/\log C)+1=\lambda$, and W follows the gamma distribution with shape parameter λ and scale parameter α_x. Hence the first integral $A\int_0^n {}_tp_x^{(\tau)}\,dt$ in the above equation is given by

$$\begin{aligned}Ae^{\alpha_x}\int_0^n \exp\big(-At-\alpha_x C^t\big)\,dt &= 1-\exp\big[-An-\alpha_x C^n+\alpha_x\big]\\&\quad - e^{\alpha_x}\Gamma(\lambda)\alpha_x^{1-\lambda}P\big[1\le W\le C^n\big].\end{aligned}$$

We use this relation to write $\bar{a}_{x:\bar{n}|}$ as

$$\bar{a}_{x:\bar{n}|} = \big\{1-\exp\big[-\delta_1 n-\alpha_x C^n+\alpha_x\big] - e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P\big[1\le W_1\le C^n\big]\big\}/\delta_1,$$

where $\delta_1 = A+\delta$, $(-\delta_1/\log C)+1=\lambda_1$, and W_1 follows the gamma distribution with shape parameter λ_1 and scale parameter α_x, provided that $\lambda_1>0$.

The annual premium payable as the whole life annuity for benefit of 1000 units if death is due to cause 1 and 2000 units if the death is due to cause 2 is then given by $P=\frac{\bar{A}}{\bar{a}_x}$. To decompose the total premium according to two causes of decrement, we divide two terms in $\bar{A}$ separately by $\bar{a}_x$. Let $P_x^{(1)}$ and $P_x^{(2)}$ denote the premiums corresponding to two causes of decrement; then these are given by

$$P_x^{(1)} = \frac{1000e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P[W_1\ge 1]}{\bar{a}_x} \quad \text{and}$$

$$P_x^{(2)} = \frac{2000A[1-e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1}P[W_1\ge 1]]}{\delta_1\bar{a}_x}.$$

Substituting the expression for $\bar{a}_x$ into $P_x^{(2)}$, we get

$$P_x^{(2)} = 2000A.$$

It is constant and does not depend on x. It seems reasonable as the force of decrement due to cause 2 is free from x. Premiums payable as the continuous n-year temporary life annuity for the whole life insurance are computed on similar lines, denominator being $\bar{a}_{x:\bar{n}|}$ instead of $\bar{a}_x$.

To find premiums for the n-year term, we have to first find the actuarial present value of the benefit for this product. The actuarial present value of the benefit to be payable at the moment of death of (x), in the n-year term insurance, denoted by $\bar{A}T$, is given by

$$\begin{aligned}
\bar{A}T &= \sum_{j=1}^{2} \int_0^n B_{x+t}^{(j)} v^t \, {}_t p_x^{(\tau)} \mu_{x+t}^{(j)} \, dt \\
&= 1000BC^x e^{\alpha_x} \int_0^n e^{-\delta t - At - \alpha_x C^t} C^t dt + 2000Ae^{\alpha_x} \int_0^n e^{-\delta t - At - \alpha_x C^t} \, dt \\
&= 1000BC^x e^{\alpha_x} \int_0^n e^{-\delta_1 t - \alpha_x t C^t} C^t dt + 2000Ae^{\alpha_x} \int_0^n e^{-\delta_1 t - \alpha_x C^t} \, dt,
\end{aligned}$$

where $\delta_1 = \delta + A$. As discussed above, the first integral in the above equation can be expressed as $1000e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1} P[1 \le W_1 \le C^n]$, where $(-\delta_1/\log C) + 1 = \lambda_1$, and W_1 follows the gamma distribution with shape parameter λ_1 and scale parameter α_x. As in the expression of $\bar{a}_{x:\bar{n}|}$, the second integral in $\bar{A}T$ can be expressed as

$$\begin{aligned}
&2000(A/\delta_1)\left\{1 - \exp\left[-\delta_1 n - \alpha_x C^n + \alpha_x\right] - e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1} P\left[1 \le W_1 \le C^n\right]\right\} \\
&\quad = 2000A\bar{a}_{x:\bar{n}|}.
\end{aligned}$$

Thus, $\bar{A}T$ is given by

$$\bar{A}T = 1000e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1} P\left[1 \le W_1 \le C^n\right] + 2000A\bar{a}_{x:\bar{n}|}.$$

We have already derived the expression for $\bar{a}_{x:\bar{n}|}$. Once we have expressions for these actuarial present values, we can compute premiums using the equivalence principle. Thus,

$$P_{x:\bar{n}|}^{1(1)} = \frac{1000e^{\alpha_x}\Gamma(\lambda_1)\alpha_x^{1-\lambda_1} P[1 \le W_1 \le C^n]}{\bar{a}_{x:\bar{n}|}} \quad \text{and}$$

$$P_{x:\bar{n}|}^{1(2)} = \frac{2000A\bar{a}_{x:\bar{n}|}}{\bar{a}_{x:\bar{n}|}} = 2000A,$$

similar to $P_x^{(2)}$.

We compute all these quantities using the following R commands:

```
a1 <- 0.0008  # A;
b <- 0.00011  # B;
a <- 1.095  # C;
m <- b/log(a, base=exp(1));
e <- exp(1);
```

Table 2.1 Premium for whole life insurance

Age x	$1000\bar{A}$	$\bar{a}_x$	$P_x^{(1)}$	$P_x^{(2)}$	P_x
30	202.77	16.2039	10.91	1.60	12.51
40	290.39	14.4229	18.53	1.60	20.13
50	406.68	12.0593	32.12	1.60	33.72
60	545.70	9.2338	57.50	1.60	59.10

```
del <- 0.05;
f <- a1+del;
p <- (-f/log(a, base=exp(1)))+1  # $\lambda_1$;
x <- c(30, 40, 50, 60);
j <- m*a^x  # $\alpha_x$;
q1 <- e^j*gamma(p)*(j^(1-p))*(1-pgamma(1, p, j))
                                          # first term in $\bar{A}$;
q2 <- (a1/f)*(1-q1)  # second integral in $\bar{A}$;
q3 <- 1000*q1+2000*q2  # $\bar{A}$;
q4 <- (1-q1)/f  # $\bar{a}_x$;
p1 <- 1000*q1/q4  # premium corresponding to cause 1;
p2 <- 1000*2*q2/q4  # premium corresponding to cause 2;
p3 <- p1+p2  # premium ;
d <- round(data.frame(q3, q4, p1, p2, p3), 4);
d1 <- data.frame(x, d);
d1  # Table 2.1;
```

The actuarial present values of benefit and annuity and the corresponding premiums for four ages are reported in Table 2.1. It is to be noted that $P_x^{(2)} = 2000A = 1.6$. Suppose that the premium are payable as the n-year temporary life annuity by (30). Then the following commands are added after the command for p in the above set to obtain the premiums:

```
x <- 30;
n <- 1:10;
j <- m*a^x;
q1 <- e^j*gamma(p)*(j^(1-p))*(1-pgamma(1, p, j));
q2 <- (a1/f)*(1-q1);
q <- e^j*gamma(p)*(j^(1-p))*(pgamma(a^n, p, j)
     -pgamma(1, p, j));
q4 <- (1-q-e^(j-f*n-j*a^n))/f  # $\bar{a}_{x:\overline{n}|}$;
p1 <- 1000*q1/q4;
p2 <- 1000*2*q2/q4;
p3 <- p1+p2;
d <- round(data.frame(p1, p2, p3), 2);
d1 <- data.frame(n, d);
d1  # Table 2.2;
```

Table 2.2 n-year premium payment

n	${}_nP_{30}^{(1)}$	${}_nP_{30}^{(2)}$	${}_nP_{30}$
1	181.53	26.61	208.14
2	93.15	13.66	106.81
3	63.72	9.34	73.07
4	49.03	7.19	56.22
5	40.24	5.90	46.14
6	34.39	5.04	39.43
7	30.23	4.43	34.66
8	27.12	3.98	31.09
9	24.71	3.62	28.33
10	22.79	3.34	26.13

Table 2.3 Premium for n-year term insurance

n	$P_{30:\bar{n}\rvert}^{1(1)}$	$P_{30:\bar{n}\rvert}^{1(2)}$	$P_{30:\bar{n}\rvert}^{1}$
1	1.75	1.6	3.35
2	1.83	1.6	3.43
3	1.92	1.6	3.52
4	2.01	1.6	3.61
5	2.10	1.6	3.70
6	2.19	1.6	3.79
7	2.29	1.6	3.89
8	2.40	1.6	4.00
9	2.51	1.6	4.11
10	2.62	1.6	4.22

Premiums payable as the n-year temporary annuity for the whole life insurance are reported in Table 2.2.

To compute the premium for the n-year term insurance, in the above set of commands, we add the following R commands:

```
q3 <- 2*a1*(1-q-e^(j-f*n-j*a^n))/f;
pr1 <- 1000*q/q4;
pr2 <- 1000*q3/q4;
pr3 <- pr1+pr2;
d <- round(data.frame(pr1, pr2, pr3), 2);
d1 <- data.frame(n, d);
d1  #Table 2.3;
```

Table 2.3 gives the premiums for the n-year term insurance.

It is to be noted that, as usual, the premium in the n-year term insurance is always less than that for the whole life insurance.

In the next example, we use the same multiple decrement model used in Example 2.3.2; however the benefit functions are taken as increasing functions and also dependent on age.

Example 2.3.3 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu_{x+t}^{(1)} = BC^{x+t}, \quad t \geq 0, \qquad \mu_{x+t}^{(2)} = A, \quad t \geq 0, \ A \geq 0, \ B \geq 0, \ C \geq 1.$$

Suppose $A = 0.0008$, $B = 0.00011$, and $C = 1.095$. Further, the force of interest is $\delta = 0.05$. The benefit to be payable at the moment of death is specified as $1000e^{b_1(x+t)}$ units if death is due to cause 1 and $1000e^{b_2(x+t)}$ units if the death is due to cause 2, where $b_1 = 0.02$, $b_2 = 0.03$.

(i) Find the actuarial present value of the benefit payable to (x) for $x = 30, 40, 50$, and 60 for the whole life insurance.
(ii) Find the premium payable as the continuous life annuity by (x) for $x = 30, 40, 50$, and 60 for the whole life insurance. Decompose the total premium according to two causes of decrement.
(iii) Find the premium payable as the continuous n-year temporary life annuity by (30) for the whole life insurance, for $n = 1, 2, \ldots, 10$. Decompose the total premium according to two causes of decrement.
(iv) Find the premium payable as the continuous n-year temporary life annuity by (30), for n-year term insurance, for $n = 1, 2, \ldots, 10$.

Solution As in Example 2.3.2, for this multiple decrement model, $\mu_{x+s}^{(\tau)} = \mu_{x+s}^{(1)} + \mu_{x+s}^{(2)} = BC^{x+s} + A$. The probability density function of T is

$$g(t) = {}_tp_x^{(\tau)} \mu_{x+t}^{(\tau)} = \exp\left[-\left(At + mC^x\left(C^t - 1\right)\right)\right]\left(A + BC^{x+t}\right) \quad \text{if } t \geq 0,$$

and the joint distribution of T and J is specified by

$$f(t, j) = \begin{cases} \exp[-(At + mC^x(C^t - 1))](BC^{x+t}) & \text{if } t \geq 0, \ j = 1, \\ \exp[-(At + mC^x(C^t - 1))](A) & \text{if } t \geq 0, \ j = 2. \end{cases}$$

Let mC^x be denoted by α_x. The actuarial present value of the benefit, omitting 1000, to be payable at the moment of death of (x), denoted by $\bar{A}$, is given by

$$\begin{aligned} \bar{A} &= \sum_{j=1}^{2} \int_0^\infty B_{x+t}^{(j)} v^t \, {}_tp_x^{(\tau)} \mu_{x+t}^{(j)} \, dt \\ &= \int_0^\infty e^{b_1(x+t)} e^{-\delta t} \, {}_tp_x^{(\tau)} \mu_{x+t}^{(1)} \, dt + \int_0^\infty e^{b_2(x+t)} e^{-\delta t} \, {}_tp_x^{(\tau)} \mu_{x+t}^{(2)} \, dt \\ &= BC^x e^{b_1 x + \alpha_x} \int_0^\infty e^{b_1 t - \delta t - At - \alpha_x C^t} C^t \, dt + Ae^{b_2 x + \alpha_x} \int_0^\infty e^{b_2 t - \delta t - At - \alpha_x C^t} \, dt \end{aligned}$$

$$= BC^x e^{b_1 x+\alpha_x} \int_0^\infty e^{-\delta_2 t-\alpha_x C^t} C^t\,dt + Ae^{b_2 x+\alpha_x} \int_0^\infty e^{-\delta_3 t-\alpha_x C^t}\,dt,$$

where $\delta_2 = \delta + A - b_1$ and $\delta_3 = \delta + A - b_2$. Suppose $(-\delta_2/\log C) + 1 = \lambda_2$ and $(-\delta_3/\log C) + 1 = \lambda_3$. If λ_1 and λ_2 are positive, then, as in Example 2.3.1, the first integral in $\bar{A}$ simplifies to $e^{b_1 x+\alpha_x}\Gamma(\lambda_2)\alpha_x^{1-\lambda_2} P[W_2 \geq 1]$, where W_2 follows the gamma distribution with shape parameter λ_2 and scale parameter α_x. Similarly, as in Example 2.3.1, the second integral in $\bar{A}$ simplifies to

$$\frac{e^{b_2 x} A\{1 - e^{\alpha_x}\Gamma(\lambda_3)\alpha_x^{1-\lambda_3} P[W_3 \geq 1]\}}{\delta_3},$$

where W_3 follows the gamma distribution with shape parameter λ_3 and scale parameter α_x.

Hence, $\bar{A}$ is given by

$$e^{b_1 x+\alpha_x}\Gamma(\lambda_2)\alpha_x^{1-\lambda_2} P[W_2 \geq 1] + \frac{e^{b_2 x} A\{1 - e^{\alpha_x}\Gamma(\lambda_3)\alpha_x^{1-\lambda_3} P[W_3 \geq 1]\}}{\delta_3}.$$

Expressions for $\bar{a}_x$ and $\bar{a}_{x:\overline{n}|}$ will remain the same as in Example 2.3.1. Once we have both components of premium calculations, we can find the premiums.

To find the premium for the n-year term insurance, we have to first find the actuarial present value of the benefit for this insurance. The actuarial present value of the benefit to be payable at the moment of death of (x), in the n-year term insurance, denoted by $\bar{A}T$, can be obtained on similar lines as in $\bar{A}T$ in Example 2.3.2. It is given by

$$\begin{aligned}
\bar{A}T &= \sum_{j=1}^{2} \int_0^n B_{x+t}^{(j)} v^t\, {}_t p_x^{(\tau)} \mu_{x+t}^{(j)}\,dt \\
&= \int_0^n e^{b_1(x+t)} e^{-\delta t}\, {}_t p_x^{(\tau)} \mu_{x+t}^{(1)}\,dt + \int_0^n e^{b_2(x+t)} e^{-\delta t}\, {}_t p_x^{(\tau)} \mu_{x+t}^{(2)}\,dt \\
&= BC^x e^{b_1 x+\alpha_x} \int_0^n e^{-\delta_2 t-\alpha_x t C^t} C^t\,dt + Ae^{b_2 x+\alpha_x} \int_0^n e^{-\delta_3 t-\alpha_x C^t}\,dt \\
&= e^{\alpha_x+b_1 x}\Gamma(\lambda_2)\alpha_x^{1-\lambda_2} P\big[1 \leq W_2 \leq C^n\big] \\
&\quad + e^{b_2 x}\frac{A}{\delta_3}\big\{1 - \exp\big[-\delta_3 n - \alpha_x C^n + \alpha_x\big] \\
&\quad - e^{\alpha_x}\Gamma(\lambda_3)\alpha_x^{1-\lambda_3} P\big[1 \leq W_3 \leq C^n\big]\big\}.
\end{aligned}$$

We have already derived the expression for $\bar{a}_{x:\overline{n}|}$. Once we have expressions for these actuarial present values, we can compute the premiums using equivalence principle. The following R commands compute all these functions:

Table 2.4 Premiums for varying benefits

Age x	$\bar{A}$	$\bar{a}_x$	$P_x^{(1)}$	$P_x^{(2)}$	P_x
30	0.6527	16.2039	37.15	3.12	40.28
40	1.0142	14.4229	66.40	3.92	70.32
50	1.5441	12.0593	123.16	4.89	128.04
60	2.2790	9.2338	240.71	6.10	246.81

```
a1 <- 0.0008  #A;
b <- 0.00011  #B;
a <- 1.095  #C;
m <- b/log(a, base=exp(1));
e <- exp(1);
del <- 0.05;
b1 <- 0.02;
b2 <- 0.03;
f <- a1+del;
p <- (-f/log(a, base=exp(1)))+1  #λ1;
f1 <- a1+del-b1;
p1 <- (-f1/log(a, base=exp(1)))+1  #λ2;
f2 <- a1+del-b2;
p2 <- (-f2/log(a, base=exp(1)))+1  #λ3;
x <- c(30, 40, 50, 60);
j <- m*a^x  #αx;
q <- e^j*gamma(p)*(j^(1-p))*(1-pgamma(1, p, j));
q1 <- e^j*gamma(p1)*(j^(1-p1))*(1-pgamma(1, p1, j));
q2 <- e^j*gamma(p2)*(j^(1-p2))*(1-pgamma(1, p2, j));
q3 <- e^(b2*x)*a1*(1-q2)/f2  #second term in Ā;
q4 <- (1-q)/f;  #āx;
q5 <- e^(b1*x)*q1+q3  #Ā;
pr1 <- 1000*e^(b1*x)*q1/q4
                         #premium corresponding to cause 1;
pr2 <- 1000*q3/q4  #premium corresponding to cause 2;
pr3 <- pr1+pr2;
d <- round(data.frame(q5, q4, pr1, pr2, pr3), 4);
d1 <- data.frame(x, d);
d1  #Table 2.4;
```

Premiums are reported in Table 2.4.

It is to be noted that the premiums are higher as compared to those reported in Table 2.1, as here the benefit function is an increasing function for both modes of decrement.

Suppose that the premiums are payable as the n-year temporary life annuity by (30). Then after the set of commands up to p_2, we add the following commands to obtain the premiums:

Table 2.5 n-year premium

n	$\bar{a}_{30:\bar{n}\vert}$	${}_nP_{30}^{(1)}$	${}_nP_{30}^{(2)}$	${}_nP_{30}$
1	0.9742	617.98	51.97	669.95
2	1.8984	317.12	26.67	343.79
3	2.7751	216.94	18.24	235.18
4	3.6066	166.92	14.04	180.96
5	4.3949	136.98	11.52	148.50
6	5.1423	117.07	9.85	126.92
7	5.8506	102.90	8.65	111.55
8	6.5216	92.31	7.76	100.08
9	7.1573	84.11	7.07	91.19
10	7.7591	77.59	6.53	84.11

```
x <- 30;
n <- 1:10
j <- m*a^x;
q <- e^j*gamma(p)*(j^(1-p))*(pgamma(a^n, p, j)
     -pgamma(1, p, j))  # ā_{x:n|};
q4 <- (1-q-e^(j-f*n-j*a^n))/f  # ā_{x:n|};
pr1 <- 1000*e^(b1*x)*q1/q4  # nP_x^(1);
pr2 <- 1000*q3/q4  # nP_x^(2);
pr3 <- pr1+pr2;
d <- round(data.frame(q4, pr1, pr2, pr3), 4);
d1 <- data.frame(n, d);
d1   # Table 2.5;
```

Table 2.5 reports the premiums, payable as the n-year temporary annuity for the whole life insurance. Here the premiums are also higher as compared to those in Table 2.2.

The following R commands compute the premium for the n-year term insurance:

```
a1 <- 0.0008  # A;
b <- 0.00011  # B;
a <- 1.095  # C;
m <- b/log(a, base=exp(1));
e <- exp(1);
del <- 0.05;
b1 <- 0.02;
b2 <- 0.03;
f <- a1+del ;
p <- (-f/log(a, base=exp(1)))+1  # λ1;
f1 <- a1+del-b1;
p1 <- (-f1/log(a, base=exp(1)))+1  # λ2;
f2 <- a1+del-b2;
```

Table 2.6 Premium for n-year term insurance

n	$\bar{a}_{30:\bar{n}\vert}$	$P^{1(1)}_{30:\bar{n}\vert}$	$P^{1(2)}_{30:\bar{n}\vert}$	$P^{1}_{30:\bar{n}\vert}$
1	0.9742	3.22	2.00	5.22
2	1.8984	3.41	2.03	5.44
3	2.7751	3.60	2.06	5.66
4	3.6066	3.81	2.09	5.89
5	4.3949	4.03	2.12	6.14
6	5.1423	4.26	2.15	6.40
7	5.8506	4.50	2.18	6.68
8	6.5216	4.76	2.21	6.96
9	7.1573	5.03	2.23	7.26
10	7.7591	5.32	2.26	7.58

```
p2 <- (-f2/log(a, base=exp(1)))+1  #λ3;
x <- 30;
j <- m*a^x  #αx;
n <- 1:10;
q <- e^j*gamma(p)*(j^(1-p))*(pgamma(a^n, p, j)
     -pgamma(1, p, j));
q1 <- e^(j+b1*x)*gamma(p1)*(j^(1-p1))*(pgamma(a^n,p1,j)
      -pgamma(1, p1, j))  #first term in ĀT;
q2 <- e^j*gamma(p2)*(j^(1-p2))*(pgamma(a^n, p2, j)
      -pgamma(1, p2, j));
q3 <- e^(b2*x)*a1*(1-q2-e^(j-f2*n-j*a^n))/f2;
                                     #second term in ĀT;
q4 <- (1-q-e^(j-f*n-j*a^n))/f  #āx:n|;
pr1 <- 1000* q1/q4;
pr2 <- 1000*q3/q4;
pr3 <- pr1+pr2;
d <- round(data.frame(q4, pr1, pr2, pr3), 2);
d1 <- data.frame(n, d);
d1   #Table 2.6;
```

The premiums for the n-year term insurance are given in Table 2.6.

Next two examples illustrate computation of discrete premiums.

Example 2.3.4 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu^{(1)}_{x+t} = BC^{x+t}, \quad t \geq 0, \qquad \mu^{(2)}_{x+t} = A, \quad t \geq 0,\ A \geq 0,\ B \geq 0,\ C \geq 1.$$

Suppose $A = 0.0008$, $B = 0.00011$, and $C = 1.095$. Further, the force of interest is $\delta = 0.05$. The benefit to be payable at the end of year of death is specified as

$1000e^{0.02(x+k+1)}$ units if death is due to cause 1 and $1000e^{0.03(x+k+1)}$ units if the death is due to cause 2.

(i) Find the premium payable as the discrete life annuity due by (30) for the whole life insurance. Decompose the total premium according to two causes of decrement.
(ii) Find the premium payable as the discrete n-year temporary life annuity due by (30) for the whole life insurance, for $n = 1, 2, \ldots, 10$. Decompose the total premium according to two causes of decrement.
(iii) Find the premium payable as the discrete n-year temporary life annuity due by (30), for the n-year term insurance, for $n = 1, 2, \ldots, 10$.

Solution To find the premiums, we have to first find out the actuarial present value of the benefits and the actuarial present value of the annuities corresponding to two modes of premium payments. The actuarial present value of the benefits in the whole life insurance is given by

$$\begin{aligned} AW_x = &\sum_{k=0}^{\infty} v^{k+1} e^{0.02(x+k+1)} P[K = k, J = 1] \\ &+ \sum_{k=0}^{\infty} v^{k+1} e^{0.03(x+k+1)} P[K = k, J = 2]. \end{aligned}$$

Similarly, the actuarial present value of the benefits in the n-year term insurance is given by

$$\begin{aligned} AT^{1}_{x:\bar{n}|} = &\sum_{k=0}^{n-1} v^{k+1} e^{0.02(x+k+1)} P[K = k, J = 1] \\ &+ \sum_{k=0}^{n-1} v^{k+1} e^{0.03(x+k+1)} P[K = k, J = 2]. \end{aligned}$$

Using the joint distribution of $K(30)$ and $J(30)$ derived in Example 1.2.5, we can obtain these actuarial present values. Further, the actuarial present value of the whole life annuity and n-year temporary life annuity is given by

$$\ddot{a}_{x:\bar{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x^{(\tau)} \quad \text{and} \quad \ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x^{(\tau)},$$

where ${}_kp_x^{(\tau)}$ for the given two decrement model is ${}_kp_x^{(\tau)} = \exp[-Ak - \alpha_x C^k + \alpha_x]$, as derived in Example 1.2.5. The following set of R commands computes all these actuarial present values and the premiums for the two insurance products. Commands at the beginning compute the joint distribution of $K(30)$ and $J(30)$ and are the same as given in Example 1.2.5.

```
a1 <- 0.0008  #A;
b <- 0.00011  #B,
a <- 1.095  #C;
m <- b/log(a, base=exp(1));
e <- exp(1);
f <- (-a1/log(a, base=exp(1)))+1
                #parameter λ as defined in Example 1.2.2;
x <- 30;
k <- 0:69;
j <- m*a^x;
j1 <- m*a^(x+k);
p <- e^(-a1*k+j-j*a^k)  #vector of kp30^(τ) for k=0 to 69;
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1))  #vector of q30+k^(1) for k=0 to 69;
q2 <- 1-e^(-a1-j1*a+j1)-q1
                          #vector of q30+k^(2) for k=0 to 69;
p1 <- p*q1  #vector of P[K(30)=k, J(30)=1] for k=0 to 69;
p2 <- p*q2  #vector of P[K(30)=k, J(30)=2] for k=0 to 69;
del <- 0.05;
v <- e^(-del);
b1 <- 0.02;
b2 <- 0.03;
x <- 30;
w130 <- e^(b1*x)*sum(p1*(v*e^b1)^(k+1))
                                     #first term in A;
w230 <- e^(b2*x)*sum(p2*(v*e^b2)^(k+1))
                                     #second term in A;
w30 <- w130+w230  #A;
w130;  w230;  w30;
wa <- e^j*sum(v^k*e^(-a1*k-j*a^k))  #ä_x;
pw1 <- 1000*w130/wa  #premium corresponding to first
                     #cause in whole life insurance;
pw2 <- 1000*w230/wa  #premium corresponding to second
                     #cause in whole life insurance;
pw <- pw1+pw2  #premium corresponding to whole life
               #insurance;
pw1;  pw2;  pw;
ta <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta <- ta[1:10]  #ä_{x:n̄|} for n=1,2,...,10;
pnw1 <- 1000*w130/nta  #n-year premium corresponding to
                  #first cause in whole life insurance;
pnw2 <- 1000*w230/nta  #n-year premium corresponding
              #to second cause in whole life insurance;
pnw <- pnw1+pnw2  #n-year premium corresponding to whole
                  #life insurance;
```

Table 2.7 n-year discrete premiums for whole life insurance

n	${}_nP^{(1)}$	${}_nP^{(2)}$	${}_nP$
1	593.07	50.12	643.20
2	304.33	25.72	330.05
3	208.18	17.59	225.77
4	160.18	13.54	173.72
5	131.44	11.11	142.55
6	112.33	9.49	121.83
7	98.73	8.34	107.07
8	88.56	7.49	96.05
9	80.69	6.82	87.51
10	74.43	6.29	80.72

```
d <- round(data.frame(pnw1, pnw2, pnw), 2);
n <- 1:10;
d1 <- data.frame(n, d);
d1  #Table 2.7;
t130 <- e^(b1*x)*cumsum(p1*(v*e^b1)^(k+1))
                                  #first term in AT^1_{x:n|};
t230 <- e^(b2*x)*cumsum(p2*(v*e^b2)^(k+1))
                                 #second term in AT^1_{x:n|};
t1 <- t130[1:10]  #AT^1_{x:n|} for n=1,2,...,10, due to cause 1;
t2 <- t230[1:10]  #AT^1_{x:n|} for n=1,2,...,10, due to cause 2;
pt1 <- 1000*t1/nta  #premium corresponding to first
                    #cause in n-year term insurance;
pt2 <- 1000*t2/nta  #premium corresponding to second
                    #cause in n-year term insurance;
pt <- pt1+pt2  #premium for n-year term insurance;
d3 <- round(data.frame(pt1, pt2, pt), 2);
d4 <- data.frame(n, d3);
d4  #Table 2.8;
```

The first term in AW_x is 0.5931, while the second term is 0.0501, and A is 0.6432. It is to be noted that these values are close and slightly smaller than the corresponding quantities in $\bar{A}$ computed in Example 2.3.2. The premiums corresponding to the first cause, the second cause and total premium in the whole life insurance, payable as the whole life annuity due, are, 35.50, 3.00, and 38.50, respectively. Table 2.7 reports the premiums for the whole life insurance when they are paid as the n-year temporary life annuity.

It is to be noted that these premiums are slightly lower than the corresponding continuous premiums. Table 2.8 gives the premiums for the n-year term insurance, with split corresponding to two causes of decrement.

Table 2.8 Discrete premium for n-year term insurance

n	${}_nPT^{(1)}$	${}_nPT^{(2)}$	${}_nPT$
1	3.10	1.93	5.02
2	3.27	1.95	5.23
3	3.46	1.98	5.44
4	3.66	2.01	5.67
5	3.86	2.04	5.90
6	4.08	2.07	6.15
7	4.32	2.10	6.42
8	4.56	2.13	6.69
9	4.83	2.15	6.98
10	5.10	2.18	7.28

Here also, these premiums are again close and slightly lower than the corresponding continuous premiums, as displayed in Table 2.6.

In the previous examples we have obtained the premiums corresponding to two causes of decrement and also the total premium. Example 2.3.5 illustrates similar computations for the whole life insurance in the presence of rider.

Example 2.3.5 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu^{(1)}_{x+t} = BC^{x+t}, \quad t \ge 0, \qquad \mu^{(2)}_{x+t} = A, \quad t \ge 0,\ A \ge 0,\ B \ge 0,\ C \ge 1.$$

Suppose $A = 0.0008$, $B = 0.00011$, and $C = 1.095$. Further, the force of interest is $\delta = 0.05$. The benefit to be payable at the end of year of death is specified as 1000 units in the whole life insurance contract issued to (30). Extra benefit of 1000 units is payable at the end of year of death if death is due to accident before (30) attains age 65. Find the premium payable as the discrete whole life annuity due by (30) for the whole life insurance. Find the extra premium to be payable as the 35-year temporary life annuity due if death occurs before age 65 due to accident.

Solution Let $J = 1$ if death is nonaccidental and $J = 2$ if death is due to accident. Thus the premium for the base policy is given by

$$P^{(\tau)}_{30} = \frac{AW_{30}}{\ddot{a}_x} = \frac{1000\sum_{k=0}^{\infty} v^{k+1} P[K = k]}{\sum_{k=0}^{\infty} v^k\, {}_kp^{(\tau)}_x} = \frac{185.13}{16.71} = 11.08.$$

The extra premium is due to cause 2. So it is given by

$$P^{(2)}_{30} = \frac{AT_{30}}{\ddot{a}_{30:\overline{35}|}} = \frac{1000\sum_{k=0}^{34} v^{k+1} P[K = k,\, J = 2]}{\sum_{k=0}^{34} v^k\, {}_kp^{(\tau)}_x} = \frac{11.97}{15.79} = 0.76.$$

These computations are done using the following set of R commands:

```
a1 <- 0.0008  # A;
b <- 0.00011  # B,
a <- 1.095  # C;
m <- b/log(a, base=exp(1));
e <- exp(1);
f <- (-a1/log(a, base=exp(1)))+1  #parameter λ as
                                    #defined in Example 1.2.2;
x <- 30;
k <- 0:69;
j <- m*a^x;
j1 <- m*a^(x+k);
p <- e^(-a1*k +j -j*a^k)  #vector of kp30^(τ) for k=0 to 69;
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1))  #vector of q30+k^(1) for k=0 to 69;
q2 <- 1-e^(-a1-j1*a+j1)-q1  #vector of q30+k^(2) for k=0 to 69;
p1 <- p*q1  #vector of P[K(30)=k, J(30)=1] for k=0 to 69;
p2 <- p*q2  #vector of P[K(30)=k, J(30)=2] for k=0 to 69;
p3 <- p1+p2;
del <- 0.05;
v <- e^(-del);
w130 <- sum(p3*v^(k+1))   #AW30;
wa <- e^j*sum(v^k*e^(-a1*k-j*a^k))  #ä_x;
pw1 <- 1000*w130/wa  #premium for base policy;
w130;  wa;  pw1;
t130 <- cumsum(p2*v^(k+1));
t1 <- t130[35]  #AT30;
ta <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta <- ta[35]  #ä_x:35|;
pt2 <- 1000*t1/nta  #premium for extra benefit;
t1;  nta;  pt2;
```

With the individual life insurance, sometimes there is a rider for disability benefits. For example, during the period of disability, the premiums for the insurance may be waived, or the policy may contain a provision for monthly income for the period of disability. The actuarial present value of the such benefit and hence the premiums can be determined using the multiple decrement model.

Apart from the premium calculation, another important computation in insurance industry is the reserve computation. We now proceed to discuss reserve calculations for the multiple decrement model in the next section.

2.4 Computation of Reserves

Reserve computations for various insurance products, in continuous and discrete setup, are thoroughly discussed in the literature on Actuarial Statistics. In this section we discuss how the reserve computation for single decrement model gets extended to the multiple decrement model. In insurance industry, usually the reserve for the base policy is computed, and a separate reserve is held for the extra benefit. Hence, we discuss reserve calculations for policies with riders, that is, when the underlying survivorship model is a multiple decrement model. The reserve for the base policy can be computed exactly on similar lines as that for single decrement model with the only modification that μ_x, q_x, and p_x will be replaced by $\mu_x^{(\tau)}$, $q_x^{(\tau)}$, and $p_x^{(\tau)}$, respectively. The reserve for the extra benefit can also be computed exactly on similar lines as that for single decrement model, replacing μ_x, q_x, and p_x by $\mu_x^{(j)}$, $q_x^{(j)}$, and $p_x^{(j)}$, respectively, for $j = 1, 2, \ldots, m$. Thus the theoretical development of the formulae for reserves remains the same as that for the single decrement model. The following example illustrates the computational procedure for discrete reserve.

Example 2.4.1 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu_{x+t}^{(1)} = BC^{x+t}, \qquad \mu_{x+t}^{(2)} = A, \quad A \geq 0,\ B \geq 0,\ C \geq 1.$$

Suppose $A = 0.0008$, $B = 0.00011$, and $C = 1.095$. Further, the force of interest is $\delta = 0.05$. The benefit to be payable at the end of year of death is specified as 1000 units in the whole life insurance contract issued to (30). Extra benefit of 1000 units is payable at the end of year of death if death is due to accident before (30) attains age 65. The premium is payable as the discrete whole life annuity due by (30) for base policy and as the 35-year temporary life annuity due for the extra benefit. Find the reserve at $t = 10, 20, 30, 35$ for the base policy and separate reserve for extra benefit.

Solution We have noted that the reserves can be computed exactly on similar lines as that for a single decrement model. Thus the reserve for the base policy and extra benefit is given by the formula

$$
\begin{aligned}
{}_tV(AW_{30}) &= AW_{30+t} - P_{30}^{(\tau)}\ddot{a}_{30+t} \quad \text{and}\\
{}_tV(AT_{30}) &= AT_{30+t:\overline{35-t}|} - P_{30}^{(2)}\ddot{a}_{30+t:\overline{35-t}|}.
\end{aligned}
$$

In Example 2.3.5 we have calculated $P_{30}^{(\tau)} = 11.08$ and $P_{30}^{(2)} = 0.76$ for the benefit of 1000 units in base policy and extra benefit of 1000 units for accidental death before age 65. The following R commands calculate the actuarial present values required for reserve calculation. In view of having a complete set of commands for reserve, the commands for premium are also included:

```
a1 <- 0.0008   # A;
b <- 0.00011   # B;
a <- 1.095   # C;
m <- b/log(a, base=exp(1));
e <- exp(1);
```

```
f <- (-a1/log(a, base=exp(1)))+1
                #parameter λ as defined in Example 1.2.2;
x <- 30;
k <- 0:69;
j <- m*a^x;
j1 <- m*a^(x+k);
p <- e^(-a1*k+j-j*a^k)  #vector of $_kp_{30}^{(\tau)}$ for $k=0$ to 69;
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1))
                           #vector of $q_{30+k}^{(1)}$ for $k=0$ to 69;
q2 <- 1-e^(-a1-j1*a+j1)-q1
                           #vector of $q_{30+k}^{(2)}$ for $k=0$ to 69;
p1 <- p*q1  #vector of $P[K(30)=k, J(30)=1]$ for $k=0$ to 69;
p2 <- p*q2  #vector of $P[K(30)=k, J(30)=2]$ for $k=0$ to 69;
p3 <- p1+p2;
del <- 0.05;
v <- e^(-del);
x <- 30;
w130 <- sum(p3*v^(k+1))  #$AW_{30}$;
wa <- e^j*sum(v^k*e^(-a1*k-j*a^k))  #$\ddot{a}_{30}$;
pw1 <- 1000*w130/wa   #premium for base policy;
t130 <- cumsum(p2*v^(k+1));
t1 <- t130[35]  #$AT_{30:\overline{35|}}$;
ta <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta <- ta[35]  #$\ddot{a}_{30:\overline{35|}}$;
pt2 <- 1000*t1/nta  #premium for extra benefit;
x <- 40;
j <- m*a^x;
k <- 0:59;
p <- e^(-a1*k+j-j*a^k);
j1 <- m*a^(x+k);
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1));
q2 <- 1-e^(-a1-j1*a+j1)-q1;
p1 <- p*q1;
p2 <- p*q2;
p3 <- p1+p2;
w40 <- sum(p3*v^(k+1))  #$AW_{40}$;
wa40 <- e^j*sum(v^k*e^(-a1*k-j*a^k));
t140 <- cumsum(p2*v^(k+1));
t40 <- t140[25]  #$AT_{40:\overline{25|}}$;
ta40 <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta40 <- ta40[25]  #$\ddot{a}_{40:\overline{25|}}$;
x <- 50;
j <- m*a^x;
```

```
k <- 0:49;
p <- e^(-a1*k+j-j*a^k);
j1 <- m*a^(x+k);
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1));
q2 <- 1-e^(-a1-j1*a+j1)-q1;
p1 <- p*q1;
p2 <- p*q2;
p3 <- p1+p2;
w50 <- sum(p3*v^(k+1))   # AW_50;
wa50 <- e^j*sum(v^k*e^(-a1*k-j*a^k));
t150 <- cumsum(p2*v^(k+1));
t50 <- t150[15]  # AT_{50:15|};
ta50 <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta50 <- ta50[15]  # ä_{50:15|};
x <- 60;
j <- m*a^x;
k <- 0:39;
p <- e^(-a1*k+j-j*a^k);
j1 <- m*a^(x+k);
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1));
q2 <- 1-e^(-a1-j1*a+j1)-q1;
p1 <- p*q1;
p2 <- p*q2;
p3 <- p1+p2;
w60 <- sum(p3*v^(k+1))   # AW_60;
wa60 <- e^j*sum(v^k*e^(-a1*k-j*a^k));
t160 <- cumsum(p2*v^(k+1));
t60 <- t160[5];
ta60 <- e^j*cumsum(v^k*e^(-a1*k-j*a^k));
nta60 <- ta60[5]  # ä_{60:5|};
x <- 65;
j <- m*a^x;
k <- 0:34;
p <- e^(-a1*k+j-j*a^k);
j1 <- m*a^(x+k);
q1 <- e^j1*gamma(f)*(j1^(1-f))*(pgamma(a, f, j1)
      -pgamma(1, f, j1));
q2 <- e^(-a1-j1*a+j1)-q1;
p1 <- p*q1;
p2 <- p*q2;
p3 <- p1+p2;
w65 <- sum(p3*v^(k+1))   # AW_65;
wa65 <- e^j*sum(v^k*e^(-a1*k-j*a^k))  # ä_65;
```

Table 2.9 Discrete reserves for a policy with rider

t	${}_tV_{30}^{(\tau)}$	${}_tV_{30}^{(2)}$	${}_tV_{30}$
10	106.58	−0.02	106.56
20	248.01	−0.04	247.97
30	417.04	−0.04	417.00
35	505.70	0.00	505.70

```
w <- c(w40, w50, w60, w65);
wa <- c(wa40, wa50, wa60, wa65);
vb <- round(1000*(w-pw1*wa), 2)   # tV(AW30);
te <- c(t40, t50, t60, 0);
ta <- c(nta40, nta50, nta60, 0);
vt <- round(1000*(te-pt2*ta), 2)   # tV(AT30);
v1 <- vb+vt;
t <- c(10, 20, 30, 35);
d <- data.frame(t, vb, vt, v1);
d  #Table 2.9;
```

Table 2.9 gives the reserves for base policy and for rider.

The reserve corresponding to extra benefit is negative, implying that the insurer does not have positive liability corresponding to the extra benefit. The negative value may be in view of the fact that the chance of claim due to accidental death is very less, and thus the actuarial present value of the outflow of the company is smaller than that of inflow. This picture is of course for the given model with given set of parameters. We will get different results for different models.

Key Terms Actuarial present value, Premium, Reserves, Term insurance, Whole life insurance.

2.5 Exercises

2.1 For the 20-year term insurance issued to (30), the following information is given.

(i) $\mu_{30+t}^{(1)} = 0.0005t$, where (1) represents death by accidental means.

(ii) $\mu_{30+t}^{(2)} = 0.0025t$, where (2) represents death by other means.

(iii) The benefit is 2000 units if death occurs by accidental means and 1000 units if death occurs by other means.

(iv) Benefits are payable at the moment of death.

Taking $\delta = 0.06$, find the premium payable as (i) the 20-year temporary continuous life annuity, (ii) 20-year temporary life annuity due, (iii) 10-year temporary continuous life annuity, and (iv) 10-year temporary life annuity due.

2.2 A multiple decrement model with 2 causes of decrement is given below in terms of the forces of decrement as

$$\mu_{x+t}^{(1)} = B_1 C_1^{x+t}, \qquad \mu_{x+t}^{(2)} = B_2 C_2^{x+t}, \quad B_i \geq 0,\ C_i \geq 1,\ i = 1, 2.$$

Suppose $B_1 = 0.00012$, $C_1 = 1.094$, $B_2 = 0.00014$, and $C_2 = 1.091$. Further, the force of interest is $\delta = 0.06$. The benefit to be payable at the moment of death is specified as 1000 units in the whole life insurance contract issued to (25). Extra benefit of 1000 units is payable at the moment of death if death is due to accident before (25) attains age 60.

(i) Find the premium payable as the whole life continuous annuity for the whole life insurance issued to (25). Find the extra premium to be payable as the 35-year temporary continuous life annuity if death occurs before age 60 due to accident.

(ii) Find the reserve at $t = 10, 20, 35$ for the base policy and separate reserve for extra benefit.

(iii) Find the premium payable as the discrete whole life annuity due by (25) for the whole life insurance. Find the extra premium to be payable as the 35-year temporary life annuity due if death occurs before age 60 due to accident.

(iv) Find the reserve at $t = 10, 20, 35$ for the base policy and separate reserve for extra benefit.

2.3 A multiple decrement model with two causes of decrement is given below in terms of the forces of decrement as

$$\mu_{x+t}^{(1)} = B_1 C_1^{x+t}, \qquad \mu_{x+t}^{(2)} = B_2 C_2^{x+t}, \quad B_i \geq 0,\ C_i \geq 1,\ i = 1, 2.$$

Suppose $B_1 = 0.00012$ and $C_1 = 1.094$, $B_2 = 0.00014$, and $C_2 = 1.091$. Further, the force of interest is $\delta = 0.06$. The benefit to be payable at the end of year of death is specified as 1000 units in the whole life insurance contract issued to (25). Extra benefit of 1000 units is payable at the end of year of death if death is due to accident before (25) attains age 60.

(i) Find the premium payable as the discrete whole life annuity due by (25) for the whole life insurance. Find the extra premium to be payable as the 35-year temporary life annuity due, if death occurs before age 60 due to accident.

(ii) Find the reserve at $t = 10, 20, 35$ for the base policy and separate reserve for extra benefit.

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