Chapter 2
Extensions of the First BCL

2.1 A Result of Barndorff-Nielsen

The first Borel–Cantelli lemma is simple and almost trivial. Yet, it is necessary to weaken its sufficient condition to tackle some problems of probability theory. The first of such extensions is due to Barndorff-Nielsen (1961), and will be stated below.

Theorem 2.1.1 *Let* $\{A_n\}_{n\geq 1}$ *be a sequence of events such that* $\liminf P(A_n) = 0$ *and* $\sum P(A_n \cap A_{n+1}^c) < \infty$. *Then*

$$P(\limsup A_n) = 0 \textit{ and } P(A_n) \to 0.$$

Proof Put $B_n = A_n \cap A_{n+1}^c$, $n \geq 1$. Then $B_n \in \mathcal{A} \forall n \geq 1$, and $P(\limsup B_n) = 0$ by the first Borel–Cantelli lemma. By (1.2.12) on page 8,

$$(\limsup A_n) \cap (\limsup A_n^c) \subset \limsup B_n.$$

Therefore, inequality (h) of Sect. 1.1 on page 2 implies that

$$P(\limsup A_n) \leq P(\limsup B_n) + P(\liminf A_n).$$

This completes the proof, since $P(\liminf A_n) = 0$ by (1.2.4) on page 7. □

The above proof suggests the following extension of Theorem 2.1.1.

Theorem 2.1.2 *Assume that*

(a) $P(\liminf A_n) = 0$ *(a fortiori,* $\liminf P(A_n) = 0$); *and*
(b) $P(\limsup(A_n \cap A_{n+1}^c)) = 0$ *or* $P(\limsup(A_n^c \cap A_{n+1})) = 0$.

Then $P(\limsup A_n) = 0$. □

The paper by Barndorff-Nielsen (1961) contains an application of Theorem 2.1.1. It is worthwhile to state the following analog of Theorem 2.1.1.

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Theorem 2.1.1′ If $\liminf P(A_n) = 0$ and $\sum_n P(A_n^c \cap A_{n+1}) < \infty$, then

$$P(\limsup A_n) = 0. \qquad \square$$

There is a short proof of the above result which is due to Balakrishnan and Stepanov (2010). This runs as follows: For each $n \geq 1$,

$$\begin{aligned} P(\limsup A_n) \leq P\left(\bigcup_{m=n}^{\infty} A_m\right) &= P(A_n) + P(A_{n+1} \cap A_n^c) \\ &\quad + P(A_{n+2} \cap A_{n+1}^c \cap A_n^c) + \cdots \\ &\leq P(A_n) + \sum_{i=n}^{\infty} P(A_i^c \cap A_{i+1}). \end{aligned}$$

As $\displaystyle\sum_n P(A_n^c \cap A_{n+1}) < \infty$, $\displaystyle\lim_{n\to\infty} \sum_{i=n}^{\infty} P(A_i^c \cap A_{i+1}) = 0$. So

$$P(\limsup A_n) \leq \liminf P(A_n) + \lim_{n\to\infty} \sum_{i=n}^{\infty} P(A_i^c \cap A_{i+1}) = 0.$$

Replacing A_n by $A \cap A_n$ for $n \geq 1$ in Theorem 2.1.2, and using

$$P(\limsup A_n) \leq P((\limsup A_n) \cap A) + P(A^c),$$

we get a further refinement of Theorem 2.1.2.

Theorem 2.1.3 *Assume that*

(a) $P(\liminf(A \cap A_n)) = 0$ *(a fortiori,* $\liminf P(A \cap A_n) = 0)$; *and*
(b) $P(\limsup(A \cap A_n \cap A_{n+1}^c)) = 0$ *or* $P(\limsup(A \cap A_n^c \cap A_{n+1})) = 0$.

Then $P(\limsup A_n) \leq 1 - P(A)$, $\limsup P(A_n) \leq 1 - P(A)$. $\square$

Theorem 2.1.4 (Balakrishnan and Stepanov 2010) *If* $P(A_n) \to 0$ *and*

$$\sum_{n=1}^{\infty} P(A_n^c \cap A_{n+1}^c \cap \cdots \cap A_{n+m-1}^c \cap A_{n+m}) < \infty \tag{2.1.1}$$

for some $m \geq 1$, *then* $P(A_n\ i.o.) = 0$.

Proof Note that for each $n \geq 1$,

$$\begin{aligned}
P(A_n \ i.o.) &\le P\left(\bigcup_{k=n}^{\infty} A_k\right) \\
&= P(A_n) + P(A_n^c \cap A_{n+1}) + P(A_n^c \cap A_{n+1}^c \cap A_{n+2}) + \cdots \\
&\le P(A_n) + P(A_{n+1}) + \cdots + P(A_{n+m-1}) \\
&\quad + \sum_{k=n}^{\infty} P(A_k^c \cap A_{k+1}^c \cap \cdots \cap A_{k+m-1}^c \cap A_{k+m}) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

□

The proof of Theorem 2.1.4 shows that

$$P(A_n \ i.o.) = \lim_{n\to\infty} \left[P(A_n) + \sum_{k=1}^{\infty} P(A_n^c \cap A_{n+1}^c \cap \cdots \cap A_{n+k-1}^c \cap A_{n+k}) \right].$$

There is a dual of Theorem 2.1.4. Before stating it, we shall give an alternative proof of Theorem 2.1.4. The rest of this section is due to Riddhipratim Basu, a graduate student of the Indian Statistical Institute.

Lemma 2.1.1 *Let for some* $m \ge 1$,

$$\begin{aligned}
C_n &= \bigcup_{j=1}^{m} A_{n+j}, \\
B_n &= A_n^c \cap A_{n+1}^c \cap \cdots \cap A_{n+m-1}^c \cap A_{n+m}, \ n \ge 1.
\end{aligned}$$

Then

$$\limsup A_n \subset (\limsup B_n) \cup (\liminf C_n).$$

Proof Let ω lie in LHS. Then $\omega \in A_n$ for infinitely many values of n. Let

$$\{n \ge 1 : \omega \in A_n\} = \{n_1, n_2, \ldots\} \quad \text{where } n_1 < n_2 < \cdots$$

Let $k = \limsup_{i\to\infty} \{n_i - n_{i-1}\}$.

Case 1 $k \ge m + 1$.

Then $\exists$ a subsequence $\{n_{i_j}\}_{j\ge1}$ of $\{n_i\}_{i\ge1}$ such that $n_{i_1} \ge m+1$ and $n_{i_j} - n_{i_j-1} \ge m + 1 \forall j \ge 1$; one can verify this by considering the three subcases '$k = \infty$', '$k = m+1$', and '$k \ge m+2$'. Then $\omega \in B_{n_{i_j-m}}, \forall j \ge 1$; for, if we fix a $j \ge 1$, $\omega \in A_{n_{i_j}}$, and as $n_{i_j-1} \le n_{i_j} - m - 1$, we get by the definition of $\{n_i\}_{n\ge1}$

$$\omega \notin A_{n_{i_j-m}}, \omega \notin A_{n_{i_j-m+1}}, \ldots, \omega \notin A_{n_{i_j-1}}.$$

So $\omega \in \limsup B_n$.

Case 2 $1 \le k \le m$.

Then $\limsup_{i\to\infty}\{n_i - n_{i-1}\} < m + \frac{1}{2}$ so that $\exists i_0 \geq 1$ such that $n_{i_0} \geq m+1$ and $n_i - n_{i-1} \leq m \forall i \geq i_0$. But then

$$\omega \in C_{n-m} \forall n \geq n_{i_0}.$$

To verify this, fix an integer $n \geq n_{i_0}.\exists j \geq 1$ such that $n_j \leq n < n_{j+1}$ for some $j \geq i_0$. Now note that that $\omega \in A_{n_j}$, and so $\omega \in C_{n-m}$ since $n - m + 1 \leq n_{j+1} - 1 - m + 1 \leq n_j$ which implies that $n_j \in \{n-m+1, \ldots, n\}$ and so $C_{n-m} \supset A_{n_j}$.

Hence $\omega \in \liminf C_n$. □

The above lemma immediately implies the following result.

Theorem 2.1.4′ If $P(\liminf C_n) = 0$ and for some $m \geq 1$, (2.1.1) holds, then

$$P(A_n \ i.o.) = 0,$$

where C_n is as in Lemma 2.1.1. □

Lemma 2.1.2 *Let C_n be as in Lemma 2.1.1, and let*

$$B_n^* = A_n \cap A_{n+1}^c \cap \cdots A_{n+m}^c \ for\ some\ m \geq 1.$$

Then

$$\limsup A_n \subset (\limsup B_n^*) \cup (\liminf C_n).$$

Proof Let $\omega \in \limsup A_n$ and $\omega \notin \limsup B_n^*$. Then $\exists$ an integer $m \geq 1$ such that $\omega \notin B_n^* \forall n > m$. Also, $\exists n_0 > m$ such that $\omega \in A_{n_0}$ and so $\omega \notin C_{n_0-1}$. We assert that $\omega \notin C_n \forall n \geq n_0 - 1$. Suppose this is false. Then $\exists$ an integer $n_1 \geq n_0$ such that $\omega \notin C_{n_1}$. Hence

$$\omega \notin A_{n_1+1}, \ldots, \omega \notin A_{n_1+m}.$$

Let n_2 be the largest integers less than $n_1 + 1$ such that $\omega \in A_{n_2}$. Then

$$\omega \in A_{n_2+1}^c, \omega \in A_{n_2+2}^c, \ldots, \omega \in A_{n_2+m}^c.$$

(Distinguish between two cases, e.g., $n_2 + m \leq n_1$ or $n_2 + m > n_1$; in the latter case, note that $n_2 + m \leq n_1 + m$.) Thus $\omega \in B_{n_2}^*$ but $n_2 \geq n_0 > m$. This is a contradiction. □

We have thus proved.

Theorem 2.1.5 *If* $P(\liminf C_n) = 0$ *and for some* $m \geq 1$,

$$\sum P(B_n^*) < \infty,$$

then $P(A_n\ i.o.) = 0$ *where* C_n *and* B_n^* *are as in Lemma 2.1.2.* □

We can combine Theorems 2.1.4′ and 2.1.5 in the following way:

Theorem 2.1.6 *If* $P(\liminf C_n) = 0$ *and*

$$P(\limsup B_n) = 0 \text{ or } P(\limsup B_n^*) = 0,$$

then $P(A_n\ i.o.) = 0$ *where* C_n, B_n *and* B_n^* *are as in Lemmas 2.1.1. and 2.1.2.* □

Remark 2.1.1 This remark is related to Theorems 2.1.4 and 2.1.5. Suppose that $P(A_n) \to 0$ and

$$\sum P(B_n \cap B_{n+1} \cap \cdots \cap B_{n+m}) < \infty$$

where each B_i is either A_i and A_i^c and at least two of the B_i for $i = n, \ldots, n+m$ are the corresponding A_i. Then it need not true that $P(A_n\ i.o.) = 0$. For, we have the counterexample : Let $\{A_n\}_{n\geq 1}$ be independent and $P(A_n) = \frac{1}{n}, n \geq 1$; then the above conditions hold, but $P(A_n\ i.o.) = 1$.

We now give two applications of Theorem 2.1.1.

Example 2.1.1 Let $\{X_n\}_{n\geq 1}$ be pairwise independent, and assume that for each $n \geq 1$,

$$P(X_n > u) = e^{-u}, \quad 0 < u < \infty.$$

(a) Show that

$$\limsup(X_n/\log n) = 1\ a.s. \text{ and } \liminf(X_n/\log n) = 0\ a.s. \tag{2.1.2}$$

(b) Let $X_{(n)} = \max(X_1, \ldots, X_n), n \geq 1$. If $\{X_n\}_{n\geq 1}$ is independent, then

$$X_{(n)}/\log n \to 1\ a.s.$$

Solution: (a) By the Borel-Cantelli lemmas, we have

$$P(X_n > a \log n\ i.o.) = 0 \text{ or } 1 \text{ according as } a > 1 \text{ or } 0 < a \leq 1.$$

This implies the first part of (2.1.2). Since for any $a > 0$, $P(X_n < a\log n) \to 1$, and so $\sum P(X_n < a \log n) = \infty$, $P(X_n < a \log n\ i.o.) = 1 \forall a > 0$. Therefore,

$$\liminf(X_n/\log n) \leq 0\ a.s.$$

which is tantamount to the second part of (2.1.2).

(b) By Example 1.6.25 (a) on page 48, the above arguments yield that $P(X_{(n)} > a \log n\ i.o.) = 0$ and hence

$$\limsup(X_{(n)}/\log n) \leq 1\ a.s.$$

We show below that $\liminf(X_{(n)}/\log n) \geq 1$ $a.s.$ For this, it suffices to show that $P(X_{(n)} > a\log n \text{ eventually}) = 1$ for each $a \in (0, 1)$. By Theorem 2.1.2, it is enough to show that, for $0 < a < 1$,

$$\sum P(A_n \cap A_{n+1}^c) < \infty \text{ and } P(A_n^c \ i.o.) = 1, \tag{2.1.3}$$

where $A_n = [X_{(n)} \leq a\log n]$, $n \geq 1$. By Example 1.6.25 (a), we have

$$P(A_n^c \ i.o) = P(X_n > a\log n \ i.o.) = 1 \text{ for } 0 < a < 1.$$

Finally,

$$\begin{aligned}
\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) &= \sum_{n=1}^{\infty} P(A_n \cap [X_{n+1} > a\log(n+1)]) \\
&= \sum_{n=1}^{\infty} P(A_n)P(X_{n+1} > a\log(n+1)) \\
&= \sum_{n=1}^{\infty} (1 - n^{-a})^n (n+1)^{-a} \\
&\leq \sum_{n=1}^{\infty} n^{-a} \exp(-n^{1-a}) \\
&\leq \sum_{n=1}^{\infty} \int_{n-1}^{n} x^{-a} \exp(-x^{1-a}) \mathrm{d}x \qquad \text{(why?)} \\
&= \int_0^{\infty} x^{-a} \exp(-x^{1-a}) \mathrm{d}x \\
&= \int_0^{\infty} e^{-y} \mathrm{d}y/(1-a) = 1/(1-a) < \infty.
\end{aligned}$$

Example 2.1.2 Let $\{X_n\}$ be pairwise independent and each X_n follow $N(0; 1)$ distribution.

(a) Show that

$$\limsup(X_n/(\sqrt{2\log n})) = 1 \ a.s., \text{ and } \liminf(X_n/(\sqrt{2\log n})) = -1 \ a.s. \tag{2.1.4}$$

(b) Show that, if $X_{(n)} = \max(X_1, \ldots, X_n)$, $n \geq 1$ and $\{X_n\}_{n \geq 1}$ is independent then

$$X_{(n)}/\sqrt{2\log n} \to 1 \ a.s.$$

Solution: We shall use the inequality (1.8) on page 175 of Feller (1968).

(a) Since $P(N(0; 1) > x) \leq \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)$ for $x \geq 1$, we have

$$P(X_n > a\sqrt{2\log n}) = 0 \text{ or } 1 \text{ according as } a > 1 \text{ or } 0 < a \leq 1$$

which implies the first part of (2.1.3). Replacing X_n by $-X_n$ for each $n \geq 1$, we get the second part of (2.1.3).

(b) Following the steps of Example 2.1.1 (b), it suffices to show that for each $a \in (0, 1)$

$$\sum P(A_n \cap A_{n+1}^c) < \infty$$

where $A_n = [X_{(n)} \leq a\sqrt{2 \log n}]$, $n \geq 1$. To do this, note that for a suitable $m \geq 1$

$$\begin{aligned}\sum_{n=m}^{\infty} P(A_n \cap A_{n+1}^c) &\leq \sum_{n=m}^{\infty} \left(1 - \frac{1}{2a\sqrt{2\pi}\sqrt{2\log n}} \exp(-a^2 \log n)\right)^n \\ &\quad \times \frac{1}{\sqrt{2\pi}} \exp(-a^2 \log(n+1)) \\ &\leq d \sum_{n=m}^{\infty} n^{-a^2} \exp(-cn^{-a^2+1}/\sqrt{\log n})\end{aligned} \tag{2.1.5}$$

where $c > 0$, $d > 0$ are suitable constants. In (2.1.5), we have used the fact that

$$P(N(0; 1) > x) \geq \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2) \left(\frac{1}{x} - \frac{1}{x^3}\right) \quad \text{for } x \geq 1$$

and m is an integer such that $(1 - 1/(2a^2 \log n)) \geq 1/2\ \forall n \geq m$. Since

$$n^{-a^2} \exp(-cn^{(1-a^2)}/\sqrt{\log n}) \leq n^{-2} \text{ for all sufficiently large } n$$

which is implied by the following fact

$$\exp\left(cn^{(1-a^2)}/\sqrt{\log n}\right)/n^{2-a^2} \to \infty \text{ for each } a \in (0, 1),$$

(*Proof* take logarithm of both sides and use the fact that

$$n^{\alpha}(\log n)^{-\beta} \to \infty \text{ if } \alpha > 0, \beta > 0.),$$

the desired result follows. □

2.2 Another Result of Barndorff-Nielsen

Theorem 2.1.1 is a special case of

Theorem 2.2.1 *If* $\{A_n\}_{n\geq 1}$ *is a sequence of events such that* $P(A_n) \to 0$ *and* $\sum P(A_n \cap A_{n+\nu_n}^c) < \infty$ *for some sequence* $\{\nu_n\}_{n\geq 1}$ *of positive integers, then* $P(A_n\ i.o.) = 0$.

Proof For every $k \geq 1$, define a sequence of integers $\{i_{k,n}\}_{n\geq 1}$ as follows:

$$i_{k,n} = \begin{cases} k & \text{if } n = 1; \\ i_{k,n-1} + \nu_{i_{k,n-1}} & \text{if } n \geq 2. \end{cases}$$

We have $P(A_n \cap A^c_{n+\nu_n}\ i.o.) = 0$. As

$$A_{i_{k,n}} \cap A_{i_{k,n+1}} = A_{i_{k,n}} \cap A^c_{i_{k,n}+\nu_{i_{k,n}}},$$

we have, by Theorem 2.1.1, $P(A_{i_{k,n}}\ i.o.(n)) = 0$. So

$$P\left(\cap_{n=1}^{\infty} A_{i_{k,n}}\right) = 0 \text{ and hence } P\left(\cup_{k=1}^{\infty} \cap_{n=1}^{\infty} A_{i_{k,n}}\right) = 0.$$

The proof will be complete if we show that

$$\limsup A_n \subset \left(\cup_{k=1}^{\infty}\ \cap_{n=1}^{\infty} A_{i_{k,n}}\right) \cup [A_n \cap A^c_{n+\nu_n}\ i.o.].$$

To this end, let $\omega \in \limsup A_n$ and $\omega \notin \limsup(A_n \cap A^c_{n+\nu_n})$. Then $\exists$ an integer $m \geq 1$ such that $\omega \notin A_n \cap A^c_{n+\nu_n} \forall n > m$. Let $p > m$ be such that $\omega \in A_p$. So $\omega \in A_{p+\nu_p} = A_{i_{p,2}}$ and hence

$$\omega \in A_{i_{p,2}} + \nu_{i_{p,2}} = A_{i_{p,3}},$$

and so on. Thus $\omega \in \bigcap\limits_{n=1}^{\infty} A_{i_{p,n}}$ and so $\omega \in \bigcup\limits_{k=1}^{\infty} \bigcap\limits_{n=1}^{\infty} A_{i_{k,n}}$. □

2.3 Results of Loève and Nash

We shall first discuss a result of Loève (1951), as stated in Nash (1954); it gives a necessary and sufficient condition for $P(\limsup A_n) = 0$.

The *necessary part* runs as follows: If $P(\limsup A_n) = 0$, then there exists an integer $m \geq 1$ such that whenever $n \geq m$,

$$P(A^c_n \cap A^c_{n+1} \cap \cdots \cap A^c_{k-1}) > 0 \quad \forall k > n \tag{2.3.1}$$

and

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} p_{nk} = 0 \tag{2.3.2}$$

where for $k > n$

$$p_{nk} = P(A_k | A^c_n \cap A^c_{n+1} \cap \cdots \cap A^c_{k-1}), \text{ and } p_{nn} = P(A_n). \tag{2.3.3}$$

For a proof, first recall from the remark after the proof of Theorem 2.1.4 that

$$P(\limsup A_n) = 0 \Leftrightarrow \lim_{n\to\infty} \left[p_{nn} + \sum_{k=n+1}^{\infty} P(A_n^c \cap \cdots \cap A_{k-1}^c \cap A_k) \right] = 0.$$

As $P(\limsup A_n) = \lim_{n\to\infty} P\left(\overset{\infty}{\underset{k=n}{\cup}} A_k\right) = 0, \exists$ an integer $m \geq 1$ such that $P\left(\overset{\infty}{\underset{k=n}{\cup}} A_k\right) < 1 \forall n \geq m$. Then (2.3.1) holds, since

$$(A_n^c \cap \cdots \cap A_{k-1}^c) \supset \left(\overset{\infty}{\underset{k=n}{\cup}} A_k\right)^c.$$

Now note that if $k > n$,

$$p_{nk} \leq P(A_n^c \cap \cdots \cap A_{k-1}^c \cap A_k) / \left[1 - P\left(\overset{\infty}{\underset{k=n}{\cup}} A_k\right)\right]$$

so that

$$\sum_{k=n}^{\infty} p_{nk} \leq \left[P(A_n) + \sum_{k=n+1}^{\infty} P(A_n^c \cap \cdots \cap A_{k-1}^c \cap A_k) \right] / \left[1 - P\left(\overset{\infty}{\underset{k=n}{\cup}} A_k\right)\right]$$

and hence (2.3.2) follows by letting $n \to \infty$ in the above inequality.

The *converse* runs as follows: If $\exists$ an integer $m \geq 1$ such that (2.3.1) holds and if (2.3.2) holds, then $P(\limsup A_n) = 0$. For a proof, we need to only note that if $k > n$,

$$P(A_n^c \cap \cdots \cap A_{k-1}^c \cap A_k) \leq p_{nk}.$$

We next turn the main result of Nash (1954). We first introduce a notation. Let $A^0 = A^c$ and $A^1 = A$, and define

$$A^{\boldsymbol{\epsilon}} = \overset{\infty}{\underset{n=1}{\cap}} A_n^{\epsilon_n}$$

where each ϵ_n is 0 or 1 and $\boldsymbol{\epsilon} = (\epsilon_1, \epsilon_2, \ldots)$; here $\{A_n\}_{n\geq 1}$ is any given sequence of events. Let

$$H = \{\boldsymbol{\epsilon} : \epsilon_i = 1 \text{ for finitely many values of } i\},$$
$$H_0 = \{\boldsymbol{\epsilon} \in H : P\left(A_1^{\epsilon_1} \cap \cdots \cap A_n^{\epsilon_n}\right) > 0 \quad \forall n \geq 1\}.$$

Then it is well known that H is countably infinite. Clearly,

$$P\left(\underset{\epsilon\in H\setminus H_0}{\cup} A^{\epsilon}\right) = 0. \tag{2.3.4}$$

Next, assume that $\boldsymbol{\epsilon} \in H_0$. Then $P(A^{\epsilon}|A_1^{\epsilon_1} \cap \cdots \cap A_n^{\epsilon_n})$ is well defined for each $n \geq 1$, and

$$P(A^{\epsilon}) = P(A_1^{\epsilon_1}) \prod_{n=2}^{\infty} \left(1 - P\left(A_n^{1-\epsilon_n}|A_1^{\epsilon_1} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right)\right).$$

Thus, using Theorem 8.52 on page 208 of Apostol (1974),

$$\begin{aligned} P(A^{\epsilon}) = 0 &\Leftrightarrow \sum_{n=2}^{\infty} P\left(A_n^{1-\epsilon_n}|A_1^{\epsilon_1} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right) = \infty \\ &\Leftrightarrow \sum_{n=2}^{\infty} P\left(A_n|A_1^{\epsilon_1} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right) = \infty \end{aligned}$$

since $\boldsymbol{\epsilon} \in H$ and hence $A_n^{1-\epsilon_n} = A_n$ for all sufficiently large n. Now observe that $\liminf A_n^c = \cup\{A^{\epsilon}|\boldsymbol{\epsilon} \in H\}$ since

$$\begin{aligned} \omega \in \liminf A_n^c &\Leftrightarrow \exists \text{ an integer } m(\omega) \geq 1 \text{ such that } \omega \notin A_n \forall n \geq m(\omega) \\ &\Leftrightarrow \omega \in A^{\epsilon} \text{ for some } \boldsymbol{\epsilon} \in H. \end{aligned}$$

Thus,

$$\begin{aligned} P(\liminf A_n^c) = 0 &\Leftrightarrow P\left(\underset{\epsilon\in H_0}{\cup} A^{\epsilon}\right) = 0 \quad \text{(by (2.3.4))} \\ &\Leftrightarrow P(A^{\epsilon}) = 0 \quad \forall \boldsymbol{\epsilon} \in H_0 \\ &\Leftrightarrow \sum P(A_n|A_1^{\epsilon_1} \cap \cdots \cap A_n^{\epsilon_n}) = \infty \forall \boldsymbol{\epsilon} \in H_0. \end{aligned}$$

We have thus proved the result of Nash, namely, $P(\limsup A_n) = 1$ iff $\sum P\left(A_n|A_1^{\epsilon_1} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right) = \infty \forall \boldsymbol{\epsilon} \in H_0$. A related result is given in Bruss (1980).

We next give an application of Nash's result. There are two urns each containing a red and b black balls. A ball is drawn at random from the first urn. This is repeated until a black ball is drawn. Each time a red ball is drawn from the first urn, the number of balls in the second run is doubled by putting in as many red balls as there are balls of either color in the second urn before. Once a black ball is drawn from the first urn, all further draws are made at random from the second urn with replacement after each draw, and no further change is made in the composition of the contents of the second urn. Let A_n be the event of drawing a black ball in the nth trial. We show below that $P(\limsup A_n) = 1$. Note that

$$P\left(A_n|A_1^{\epsilon_1} \cap A_2^{\epsilon_2} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right) = \begin{cases} 1/2 & \text{if } n < k; \\ 2^{-k} & \text{if } n \geq k, \end{cases}$$

where $k > 1$ is such that a black ball is drawn for the first time at $(k-1)$th trial. (This is true, because the second urn will contain 2^k balls at the kth trial and thereafter, $2^k - 1$ red and 1 black balls.) Then $p'_n := \inf_{n>1} P(A_n | A_1 \cap \cdots \cap A_{n-1}) = 2^{-n} > 0$; but $\sum p'_n = 1 < \infty$, so that the hypothesis of Borel's criterion (stated in the historical remarks on page 18) does not hold. However,

$$\sum P\left(A_n | A_1^{\epsilon_1} \cap \cdots \cap A_{n-1}^{\epsilon_{n-1}}\right) = \infty \forall \epsilon,$$

and so $P(\limsup A_n) = 1$.

We conclude this section with a result of Martikainen and Petrov (1990).

Theorem 2.3.1 (Martikainen and Petrov 1990) *Let* $0 < \alpha \leq 1$.

(a) The following are equivalent:

- *(i)* $P(A_n \; i.o.) \geq \alpha$.
- *(ii)* $\sum P(A_n \cap B) = \infty$ *for any event* B *satisfying* $P(B) > 1 - \alpha$.
- *(iii)* $P(A_n \cap B) > 0$ *for infinitely many values of* n *for every event* B *satisfying* $P(B) > 1 - \alpha$.

(b) The following are equivalent:

- *(iv)* $P(A_n \; i.o.) = \alpha$
- *(v) Statement (ii) holds and for each* $\epsilon > 0, \exists$ *an event* B_0 *such that* $P(B_0) > 1 - \alpha - \epsilon$ *and* $\sum P(A_n \cap B_0) < \infty$.
- *(vi) Statement (iii) holds and for each* $\epsilon > 0, \exists$ *an event* B_0 *such that* $P(B_0) > 1 - \alpha - \epsilon$ *and* $P(A_n \cap B_0) = 0$ *for all sufficiently large* n.

Proof (a) See Petrov (1995, p. 201).

(b) (v) $\Rightarrow$ (iv): Clearly, $P(A_n \; i.o.) \geq \alpha$ by (a). If possible, let $P(A_n \; i.o.) > \alpha$. Then $\exists$ an $\epsilon > 0$ such that $P(A_n \; i.o.) \geq \alpha + \epsilon$. By (a), $\sum P(A_n \cap B) = \infty$ for every event B satisfying $P(B) > 1 - \alpha - \epsilon$. This contradicts the second condition of (v).

(vi) $\Rightarrow$ (v): This is clear.

(iv) $\Rightarrow$ (vi): Clearly, Statement (iii) holds by (c). If possible, let the second condition of (vi) fail. Then $\exists$ an $\epsilon > 0$ such that $P(A_n \cap B) > 0$ for infinitely many value of n for every event B satisfying $P(B) > 1 - \alpha - \epsilon$. Then $P(A_n \; i.o.) \geq \alpha + \epsilon$ by (a), contradicting (iv). □

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