

---

# Contents

<b>1</b>	<b>Introduction</b>	1
1.1	Virtual velocity laws	2
1.2	Virtual displacement laws	4
1.3	Virtual work laws as principles of mechanics	5
1.4	Virtual work laws as theorems of mechanics	9
1.5	Contemporary tendencies	10
1.6	Final remarks. The rational justification of virtual work laws	12
<b>2</b>	<b>Logic status of virtual work laws</b>	15
2.1	The theorem of virtual work	17
2.1.1	Proofs of the virtual work theorems in the literature	23
2.1.1.1	Physics and rational mechanics treatises	23
2.1.1.2	Statics handbooks	24
2.1.1.3	Poinsot's proof	26
2.2	The principle of virtual work	27
2.2.1	Force as a primitive concept	28
2.2.1.1	Equilibrium case	28
2.2.1.2	Motion case	29
2.2.2	Work as a primitive concept	31
2.2.2.1	Equilibrium case	31
2.2.2.2	Motion case	32
<b>3</b>	<b>Greek origins</b>	33
3.1	Different approaches to the law of the lever	34
3.1.1	Aristotelian mechanics	34
3.1.1.1	<i>Physica</i> and <i>De caelo</i>	35
3.1.1.2	<i>Mechanica problemata</i>	38
3.1.1.3	A law of virtual work	43
3.1.2	Archimedean mechanics	45
3.1.2.1	Proof of the law of the lever	48
3.2	The mechanics of Hero of Alexandria	51

3.2.1	The principles of Hero's mechanics .....	53
3.2.1.1	A law of virtual work .....	55
3.2.1.2	Hero's inclined plane law .....	58
3.3	The mechanics of Pappus of Alexandria .....	59
3.3.1	Pappus' inclined plane law .....	60
<b>4</b>	<b>Arabic and Latin science of weights .....</b>	<b>63</b>
4.1	Arabic mechanics .....	66
4.1.1	Weight as an active factor in Arabic mechanics .....	68
4.1.1.1	<i>Liber karastonis</i> .....	69
4.1.1.2	<i>Kitab al-Qarastun</i> .....	73
4.1.2	Comments on the Arabic virtual work law .....	74
4.2	Latin mechanics .....	75
4.2.1	Weight as a passive factor in the Latin mechanics .....	80
4.2.2	Propositions .....	80
4.2.2.1	Proposition I .....	81
4.2.2.2	Proposition II .....	84
4.2.2.3	Proposition VI. The law of the Lever .....	86
4.2.2.4	Proposition VIII .....	86
4.2.2.5	Proposition X. The law of the inclined plane ..	88
4.2.3	Comments on the Latin virtual work law .....	89
<b>5</b>	<b>Italian Renaissance statics .....</b>	<b>91</b>
5.1	Renaissance engineering .....	95
5.1.1	Daniele Barbaro and Buonaventura Lorini .....	96
5.2	Niccolò Tartaglia .....	97
5.2.1	Definitions and petitions .....	98
5.2.2	Propositions .....	98
5.2.2.1	Proof of propositions I–IV .....	100
5.2.2.2	The law of the lever .....	101
5.2.2.3	The law of the inclined plane .....	103
5.3	Girolamo Cardano .....	104
5.3.1	<i>De subtilitate</i> .....	105
5.3.2	<i>De opus novum</i> .....	107
5.4	Guidobaldo dal Monte .....	108
5.4.1	The centre of gravity .....	109
5.4.2	The balance .....	109
5.4.3	The virtual work law .....	115
5.5	Giovanni Battista Benedetti .....	116
5.5.1	Effect of the position of a weight on its heaviness .....	116
5.5.2	Errors of Tartaglia and Jordanus .....	118
5.6	Galileo Galilei .....	120
5.6.1	The concept of moment. A law of virtual velocities ....	121
5.6.2	A law of virtual displacements .....	127
5.6.3	Proof of the law of the inclined plane .....	131

<b>6</b>	<b>Torricelli's principle</b>	135
6.1	The centrobaric	135
6.2	Galileo's centrobaric	138
6.3	Torricelli's joined heavy bodies	140
6.3.1	Torricelli's fundamental concepts on the centre of gravity	141
6.4	Torricelli's principle	144
6.4.1	Analysis of the aggregate of two bodies	146
6.4.2	Torricelli's principle as a criterion of equilibrium	148
6.5	Evolution of Torricelli's principle. Its role in virtual work laws	153
6.5.1	A restricted form of Torricelli's principle	154
<b>7</b>	<b>European statics during the XVI and XVII centuries</b>	157
7.1	French statics	157
7.1.1	Gille Personne de Roberval	160
7.1.1.1	The inclined plane law	160
7.1.1.2	The rule of the parallelogram	161
7.1.2	René Descartes	164
7.1.2.1	The concept of force	164
7.1.2.2	The application to simple machines	167
7.1.2.3	The refusal of virtual velocities	170
7.1.2.4	Displacements at the very beginning of motion	171
7.1.2.5	A possible precursor	173
7.1.3	Blaise Pascal	175
7.1.4	Post Cartesians	176
7.2	Nederland statics	177
7.2.1	Simon Stevin	178
7.2.1.1	The rule of the parallelogram of forces	180
7.2.1.2	The law of virtual work	184
7.2.2	Christiaan Huygens	187
7.3	British statics	189
7.3.1	John Wallis	190
7.3.2	Isaac Newton	193
<b>8</b>	<b>The principle of virtual velocities</b>	195
8.1	The concept of force in the XVIII century	195
8.1.1	Newtonian concept of force	195
8.1.2	Leibnizian concept of force	197
8.2	Johann Bernoulli mechanics	199
8.2.1	Dead and living forces according to Bernoulli	199
8.2.2	The rule of energies	201
8.3	Varignon: the rule of energies and the law of composition of forces	210
8.3.1	Elements of Varignon's mechanics	210
8.3.2	The rule of the parallelogram versus the rule of energies	213

<b>9</b>	<b>The Jesuit school of the XVIII century</b>	217
9.1	Vincenzo Angiulli and Vincenzo Riccati	218
9.1.1	The principle of actions of Vincenzo Angiulli	218
9.1.1.1	The action of a force	219
9.1.1.2	The principle of actions	221
9.1.1.3	The measure of actions	223
9.1.1.4	The principle of action and the principles of statics	225
9.1.1.5	The applications to simple machines	228
9.1.2	The principle of actions of Vincenzo Riccati	230
9.2	Ruggiero Giuseppe Boscovich	233
9.2.1	A virtual work law for Saint Peter's dome	234
9.2.1.1	The mechanism of failure and the forces	235
<b>10</b>	<b>Lagrange's contribution</b>	237
10.1	First introduction of the virtual velocity principle	240
10.1.1	The first ideas about a new principle of mechanics	240
10.1.2	<i>Recherches sur la libration de la Lune</i>	242
10.1.2.1	Setting of the astronomical problem	245
10.1.2.2	The symbolic equation of dynamics	247
10.1.2.3	The virtual velocity principle	250
10.1.3	The <i>Théorie de la libration de la Lune</i>	251
10.2	<i>Mécanique analitique and Mécanique analytique</i>	252
10.2.1	<i>Mécanique analitique</i>	253
10.2.1.1	Constraint reactions	258
10.2.2	<i>Mécanique analytique</i>	259
10.2.2.1	Criticisms of Lagrange's proof	263
10.3	The <i>Théorie des fonctions analytiques</i>	264
10.4	Generalizations of the virtual velocity principle to dynamics	268
10.4.1	The calculus of variations	273
10.4.2	Elements of D'Alembert's mechanics	274
10.4.2.1	D'Alembert principle	277
<b>11</b>	<b>Lazare Carnot's mechanics of collision</b>	281
11.1	Carnot's laws of mechanics	285
11.1.1	The first fundamental equation of mechanics	287
11.1.2	Geometric motions	289
11.1.3	The second fundamental equation of mechanics	291
11.2	Gradual changing of motion. A law of virtual work	293
11.3	The moment of activity	295
<b>12</b>	<b>The debate in Italy</b>	299
12.1	The criticisms on the evidence of the principle	300
12.1.1	Vittorio Fossombroni	300
12.1.1.1	Invariable distance systems	301

12.1.1.2	The equation of forces . . . . .	302
12.1.1.3	The equation of moments . . . . .	304
12.1.2	Girolamo Saladini . . . . .	306
12.1.3	François Joseph Servois . . . . .	308
12.2	The criticisms on the use of infinitesimals . . . . .	311
12.2.1	Gabrio Piola . . . . .	312
12.2.1.1	Piola's principles of material point mechanics . . . . .	312
12.2.1.2	System of free material points . . . . .	314
12.2.1.3	System of constrained material points . . . . .	315
<b>13</b>	<b>The debate at the École polytechnique . . . . .</b>	<b>317</b>
13.1	One of the first professor of mechanics, Gaspard de Prony . . . . .	319
13.1.1	Proof from the composition of forces rule . . . . .	320
13.2	Joseph Fourier . . . . .	321
13.2.1	First proof . . . . .	323
13.2.2	Second proof . . . . .	325
13.2.3	Third proof . . . . .	326
13.3	André Marie Ampère . . . . .	328
13.4	Pierre Simon Laplace . . . . .	332
<b>14</b>	<b>Poinsot's criticism . . . . .</b>	<b>335</b>
14.1	<i>Considérations sur le principe des vitesses virtuelles</i> . . . . .	336
14.2	<i>Théorie générale de l'équilibre et du mouvement des systèmes</i> . . . . .	339
14.2.1	Poinsot's principles of mechanics . . . . .	342
14.2.1.1	System of material points constrained by a unique equation . . . . .	344
14.2.1.2	System of material points constrained by more equations . . . . .	346
14.3	Demonstration of the virtual velocity principle . . . . .	348
<b>15</b>	<b>Complementary virtual work laws . . . . .</b>	<b>353</b>
15.1	Augustin Cauchy formulation . . . . .	354
15.1.1	Kinematics of plane rigid bodies . . . . .	356
<b>16</b>	<b>The treatises of mechanics . . . . .</b>	<b>361</b>
16.1	Siméon Denis Poisson . . . . .	362
16.2	Jean Marie Duhamel . . . . .	365
16.3	Gaspard Gustave Coriolis . . . . .	367
<b>17</b>	<b>Virtual work laws and continuum mechanics . . . . .</b>	<b>375</b>
17.1	First applications . . . . .	375
17.1.1	Joseph Louis Lagrange . . . . .	375
17.1.1.1	Mono-dimensional continuum . . . . .	376
17.1.1.2	Three-dimensional continuum . . . . .	377
17.1.2	Navier's equations of motion . . . . .	381
17.2	Applications in the theory of elasticity . . . . .	383

17.2.1	Alfred Clebsch . . . . .	383
17.3	The Italian school . . . . .	387
17.3.1	Gabrio Piola . . . . .	388
17.3.2	Eugenio Beltrami . . . . .	390
17.3.3	Enrico Betti . . . . .	392
<b>18</b>	<b>Thermodynamical approach . . . . .</b>	<b>395</b>
18.1	Pierre Duhem's concept of oeuvre . . . . .	396
18.1.1	Virtual transformations . . . . .	397
18.1.2	Activity, energy and work . . . . .	398
18.1.3	Rational mechanics . . . . .	401
18.1.3.1	Free systems . . . . .	401
18.1.3.2	Constrained systems . . . . .	402
<b>A</b>	<b>Quotations . . . . .</b>	<b>405</b>
A.1	Chapter 1 . . . . .	405
A.2	Chapter 2 . . . . .	406
A.3	Chapter 3 . . . . .	407
A.4	Chapter 4 . . . . .	409
A.5	Chapter 5 . . . . .	412
A.6	Chapter 6 . . . . .	423
A.7	Chapter 7 . . . . .	426
A.8	Chapter 8 . . . . .	433
A.9	Chapter 9 . . . . .	437
A.10	Chapter 10 . . . . .	441
A.11	Chapter 11 . . . . .	448
A.12	Chapter 12 . . . . .	452
A.13	Chapter 13 . . . . .	454
A.14	Chapter 14 . . . . .	457
A.15	Chapter 15 . . . . .	463
A.16	Chapter 16 . . . . .	464
A.17	Chapter 17 . . . . .	467
A.18	Chapter 18 . . . . .	471
	<b>References . . . . .</b>	<b>473</b>
	<b>Index . . . . .</b>	<b>489</b>

History of Virtual Work Laws

A History of Mechanics Prospective

Capecchi, D.

2012, XII, 492 p., Hardcover

ISBN: 978-88-470-2055-9