
Preface

This book contains an exposition of the basic ideas and techniques of the theory of groups, starting from scratch and gradually delving into more profound topics. It is not intended as a treatise in group theory, but rather as a text book for advanced undergraduate and graduate students in mathematics, physics and chemistry, that will give them a solid background in the subject and provide a conceptual starting point for further developments. Both finite and infinite groups are dealt with; in the latter case, the relationship with logical and decision problems is part of the exposition.

The general philosophy of the book is to stress what according to the author is one of the main features of the group structure: groups are a means of classification, via the concept of the action on a set, thereby inducing a partition of the set into classes. But at the same time they are the object of a classification. The latter feature, at least in the case of finite groups, aims at answering the following general problem: how many groups of a given type are there, and how can they be described? We have tried to follow, as far as possible, the road set for this purpose by Hölder in his program. It is possible to give a flavor of this research already at an elementary level: we formulate Brauer's theorem on the bound of the order of a group as a function of the order of the centralizer of an involution immediately after the introduction of the basic concepts of conjugacy and that of centralizer.

The relationship with Galois theory is often taken into account. For instance, after proving the Jordan-Hölder theorem, which is the starting point for the classification problem, we show that this theorem arises in Galois theory when one considers the extensions of a field by means of the roots of two polynomials. When dealing with the theory of permutation groups we will have an opportunity to stress this relationship.

Groups of small order can be classified using the concepts of direct and semidirect product; moreover, these concepts provide a first introduction to the theory of extensions that will be considered in greater detail in the chapter on the cohomology of groups. The problem of finding all the extensions of a

group by another is one half of Hölder's program, the other half being the classification of all finite simple groups.

A large section is devoted to the symmetric and the alternating groups, and to the relevance of the cycle structure for the determination of the automorphism group of the symmetric group. The notion of an action of a group on a set is thoroughly dealt with. An action provides a first example of a representation of a group, in the particular case of permutation matrices, and of notions like that of induced representation (Mackey's theorem is considered in this framework). These notions are usually introduced in the case of linear representations over the complex numbers (a whole chapter is devoted to this topic); we think that important concepts can already be introduced in the case of permutation representations, allowing a better appreciation of what is implied by the introduction of linearity. For instance, two similar actions have the same character, but the converse is false in the case of permutation representations and true in the case of the linear ones.

Among other things, the notion of an action is used to prove Sylow's theorem. Always in the vein of the classification problem, this theorem can be used to prove that many groups of small order, or whose order has a special arithmetic structure, cannot be simple. A section is devoted to prove that some projective groups are simple, and to the proof of the simplicity of the famous Klein group of order 168.

The classification of finite abelian groups has been one of the first achievements of abstract group theory. We show that the classification of finitely generated abelian groups is in fact a result in linear algebra (the reduction of an integer matrix to the Smith normal form). This allows us to introduce the concepts of a group given by generators and relations, first for abelian groups and then in the general case.

Infinite groups play an important role both in group theory and in other fields of mathematics, like geometry and topology. The concept of a free group is especially important: a large section is devoted to it, including the Nielsen-Schreier theorem on the freeness of a subgroup of a free group. As already mentioned, infinite groups allow the introduction of logical and decision problems, like the word problem. Among other things, we prove that the word problem is solvable for a finitely presented simple group; Malcev's theorem on hopfian groups is also proved. The analogy between complete theories in logic, simple groups and Hausdorff spaces in topology is explained.

This kind of problem is also considered in the case of nilpotent groups; the word problem is solvable for these groups. A detailed analysis of nilpotent groups is given, also in the finite case. The notion of nilpotence is shown to be the counterpart of the same notion in the theory of rings. In the finite case, nilpotent groups are generalized to p -nilpotent groups: the transfer technique is employed to prove Burnside's criterion for p -nilpotence. The relationship between the existence of fixed-point-free automorphisms and p -nilpotence is also illustrated.

A whole area of problems concerns the so-called local structure of groups, a local subgroup being the normalizer of a p -subgroup. A major breakthrough in this area is Alperin's theorem, which we prove; it shows that conjugacy has a local character.

Solvable groups are a natural generalization of nilpotent groups, although their interest goes far beyond this property, especially for the role they play in Galois theory. Among other things, we explain what is the meaning in Galois theory of the result on the solvability of the transitive subgroups of the symmetric group on a prime number of elements.

The technique of using a minimal normal subgroup for proving results on solvable groups is fully described and applied, as well as the meaning of the Fitting subgroup as a counterpart of the center in the case of nilpotent groups. Carter subgroups are also introduced, and the analogy with the Cartan subalgebras of Lie algebras is emphasized.

The last two chapters are more or less independent of the rest of the book. Representation theory is a subject in itself, but its applications in group theory are often unavoidable. For instance, we give the standard proof of Burnside's theorem on the solvability of groups divisible by at most two primes, a proof that only needs the first rudiments of the theory, but that is very difficult and involved without representation theory.

Finally, the cohomology theory of groups is dealt with. We give a down to earth treatment mainly considering the aspects related to the extensions of groups. We prove the Schur-Zassenhaus theorem, both for the interest it has in itself, and to show how cohomological methods may be used to prove structural results. Schur's multiplier and its relationship with projective representations close the section and the book.

Every section of the book contains a large number of exercises, and some interesting results have been presented in this form; the text proper does not make use of them, except where specifically indicated. They amount to more than 400. Hints to the solution are given, but for almost all of them a complete solution is provided.

Rome, September 2011

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An Introduction to Ideas and Methods of the Theory of
Groups

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2012, XIII, 371 p., Softcover

ISBN: 978-88-470-2420-5