

Preface

As is well-known, the classical Brownian motion is a stochastic process which is self-similar of index $1/2$ and has stationary increments. It is actually the only continuous Gaussian process (up to a constant factor) to have these two properties that are often observed in the ‘real life’, for instance in the movement of particles suspended in a fluid, or in the behavior of the logarithm of the price of a financial asset. More generally, it is natural to wonder whether there exists a stochastic process which would be at the same time Gaussian, with stationary increments and selfsimilar, but not necessarily with an index $1/2$ as in the Brownian motion case. Such a process happens to exist, and was introduced by Kolmogorov [27] in the early 1940s for modeling turbulence in liquids. The name *fractional Brownian motion* (fBm in short), which is the terminology everyone uses nowadays, comes from the paper by Mandelbrot and Van Ness [29].

The law of fBm relies on a single parameter H between 0 and 1, the so-called Hurst parameter or selfsimilarity index. Fractional Brownian motion is interesting for modeling purposes, as it allows the modeler to adjust the value of H to be as close as possible to its observations. It is worthwhile noting at this stage, however, that the picture is not as rosy as it seems. Indeed, except when its selfsimilarity index is $1/2$, fBm is neither a semimartingale, nor a Markov process. As a consequence, its toolbox is limited, so that solving problems involving fBm is often a non-trivial task. On the positive side, fBm offers new challenges for the specialists of stochastic calculus!

The goal of this book is to develop some aspects of fBm (as well as related topics), without seeking for completeness at all. To be comprehensive would have been an impossible task to fulfill anyway, given the huge amount of works that are nowadays dedicated to fBm¹. Instead, my guiding thread was to develop the topics I found the most aesthetic (with all the subjectivity it implies!) by trying to avoid technicalities as much as possible, in order to show the reader that solving questions involving fBm may lead to beautiful mathematics. In fact, it was often an excuse for the development of a more general theory, for which the fBm then becomes a concrete and significant example.

¹ According to MathSciNet, at the time this book is written there is more than one thousand papers dealing with fBm.

The plan of the book, which is directed at an audience having a reasonable probability background (at the level of any of the standard texts) is as follows. Chapter 1 gathers the needed preliminary results; in particular, it recalls the exact definitions of Gaussian vectors, sequences and processes, as well as their basic properties. It also contains a proof of the existence of the fBm. Chapter 2 introduces the fBm and provides some of its main properties. Chapter 3 develops an integration theory with respect to fBm. When the Hurst parameter is greater than $1/2$, it also gives a framework allowing to solve integral equations involving fBm. Chapter 4 is devoted to the study of the asymptotic behavior of the cumulative distribution function of the supremum of fBm. In passing, we prove that the supremum of any Gaussian process roughly behaves like a single Gaussian variable with variance equal to the largest variance achieved by the entire process. Chapter 5 contains all the definitions and results on Malliavin calculus that are relevant throughout the sequel. Chapter 6 gives a complete characterization of CLTs on the Wiener space in terms of “fourth moments conditions”, by combining Stein’s method with Malliavin calculus. The asymptotic behavior of the quadratic variation of fBm is then studied in detail. Chapter 7 shows how fBm (as well as another related process) arises naturally in the large limit of partial sums associated to time series with long-range dependence. Finally, Chapter 8 extends some of the results of Chapter 7 to the free probability context; in particular, it introduces the reader to a non-commutative counterpart of the fBm.

It is fair to mention that this book is not the only treatise devoted to fBm. Other references (focusing mainly on different aspects of fBm) include the books by Biagini, Hu, Øksendal and Zhang [3], Cohen and Istas [11], Embrechts and Maejima [17], Hu [22], Mishura [30], Nualart [45], Pipiras and Taqqu [49], Prakasa Rao [50] and Samorodnitsky and Taqqu [56]. Also, it is worthwhile noting that the present book complements in many respects the recent monograph [39] by the author with Peccati (and viceversa), that only tangentially deals with fBm.

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