Chapter 2
Weyl Quantization and Coherent States

Abstract It is well known from the work of Berezin (Commun. Math. Phys. 40:153–174, 1975) in 1975 that the quantization problem of a classical mechanical system is closely related with coherent states. In particular coherent states help to understand the limiting behavior of a quantum system when the Planck constant $\hbar$ becomes negligible in macroscopic units. This problem is called the semi-classical limit problem.

In this chapter we discuss properties of quantum systems when the configuration space is the Euclidean space $\mathbb{R}^n$, so that in the Hamiltonian formalism, the phase space is $\mathbb{R}^n \times \mathbb{R}^n$ with its canonical symplectic form σ. The quantization problem has many solutions, so we choose a convenient one, introduced by Weyl (The Classical Groups, 1997) and Wigner (Group Theory and Its Applications to Quantum Mechanics of Atomic Spectra, 1959).

We study the symmetries of Weyl quantization, the operational calculus and applications to propagation of observables.

We show that Wick quantization is a natural bridge between Weyl quantization and coherent states. Applications are given of the semi-classical limit after introducing an efficient modern tool: semi-classical measures.

We illustrate the general results proved in this chapter by explicit computations for the harmonic oscillator. More applications will be given in the following chapters, in particular concerning propagators and trace formulas for a large class of quantum systems.

2.1 Classical and Quantum Observables

The quantization problem comes from quantum mechanics and is a mathematical setting for the Bohr correspondence principle between the classical world and the quantum world.

Let us consider a system with n degrees of freedom. According the Bohr correspondence principle, it is natural to check a way to associate to every real function A on the phase space $\mathbb{R}^{2n}$ (classical observable) a self-adjoint operator $\hat{A}$ in the Hilbert space $L^2(\mathbb{R}^n)$ (quantum observable). According the quantum mechanical principles, the map $A \to \hat{A}$ has to satisfy some properties.

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(1) $A \to \hat{A}$ is *linear*, $\hat{A}$ is self-adjoint if A is real and $\hat{1} = \mathbb{1}_{L^2(\mathbb{R}^n)}$.
(2) *position observables*: $x_j \to \hat{x}_j := \hat{Q}_j$ where $\hat{Q}_j$ is the multiplication operator by x_j.
(3) *momentum observables* : $\xi_j \to \hat{\xi}_j := \hat{P}_j$ where $\hat{P}_j$ is the differential operator $\frac{\hbar}{i}\frac{\partial}{\partial x_j}$.
(4) *commutation rule and classical limit*: for every classical observables A, B we have

$$\lim_{\hbar \to 0}\left(\frac{i}{\hbar}[\hat{A}, \hat{B}] - \widehat{\{A, B\}}\right) = 0.$$

Let us recall that $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ is the commutator of $\hat{A}$ and $\hat{B}$, $\{A, B\}$ is the Poisson bracket defined as follows:

$$\{A, B\}(x, \xi) = (\partial_x A \cdot \partial_\xi B - \partial_x B \cdot \partial_\xi A)(x, \xi), \quad x, \xi \in \mathbb{R}^n.$$

Let us remark that if we introduce $\nabla A = (\partial_x A, \partial_\xi A)$ then we have $\{A, B\}(x, \xi) = \sigma(\nabla A(x, \xi), \nabla B(x, \xi))$ (σ is the symplectic bilinear form).

If the observables A, B depend only on the position variable (or on the momentum variables) then $\hat{A} \cdot \hat{B} = \widehat{A.B}$ but, this is no longer true for a mixed observable. This is related to the non-commutativity for product of quantum observables and the identity: $[\hat{x}_j, \hat{\xi}_j] = i\hbar$ so, the quantum observable corresponding to $x_1\xi_1$ is not determined by the rules (1) to (4).

We do not want to discuss here the quantization problem in its full generality (see for example [77]). One way to choose a reasonable and convenient quantization procedure is the following, which is called Weyl quantization (see [117] for more details). Let L_z be a real linear form on the phase space $\mathbb{R}^{2n}$, where $z = (p, q)$, $L_z(x, \xi) = \sigma(z, (x, \xi))$ (every linear form on $\mathbb{R}^{2n}$ is like this). It is not difficult to see that $\hat{L}_z$ is a well defined quantum Hamiltonian (i.e. an essentially self-adjoint operator in $L^2(\mathbb{R}^n)$). Its propagator $\mathrm{e}^{\frac{-it}{\hbar}\widehat{L_z}}$ has been studied in Chap. 1.

Remark that we have $\hat{L}_z = -\hat{L}(z)$, with the notation of Chap. 1.

For $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have explicitly

$$\mathrm{e}^{\frac{-it}{\hbar}\widehat{L_z}}\psi(x) = \mathrm{e}^{-\frac{i}{2\hbar}t^2 q\cdot p}\mathrm{e}^{\frac{it}{\hbar}x\cdot p}\psi(x - tq). \tag{2.1}$$

So, the Weyl prescription is defined by the conditions (1) to (4) and the following:

(5)

$$\mathrm{e}^{-iL_z(x,\xi)} \to \widehat{\mathrm{e}^{-iL_z}} = \hat{T}(z)$$

We shall use freely the Schwartz space $\mathcal{S}(\mathbb{R}^n)$[1] and its dual $\mathcal{S}'(\mathbb{R}^n)$ (temperate distributions space).

[1]Recall that $f \in \mathcal{S}(\mathbb{R}^n)$ means that f is a smooth function in $\mathbb{R}^n$ and for every multiindices α, β, $x^\alpha \partial_x^\beta u$ is bounded in $\mathbb{R}^n$. It has a natural topology. $\mathcal{S}'(\mathbb{R}^n)$ is the linear space of continuous linear form on $\mathcal{S}(\mathbb{R}^n)$.

Proposition 11 *There exists a unique continuous map $A \to \hat{A}$ from $\mathcal{S}'(\mathbb{R}^{2n})$ into $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ satisfying conditions* (1) *to* (5).

Moreover if $A \in \mathcal{S}(\mathbb{R}^{2n})$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ we have the familiar formula

$$\hat{A}\psi(x) = (2\pi\hbar)^{-n} \iint_{\mathbb{R}^{2n}} A\left(\frac{x+y}{2}, \xi\right) e^{i\hbar^{-1}(x-y)\cdot\xi} \psi(y)\, dy\, d\xi, \tag{2.2}$$

and $\hat{A}$ is a continuous map from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$.

The hermitian conjugate of $\hat{A}$ is the quantization of the complex conjugate of A i.e. $(\hat{A})^* = \hat{\bar{A}}$. In particular $\hat{A}$ is Hermitian if and only if A is real.

Proof Here it is enough to assume that $\hbar = 1$.

Let us consider the symplectic Fourier transform in $\mathcal{S}'(\mathbb{R}^{2n})$. Assume first that $A \in \mathcal{S}(\mathbb{R}^{2n})$.

$$\tilde{A}(z) = \int_{\mathbb{R}^{2n}} A(\zeta) e^{-i\sigma(z,\zeta)}\, d\zeta. \tag{2.3}$$

We have the inverse formula

$$A(X) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{A}(z) e^{i\sigma(z,X)}\, dz. \tag{2.4}$$

For $\psi, \eta \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\langle \psi, \hat{A}\eta \rangle = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{A}(z) \left\langle e^{i\hat{L}_z}\psi, \eta \right\rangle dz. \tag{2.5}$$

In other words we get

$$\hat{A}\psi = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \tilde{A}(z) \hat{T}(z)\psi\, dz. \tag{2.6}$$

□

Definition 2 For a given operator $\hat{A}$, the function A is called the contravariant symbol of $\hat{A}$ and the function $\tilde{A}$ is the covariant symbol of $\hat{A}$.

Let us remark that we have the inverse formula

Proposition 12 *If $\hat{A}$ is a continuous map from $\mathcal{S}'(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^n)$ then we have for every $X \in \mathbb{R}^{2n}$,*

$$\tilde{A}(X) = \mathrm{Tr}\big(\hat{A}\hat{T}(-X)\big). \tag{2.7}$$

Proof For $X = 0$ the formula is a consequence of the Fourier inversion formula.

For any X we use that the Weyl symbol of $\hat{T}(-X)$ is $z \mapsto e^{-i\sigma(z,X)}$. □

As a consequence we have a first norm operator estimate. If $\tilde{A} \in L^1(\mathbb{R}^{2n})$ we have

$$\|\hat{A}\| \leq (2\pi)^{-n} \int_{\mathbb{R}^{2n}} |\tilde{A}(z)| \, dz. \tag{2.8}$$

The r.h.s. in formula (2.2) can be extended by continuity in A to the distribution space $\mathcal{S}'(\mathbb{R}^{2n})$.

Let us compute now the Schwartz kernel K_A of the operator $\hat{A}$ defined in formula (2.6). We have

$$K_A(x, y) = \int_{\mathbb{R}^n} \tilde{A}(x - y, p) \mathrm{e}^{ip\cdot(x+y)/2} \, dp. \tag{2.9}$$

Using inverse Fourier transform in p variables, we get

$$K_A(x, y) = (2\pi)^{-n} \int_{\mathbb{R}^n} A\left(\frac{x+y}{2}, \xi\right) \mathrm{e}^{i(x-y)\cdot\xi} \, d\xi \tag{2.10}$$

this gives (2.2). The other properties are easy to prove and left to the reader.

Let us first remark that from (2.10) we get a formula to compute the $\hbar$-Weyl symbol of $\hat{A}$ if we know its Schwartz kernel K

$$A(x, \xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{i}{\hbar} u\cdot\xi} K\left(x + \frac{u}{2}, x - \frac{u}{2}\right) du. \tag{2.11}$$

Sometimes, we shall use also the notation $\hat{A} = \mathrm{Op}_\hbar^w A$ ($\hbar$-Weyl quantization of A). Hence we shall say that $\hat{A}$ is an $\hbar$-pseudodifferential operators and that A is its Weyl symbol. For applications it is useful to be able to read properties of the operator $\hat{A}$ on its Weyl symbol A. A first example is the Hilbert–Schmidt property.

Proposition 13 *Let $\hat{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$. Then $\hat{A}$ is Hilbert–Schmidt in $L^2(\mathbb{R}^n)$ if and only if $A \in L^2(\mathbb{R}^{2n})$ and we have*

$$\|\hat{A}\|_{HS}^2 = (2\pi\hbar)^{-n} \iint_{\mathbb{R}^{2n}} |A(x, \xi)|^2 \, dx \, d\xi. \tag{2.12}$$

In particular if $\hat{A}$ and $\hat{B}$ are two Hilbert–Schmidt operators then $\hat{A}.\hat{B}$ is a trace operator and we have

$$\mathrm{Tr}(\hat{A}.\hat{B}) = (2\pi\hbar)^{-n} \iint_{\mathbb{R}^{2n}} A(x, \xi) B(x, \xi) \, dx \, d\xi. \tag{2.13}$$

Proof We know that

$$\|\hat{A}\|_{HS}^2 = \iint_{\mathbb{R}^{2n}} |K_A(x, y)|^2 \, dx \, dy.$$

Then we get the proposition using formula (2.10) and Plancherel theorem. □

We shall see later many other properties concerning Weyl quantization but most of time we only have sufficient conditions on A to have some property of $\hat{A}$, like for example L^2 continuity or trace-class property.

Let us give a first example of computation of a Weyl symbol starting from an integral kernel. We consider the heat semi-group $e^{-t\hat{H}_{os}}$, of the harmonic oscillator $\hat{H}_{os}$. Let us denote $K_w(t; x, \xi)$ the Weyl symbol of $e^{-t\hat{H}_{os}}$ and $K(t; x, y)$ its integral kernel. From formula (2.11) we get

$$K_w(t; x, \xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} u\cdot\xi} K\left(t; x + \frac{u}{2}, x - \frac{u}{2}\right) du. \tag{2.14}$$

Using Mehler formula (1.72) we have to compute the Fourier transform of a generalized Gaussian function, so after some computations, we get the following nice formula:

$$K_w(t; x, \xi) = \big(\cos(t/2)\big)^{-n/2} e^{-\tanh(t/2)(x^2+\xi^2)}. \tag{2.15}$$

Recall that $x^2 = x \cdot x = |x|^2$.

2.1.1 Group Invariance of Weyl Quantization

Let us first remark that an easy consequence of the definition of Weyl quantization is the invariance by translations in the phase space. More precisely, we have, for any classical observable A and any $z \in \mathbb{R}^{2n}$,

$$\hat{T}(z)^{-1} \hat{A} \hat{T}(z) = \widehat{A \cdot T(z)}, \quad \text{where } A \cdot T(z)(z') = A(z' - z). \tag{2.16}$$

Hamiltonian classical mechanics is invariant by the action of the group $\mathrm{Sp}(n)$ of symplectic transformations of the phase space $\mathbb{R}^{2n}$. A natural question to ask is to quantize linear symplectic transformations. We shall see later how it is possible. In this section we state the main results.

Recall that the symplectic group $\mathrm{Sp}(n)$ is the group of linear transformations of $\mathbb{R}^{2n}$ which preserves the symplectic form σ. So $F \in \mathrm{Sp}(n)$ means that $\sigma(FX, FY) = \sigma(X, Y)$ for all $X, Y \in \mathbb{R}^{2n}$. If we introduce the matrix

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

then

$$F \in \mathrm{Sp}(n) \quad \Longleftrightarrow \quad F^t J F = J, \tag{2.17}$$

where F^t is the transposed matrix of F.

If $n = 1$ then F is symplectic if and only if $\det(F) = 1$.

Linear symplectic transformations can be quantized as unitary operators in $L^2(\mathbb{R}^n)$

Theorem 2 *For every linear symplectic transformation $F \in \mathrm{Sp}(n)$ and every symbol $A \in \Sigma(1)$ we have*

$$\hat{R}(F)^{-1}\hat{A}\hat{R}(F) = \widehat{A \cdot F}. \tag{2.18}$$

Moreover $\hat{R}(F)$ is unique up to multiplication by a complex number of modulus 1

Definition 3 The *metaplectic group* is the group $\mathrm{Met}(n)$ generated by $\hat{R}(F)$ and $\lambda\mathbb{1}$, $\lambda \in \mathbb{C}$, $|\lambda| = 1$.

Remark 4 A consequence of Theorem 2 is that $\hat{R}$ is a projective representation of the symplectic group $\mathrm{Sp}(n)$ in the Hilbert space $L^2(\mathbb{R}^n)$. It is a particular case of a more general setting [193].

More properties of the metaplectic group will be studied in the next chapter. Let us give here some examples of the metaplectic transform.

- The Fourier transform $\mathcal{F}$ is associated with the symplectic transformation $(x, \xi) \mapsto (\xi, -x)$.
- The partial Fourier transform $\mathcal{F}_j$, in variable x_j, is associated with the symplectic transform:

$$(x_j, \xi_j) \mapsto (\xi_j, -x_j), \qquad (x_k, \xi_k) \mapsto (x_k, \xi_k), \quad \text{if } k \neq j.$$

- Let A be a linear transformation on $\mathbb{R}^n$, the transformation $\psi \mapsto |\det(A)|^{1/2} \times \psi(Ax)$ is associated with the symplectic transform

$$F_A\begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{pmatrix} Ax \\ (A^t)^{-1}\xi \end{pmatrix}.$$

- Let A be a real symmetric matrix, the transformation $\psi \mapsto \mathrm{e}^{iAx\cdot x/2}\psi$ is associated with the symplectic transform

$$F = \begin{pmatrix} \mathbb{1} & 0 \\ A & \mathbb{1} \end{pmatrix}.$$

2.2 Wigner Functions

Let $\varphi, \psi \in L^2(\mathbb{R}^n)$. They define a rank one operator $\Pi_{\psi,\varphi}\eta = \langle \psi, \eta \rangle \varphi$. Its Weyl symbol can be computed using (2.11).

Definition 4 The Wigner function of the pair (ψ, φ) is the Weyl symbol of the rank one operator $\Pi_{\psi,\varphi}$. It will be denoted $\mathcal{W}_{\varphi,\psi}$. More explicitly we have

$$\mathcal{W}_{\varphi,\psi}(x,\xi) = \int_{\mathbb{R}^n} \mathrm{e}^{-\frac{i}{\hbar}u\cdot\xi} \varphi\left(x + \frac{u}{2}\right) \overline{\psi\left(x - \frac{u}{2}\right)}\, du. \tag{2.19}$$

An equivalent definition of the Wigner function is the following:

$$\mathcal{W}_{\varphi,\psi}(z) = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} \langle \varphi, \hat{T}(z')\psi \rangle \mathrm{e}^{-i\sigma(z,z')/\hbar}\, dz', \tag{2.20}$$

where $\hat{T}(z) = \mathrm{e}^{-i\hat{L}_z}$.

We can easily see that (2.19) and (2.20) are equivalent using formula (2.6) and Plancherel formula for symplectic Fourier transform.

The Wigner functions are very convenient to use. In particular we have the following nice property:

Proposition 14 *Let us assume that $\hat{A}$ is Hilbert–Schmidt and $\psi, \varphi \in L^2(\mathbb{R}^n)$. Then we have*

$$\langle \psi, \hat{A}\varphi \rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A(X) \mathcal{W}_{\psi,\varphi}(X)\, dX. \tag{2.21}$$

If $A \in \mathcal{S}'(\mathbb{R}^{2n})$ and if $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$, the formula (2.21) is still true in the weak sense of temperate distributions.

Proof Let us first remark that $\langle \psi, \hat{A}\varphi \rangle = \mathrm{Tr}(\hat{A}\Pi_{\psi,\varphi})$. Hence the first part of the proposition comes from (2.13).

Now if $\psi, \varphi \in \mathcal{S}(\mathbb{R}^n)$ then we easily get $\mathcal{W}_{\psi,\varphi} \in \mathcal{S}(\mathbb{R}^{2n})$. On the other side there exists $A_j \in \mathcal{S}(\mathbb{R}^{2n})$ such that $A_j \to A$ in $\mathcal{S}'(\mathbb{R}^{2n})$. So we apply (2.21) to A_j and we go to the limit in j. □

What Wigner was looking for was an equivalent of the classical probability distribution in the phase space $\mathbb{R}^{2n}$. That is, associated to any quantum state a distribution function in phase space that imitates a classical distribution probability in phase space. Recall that a classical probability distribution is a non-negative Borel function ρ; $Z \to \mathbb{R}^+$, $Z := \mathbb{R}^{2n}$, normalized to unity:

$$\int_Z \rho(z)\, dz = 1,$$

and such that the average of any observable $A \in \mathcal{C}^\infty$ is simply given by

$$\rho(A) = \int_Z A(z)\rho(z)\, dz.$$

From Proposition 14 we see that a possible candidate is

$$\rho(z) = (2\pi\hbar)^{-n}\mathcal{W}_{\varphi,\varphi}.$$

Actually in the physical literature the expression above (with the factor $(2\pi\hbar)^{-n}$) is taken as the definition of the Wigner function but we do not take this convention.

In the following we denote by $\mathcal{W}_\varphi$ the Wigner transform for φ, φ.

What about the expected properties of $(2\pi\hbar)^{-n}\mathcal{W}_\varphi$ as a possible probability distribution in phase space? Namely:

- positivity
- normalization to 1
- correct marginal distributions

Proposition 15 *Let $z = (x, \xi) \in \mathbb{R}^{2n}$ and $\varphi \in L^2(\mathbb{R}^n)$ with $\|\varphi\| = 1$. We have*

(i)

$$(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\mathcal{W}_\varphi(x,\xi)\,d\xi = \left|\varphi(x)\right|^2,$$

which is the probability amplitude to find the quantum particle at position x.

(ii)

$$(2\pi\hbar)^{-n}\int_{\mathbb{R}^n}\mathcal{W}_\varphi(x,\xi)\,dx = \left|\tilde{\varphi}(\xi)\right|^2,$$

which is the probability amplitude to find the quantum particle at momentum ξ.

(iii)

$$(2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}}\mathcal{W}_\varphi(x,\xi)\,dx\,d\xi = 1.$$

(iv) $\mathcal{W}_\varphi(x,\xi) \in \mathbb{R}$.

Proof

(i) Let $f \in \mathcal{S}$ be an arbitrary test function. We have

$$\begin{aligned}
&\int_{\mathbb{R}^n}\mathcal{W}_\varphi(x,\xi)f(\xi)\,d\xi \\
&\quad= \int dy\,\bar{\varphi}\left(x+\frac{y}{2}\right)\varphi\left(x-\frac{y}{2}\right)\int d\xi\,\mathrm{e}^{-i\xi\cdot y/\hbar}f(\xi) \\
&\quad= (2\pi\hbar)^n\int_{\mathbb{R}^n}dy\,\bar{\varphi}\left(x+\frac{y}{2}\right)\varphi\left(x-\frac{y}{2}\right)(\mathcal{F}f)(y).
\end{aligned} \tag{2.22}$$

By taking for the usual Fourier transform $\mathcal{F}f$ an approximation of the Dirac distribution at $y = 0$ we get the result.

(ii) Is proven similarly.
(iii) Follows from the normalization to unity of the state φ.
(iv) We have

$$\mathcal{W}_\varphi(z)^* = (2\pi\hbar)^{-n}\int dz'\,\langle\varphi, \hat{T}(-z')\varphi\rangle e^{i\sigma(z,z')/\hbar}$$

and the result follows by change of the integration variable $z' \to -z'$ and by noting that $\sigma(z,-z') = -\sigma(z,z')$. □

Let us now compute the Wigner function $\mathcal{W}_{z,z'}$ for a pair $(\varphi_z, \varphi_{z'})$ of coherent states.

Proposition 16 *For every $X, z, z' \in \mathbb{R}^{2n}$ we have*

$$\mathcal{W}_{z,z'}(X) = 2^n \exp\left(-\frac{1}{\hbar}\left|X - \frac{z+z'}{2}\right|^2 - \frac{i}{\hbar}\sigma\left(X - \frac{1}{2}z', z - z'\right)\right). \tag{2.23}$$

Proof It is enough to consider the case $\hbar = 1$. Let us apply formula (2.20):

$$\mathcal{W}_{z,z'}(X) = (2\pi)^{-n}\int_{\mathbb{R}^{2n}} \langle\varphi_z, \hat{T}(z'')\varphi_{z'}\rangle e^{-i\sigma(X,z'')}\,dz''. \tag{2.24}$$

Using formula (1.7) from Chap. 1, we have

$$\begin{aligned}\langle\varphi_z, \hat{T}(z'')\varphi_{z'}\rangle &= \langle\varphi_z, \varphi_{z'+z''}\rangle e^{\frac{i}{2}\sigma(z',z'')}\\ &= e^{-\frac{1}{4}|z-z'-z''|^2} e^{\frac{i}{2}\sigma(z,z'+z'')+\sigma(z',z'')}.\end{aligned} \tag{2.25}$$

Using the change of variables $z'' = z - z' + u$, we have to compute the Fourier transform of the standard Gaussian $e^{-|u|^2/4}$ and (2.23) follows. □

We have the following properties of the Wigner transform:

Proposition 17 *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$ be two quantum states. Then $\mathcal{W}_{\varphi,\psi} \in L^2(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ and we have*

(i)

$$\|\mathcal{W}_{\varphi,\psi}\|_{L^\infty} \le 2^n \|\varphi\|_2 \|\psi\|_2.$$

(ii)

$$\|\mathcal{W}_{\varphi,\psi}\|_{L^2} \le (2\pi\hbar)^{n/2} \|\varphi\|_2 \|\psi\|_2.$$

(iii) *Let $\varphi, \psi \in L^2(\mathbb{R}^n)$. Then we have*

$$\left|\langle\varphi, \psi\rangle\right|^2 = (2\pi\hbar)^{-n}\langle\mathcal{W}_\varphi, \mathcal{W}_\psi\rangle_{L^2(\mathbb{R}^{2n})}.$$

Proof (i) is a simple consequence of the definition of the Wigner transform and of the Cauchy–Schwartz inequality. For the proof of (ii) we note that

$$\int dz \left|\mathcal{W}_{\varphi,\psi}(z)\right|^2 = \int dx\, d\xi \left|\int dy\, \mathrm{e}^{i\xi\cdot y/\hbar}\bar{\varphi}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\right|^2.$$

Using an approximation argument, we can assume that $\varphi, \psi \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. So we have $\bar{\varphi}(x+\frac{y}{2})\psi(x-\frac{y}{2}) \in L^2(\mathbb{R}^n, dy)$. According to the Plancherel theorem we have

$$\begin{aligned}(2\pi\hbar)^{-n} &\int d\xi \left|\int dy\, \mathrm{e}^{i\xi\cdot y/\hbar}\bar{\varphi}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\right|^2 \\ &= \int dy \left|\bar{\varphi}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\right|^2\end{aligned}$$

so that

$$\begin{aligned}\int dz \left|\mathcal{W}_{\varphi,\psi}(z)\right|^2 &= (2\pi\hbar)^n \int dx \int dy \left|\bar{\varphi}\left(x+\frac{y}{2}\right)\psi\left(x-\frac{y}{2}\right)\right|^2 \\ &= (2\pi\hbar)^n \|\varphi\|^2 \|\psi\|^2. \end{aligned} \tag{2.26}$$

□

The Wigner transform operate "as one wishes" in phase space, namely according to the scheme of classical mechanics:

Proposition 18 *Let* $\varphi, \psi \in L^2(\mathbb{R}^n)$ *and* $\hat{T}(z)$, $\hat{R}(F)$ *be, respectively, operators of the Weyl–Heisenberg and metaplectic groups, corresponding, respectively, to*

- *a phase-space translation by vector* $z \in \mathbb{R}^{2n}$
- *a symplectic transformation in phase space*

We have

$$\mathcal{W}_{\hat{T}(z')\varphi,\hat{T}(z')\psi}(z) = \mathcal{W}_{\varphi,\psi}(z-z'), \tag{2.27}$$

$$\mathcal{W}_{\hat{R}(F)\varphi,\hat{R}(F)\psi}(z) = \mathcal{W}_{\varphi,\psi}\left(F^{-1}z\right). \tag{2.28}$$

Proof We have the nice group property of the Weyl–Heisenberg translation operator:

$$\hat{T}(-z')\hat{T}(X)\hat{T}(z') = \exp\left(-\frac{i}{\hbar}\sigma(X,z')\right)\hat{T}(X)$$

so that

$$\begin{aligned}\mathcal{W}_{\hat{T}(z')\varphi,\hat{T}(z')\psi}(z) &= (2\pi\hbar)^{-n}\int dX \exp\left(-\frac{i}{\hbar}\sigma(z-z',X)\right)\left\langle\varphi, \hat{T}(X)\psi\right\rangle \\ &= \mathcal{W}_{\varphi,\psi}(z-z').\end{aligned}$$

As a result of the property of the metaplectic transformation we have

$$\hat{R}(F)^{-1}\hat{T}(z')\hat{R}(F) = \hat{T}\big(F^{-1}z'\big).$$

Therefore

$$\begin{aligned}
\mathcal{W}_{\hat{R}(F)\varphi,\hat{R}(F)\psi}(z) &= (2\pi\hbar)^{-n}\int dz'\,\big\langle\varphi,\hat{T}(Fz')\psi\big\rangle e^{-i\sigma(z,z')/\hbar} \\
&= (2\pi\hbar)^{-n}\int dz''\,\big\langle\varphi,\hat{T}(z'')\psi\big\rangle e^{-i\sigma(z,Fz'')/\hbar} \\
&= (2\pi\hbar)^{-n}\int dz'\,\big\langle\varphi,\hat{T}(z')\psi\big\rangle e^{-i\sigma(F^{-1}z,z')/\hbar},
\end{aligned}$$

where we have used the change of variable $Fz' = z''$ and the fact that a symplectic matrix has determinant one. □

Now we get a formula to recover the Weyl symbol of any operator $\hat{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$.

Proposition 19 *Every operator $\hat{A} \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ has a contravariant Weyl symbol A and a covariant Weyl symbol $\tilde{A}$ in $\mathcal{S}'(\mathbb{R}^{2n})$.*

We have, in the distribution sense in general, in the usual sense if $\hat{A}$ is bounded in $L^2(\mathbb{R}^n)$,

$$A(X) = (2\pi\hbar)^{-2n}\iint_{\mathbb{R}^{4n}}\big\langle\varphi_{z'},\hat{A}\varphi_z\big\rangle\mathcal{W}_{z',z}(X)\,dz\,dz', \tag{2.29}$$

$$\tilde{A}(X) = (2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}}\big\langle\varphi_{z+X},\hat{A}\varphi_z\big\rangle e^{-\frac{i}{\hbar}\sigma(X,z)}\,dz. \tag{2.30}$$

Proof We compute formally. It is not very difficult to give all the details for a rigorous proof.

We apply inverse formula for the Fourier–Bargmann transform (see Chap. 1). So for any $\psi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\hat{A}\psi(x) = (2\pi\hbar)^{-2n}\iint_{\mathbb{R}^{4n}}\big\langle\varphi_{z'},\hat{A}\varphi_z\big\rangle\langle\varphi_z,\psi\rangle\varphi_{z'}(x)\,dz\,dz'. \tag{2.31}$$

So we get a formula for the Schwartz kernel K_A for $\hat{A}$,

$$K_A(x,x') = (2\pi\hbar)^{-2n}\iint_{\mathbb{R}^{4n}}\big\langle\varphi_{z'},\hat{A}\varphi_z\big\rangle\overline{\varphi_z(x')}\varphi_{z'}(x)\,dz\,dz'. \tag{2.32}$$

Then we apply formula (2.11) to get the contravariant symbol A.

The formula for the covariant symbol follows from (2.7) and trace computation with coherent states. □

The only but important missing property to have a nice probabilistic setting with the Wigner functions is *positivity* which is unfortunately not satisfied because we have the following result, proved by Hudson [120] for $n = 1$, then extended to $n \geq 2$ by Soto–Claverie [181].

Theorem 3 *$\mathcal{W}_\psi(X) \geq 0$ on $\mathbb{R}^{2n}$ if and only if $\psi = C\varphi_z^{(\Gamma)}$ where C is a complex number, Γ a complex, symmetric $n \times n$ matrix with a positive non degenerate imaginary part $\Im\Gamma$, $z \in \mathbb{R}^{2n}$, where we define the Gaussian*

$$\varphi^{(\Gamma)}(x) = (\pi\hbar)^{-n/4} \det{}^{1/4} \Im\Gamma \exp\left(\frac{i}{2\hbar}\Gamma x \cdot x\right). \tag{2.33}$$

Proof We more or less follow the paper of Soto–Claverie [181].

We can check by direct computation that the Wigner density of ${\varphi_z}^{(\Gamma)}$ is positive (according the definition we have to compute the Fourier transform of the exponent of a quadratic form). We can also give the following more elegant proof. First, it is enough to consider the case $z = (0, 0)$. Second, it is possible to find a metaplectic transformation F such that ${\varphi_z}^{(\Gamma)} = \hat{R}(F)\varphi_0$ (see the section on symplectic invariance and Chap. 3 for more properties on the metaplectic group). Hence we get $\mathcal{W}_{\hat{R}(F)\varphi_0}(X) = \mathcal{W}_{\varphi_0}(F^{-1}(X))$. But we have computed above $\mathcal{W}_{\varphi_0}$, which is a standard Gaussian, so it is positive.

Conversely, assume now that $\mathcal{W}_\psi(X) \geq 0$ on $\mathbb{R}^{2n}$. We shall prove that the Fourier–Bargmann transform $\psi^\#(z)$ is a Gaussian function on the phase space. Hence using the inverse Bargmann transform formula, we shall see that ψ is a Gaussian.

Let us first prove the two following properties:

$$\psi^\#(z) \neq 0, \quad \forall z \in \mathbb{R}^{2n}, \tag{2.34}$$

$$\left|\psi^\#(z)\right| \leq C\mathrm{e}^{\delta|z|^2}, \quad \forall z \in \mathbb{R}^{2n}, \text{ for some } C, \delta > 0. \tag{2.35}$$

We have seen that

$$\begin{aligned} \left|\langle\psi, \varphi_z\rangle\right|^2 &= (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} \mathcal{W}_\psi(X)\mathcal{W}_{\varphi_z}(X)\, dX \\ &= 2^n \int_{\mathbb{R}^{2n}} \mathcal{W}_\psi(X)\mathrm{e}^{-\frac{1}{\hbar}|X-z|^2}\, dX. \end{aligned} \tag{2.36}$$

The last integral is positive because by assumption $\mathcal{W}_\psi(X) \geq 0$ and $\int \mathcal{W}_\psi(X)\, dX = 1$.

Using again (2.36) we easily get (2.34). The second step is to use a property of entire functions in $\mathbb{C}^n$. Let us recall that in Chap. 1, we have seen that the function

$$\psi_a^\#(\zeta) := \exp\left(\frac{p^2 + ip \cdot q}{2\hbar}\right)\psi^\#(q, p) \tag{2.37}$$

is an entire function in the variable $\zeta = q - ip \in \mathbb{C}^n$. Moreover we get easily that $\psi_a^\#(\zeta)$ satisfies properties (2.34). To achieve the proof of Theorem 3 we apply the following lemma, which is a particular case of Hadamard factorization theorem for $n = 1$, extended for $n \geq 2$ in [181]. $\square$

Lemma 11 *Let f be an entire function in $\mathbb{C}^n$ such that $f(\zeta) \neq 0$ for all $\zeta \in \mathbb{C}^n$ and for some $C > 0$, $\delta > 0$,*

$$\left|f(\zeta)\right| \leq C e^{\delta|\zeta|^m}, \quad \forall \zeta \in \mathbb{C}^n. \tag{2.38}$$

Then $f(\zeta) = e^{P(\zeta)}$, where P is a polynomial of degree $\leq m$.

2.3 Coherent States and Operator Norms Estimates

Let us give now a first application of coherent states to Weyl quantization. We assume first that $\hbar = 1$.

Theorem 4 (Calderon–Vaillancourt) *There exists a universal constant C_n such that for every symbol $A \in C^\infty(\mathbb{R}^{2n})$ we have*

$$\|\hat{A}\|_{\mathcal{L}(L^2,L^2)} \leq C_n \sup_{|\gamma| \leq 2n+1, X \in \mathbb{R}^{2n}} \left|\partial_X^\gamma A(X)\right|. \tag{2.39}$$

Beginning of the Proof From (2.32) we get the formula

$$\langle \psi, \hat{A}\eta \rangle = (2\pi)^{-n} \iint_{\mathbb{R}^{4n}} \langle \varphi_{z'}, \hat{A}\varphi_z \rangle \psi^\#(z') \overline{\eta^\#(z)}\, dz\, dz'. \tag{2.40}$$

We shall get (2.39) by proving that the Bargmann kernel $K_A^{\mathcal{B}}(z, z') := \langle \varphi_{z'}, \hat{A}\varphi_z \rangle$ is the kernel of a bounded operator in $L^2(\mathbb{R}^{2n})$. Let us first recall a classical lemma

Lemma 12 *Let (Ω, μ) be a measured (σ-finite) space, K a measurable function on $\Omega \times \Omega$ such that*

$$m_K := \max\left\{\sup_{z \in \Omega} \int_\Omega \left|K(z, z')\right| dz', \sup_{z' \in \Omega} \int_\Omega \left|K(z, z')\right| dz\right\}.$$

Then K is the integral kernel of a bounded operator T_K on $L^2(\Omega)$ and we have

$$\|T_K\| \leq m_K.$$

So the Calderon–Vaillancourt theorem will be a consequence of the following.

Lemma 13 *There exists a universal constant C_n such that for every symbol $A \in C^\infty(\mathbb{R}^{2n})$ we have*

$$\left|K_A^{\mathcal{B}}(z,z')\right| \le C_n\big(1+|z-z'|\big)^{-2n-1} \sup_{|\gamma|\le 2n+1, X\in\mathbb{R}^{2n}} \left|\partial_X^\gamma A(X)\right|. \tag{2.41}$$

Proof We have already seen that

$$\begin{aligned} K_A^{\mathcal{B}}(z,z') &= \int_{\mathbb{R}^n} A(X)\mathcal{W}_{z',z}(X)\,dX \\ &= 2^n \int_{\mathbb{R}^n} A(X)\exp\left(-\left|X-\frac{z+z'}{2}\right|^2 - i\sigma\left(X-\frac{1}{2}z', z-z'\right)\right)dX. \end{aligned} \tag{2.42}$$

First remark that we have

$$\left|\langle\varphi_{z'}, \hat{A}\varphi_z\rangle\right| \le \sup_{X\in\mathbb{R}^{2n}} \left|A(X)\right|. \tag{2.43}$$

So we only have to consider the case $|z'-z|\ge 1$. The estimate is proved by integration by parts (as is usual for an oscillating integral).

Let us introduce the phase function

$$\Phi = -\left|X-\frac{z+z'}{2}\right|^2 - i\sigma\left(X-\frac{1}{2}z', z-z'\right). \tag{2.44}$$

We have $|\partial_X\Phi|\ge|z-z'|$ hence

$$\frac{\overline{\partial_X\Phi}\cdot\partial_X}{|\partial_X\Phi|^2}\mathrm{e}^{\Phi} = \mathrm{e}^{\Phi}. \tag{2.45}$$

So we get the wanted estimates performing $2n+1$ integrations by parts in the integral (2.42) using formula (2.45).

This achieves the proof of the Calderon–Vaillancourt theorem. □

Corollary 4 *$\hat{A}$ is a compact operator in $L^2(\mathbb{R}^n)$ if A is C^∞ on $\mathbb{R}^{2n}$ and satisfies the following condition*:

$$\lim_{|z|\to+\infty}\left|\partial_z^\gamma A(z)\right| = 0, \quad \forall\gamma\in\mathbb{N}^{2d},\ |\gamma|\le 2n+1. \tag{2.46}$$

Proof Let us introduce $\chi\in C^\infty(\mathbb{R}^{2n})$ such that $\chi(X)=1$ if $|X|\le\frac{1}{2}$ and $\chi(X)=0$ if $|X|\ge 1$. Let us define $A_R(X)=\chi(X/R)A(X)$. For every $R>0$, $\hat{A}_R$ is Hilbert–

Schmidt hence compact. Using the Calderon–Vaillancourt estimate, we get

$$\lim_{|R|\to+\infty} \|\hat{A} - \widehat{A_R}\| = 0.$$

So $\hat{A}$ is compact. □

Using the same idea as for proving Calderon–Vaillancourt theorem, we get now a sufficient trace-class condition.

Theorem 5 *There exists a universal constant τ_n such that for every $A \in C^\infty(\mathbb{R}^{2n})$ we have*

$$\|\hat{A}\|_{Tr} \leq \tau_n \sum_{|\gamma|\leq 2n+1} \int_{\mathbb{R}^{2n}} \left|\partial_X^\gamma A(X)\right| dX. \tag{2.47}$$

In particular if the r.h.s. is finite then $\hat{A}$ is in the trace class and we have

$$\mathrm{Tr}\,\hat{A} = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} A(X)\, dX. \tag{2.48}$$

Proof Recall that $\hbar = 1$. From (2.29) we know that $\hat{A}$ has the following decomposition into rank one operators:

$$\hat{A} = (2\pi)^{-n} \iint_{\mathbb{R}^{4n}} \langle \varphi_{z'}, \hat{A}\varphi_z \rangle \Pi_{z,z'}\, dz\, dz'. \tag{2.49}$$

But we know that $\|\Pi_{z,z'}\|_{TR} = 1$. So we have

$$\|\hat{A}\|_{TR} \leq (2\pi)^{-n} \iint_{\mathbb{R}^{4n}} \left|\langle \varphi_{z'}, \hat{A}\varphi_z \rangle\right| dz\, dz'. \tag{2.50}$$

Using integration by parts as in the proof of Calderon–Vaillancourt, we have

$$\left|\langle \varphi_{z'}, \hat{A}\varphi_z \rangle\right| \leq C_N \big(1 + |z - z'|\big)^{-N} \sum_{|\gamma|\leq N} \int_{\mathbb{R}^{2n}} \mathrm{e}^{-|X-(z+z')/2|^2} \left|\partial_X^\gamma A(X)\right| dX \tag{2.51}$$

with $N = 2n + 1$. Now perform the change of variables $u = (z + z')/2$, $v = z - z'$ and using Young inequality we get

$$\iint_{\mathbb{R}^{4n}} \left|\langle \varphi_{z'}, \hat{A}\varphi_z \rangle\right| dz\, dz' \leq \tau_n \sum_{|\gamma|\leq N} \int_{\mathbb{R}^{2n}} \left|\partial_X^\gamma A(X)\right| dX \tag{2.52}$$

hence (2.47) follows.

We can get (2.48) by using approximations with compact support A_R like in the proof of Corollary 4. □

Remark 5 Using interpolation results it is possible to get similar estimates for the Schatten norm $\|\hat{A}\|_p$ for $1 < p < +\infty$.

Let us now compute the action of Weyl quantization on Gaussian coherent states.

Lemma 14 *Assume that* $A \in \Sigma(m)$ *(*m *is temperate weight). Then for every* $N \geq 1$, *we have*

$$\hat{A}\varphi_z = \sum_{|\gamma|\leq N} \hbar^{\frac{|\gamma|}{2}} \frac{\partial^\gamma A(z)}{\gamma!} \Psi_{\gamma,z} + \mathcal{O}\big(\hbar^{(N+1)/2}\big), \tag{2.53}$$

the estimate of the remainder is uniform in $L^2(\mathbb{R}^n)$ *for* z *in every bounded set of the phase space and*

$$\Psi_{\gamma,z} = \hat{T}(z)\Lambda_\hbar \mathrm{Op}_1^w\big(z^\gamma\big)g, \tag{2.54}$$

where $g(x) = \pi^{-n/4}\mathrm{e}^{-|x|^2/2}$, $\mathrm{Op}_1^w(z^\gamma)$ *is the* 1*-Weyl quantization of the monomial*: $(x,\xi)^\gamma = x^{\gamma'}\xi^{\gamma''}$, $\gamma = (\gamma', \gamma'') \in \mathbb{N}^{2d}$. *In particular* $\mathrm{Op}_1^w(z^\gamma)g = P_\gamma g$ *where* P_γ *is a polynomial of the same parity as* $|\gamma|$.

Proof Let us write

$$\hat{A}\varphi_z = \hat{A}\Lambda_\hbar\hat{T}_1(z)g = \Lambda_\hbar\hat{T}_1(z)\big(\Lambda_\hbar\hat{T}_1(z)\big)^{-1}\hat{A}\Lambda_\hbar\hat{T}_1(z)g,$$

where $\Lambda_\hbar$ is the dilation: $\Lambda_\hbar\psi = \hbar^{-n/4}\psi(\hbar^{-1/2}x)$ and $\hat{T}_1$ is $\hat{T}$ for $\hbar = 1$.

Let us remark that $(\Lambda_\hbar\hat{T}(z))^{-1}\hat{A}\Lambda_\hbar\hat{T}(z) = \mathrm{Op}_1^w[A_{\hbar,z}]$ where $A_{\hbar,z}(X) = A(\sqrt{\hbar}X + z)$. So we prove the lemma by expanding $A_{\hbar,z}$ in X, around z, with the Taylor formula with integral remainder term to estimate the error term. □

The following Lemma allows to localized observables acting on coherent states.

Lemma 15 *Let* A *be a smooth observable with compact support in the ball* $B(X_0, r_0)$ *of the phase space. Then there exists* $R > 0$ *and for all* $N \geq 1$ *there exists* C_N *such that for* $|z - X_0| \geq 2r_0$ *we have*

$$\big\|\hat{A}\varphi_z\big\| \leq C_N\hbar^N\langle z\rangle^{-N}, \quad \text{for } |z| \geq R. \tag{2.55}$$

Proof It is convenient here to work on Fourier–Bargmannn side. So we estimate

$$\big\langle\varphi_z, \hat{A}\varphi_{z'}\big\rangle = (2\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}} A(Y)\mathcal{W}_{z,z'}(Y)\,dY. \tag{2.56}$$

As we have already seen, we have

$$\int_{\mathbb{R}^{2n}} A(Y)\mathcal{W}_{z,z'}(Y)\,dY$$
$$= 2^n \int_{\mathbb{R}^{2n}} \exp\left(-\frac{1}{\hbar}\left|Y - \frac{z+z'}{2}\right|^2 - \frac{i}{\hbar}\sigma\left(Y - \frac{1}{2}z', z - z'\right)\right) A(Y)\,dY. \quad (2.57)$$

Using integrations by parts as above, considering the phase function $\Psi(Y) = -|Y - \frac{z+X}{2}|^2 - i\sigma(Y - \frac{1}{2}X, z - X)$ and the differential operator $\frac{\overline{\partial_Y \Psi}}{|\partial_Y \Psi|^2}\partial_Y$, we get for every M, M' large enough,

$$\left|\langle \hat{A}\varphi_z, \varphi_{z'}\rangle\right| \le C_{M,M'} \int_{[|Y|\le r_0]} \left(1 + \frac{|Y-z|}{\sqrt{\hbar}}\right)^{-M} \left(1 + \frac{|z-z'|}{\sqrt{\hbar}}\right)^{M-M'} dY. \quad (2.58)$$

Therefore we easily get the estimate choosing M, M' conveniently and using that the Fourier–Bargmannn transform is an isometry. □

We need to introduce some properties for the Weyl symbols A.

Definition 5 A positive function m on $\mathbb{R}^d$ is a temperate weight if it satisfies the following property. There exist N, C such that

$$m(X+Y) \le m(X)\big(1 + |X-Y|\big)^N, \quad \forall X, Y \in \mathbb{R}^d. \quad (2.59)$$

A symbol A is a classical observable of weight m if for every multiindex α there exists C_α such that

$$\left|\partial_X^\alpha A(X)\right| \le C_\alpha m(X), \quad \forall X \in \mathbb{R}^{2n}.$$

The space of symbols of weight m is denoted $\Sigma(m)$.

A basic example of temperate weight is $m_\mu(X) = (1 + |X|)^\mu$, $\mu \in \mathbb{R}$. We shall denote $\Sigma^\mu = \Sigma(m_\mu)$. For example $\Sigma^0 = \Sigma(1)$.

Remark 6 The product of two temperate weights is a temperate weight and if m is a temperate weight then m^{-1} is also a temperate weight.

As proved by Unterberger [186] and rediscovered by Tataru [183], it is possible to characterize the operator class $\widehat{\Sigma(1)}$ on the matrix element $\langle\varphi_{z'}, \hat{A}\varphi_z\rangle$. We state now a semi-classical version of Unterberger result.

Theorem 6 *Let $\hat{A}_\hbar$ be a $\hbar$-dependent family of operators from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$. Then $\hat{A} = \mathrm{Op}_\hbar^w(A_\hbar)$ with $A_\hbar \in \Sigma(1)$ with uniform estimate*[2] *if and only if for every*

[2]This means that for every γ, $\sup_{\hbar\in]0,1]} \|\partial^\gamma A\|_\infty < +\infty$.

N there exists C_N such that we have

$$\left|\langle\varphi_{z'}, \hat{A}\varphi_z\rangle\right| \le C_N\left(1+\frac{|z-z'|}{\sqrt{\hbar}}\right)^{-N}, \quad \forall\hbar\in]0,1),\ z,z'\in\mathbb{R}^{2n}. \tag{2.60}$$

Proof Suppose that $\hat{A} = \mathrm{Op}_\hbar^w(A_\hbar)$, with $A_\hbar \in \Sigma(1)$ is a bounded family. We get estimate (2.60) by integrations by parts as above.

Conversely if we have estimates (2.60), using (2.23) and (2.29) we have

$$A_\hbar(X) = (\pi\hbar)^{-n}\iint_{\mathbb{R}^{4n}}\langle\varphi_{z'}, \hat{A}_\hbar\varphi_z\rangle\exp\left(-\frac{1}{\hbar}\left(\left|X-\frac{z+z'}{2}\right|^2 + iJ\left(X-\frac{z'}{2}\right)\cdot(z-z')\right)\right)dz\,dz'. \tag{2.61}$$

Using the change of variables $\frac{z+z'}{2} = u$ and $z-z' = \sqrt{\hbar}v$ we get easily that there exists $C>0$ such that

$$\left|A_\hbar(X)\right| \le C, \quad \forall X\in\mathbb{R}^{2n},\ \hbar\in]0,1]. \tag{2.62}$$

In the same way we can estimate every derivatives of $A_\hbar$, after derivation in X in the integral (2.61). $\square$

The other main fact in Weyl quantization is existence of an operational calculus. We shall recall its properties in the next section.

2.4 Product Rule and Applications

2.4.1 The Moyal Product

One of the most useful properties of Weyl quantization is that we have an operational calculus defined by:

The Product Rule for Quantum Observables Let us start with $A, B \in \mathcal{S}(\mathbb{R}^{2n})$. We look for a classical observable C such that $\hat{A}\cdot\hat{B} = \hat{C}$. Let us first remark that the integral kernel of $\hat{C}$ is

$$K_C(x,y) = \int_{\mathbb{R}^n} K_A(x,s)K_B(s,y)\,ds. \tag{2.63}$$

Using relationship between integral kernels and Weyl symbols, we get

$$C(X) = (\pi\hbar)^{-2n}\iint_{\mathbb{R}^{4n}} \mathrm{e}^{2i\hbar\sigma(Y,Z)}A(X+Z)B(X+Y)\,dY\,dZ, \tag{2.64}$$

where σ is the symplectic bilinear form introduced above.

Now let us apply Plancherel formula in $\mathbb{R}^{4n}$ and the following Fourier transform formula:

Lemma 16 *Let* $f(T) = \mathrm{e}^{\frac{i}{2}\langle B.T,T\rangle}$, *for* $T \in \mathbb{R}^m$ *where* B *is a non degenerate symmetric* $m \times m$ *matrix. Then the Fourier transform* $\tilde{f}$ *is*

$$\tilde{f}(\zeta) = (2\pi)^{m/2}|\det B|^{-1/2}\mathrm{e}^{i\pi \operatorname{sgn} B}\mathrm{e}^{-\frac{i}{2}\langle B^{-1}\zeta,\zeta\rangle}, \tag{2.65}$$

where $\operatorname{sgn} B$ *is the signature of the matrix* B.

Proof See [117, 163]. □

Hence we get

$$C(x,\xi) = \exp\left(\frac{i\hbar}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)A(x,\xi)B(y,\eta)\Big|_{(x,\xi)=(y,\eta)}. \tag{2.66}$$

We can see easily on formula (2.66) that $C \in \mathcal{S}(\mathbb{R}^{2n})$. So that (2.64) defines a non-commutative product on classical observables. We shall denote this product $C = A \star B$ (Moyal product).

In semi-classical analysis, it is useful to expand the exponent in (2.66), so we get the formal series in $\hbar$:

$$C(x,\xi) = \sum_{j\geq 0} C_j(x,\xi)\hbar^j, \quad \text{where}$$

$$C_j(x,\xi) = \frac{1}{j!}\left(\frac{i}{2}\sigma(D_x, D_\xi; D_y, D_\eta)\right)^j A(x,\xi)B(y,\eta)\Big|_{(x,\xi)=(y,\eta)}. \tag{2.67}$$

We can easily see that in general C is not a classical observable because of the $\hbar$ dependence. It can be proved that it is a *semi-classical observable* in the following sense.

Definition 6 We say that A is a semi-classical observable of weight m, where m is temperate weight on $\mathbb{R}^{2n}$, if there exist $\hbar_0 > 0$ and a sequence $A_j \in \Sigma(m)$, $j \in \mathbb{N}$, so that A is a map from $]0, \hbar_0]$ into $\Sigma(m)$ satisfying the following asymptotic condition: for every $N \in \mathbb{N}$ and every $\gamma \in \mathbb{N}^{2n}$ there exists $C_N > 0$ such that for all $\hbar \in]0, 1[$ we have

$$\sup_{\mathbb{R}^{2n}} m^{-1}(z)\left|\frac{\partial^\gamma}{\partial z^\gamma}\left(A(\hbar, z) - \sum_{0\leq j\leq N}\hbar^j A_j(z)\right)\right| \leq C_N \hbar^{N+1}, \tag{2.68}$$

A_0 is called the principal symbol, A_1 the sub-principal symbol of $\hat{A}$.

The set of semi-classical observables of weight m is denoted by $\Sigma_{sc}(m)$. Its range in $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ is denoted $\widehat{\Sigma}_{sc}(m)$.

We may use the notation $\Sigma_{sc}^{\mu} = \Sigma_{sc}(m_{\mu})$.

Now we state the product rule for Weyl quantization.

Theorem 7 *Let m, m' be two temperate weights in $\mathbb{R}^{2n}$. For every $A \in \Sigma(m)$ and $B \in \Sigma(m')$, there exists a unique $C \in \Sigma_{sc}(mp)$ such that $\hat{A} \cdot \hat{B} = \hat{C}$ with $C \asymp \sum_{j\geq 0} \hbar^j C_j$. The C_j are given by*

$$C_j(x,\xi) = \frac{1}{2^j} \sum_{|\alpha+\beta|=j} \frac{(-1)^{|\beta|}}{\alpha!\beta!} \big(D_x^{\beta}\partial_{\xi}^{\alpha} A\big) \cdot \big(D_x^{\alpha}\partial_{\xi}^{\beta} B\big)(x,\xi).$$

Proof The main technical point is to control the remainder terms uniformly in the semi-classical parameter $\hbar$. This is detailed in the appendix of the paper [31]. □

Corollary 5 *Under the assumption of the theorem, we have the well known correspondence between the commutator for quantum observables and the Poisson bracket for classical observables, $\frac{i}{\hbar}[\hat{A}, \hat{B}] \in \widehat{\Sigma_{sc}(mm')}$ and its principal symbol is the Poisson bracket $\{A, B\}$.*

A very useful application of the Moyal product is the possibility to get semi-classical approximations for inverse of elliptic symbol.

Definition 7 Let $A(\hbar)$ be a semi-classical observable in $\Sigma_{sc}(m)$ and $X_0 \in \mathbb{R}^{2n}$. We shall say that A is elliptic at X_0 if $A_0(X_0) \neq 0$.

We shall say that A is uniformly elliptic if there exists $c > 0$ such that

$$\big|A(X)\big| \geq cm(X), \quad \forall X \in \mathbb{R}^{2n}. \tag{2.69}$$

Theorem 8 *Let $A \in \Sigma_{sc}(m)$ be an uniformly elliptic semi-classical symbol. Then there exists $B \in \Sigma_{sc}(m^{-1})$ such that $B \star A = 1$ (in the sense of asymptotic expansion in $\Sigma_{sc}(1)$). Moreover, we have*

$$\hat{B} \cdot \hat{A} = \mathbb{1} + \mathcal{O}\big(\hbar^{\infty}\big), \tag{2.70}$$

where the remainder is estimated in the L^2norm of operators.

Moreover the semi-classical symbol B of $\hat{B}$ is $B = \sum_{j\geq 0} \hbar^j B_j$ with

$$B_0 = A_0^{-1}, \qquad B_1 = -A_1 A_0^{-2}. \tag{2.71}$$

Proof Let us denote by $C_j(E, F)$ the jth term in the Moyal product $E \star F$. The method consists to compute by induction $B_0, \dots, B_N$ such that

$$\Big(\sum_{0\leq j\leq N} \hbar^j B_j\Big) \star A(h) = \mathcal{O}\big(\hbar^{N+1}\big). \tag{2.72}$$

We start with $B_0 = \frac{1}{A_0}$. The next step is to compute B_1 such that $B_1 A_0 + A_1 B_0 = 0$. Then to compute B_2 such that

$$C_2(A_0, B_0) + C_1(A_1, B_1) + B_2 A_0 = 0.$$

So we get all the B_j by induction using the asymptotic expansion for the Moyal product.

The remainder term in (2.70) is estimated using the Calderon–Vaillancourt theorem. □

We give now a local version of the above theorem, which can be proved by the same method.

Theorem 9 *Let $A \in \Sigma_{sc}(m)$ be an elliptic symbol in an open bounded set Ω of $\mathbb{R}^{2n}$. Then for every $\chi \in C_0^\infty(\Omega)$ there exists $B_\chi \in \Sigma_{sc}^{-\infty}$ such that*

$$\hat{B}_\chi \hat{A} = \hat{\chi} + \mathcal{O}\big(\hbar^\infty\big). \tag{2.73}$$

Remark 7 For application it is useful to note that if A depends in a uniform way of some parameter $\varepsilon \in [0, 1]$ then B also depends uniformly in ε. In particular ε may depend on $\hbar$.

2.4.2 Functional Calculus

An useful consequence of the algebraic properties of symbolic quantization is a functional calculus: under suitable assumptions if $\hat{H}$ is an Hermitian semi-classical observable then for every smooth function f, $f(\hat{H})$ is also a semi-classical observable. The technical statement is

Theorem 10 *Let $\hat{H}$ be a uniformly elliptic semi-classical Hamiltonian. Let f be a smooth real valued function such that, for some $r \in \mathbb{R}$, we have*

$$\forall k \in \mathbb{N},\ \exists C_k, \quad \big|f^{(k)}(s)\big| \le C_k \langle s\rangle^{r-k}, \quad \forall s \in \mathbb{R}.$$

Then $f(\hat{H})$ is a semi-classical observable with a semi-classical symbol $H_f(\hbar, z)$ given by

$$H_f(\hbar, z) \asymp \sum_{j\ge 0} \hbar^j H_{f,j}(z). \tag{2.74}$$

In particular we have

$$H_{f,0}(z) = f\big(H_0(z)\big), \tag{2.75}$$

$$H_{f,1}(z) = H_1(z) f'\big(H_0(z)\big), \tag{2.76}$$

$$\textit{and for}, \ j \geq 2, \quad H_{f,j} = \sum_{1 \leq 1 \leq 2j-1} d_{j,k}(H) f^{(k)}(H_0), \tag{2.77}$$

where $d_{j,k}(H)$ are universal polynomials in $\partial_z^\gamma H_\ell(z)$ with $|\gamma| + \ell \leq j$.

A proof of this theorem can be found in [68, Chap. 8], [107]. In particular we can take $f(s) = (\lambda + s)^{-1}$ for $\Im\lambda \neq 0$ (the proof begins with this case) or f with a compact support.

From this theorem we can get the following consequences on the spectrum of $\hat{H}$ (see [107]).

Theorem 11 *Let $\hat{H}$ be like in Theorem 10. Assume that $H_0^{-1}[E_-, E_+]$ is a compact set in $\mathbb{R}^n \times \mathbb{R}^n$. Consider a closed interval $I \subset [E_-, E_+]$. Then we have the following properties.*

(i) $\forall \hbar \in]0, \hbar_0]$, $\hbar_0 > 0$, *the spectrum of $\hat{H}$ is discrete and is a finite sequence of eigenvalues $E_1(\hbar) \leq E_2(\hbar) \leq \cdots \leq E_{N_I}(\hbar)$ where each eigenvalue is repeated according its multiplicity.*

Moreover $N_I = O(\hbar^{-n})$ as $\hbar \searrow 0$.

(ii) *For all $f \in C_0^\infty(I)$, $f(\hat{H})$ is a trace-class operator and we have*

$$\mathrm{Tr}\big[f\big(\hat{H}\big)\big] \asymp \sum_{j \geq 0} \hbar^{j-d} \tau_j(f), \tag{2.78}$$

where τ_j are distributions supported in $H_0^{-1}(I)$. In particular, we have

$$\tau_0(f) = (2\pi)^{-d} \int_{\mathbb{R}^{2n}} f\big(H_0(z)\big)\, dz, \tag{2.79}$$

$$\tau_1(f) = (2\pi)^{-d} \int_{\mathbb{R}^{2n}} f'\big(H_0(z)\big) H_1(z)\, dz. \tag{2.80}$$

An easy consequence of this is the following Weyl asymptotic formula:

Corollary 6 *If $I = [\lambda_-, \lambda_+]$ such that $\lambda_\pm$ are non critical values for H_0*[3] *then we have*

$$\lim_{\hbar \to 0} (2\pi\hbar)^n N_I = \int_{[H_0(q,p) \in I]} dq\, dp. \tag{2.81}$$

Remark 8 Formula (2.81) is very well known and can be proved in many ways, under much weaker assumptions.

For a proof using the functional calculus see [163, pp. 283–287].

[3] That λ is a non-critical value for H means that $\nabla H(z) \neq 0$ if $H(z) = \lambda$.

Under our assumptions we shall see in Chap. 4 that we have a Weyl asymptotic with an accurate remainder estimate:

$$N_I = (2\pi\hbar)^{-n} \int_{[H(q,p)\in I]} dq\,dp + O\big(\hbar^{1-n}\big),$$

using a time dependent method due to Hörmander and Levitan ([116] and its bibliography). For more accurate results about spectral asymptotics see [122].

2.4.3 *Propagation of Observables*

Now we come to the main application of the results of this section. We shall give a proof of the correspondence (in the sense of Bohr) between quantum and classical dynamics. As we shall see this theorem is a useful tool for semi-classical analysis although its proof is an easy application of Weyl calculus rules stated above. The microlocal version of the following result is originally due to Egorov [73]. R. Beals [18] found a nice simple proof.

Theorem 12 (The Semi-classical Propagation Theorem) *Let us consider a time dependent Hamiltonian $H(t) \in \Sigma^2_{sc}$ satisfying*:

$$\big|\partial_z^\gamma H_j(t,z)\big| \le C_\gamma, \quad \text{for } |\gamma| + j \ge 2; \tag{2.82}$$

$$\hbar^{-2}\big(H(t) - H_0(t) - \hbar H_1(t)\big) \in \Sigma^0_{sc}. \tag{2.83}$$

We assume that $H(t,z)$ is continuous for $t \in \mathbb{R}$ and that all the estimates are uniform in t for $t \in [-T,T]$.

Let us introduce an observable $A \in \Sigma^1$, such that $\partial_X^\gamma A \in \Sigma^0$ if $|\gamma| \ge 1$. Then we have the following.

(a) *For $\hbar$ small enough and for every $\psi \in \mathcal{S}(\mathbb{R}^n)$, the Schrödinger equation*

$$i\hbar\partial_t\psi_t = \hat{H}(t)\psi_t, \quad \psi_{t=s} = \psi \tag{2.84}$$

has a unique solution which we denote $\psi_t = \hat{U}(t,s)\psi$. Moreover $\hat{U}(t,s)$ can be extended as a unitary operator in $L^2(\mathbb{R}^n)$.

(b) *The time evolution $\hat{A}(t,s)$ of $\hat{A}$, from the initial time s is $\hat{A}(t,s) = \hat{U}(s,t)\hat{A} \times \hat{U}(t,s)$ and has a semi-classical Weyl symbol $A_\hbar(t,s)$ such that $A_\hbar(t,s) \in \Sigma^1_{sc}$. More precisely we have $A(t,s) \asymp \sum_{j\ge 0} \hbar^j A_j(t,s)$, in Σ^0_{sc}, which is uniform in t,s, for $t,s \in [-T,T]$. Moreover $A_j(t,s)$ can be computed by the following formulas*:

$$A_0(t,s;z) = A\big(\Phi^{t,s}(z)\big), \tag{2.85}$$

$$A_1(t,s;,z) = \int_s^t \big\{A\big(\Phi^{\tau,t}\big), H_1(\tau)\big\}\big(\Phi^{t,\tau}(z)\big)\,d\tau \tag{2.86}$$

and for $j \ge 2$, $A_j(t,s;z)$ can be computed by induction on j.

Proof Property (a) will be proved later. It is easier to prove it if H is time independent because we can prove in this case that $\hat{H}$ is essentially self-adjoint (for a proof see [163]). Then we have

$$\hat{U}(t) := \hat{U}(t,0) = \exp\left(-\frac{it}{\hbar}\hat{H}\right).$$

Let us remark that, under the assumption of the theorem, the classical flow for H_0 exists globally. Indeed, the Hamiltonian vector field $(\partial_\xi H_0, -\partial_x H_0)$ has a sublinear growing at infinity so, no classical trajectory can blow up in a finite time. Moreover, using usual methods in non linear O.D.E. (variation equation) we can prove that $A(\Phi^{t,s}) \in \Sigma(1)$ with semi-norm uniformly bounded for t, s bounded.

Now, from the Heisenberg equation and the classical equations of motion we get

$$\begin{aligned}&\frac{\partial}{\partial\tau}\hat{U}(s,\tau)\widehat{A_0}(t,\tau)\hat{U}(\tau,s)\\ &\quad = \hat{U}(s;\tau)\left\{\frac{i}{\hbar}\left[\hat{H}(\tau), \widehat{A_0}(t,\tau)\right] - \left\{\widehat{H(\tau), A_0}\right\}\left(\Phi^{t,\tau}\right)\right\}\hat{U}(\tau,s),\end{aligned} \tag{2.87}$$

where $A_0(t,s) = A(\Phi^{t,s})$. But, from the corollary of the product rule, the principal symbol of

$$\frac{i}{\hbar}\left(\left[\hat{H}(\tau), \widehat{A_0(t,\tau)}\right] - \left\{\widehat{H(\tau), A_0}\right\}\right)\left(\Phi^{t,\tau}\right)$$

vanishes. So, in the first step, using the product rule formula, we get the approximation

$$\begin{aligned}&\hat{U}(s,t)\hat{A}\hat{U}(t,s) - \widehat{A_0(t,s)}\\ &\quad = \int_s^t \hat{U}(s,\tau)\left(\frac{i}{\hbar}\left[\hat{H}(\tau), \widehat{A_0(t,\tau)}\right] - \left\{\widehat{\hat{H}(\tau), A_0}\right\}\Phi^{t,\tau}\right)\hat{U}(\tau,s)\,d\tau.\end{aligned} \tag{2.88}$$

Now, it is not difficult to obtain, by induction, the full asymptotics in $\hbar$. For $j \geq 2$,

$$A_j(t,s;z) = \sum_{\substack{|(\alpha,\beta)|+k=j+1\\ 0\le\ell\le j-1}} \Gamma(\alpha,\beta)\int_s^t \left[\left(\partial_\xi^\alpha\partial_x^\beta H_k(\tau)\right)\cdot\left(\partial_\xi{}^\alpha\partial_x{}^\beta A_\ell\right)\right]\left(\Phi^{t,\tau}(z)\right)d\tau, \tag{2.89}$$

with

$$\Gamma(\alpha,\beta) = \frac{(-1)^{|\beta|} - (-1)^{|\alpha|}}{\alpha!\beta!2^{|\alpha|+|\beta|}} i^{-1-|(\alpha,\beta)|}.$$

The main technical point is to estimate the remainder terms. For a proof with more details see [31] where the authors get a uniform estimate up to Ehrenfest time (of order $\log\hbar^{-1}$). We give in Appendix B the necessary details for uniform estimates on finite times intervals. □

Remark 9 If $H(t) = H_0(t)$ is a polynomial function of degree ≤ 2 in z on the phase space $\mathbb{R}^{2n}$ then the propagation theorem assumes a simpler form: $A(t, s) = A(\Phi^{t,s})$ and the remainder term is null. This is a consequence of the following exact formula:

$$\frac{i}{\hbar}\big[\hat{H}, \hat{B}\big] = \widehat{\{H, B\}}, \tag{2.90}$$

where $B \in \Sigma^{+\infty}$.

Now we give an application of the propagation theorem and coherent states in semi-classical analysis: we recover the classical evolution from the quantum evolution, in the classical limit $\hbar \searrow 0$.

Corollary 7 *For every observable $A \in \Sigma^0$ and every $z \in \mathbb{R}^{2n}$, we have*

$$\lim_{\hbar \searrow 0}\big\langle \hat{U}(t, s)\varphi_z, \hat{A}\hat{U}(t, s)\varphi_z\big\rangle = A\big(\Phi^{t,s}(z)\big) \tag{2.91}$$

and the limit is uniform in $(t, s; z)$ on every bounded set of $\mathbb{R}_t \times \mathbb{R}_s \times \mathbb{R}^{2n}_z$.

Proof

$$\begin{aligned}\big\langle \hat{U}(t, s)\varphi_z, \hat{A}\hat{U}(t, s)\varphi_z\big\rangle &= \big\langle \varphi_z, \hat{U}(s, t)\hat{A}\hat{U}(t, s)\varphi_z\big\rangle \\ &= \int_{\mathbb{R}^{2n}} A(t, s; X)\mathcal{W}_{z,z}(X)\, dX \\ &= (\pi\hbar)^{-n}\int_{\mathbb{R}^{2n}} A(t, s; X)\mathrm{e}^{-\frac{|X-z|^2}{\hbar}}\, dX. \end{aligned} \tag{2.92}$$

So by the propagation theorem we know that $A(t, s; X) = A(\Phi^{t,s}(X)) + \mathcal{O}(\hbar)$. Hence the corollary follows. □

Remark 10 The last result has a long history beginning with Ehrenfest [74] and continuing with Hepp [113], Bouzouina–Robert [31]. In this last paper it is proved that the corollary is still valid for times smaller than the Ehrenfest time $T_E := \gamma_E |\log\hbar|$, for some constant $\gamma_E > 0$.

2.4.4 Return to Symplectic Invariance of Weyl Quantization

Let us give now a first construction of metaplectic transformations. Other equivalent constructions and more properties will be given later (chapter on quadratic hamiltonians).

Lemma 17 *For every $F \in \mathrm{Sp}(n)$ we can find a C^1-smooth curve F_t, $t \in [0, 1]$, in $\mathrm{Sp}(n)$, such that $F_0 = \mathbb{1}$ and $F_1 = F$.*

Proof An explicit way to do that is to use the polar decomposition of F, $F = V|F|$ where V is a symplectic orthogonal matrix and $|F| = \sqrt{F^t F}$ is positive symplectic matrix. Each of these matrices have a logarithm, so $F = \mathrm{e}^K \mathrm{e}^L$ with K, L Hamiltonian matrices, and we can choose $F_t = \mathrm{e}^{tK}\mathrm{e}^{tL}$. F_t is clearly the linear flow defined by the quadratic Hamiltonian $H_t(z) = \frac{1}{2} S_t z \cdot z$ where $S_t = -J\dot{F}_t F_t^{-1}$. □

Now we use the (exact) propagation theorem. $\hat{U}(t,s)$ denotes the propagator defined by the quadratic Hamiltonian built in the proof of Lemma 17 and Theorem 12. Then *we define* $\hat{R}(F) = \hat{U}(1,0)$. Recall that $\hat{U}(t,0)$ is the solution of the Schrödinger equation

$$i\hbar \frac{d}{dt}\hat{U}(t,0) = \hat{H}(t)\hat{U}(t,0), \quad \hat{U}(0,0) = \mathbb{1}. \tag{2.93}$$

The following theorem translates the symplectic invariance of the Weyl quantization.

Theorem 13 *For every linear symplectic transformation $F \in \mathrm{Sp}(n)$ and every symbol $A \in \Sigma(1)$ we have*

$$\hat{R}(F)^{-1}\hat{A}\hat{R}(F) = \widehat{A \cdot F}. \tag{2.94}$$

Proof This is a direct consequence of the exact propagation formula for quadratic Hamiltonians

$$\hat{U}(0,t)\hat{A}\hat{U}(t,0) = \widehat{A\Phi^{t,0}}. \tag{2.95}$$

□

We can get another proof of the following result (see formulas (2.27)).

Corollary 8 *Let $\psi, \eta \in L^2(\mathbb{R}^n)$. For every linear symplectic transformation $F \in \mathrm{Sp}(n)$, we have the following transformation formula for the Wigner function*:

$$\mathcal{W}_{\hat{R}(F)\psi, \hat{R}(F)\eta}(z) = \mathcal{W}_{\psi,\eta}\big(F^{-1}(z)\big), \quad \forall z \in \mathbb{R}^{2n}. \tag{2.96}$$

Proof For every $A \in \mathcal{S}(\mathbb{R}^{2n})$, we have

$$\begin{aligned}
\big\langle \hat{R}(F)\eta, \hat{A}\hat{R}(F)\psi \big\rangle &= \int_{\mathbb{R}^{2n}} A(z)\mathcal{W}\big(\hat{R}(F)\psi, \hat{R}(F)\eta\big)(z)\,dz \\
&= \big\langle \eta, \hat{R}(F)^{-1}\hat{A}\hat{R}(F)\psi \big\rangle \\
&= \int_{\mathbb{R}^{2n}} A(F \cdot z)\mathcal{W}_{\psi,\eta}(z)\,dz.
\end{aligned} \tag{2.97}$$

The corollary follows. □

We have the following uniqueness result.

Proposition 20 *Given the linear symplectic transformation $F \in \mathrm{Sp}(n)$, there exists a unique transformation $\hat{R}(F)$, up to a complex number of modulus* 1, *satisfying* (2.18).

Proof If $\hat{V}$ satisfies $\hat{V}^{-1}\hat{A}\hat{V} = \widehat{A \cdot F}$ then if $\hat{B} = \hat{V}^{-1} \cdot \hat{R}(F)$, we see that $\hat{B}$ commutes with every $\hat{A}$, $A \in \Sigma(1)$. In particular $\hat{B}$ commutes with the Heisenberg–Weyl translations $\hat{T}(z)$, hence $\hat{T}(z)^{-1}\hat{B}\hat{T}(z) = \hat{B}$. But we knows that $\hat{T}(z)^{-1}\hat{B}\hat{T}(z) = \widehat{B(\cdot + z)}$. So the Weyl symbol of $\hat{B}$ (it is a temperate distribution) is a constant complex number λ. But here $\hat{B}$ is unitary, so $|\lambda| = 1$. □

2.5 Husimi Functions, Frequency Sets and Propagation

2.5.1 Frequency Sets

The Husimi transform of some temperate distributions $u \in \mathcal{S}'(\mathbb{R}^n)$ is defined as follows:

Definition 8 The Husimi transform of $u \in \mathcal{S}'(\mathbb{R}^n)$ is the function $\mathcal{H}_u(z)$ defined on the phase space $\mathbb{R}^{2n}$ by

$$\mathcal{H}_u(z) = (2\pi\hbar)^{-n} \left|\langle u, \varphi_z\rangle\right|^2, \quad z \in \mathbb{R}^{2n}. \tag{2.98}$$

The Husimi transform in contrast with the Wigner transform is always nonnegative. We shall see below that the Husimi distribution is a "regularization" of the Wigner distribution.

Proposition 21 *For every $\varphi \in L^2(\mathbb{R}^n)$ we have*

$$\mathcal{H}_\varphi = \mathcal{W}_\varphi * G_0,$$

where G_0 is a gaussian function in phase space namely

$$G_0(z) = (\pi\hbar)^{-n} \mathrm{e}^{-|z|^2/\hbar}.$$

One has $\int_{\mathbb{R}^{2n}} G_0(z)\,dz = 1$. This means that the Husimi distribution is a "regularization" of the Wigner distribution.

Proof According to the Proposition 17(iii) we have

$$\mathcal{H}_\varphi(z) = (2\pi\hbar)^{-n} \langle \mathcal{W}_{\varphi_z}, \mathcal{W}_\varphi\rangle_{L^2(\mathbb{R}^{2n})}.$$

But we know that

$$\mathcal{W}_{\varphi_z}(X) = \mathcal{W}_{\varphi_0}(X - z).$$

We use Proposition 16:

$$\langle \mathcal{W}_{\varphi_z}, \mathcal{W}_\varphi \rangle = 2^n \int_{\mathbb{R}^{2n}} \exp\left(-\frac{|X-z|^2}{\hbar}\right) \mathcal{W}_\varphi(X)\, dX.$$

This yields the result. □

In semi-classical analysis (or in high frequency analysis) it is important to understand what is the region of the phase space $\mathbb{R}^{2n}$ where some states $\psi \in L^2(\mathbb{R}^n)$ depending on $\hbar$, essentially lives when $\hbar$ is small. For that purpose let us introduce the *frequency set* of ψ.

Definition 9 Let $\psi_\hbar \in L^2(\mathbb{R}^n)$, depending on $\hbar$, such that $\|\psi_\hbar\| \le 1$. We say that $\psi_\hbar$ is negligible near a point $X_0 \in \mathbb{R}^{2n}$, if there exists a neighborhood $\mathcal{V}_{X_0}$ such that

$$\mathcal{H}_{\psi_\hbar}(z) = \mathcal{O}\big(\hbar^\infty\big), \quad \forall z \in \mathcal{V}_{X_0}. \tag{2.99}$$

Let us denote $\mathcal{N}[\psi_\hbar]$ the set $\{X \in \mathbb{R}^{2n},\ \psi_\hbar \text{ is negligible near } X\}$. The frequency set $\mathrm{FS}[\psi_\hbar]$ is defined as the complement of $\mathcal{N}[\psi_\hbar]$ in $\mathbb{R}^{2n}$.

Example 1

- If $\psi_\hbar = \varphi_z$ then $\mathrm{FS}[\varphi_z] = \{z\}$.
- Let $\psi = a(x)\mathrm{e}^{\frac{i}{\hbar}S(x)}$ where a and S are smooth functions, $a \in \mathcal{S}(\mathbb{R}^n)$, S real. Then we have the inclusion

$$\mathrm{FS}[\psi] \subseteq \big\{(x,\xi) | \xi = \nabla S(x)\big\}. \tag{2.100}$$

There are several equivalent definitions of the frequency set that we now give.

Proposition 22 *Let $\psi_\hbar$ be such that $\|\psi_\hbar\| \le 1$ and $X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n}$. The following properties are equivalent*:

(i)

$$\mathcal{H}_{\psi_\hbar}(X) = \mathcal{O}\big(\hbar^{+\infty}\big), \quad \forall X \in \mathcal{V}_{X_0}.$$

(ii) *There exists $A \in \mathcal{S}(\mathbb{R}^{2n})$, such that $A(X_0) = 1$ and*

$$\big\|\hat{A}\psi_\hbar\big\| = \mathcal{O}\big(\hbar^{+\infty}\big). \tag{2.101}$$

(iii) *There exists a neighborhood $\mathcal{V}_{X_0}$ of X_0 such that for all $A \in C_0^\infty(\mathcal{V}_{X_0})$*,

$$\big\|\hat{A}\psi_\hbar\big\| = \mathcal{O}\big(\hbar^{+\infty}\big). \tag{2.102}$$

(iv) *There exist $\chi \in C_0^\infty(\mathbb{R}^n)$ such that $\chi(x_0) = 1$ and a neighborhood V_{ξ_0} of ξ_0 such that*

$$\big\langle \chi(x)\mathrm{e}^{\frac{i}{\hbar}x\cdot\xi}, \psi_\hbar\big\rangle = \mathcal{O}\big(\hbar^{+\infty}\big) \tag{2.103}$$

for all $\xi \in V_{\xi_0}$.

Proof Let us assume (i). Then we have

$$\mathcal{H}_{\psi_\hbar}(z) = \mathcal{O}\big(\hbar^\infty\big), \quad |z - X_0| < r_0. \tag{2.104}$$

Using Lemma 15 we have

$$\big\|\hat{A}\varphi_z\big\| \le C_N \hbar^{N/2} \langle z\rangle^{-N}, \quad \text{if } |z - X_0| > r_0/2. \tag{2.105}$$

We have, using linearity of integration,

$$\hat{A}\psi_\hbar = (2\pi\hbar)^{-n} \int dz \langle \varphi_z, \psi_\hbar\rangle \hat{A}\varphi_z.$$

From the triangle inequality, we have

$$\begin{aligned}\big\|\hat{A}\psi_\hbar\big\| &\le (2\pi\hbar)^{-n} \int dz \big|\langle \psi_\hbar, \varphi_z\rangle\big| \big\|\hat{A}\varphi_z\big\| \\ &\le (2\pi\hbar)^{-n} \bigg(\int_{[|z-X_0|<r_0]} dz + \int_{[|z-X_0|\ge r_0]} dz\bigg).\end{aligned} \tag{2.106}$$

Then we get (iii):

$$\big\|\hat{A}\psi_\hbar\big\|^2 = \mathcal{O}\big(\hbar^{+\infty}\big).$$

Let us now assume (iii); we want to prove (i).

Let us introduce $\chi \in C_0^\infty(B(X_0, r_0))$, $\chi(X) = 1$ if $|X - X_0| \le r_0/2$. Using Theorem 9 we have $\hat{B}\hat{A} = \hat{\chi} + \mathcal{O}(\hbar^{+\infty})$. Hence $\hat{\chi}\psi_\hbar = \mathcal{O}(\hbar^{+\infty})$. But using Lemma 15 we have $\langle (1 - \hat{\chi})\psi, \varphi_z\rangle = \mathcal{O}(\hbar^{+\infty})$ for $|z - X_0| \le r_0/4$. So we have proved $\langle \psi_\hbar, \varphi_z\rangle = \mathcal{O}(\hbar^{+\infty})$ for $|z - X_0| \le r_0/4$. □

A consequence of this proposition is that Weyl quantization does not increase the frequency set.

Corollary 9 *Let $\psi_\hbar$ be such that $\|\psi_\hbar\| \le 1$, $A \in \Sigma(1)$, then we have*

$$\mathrm{FS}\big[\hat{A}(\psi_\hbar)\big] \subseteq \mathrm{FS}[\psi_\hbar]. \tag{2.107}$$

Moreover if A is elliptic at X_0 then we have

$$X_0 \in \mathrm{FS}\big[\hat{A}(\psi_\hbar)\big] \iff X_0 \in \mathrm{FS}[\psi_\hbar]. \tag{2.108}$$

Proof Let us assume that $X_0 \notin \mathrm{FS}[\psi_\hbar]$. If χ is like in the proof of the proposition, we have $\widehat{\chi A}\psi_\hbar = \mathcal{O}(\hbar^\infty)$. Applying Lemma 15 we have, for z near X_0,

$$\big\langle \varphi_z, \widehat{(1 - \chi)A}\psi_\hbar\big\rangle = \mathcal{O}\big(\hbar^\infty\big)$$

so we get, z near X_0,

$$\big\langle \varphi_z, \widehat{\chi A}\psi_\hbar\big\rangle = \mathcal{O}\big(\hbar^\infty\big).$$ □

2.5.2 About Frequency Set of Eigenstates

Let us consider a quantum Hamiltonian $\hat{H}$. Assume that $H \in \Sigma(m)$. Let us consider the stationary Schrödinger equation

$$\hat{H}\psi_\hbar = E_\hbar \psi_\hbar, \tag{2.109}$$

where $\|\Psi_\hbar\| = 1$, $\lim_{\hbar \to 0} E_\hbar = E$.

Proposition 23 *The frequency set of $\psi_\hbar$ is in the energy level set $S_E = \{X \in \mathbb{R}^{2n},\ H(X) = E\}$.*

Proof Let $X_0 \in \mathbb{R}^{2n}$ such that $H(X_0) \neq E$. There exist $\delta > 0$, $r_0 > 0$ such that $|H(X) - E| \geq \delta$, for every $X \in B(X_0, r_0)$. Let us choose some $\chi \in C_0^\infty(B(X_0, r_0))$, $\chi(X_0) = 1$. Using theorem 9 and the remark following this theorem (here at the end $\varepsilon = \hbar$), we can find B such that

$$\hat{B}\big(\hat{H} - E_\hbar\big) = \hat{\chi} + \mathcal{O}\big(\hbar^{+\infty}\big), \tag{2.110}$$

so we get $\hat{\chi}\psi_\hbar = \mathcal{O}(\hbar^{+\infty})$ hence $X_0 \notin \mathrm{FS}[\psi_\hbar]$. □

Assume now that $\hat{H}$ satisfies the assumptions of the Propagation theorem and $\psi_\hbar$ satisfies the Schrödinger equation (2.109).

Proposition 24 *The frequency set* $\mathrm{FS}[\psi_\hbar]$ *is invariant under the classical flow Φ^t, for every $t \in \mathbb{R}$.*

Proof Let $X_0 \notin \mathrm{FS}[\psi_\hbar]$. There exists a compact support symbol A elliptic at X_0 such that $\hat{A}\psi_\hbar = \mathcal{O}(\hbar^{+\infty})$.

For every t we have

$$\hat{U}(-t)\hat{A}\psi_\hbar = \mathcal{O}\big(\hbar^{+\infty}\big) = \mathrm{e}^{\frac{itE_\hbar}{\hbar}}\hat{A}(t)\psi_\hbar.$$

Recall that the principal symbol of $\hat{A}(t)$ is $A \cdot \Phi^t$. So we find that if z is near $\Phi^{-t}(X_0)$, then $\hat{A}(t)\psi_\hbar = \mathcal{O}(\hbar^{+\infty})$, hence $\Phi^{-t}X_0 \notin \mathrm{FS}[\psi_\hbar]$. So we see that $\mathrm{FS}[\psi_\hbar]$ is invariant. □

2.6 Wick Quantization

2.6.1 General Properties

Following Berezin–Shubin [23] we start with the following general setting.

Let M be a locally compact metric space, with a positive Radon measure μ and $\mathcal{H}$ an Hilbert space. For each $m \in M$ we associate a unit vector $e_m \in \mathcal{H}$ such that the map $m \mapsto e_m$ is strongly continuous from M into $\mathcal{H}$. Moreover we assume that the following Plancherel formula is satisfied, for all $\psi \in \mathcal{H}$,

$$\|\psi\|^2 = \int_M \left|\langle e_m, \psi\rangle\right|^2 d\mu(m). \tag{2.111}$$

Let us denote $\psi^\#(m) = \langle e_m, \psi\rangle$. The map $\psi \mapsto \psi^\#(m) := \mathcal{I}\psi(m)$ is an isometry from $\mathcal{H}$ into $L^2(M)$. The canonical coherent states introduced in Chap. 1 are examples of this setting where $M = \mathbb{R}^{2n}$, $\mathcal{H} = L^2(\mathbb{R}^n)$, $z \mapsto \varphi_z$, with the measure $d\mu(z) = (2\pi\hbar)^{-n}\, dq\, dp$, $z = (q, p) \in \mathbb{R}^{2n}$.

Definition 10 Let $\hat{A} \in \mathcal{L}(\mathcal{H})$.

(i) The covariant symbol of $\hat{A}$ is the function on M defined by $A_c(m) = \langle e_m, \hat{A} e_m\rangle$.
(ii) The contravariant symbol of $\hat{A}$ is the function on M, if it exists, such that

$$\hat{A}\psi = \int_M A^c(m) \Pi_m \psi\, dm, \quad \psi \in \mathcal{H}. \tag{2.112}$$

For the standard coherent states example, the covariant symbol is called Wick symbol and the contravariant symbol the anti-Wick symbol.

The covariant symbol satisfies the equality $A_c(m) = \mathrm{Tr}(\hat{A}\Pi_m)$.

Let us compute the anti-Wick symbol of some operator $\hat{A}$ with Weyl symbol A.

We know that the $\hbar$-Weyl symbol of the projector Π_z is the Gaussian $(\pi\hbar)^{-n} \mathrm{e}^{-\frac{|X-z|^2}{\hbar}}$. So we find that the Weyl symbol of $\hat{A}$ is the convolution of its anti-Wick symbol and a standard Gaussian function:

$$A(X) = (\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A^c(X) \mathrm{e}^{-\frac{|X-z|^2}{\hbar}}\, dz. \tag{2.113}$$

This formula shows that if $\hat{A}$ has a bounded anti-Wick symbol ($A^c \in L^\infty(\mathbb{R}^{2n})$) then its Weyl symbol is an entire function in $\mathbb{C}^{2n}$, which is a restriction for a given operator to have an anti-Wick symbol.

Let us remark that the Wick symbol is an inverse formula associated with (2.113):

$$A_c(z) = 2^n \int_{\mathbb{R}^{2n}} A(X) \mathrm{e}^{-\frac{|X-z|^2}{\hbar}}\, dX. \tag{2.114}$$

Now we give another interpretation of the contravariant symbol. Let us first remark that we have

$$\mathcal{I}^* \cdot \mathcal{I} = \mathbb{1}_{\mathcal{H}}, \tag{2.115}$$

$$\mathcal{I} \cdot \mathcal{I}^* = \Pi_{\mathcal{H}}, \tag{2.116}$$

where $\Pi_{\mathcal{H}}$ is the orthogonal projector in $L^2(M)$ on $\mathcal{H}$ identified with $\mathcal{I}(\mathcal{H})$.

Proposition 25 *Let us assume that $\hat{A}$ has a contravariant symbol A^c such that $A^c \in L^\infty(M)$. Then we have*

$$\hat{A} = \mathcal{I}^* \cdot A^c \cdot \mathcal{I}, \tag{2.117}$$

where A^c is here the multiplication operator in $L^2(M)$.

Proof For every $\psi, \eta \in \mathcal{H}$ we have

$$\langle \eta, \hat{A}\psi \rangle = \int_M \langle \eta, e_m \rangle \langle e_m, \hat{A}\psi \rangle \, d\mu(m) \tag{2.118}$$

and

$$\begin{aligned} \langle e_m, \hat{A}\psi \rangle &= \int_M A^c(m') \langle e_m, \Pi_{m'}\psi \rangle \, d\mu(m') \\ &= \int_M A^c(m') \langle \Pi_{m'} e_m, \Pi_{m'}\psi \rangle \, d\mu(m'). \end{aligned} \tag{2.119}$$

So we get

$$\langle \eta, \hat{A}\psi \rangle = \iint_{M\times M} A^c(m') \langle \Pi_{m'} e_m, \Pi_{m'}\psi \rangle \langle \eta, e_m \rangle \, d\mu(m') \, d\mu(m). \tag{2.120}$$

We get the conclusion using the equality

$$\langle \eta, e_{m'} \rangle = \int_M \langle e_{m'}, e_m \rangle \langle \eta, e_m \rangle \, d\mu(m). \tag{2.121}$$

□

Estimates on operators with covariant and contravariant symbols are easier to prove than for Weyl symbols. Moreover they can be used as a first step to get estimates in the setting of Weyl quantization as we shall see for positivity. The following proposition is easy to prove.

Proposition 26 *Let $\hat{A}$ be an operator in $\mathcal{H}$ with a contravariant symbol A^c. Suppose that $A^c \in L^\infty(M)$. Then $\hat{A}$ is bounded in $\mathcal{H}$ and we have*

$$\|A_c\|_\infty \le \|\hat{A}\| \le \|A^c\|_\infty. \tag{2.122}$$

Moreover $\hat{A}$ is self-adjoint if and only if A^c is real and $\hat{A}$ is non-negative if A^c is μ-almost everywhere non-negative on M.

For our basic example $\mathcal{H} = L^2(\mathbb{R}^n)$, it is convenient to use the following notation. If A is a classical observable, $A \in \Sigma(1)$, $\mathrm{Op}_\hbar^w(A)$ denotes the Weyl quantization of A and $\mathrm{Op}_\hbar^{aw}(A)$ denotes the anti-Wick quantization of A. In other words $\mathrm{Op}_\hbar^{aw}(A)$ admits A as an anti-Wick symbol. The following proposition is an easy consequence of the above results.

Proposition 27 *Let $A \in \Sigma(1)$ (more general symbols could be considered). Then we have*

$$\mathrm{Op}_\hbar^{aw}(A) = \mathrm{Op}_\hbar^{w}(A * G), \quad \text{where } G(X) = (\pi\hbar)^{-n} \mathrm{e}^{-\frac{|X|^2}{\hbar}}, \tag{2.123}$$

$$\langle \psi, \mathrm{Op}_\hbar^{aw}(A)\psi \rangle = (2\pi\hbar)^{-n} \int_{\mathbb{R}^{2n}} A(z)\mathcal{H}_\psi(z)\,dz, \tag{2.124}$$

where $\mathcal{H}_\psi(z)$ is the Husimi function of ψ.

We get now the following useful consequence for Weyl quantization.

Proposition 28 (Semi-classical Garding inequality) *Let $A \in \Sigma(1)$, $A \geq 0$ on $\mathbb{R}^{2n}$. Then there exists $C \in \mathbb{R}$ such that for every $\hbar \in]0, 1]$ we have*

$$\langle \psi, \hat{A}\psi \rangle \geq C\hbar, \quad \forall \psi \in L^2(\mathbb{R}^n). \tag{2.125}$$

Proof We know that $\mathrm{Op}_\hbar^w(A * G)$ is a non-negative bounded operator. So the proposition will be proved if

$$\|\mathrm{Op}_\hbar^w(A * G - A)\| = \mathcal{O}(\hbar). \tag{2.126}$$

Using a standard argument for smoothing with convolution, we get $\hbar^{-1}(A * G - A) \in \Sigma(1)$, with uniform estimates in $\hbar \in]0, 1]$. Hence we get (2.126) as a consequence of the Calderon–Vaillancourt theorem. □

These results are useful to study the matrix elements $\langle \psi_\hbar, \hat{A}\psi_\hbar \rangle$, for a family $\{\psi_\hbar\}_\hbar$ in the semi-classical regime [106]. This subject is related with an efficient tool introduced by Lions–Paul [137] and P. Gérard [82] (see also [35]): the semi-classical measures. This is an application of anti-Wick quantization as we shall see now.

2.6.2 Application to Semi-classical Measures

Semi-classical measures were introduced to describe localization and oscillations of families of states $\{\psi_\hbar\}_\hbar$, $\|\psi_\hbar\| = 1$ (or at least bounded in $L^2(\mathbb{R}^n)$).

Let us first remark that

$$A \mapsto \langle \psi, \mathrm{Op}_\hbar^{aw} A\psi \rangle$$

is a probability measure $\mu^\hbar$ in $\mathbb{R}^{2n}$. Moreover this probability measure has a density given by the Husimi function of $\psi_\hbar$,

$$d\mu^\hbar = (2\pi\hbar)^{-n}\mathcal{H}_{\psi_\hbar}(z)\,dz.$$

In particular we have

$$\left|\langle\psi, \mathrm{Op}_\hbar^{aw} A\psi\rangle\right| \le \|A\|_\infty$$

for every $A \in C_b(\mathbb{R}^{2n})$ (space of continuous, bounded functions on $\mathbb{R}^{2n}$).

Definition 11 A semi-classical measure for the family of normalized states $\{\psi_\hbar\}_\hbar$ is a probability measure μ on the phase space $\mathbb{R}^{2n}$ for which there exists at least one sequence $\{\hbar_k\}$, $\lim_{k\to+\infty} \hbar_k = 0$ such that for every $A \in \Sigma(1)$, we have

$$\lim_{k\to+\infty}\langle\psi_{\hbar_k} \mathrm{Op}_{\hbar_k}{}^{aw} A\psi_{\hbar_k}\rangle = \int_{\mathbb{R}^{2n}} A\, d\mu. \tag{2.127}$$

In other words, the measure sequence $\mu^{\hbar_k}$ weakly converges toward the measure μ.

Remark 11 Semi-classical measures can also be defined for states $\psi_\hbar \in L^2(\mathbb{R}^n, \mathcal{K})$ where $\mathcal{K}$ is an Hilbert space. By the way in this setting Weyl symbols and anti-Wick symbols are operators in $\mathcal{K}$.

We can also define semi-classical measures for statistical mixed states $\hat\rho$, where $\hat\rho$ is a non-negative operator such that $\mathrm{Tr}\,\hat\rho = 1$.

For more applications and properties of these extensions see the huge literature on this subject; for example see [135].

The following proposition is a straightforward application of the properties of the Husimi function.

Proposition 29 *Let μ be a semi-classical measure for $\{\psi_\hbar\}_\hbar$. Then the support* $\mathrm{supp}(\mu)$ *of the measure μ is included in the frequency set* $\mathrm{FS}[\psi_\hbar]$, $\mathrm{supp}(\mu) \subseteq \mathrm{FS}[\psi_\hbar]$.

Example 2

(i) Let $\psi_\hbar = \varphi_z$, a standard coherent state. Then this family has one semi-classical measure, $\mu = \delta_z$ (Dirac probability).
(ii) Let us assume that the states family $\{\psi_\hbar\}_\hbar$ is tight in the following sense. There exists a smooth symbol χ, with compact support, such that $\hat\chi\psi_\hbar = \psi_\hbar + \mathcal{O}(\hbar)$. Then using Lemma 15, we can see that the family of probabilities $\{\mu^\hbar\}$ is tight, so applying the Prokhorov compacity theorem, there exists at least one semi-classical measure. One of a challenging problem in quantum mechanics is to compute these semi-classical measures for family of bound states satisfying (2.109). If for some $\varepsilon > 0$, $H^{-1}[E-\varepsilon, E+\varepsilon]$ is a bounded set, this family is tight. For classically ergodic systems it is conjectured that there exists only one semi-classical measure, which is the Liouville measure [106].

One important property of semi-classical measures is the following propagation result.

Let us consider the time dependent Schrödinger equation

$$i\hbar\partial_t\psi_\hbar(t) = \hat{H}\psi_\hbar(t), \quad \psi_\hbar(0) = \psi_\hbar, \tag{2.128}$$

where H is a time independent Hamiltonian. We assume that H is real, subquadratic and $\hbar$ independent (for simplicity).

$$\partial_X^\gamma H \in L^\infty\big(\mathbb{R}^{2n}\big), \quad \text{for all } \gamma \text{ such that } |\gamma| \geq 2. \tag{2.129}$$

Let μ be a semi-classical measure for $\{\psi_\hbar\}$.

Theorem 14 *For every $t \in \mathbb{R}$, $\{\psi_\hbar(t)\}$ has a semi-classical measure $d\mu_t$ for the same subsequence $\hbar_k$ given by the transport of $d\mu$ by the classical flow: Φ^t, $\mu(t) = (\Phi^t)^*\mu$.*

Proof For every $A \in C_0{}^\infty(\mathbb{R}^{2n})$, the semi-classical Egorov theorem and comparison between anti-Wick and Weyl quantization give

$$\big\langle\psi_\hbar(t), \mathrm{Op}_\hbar^{aw}(A)\psi_\hbar(t)\big\rangle = \int_{\mathbb{R}^{2n}} A\cdot\Phi^t\, d\mu_{\psi_\hbar} + \mathcal{O}(\hbar). \tag{2.130}$$

Hence we get the result going to the limit for the sequence $\hbar_k$. □

We have the following consequence for the stationary Schrödinger equation.

Corollary 10 *Let μ be semi-classical measure for a family of bound states $\{\psi_\hbar\}$, satisfying $\hat{H}\psi_\hbar = E_\hbar\psi_\hbar$. Then μ is invariant by the classical flow Φ^t for every $t \in \mathbb{R}$.*

Proof $\psi_\hbar(t) = \mathrm{e}^{-\frac{it}{\hbar}E_\hbar}\psi_\hbar$ satisfies the time dependent Schrödinger equation so using the Theorem we get $(\Phi^t)^*\mu = \mu$. □

Now we illustrate Corollary 10 on Hermite bound states of the harmonic oscillator.

We assume $n = 1$. We can easily compute Husimi function $\mathcal{H}_j$ of the Hermite function ϕ_j.

$$\mathcal{H}_j(q,p) = \big|\langle\varphi_X, \phi_j\rangle\big|^2 = \frac{(q^2+p^2)^j}{2^j j!}\mathrm{e}^{-\frac{1}{2\hbar}(q^2+p^2)}. \tag{2.131}$$

We want to study the quantum measures $d\mu_j = (2\pi\hbar)^{-1}\mathcal{H}_j(q,p)\,dq\,dp$ when the energies $E_j = (j+\frac{1}{2})\hbar$ have a limit $E > 0$. So we have $\hbar \to 0$ and $j \to +\infty$. For simplicity we fix $E > 0$ and choose $\hbar = \hbar_j = \frac{E}{j}$.

Let f be in the Schwartz class $\mathcal{S}(\mathbb{R}^2)$. We have to compute the limit of $\int f(X)\,d\mu_j(X)$ for $j \to +\infty$. Using polar coordinates and a change of variables we have to study the large k limit for the Laplace integral

$$I(j) := \frac{1}{(j+1)!}\int_0^\infty u^j \mathrm{e}^{-\frac{j}{E}u} f\big(\sqrt{2u}\cos\theta, \sqrt{2u}\sin\theta\big)\,du, \quad \theta \in [0, 2\pi[.$$

We can assume that f has a bounded support and $(0,0)$ is not in the support of f.

Using the Laplace method we get

$$\lim_{j\to+\infty} I(j) = f\big(\sqrt{2E}(\cos\theta, \sin\theta)\big). \tag{2.132}$$

So, we have

$$\lim_{j\to+\infty}\int f(X)\,d\mu_j(X) = \frac{1}{2\pi\sqrt{2E}}\int_0^{2\pi} f\big(\sqrt{2E}(\cos\theta, \sin\theta)\big)\,d\theta. \tag{2.133}$$

On the r.h.s. of (2.133) we recognize the uniform probability measure on the circle of radius $\sqrt{2E}$. This measure is a semi-classical measure for the quantum harmonic oscillator. Let us remark that the classical oscillator of energy $\sqrt{2E}$ moves on the circle of radius $\sqrt{2E}$ in the phase space.

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