2 All-Pole Approximations

2.1 Filter Specifications and Approximations

The behavior of analogue filters can be described in the time or in the frequency domain and filters can be designed from time or frequency domain specifications. However, it is more often the case that filters are designed from frequency domain specifications from either amplitude or phase requirements.

Filter specifications are usually given in terms of magnitude characteristics and describe the desirable gain or attenuation in the passband and stopband within specified tolerances. As we will see and prove, the complexity of the transfer function of the filter and of the filter itself depends heavily on these tolerances and increases as these become stricter.

All filter approximations, tables and nomographs refer to normalized lowpass specifications (see Sect. 1.6) since they are not only easily denormalized but can be transformed to any other filter type, such as highpass, band-reject and bandpass filters.

Approximations are mathematical procedures used to translate given magnitude specifications into realizable transfer functions. Employment of well known approximations like Butterworth, Chebyshev, elliptic etc. to determine the transfer function of a filter ensures that the magnitude response will satisfy the specifications and will be realizable with passive or active circuits.

The synthesis of a normalized lowpass filter with given magnitude specifications starts with finding a gain function $G(\Omega)$ that satisfies the specifications with $G(\Omega)^2$ being a rational even function of Ω. This procedure, *the approximation*, yields theoretically infinite solutions and some of these are not suitable since they do not satisfy certain realizability conditions. This means that simply finding a mathematical function, the plot of which does not violate the specifications, does not necessarily assure its realizability as a gain function of a realizable circuit.

Fortunately, as it has been mentioned in Chap. 1, the filter designer can choose from some established approximations that lead to realizable transfer functions. Among these, the most popular are the Butterworth, Chebyshev and Cauer (or elliptic) approximations. The manner in which each of them approximates the ideal lowpass gain specifications is shown in Fig. 2.1. From these approximations, Butterworth, Chebyshev and Pascal, (a), (b) and (d) in Fig. 2.1 are monotonic in the stopband. Inverse Chebyshev (Fig. 2.1c), inverse Pascal (Fig. 2.1e) and Cauer (Fig. 2.1f) have transmission zeros, i.e. frequencies at which the plain gain becomes zero.

In order to synthesize and finally implement a passive or active filter from magnitude specifications, it is necessary to calculate its transfer function $H(s) = \frac{X_{out}(s)}{X_{in}(s)}$. The approximation gives the gain function $G(\omega) = |H(s)|_{s=j\omega}$. The poles and zeros of the required and stable transfer function are obtained by a factorization process, the basis of which is as indicated in (2.1)

$$H(s)H(-s) = |H(j\omega)|^2_{\omega=-js} = |H(j\omega)|^2_{\omega^2=-s^2} \tag{2.1}$$

H.G. Dimopoulos, *Analog Electronic Filters*, Analog Circuits and Signal Processing,
DOI 10.1007/978-94-007-2190-6_2, © Springer Science+Business Media B.V. 2012

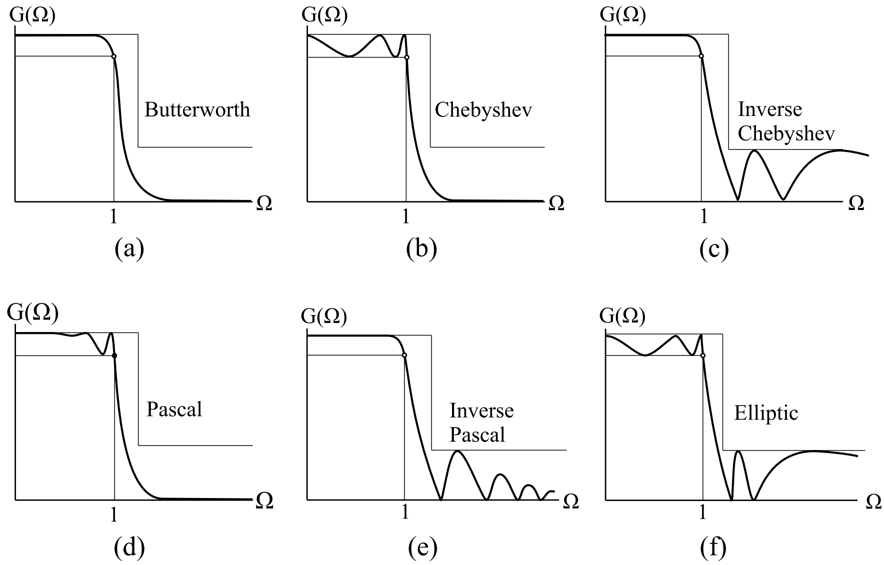


Fig. 2.1 Butterworth, Chebyshev, Pascal and elliptic approximations

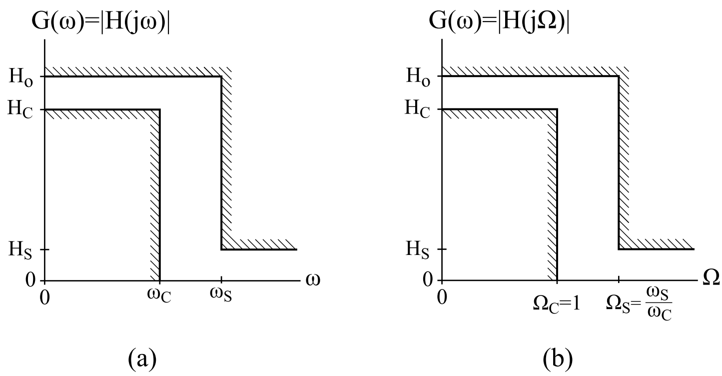


Fig. 2.2 The lowpass specifications and the normalized lowpass specifications

The approximation procedure is completed when we have constructed the transfer function $H(s)$, the magnitude of which $|H(s)|_{s=j\omega} = G(\omega)$ satisfies the filter specifications.

If the lowpass filter specifications are not normalized with $\Omega_C = 1$, as in Fig. 2.2a, they can be normalized by scaling the angular frequencies by the given ω_C so that the normalized stopband edge frequency becomes $\Omega_S = \frac{\omega_S}{\omega_C} > 1$ (Fig. 2.2b).

As shown in Chap. 1, frequency scaling does not affect the magnitude characteristics (gain or attenuation). Moreover, a filter designed with normalized specifications can easily be made to operate at any desired frequency, within the operational frequency range of the devices used in the implementation, by denormalizing accordingly the circuit elements.

2.1.1 All-Pole Transfer Functions and Approximations

The term *all-pole* refers to transfer functions with constant numerators of the form $H(s) = \frac{k}{D(s)}$, where $D(s)$ is in general a polynomial in s of degree N:

$$H(s) = \frac{k}{D(s)} = \frac{k}{s^N + B_{N-1}s^{N-1} + \cdots + B_1 s + B_0} \tag{2.2}$$

Filter transfer functions have to be stable with all their poles in the left hand half s-plane. Poles on the $j\omega$-axis would make the transfer function *marginally stable* and would create infinite gain at their frequencies, a situation which is neither desirable nor useful in filter applications. Therefore, the only acceptable kinds of poles in the filter transfer functions are negative real poles $s_p = -c$ $(c > 0)$ and complex conjugate pole pairs $s_p = -a \pm jb$, with negative real part, i.e. $a > 0$. This means that only terms of the form $(s + c)$ and $(s^2 + 2as + a^2 + b^2)$ will be present in the denominator polynomial $D(s)$ of (2.2) which, due to the exclusive presence of such terms, will be complete, i.e. without missing terms.

From (2.2) we get

$$\begin{aligned} &H(j\Omega) = \frac{k}{D(j\Omega)} \\ \rightarrow \quad &|H(j\Omega)| = \frac{|k|}{\sqrt{1 + [|D(j\Omega)|^2 - 1]}} = \frac{|k|}{\sqrt{1 + Q(\Omega)}} \end{aligned} \tag{2.3}$$

This is the general form of the magnitude response of all-pole lowpass filters with a constant numerator and a denominator which is the square root of a factor of the form $1 + [|D(j\Omega)|^2 - 1]$. The term $|D(j\Omega)|$, as a magnitude function, is an even function of Ω and therefore, the bracketed term

$$\left[|D(j\Omega)|^2 - 1\right] = Q(\Omega) \tag{2.4}$$

is a real even polynomial in Ω of even degree $2N$ of the form:

$$Q(\Omega) = \Omega^{2N} + b_{2N-2}\Omega^{2N-2} + b_{2N-4}\Omega^{2N-4} + \cdots + b_2\Omega^2 + b_0 \tag{2.5}$$

with all $b_i \neq 0$, except the constant term b_0, which can be zero only in the case when the constant term B_0 of $D(s)$ is equal to 1, i.e. only if $D(0) = 1$.

The term *complete polynomial* is used for a polynomial of order N in which all coefficients are $\neq 0$. The terms *complete odd* and *complete even* polynomial will also be used for odd and even polynomials respectively with no missing terms. In complete even polynomials, the constant term can be zero but in any case they can be expressed as polynomials in $\Omega_{sq} = \Omega^2$:

$$\begin{aligned} &Q(\Omega) = \Omega^{2N} + b_{2N-2}\Omega^{2N-2} + \cdots + b_2\Omega^2 + b_0 \\ \Rightarrow \quad &Q(\Omega) = Q(\Omega_{sq}) = \Omega_{sq}^N + b_{2N-2}\Omega_{sq}^{N-1} + \cdots + b_2\Omega_{sq} + b_0 \end{aligned}$$

Bearing in mind the expression (2.3) for the magnitude response of all-pole transfer functions as well as that $Q(\Omega)$ is a complete even polynomial, in all polynomial approximations the gain is expressed with a similar function of the form

$$G(\Omega) = |H(j\Omega)| = \frac{H_o}{\sqrt{1 + P_a(\Omega)}} \tag{2.6}$$

where $P_a(\Omega)$ must meet the conditions set for $Q(\Omega)$, i.e. it must be a complete even polynomial, as defined above. In fact any complete even polynomial can be used in (2.6) in order to express the lowpass gain characteristics, provided it assumes low values in the passband and high values in the stopband.

Equation (2.6) is the basis of the all-pole approximations which differ only in the polynomial used. In fact, all known all-pole approximations use (2.6) with $P_a(\Omega) = \gamma^2 P_N^2(\Omega)$:

$$G(\Omega) = \frac{H_o}{\sqrt{1 + \gamma^2 P_N^2(\Omega)}} \tag{2.7}$$

with $P_N(\Omega)$ being the *approximating polynomial* (complete even or odd) and γ a design parameter.

Although it is not necessary, $P_a(\Omega)$ is taken to be equal to the square of an even or odd polynomial of degree N, i.e. $P_a(\Omega) = \gamma^2 P_N^2(\Omega)$ in order to simplify the related mathematical complexity.

For lowpass gain characteristics to be achieved, $P_N(\Omega)$ must assume low values for $0 \le \Omega \le 1$ and high values for $\Omega > 1$. For practical reasons, the approximating polynomial $P_N(\Omega)$ is made to assume unity value at $\Omega = 1$, i.e. $P_N(1) = 1$.

In the Butterworth approximation, $P_N(\Omega) = \Omega^N$ is used, as will be seen in the next section. If Chebyshev polynomials are used instead, the Chebyshev approximation is derived. In a similar way, Pascal, Legendre or Bessel polynomials can be used to produce the approximation with the corresponding name and properties. All-pole approximations are also referred to as *polynomial approximations* due to the fact that the approximating functions are polynomials.

It should be noted that all-pole approximations using in (2.6) a complete even polynomial $P_a(\Omega)$ of order $2N$ which is not the square of another polynomial of degree N have not been reported in the related literature. For example, the polynomial of degree $2 \cdot 6 = 12$

$$P_a(\Omega) = 47.346554 \prod_{m=1}^{6} (\Omega^2 - \Omega_{zm}^2) \quad \text{where } \Omega_{zm} = \frac{m-1}{0.725806(m+1)}$$

which is not the square of another polynomial of degree $N = 6$ can be used in (2.6) with $H_o = 0.958833$ to give the logarithmic gain shown in Fig. 2.3. The figure also shows the corresponding 6th order Chebyshev filter with 0.5 dB passband ripple, for comparison. This approximation exhibits a non-equiripple behavior in the passband and its advantage over the corresponding Chebyshev response might be the inherent diminishing ripple [11] and the lower pole Qs.

2.2 The Butterworth Approximation

Assuming that the type of response is selected a priori (e.g. Butterworth type etc.) the following four parameters describe the filter transfer function completely:

$$\{H_o, H_C, H_S, \Omega_S\} \qquad (\Omega_C = 1)$$

In terms of logarithmic gain, the normalized lowpass specifications can be described by

$$\{\alpha_o, \alpha_{\max}, \alpha_{\min}, \Omega_S\} \qquad (\Omega_C = 1)$$

where $\alpha_o = 20\log(H_o)$, $\alpha_{\max} = 20\log(H_o/H_C)$ and $\alpha_{\min} = 20\log(H_o/H_S)$.

The parameter H_o can be taken equal to 1 (i.e. $\alpha_o = 0$ dB) without loss of generality and therefore the filter requirements can in practice be determined by three parameters:

$$\Omega_S \quad \text{and} \quad \{H_C, H_S\} \quad \text{or} \quad \{a_{\max}, \alpha_{\min}\} \tag{2.8}$$

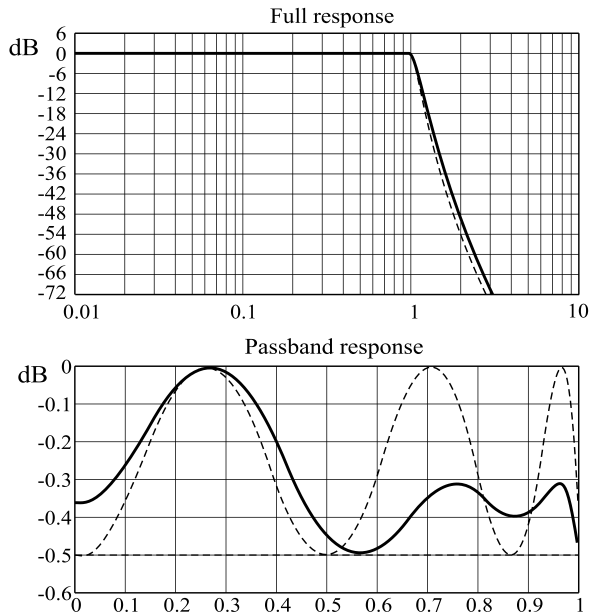


Fig. 2.3 Logarithmic gain of a 6th order non-square approximation compared to the corresponding Chebyshev response (*dotted*)

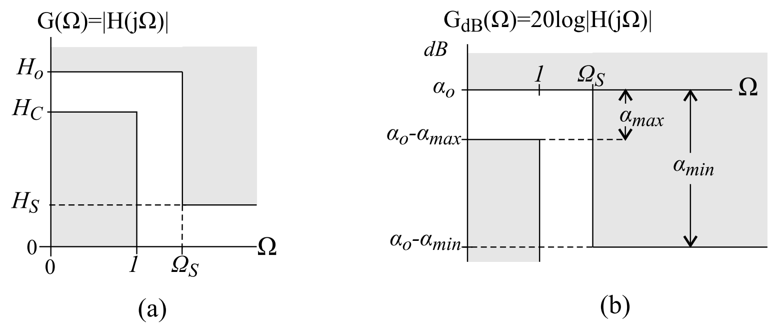


Fig. 2.4 Normalized lowpass specifications: (**a**) Plain gain. (**b**) Logarithmic gain

referred to as the normalized lowpass filter specifications.

Thus, the design process seeks to find a function G(Ω) which stays within the limits defined by the specifications. In other words, the response does not stray into the dashed areas of Fig. 2.4. S. Butterworth proposed [1] the monotonic function:

$$G(\Omega) = \frac{H_o}{\sqrt{1 + \beta^2 \Omega^{2N}}} \tag{2.9}$$

with N, the *order* of the approximation, a positive integer, and β a design parameter related to the passband tolerance. Notice that the Butterworth gain function of (2.9) is in accordance with (2.7) with $P_N(\Omega) = \Omega^N$ with $Q_{2N}(\Omega) = \beta^2 \Omega^{2N}$.

Equation (2.9) for $\Omega = 0$ gives by definition $G(0) = H_o$ while the *integer order* N and the design parameter β remain to be determined.

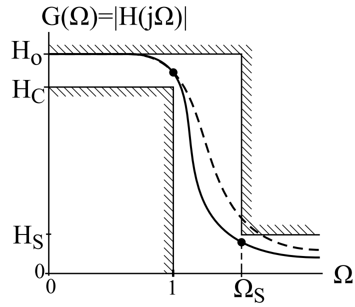


Fig. 2.5 Butterworth approximation stopband scenarios

The order N is selected to be the lowest integer consistent with the specifications. Its selection will be examined shortly. For the present we select β such that

$$G(1) = \frac{H_o}{\sqrt{1+\beta^2}} \geq H_C \quad \Leftrightarrow \quad \beta^2 \leq (H_o/H_C)^2 - 1$$

$$\Leftrightarrow \quad \beta \leq \beta_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{10^{\frac{a_{\max}}{10}} - 1} \tag{2.10}$$

Thus any value of $\beta \leq \beta_{\max}$ ensures that the gain for $0 \leq \Omega \leq 1$ will remain $\geq H_C$. For $\beta = \beta_{\max}$ we make the gain $G(1) = H_C$ and the gain curve passes from the point $\{1, H_C\}$. It is almost exclusively the case in the literature that β is taken to be $\beta_{\max}$ and only in [5] is it allowed to be a design parameter and can assume any value $\beta_{\min} \leq \beta \leq \beta_{\max}$. The minimum value $\beta_{\min}$ depends on the order N which must be first calculated. Before the order calculation, let us examine the derivative of the Butterworth gain function of (2.9).

$$\frac{d}{d\Omega} G(\Omega) = \frac{d}{d\Omega}\left(\frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}\right) = -\frac{N\beta^2 H_o \Omega^{2N-1}}{(1+\beta^2\Omega^{2N})^{\frac{3}{2}}} \tag{2.11}$$

The derivative is non-positive and henceforth $G(\Omega)$ is monotonically decaying. It can also be shown that all $(N-1)$ derivatives of $G(\Omega)$ assume zero value for $\Omega = 0$ and for this reason Butterworth approximation is called *maximally flat*.

The Butterworth gain function of (2.9) with $\beta \leq \beta_{\max}$ satisfies the passband specifications ($0 \leq \Omega \leq 1$) but may violate the specifications for $\Omega \geq 1$ as shown in Fig. 2.5 with a dashed curve.

In order for the monotonic gain function to satisfy the specifications for $\Omega \geq 1$, it is sufficient to ensure that $G(\Omega_S) \leq H_S$. This will lead to the calculation of the order N of the approximation:

$$G(\Omega_S) = \frac{H_o}{\sqrt{1+\beta^2\Omega_S^{2N}}} \leq H_S \quad \Leftrightarrow \quad \Omega_S^{2N} \geq \frac{(H_o/H_S)^2 - 1}{\beta^2}$$

$$\Leftrightarrow \quad 2N \log \Omega_S \geq \log\left(\frac{(H_o/H_S)^2 - 1}{\beta^2}\right)$$

$$\Leftrightarrow \quad N \geq \frac{\log(\frac{(H_o/H_S)^2-1}{\beta^2})}{2\log \Omega_S}$$

From the above relation, it can be concluded that the integer order N of the approximation must be greater than the *fractional order* N_d of the approximation:

$$N \geq N_d = \frac{\log(\frac{(H_o/H_S)^2-1}{\beta^2})}{2\log \Omega_S} \tag{2.12}$$

Since N must be integer, it is taken as the minimum integer that satisfies $N \geq N_d$, i.e. the integer that results from rounding up N_d. The complexity of a filter is directly proportional to the order of the approximation, and for this reason we try to keep N as low as possible. The fractional order N_d, if seen as a function of β, assumes its minimum value for $\beta = \beta_{\max}$ and the *order equation* (2.12) becomes:

$$n_{f\,\min} = \frac{\log(\frac{\frac{H_o^2}{H_S^2}-1}{\frac{H_o^2}{H_C^2}-1})}{2\log \Omega_S} = \frac{\log(\frac{10^{\frac{a_{\min}}{10}}-1}{10^{\frac{a_{\max}}{10}}-1})}{2\log \Omega_S} \tag{2.13}$$

The order equation gives the fractional order of the approximation as a function of the specifications. The integer order N of the approximation will of course be the next integer value greater than N_d. It is clear from (2.13) that the order is increasing as Ω_S approaches unity and becomes infinite for $\Omega_S = 1$. This means that as we try to make a narrow transition band, we increase the order or that higher order filters realize narrower transition bands. The order also increases as the passband tolerance decreases (H_C approaches H_o) and as the maximum allowed stopband gain H_S decreases. Thus, as we try to make the response closer to the ideal brick wall characteristic the order becomes increasingly large until it becomes infinite for the ideal case.

Filters designed to have the monotonic maximally flat gain function of (2.9) are referred to as Butterworth filters and the name implies only that they realize this type of response, no matter how they are implemented. Butterworth filters can be implemented under any suitable technology.

2.2.1 Optimization Using β as a Design Parameter

Returning to (2.12), we saw that N_d is minimized for $\beta = \beta_{\max}$. Decreasing β from its maximum value, the fractional order increases until it takes integer value and $G(\Omega_S) = H_S$. This minimum value of β is given by

$$\beta_{\min} = \frac{\sqrt{(H_o/H_S)^2-1}}{\Omega_S^N} = \frac{\sqrt{10^{\frac{a_{\min}}{10}}-1}}{\Omega_S^N} \tag{2.14}$$

Decreasing β to a value lower than $\beta_{\min}$, N_d exceeds the integer order N and the new integer order of the filter will increase by 1. This is shown in Fig. 2.6 for a filter designed with normalized specifications $H_o = 1$, $H_C = 0.95$, $H_S = 0.05$ and $\Omega_S = 2.7$ which give $N_d = 4.135 \Rightarrow N = 5$.

When the minimum fractional order N_d has been calculated from (2.13) and has been rounded up to the next integer value N, the coefficient β becomes a design parameter and its value can be selected from

$$\beta_{\min} = \frac{\sqrt{\frac{H_o^2}{H_S^2}-1}}{\Omega_S^N} \leq \beta \leq \sqrt{\frac{H_o^2}{H_C^2}-1} = \beta_{\max} \tag{2.15}$$

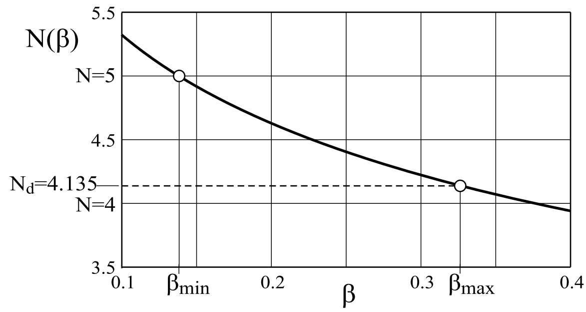


Fig. 2.6 Plot of N_d as a function of β

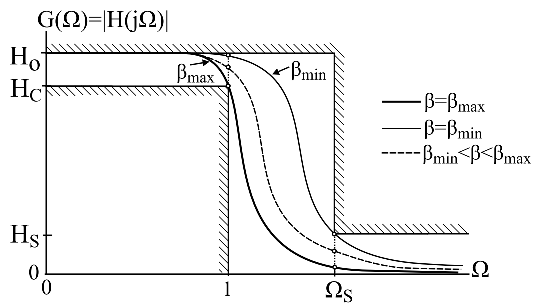


Fig. 2.7 Butterworth gain plot for the various values of β

or, in terms of logarithmic gain specifications

$$\beta_{\min} = \frac{\sqrt{10^{\frac{\alpha_{\min}}{10}} - 1}}{\Omega_S^N} \le \beta \le \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} = \beta_{\max} \tag{2.16}$$

For the calculated N, any value of β within the permitted range $\beta_{\min} \le \beta \le \beta_{\max}$ leads to a Butterworth gain function $G(\Omega) = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$ which satisfies the specifications in a different manner, as shown in Fig. 2.7.

2.2.1.1 Case I: $\beta = \beta_{\max}$

Using $\beta = \beta_{\max}$ in (2.9) we get $G(1) = H_C$ and $G(\Omega_S) < H_S$. In this case, the passband tolerance is fully used and the gain is optimized at the stopband edge frequency Ω_S, where it takes the value

$$H_{S\min} = G(\Omega_S) = \frac{H_o}{\sqrt{1 + \beta_{\max}^2 \Omega_S^{2N}}} \le H_S \tag{2.17}$$

This case is referred to as *stopband edge frequency gain optimized.*

2.2.1.2 Case II: $\beta = \beta_{\min}$

Using $\beta = \beta_{\min}$ in (2.9), we get $G(\Omega_S) = H_S$ and $G(1) = H_{C\max} > H_C$. In this case, the stopband tolerance is fully used with $G(\Omega_S) = H_S$ and the gain is optimized in the passband, where it varies from H_o down to $H_{C\max} > H_C$:

$$H_{C\max} = G(1) = \frac{H_o}{\sqrt{1 + \beta_{\min}^2}} = \frac{H_o}{\sqrt{1 + \frac{(H_o/H_S)^2 - 1}{\Omega_S^{2N}}}} > H_C \tag{2.18}$$

This case, in which β takes the minimum value, is referred to as *passband gain optimized.*

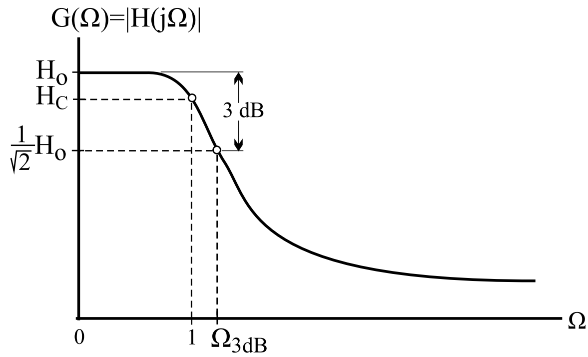


Fig. 2.8 The 3 dB frequency Ω_{3dB}

2.2.1.3 Case III: $\beta_{\min} \le \beta \le \beta_{\max}$

Using any other permitted value of β in (2.9), the specifications are satisfied with $G(1) \ge H_C$ and $G(\Omega_S) \le H_S$, namely

$$G(\Omega_S) = \frac{H_o}{\sqrt{1+\beta^2\Omega_S^{2N}}} < H_S \quad \text{and} \quad G(1) = \frac{H_o}{\sqrt{1+\beta^2}} > H_C \tag{2.19}$$

2.2.2 The 3 dB Frequency of Butterworth Filters

The characteristic frequency Ω_{3dB} of Fig. 2.8 at which the gain drops by 3 dB from H_o to $H_o/\sqrt{2}$ is such that:

$$G(\Omega_{3dB}) = \frac{H_o}{\sqrt{1+\beta^2\Omega_{3dB}^{2N})}} = \frac{H_o}{\sqrt{2}} \quad \Leftrightarrow \quad \beta\Omega_{3dB}^N = 1 \quad \Leftrightarrow \quad \Omega_{3dB}^N = \frac{1}{\beta}$$

$$\Omega_{3dB} = \left(\frac{1}{\beta}\right)^{\frac{1}{N}} = \beta^{-\frac{1}{N}} \tag{2.20}$$

The Ω_{3dB} frequency is shown in Fig. 2.8.

2.2.3 The Cut-off Rate

Figure 2.9 shows the gain plots of Butterworth approximation for various values of N and $\Omega \ge 1$. The normalized frequency Ω is plotted in logarithmic scale and it is obvious that the curves become almost straight lines for $\Omega \gg 1$. The slope of these lines increases with N and it will be shown that it depends only on N. The slope of the logarithmic gain curves is measured in dB/octave. The slope of the logarithmic gain at frequencies where it is constant ($\Omega \gg 1$) is referred to as *cut-off rate* and can be determined by taking the difference of the logarithmic gain at a frequency Ω_1 and at the double frequency $2\Omega_1$:

$$r = 20\log(G(\Omega_1)) - 20\log(G(2\Omega_1)) = 20\log\left(\frac{\sqrt{1+\beta^2 2^{2N}\Omega_1^{2N}}}{\sqrt{1+\beta^2\Omega_1^{2N}}}\right)$$

$$\text{since } \Omega_1 \gg 1 \Rightarrow \beta^2\Omega_1^{2n} \gg 1 \quad \text{and}$$

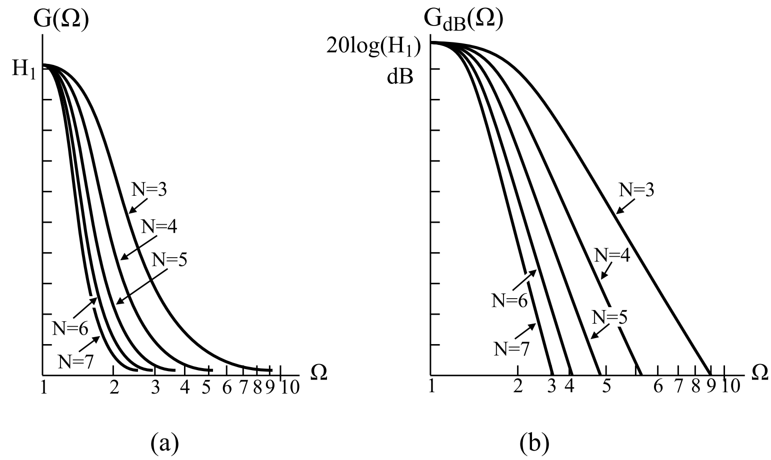


Fig. 2.9 Plots of plain and logarithmic Butterworth gain

$$r \approx 20\log\left(\frac{\sqrt{\beta^2 2^{2N}\Omega_1^{2N}}}{\sqrt{\beta^2\Omega_1^{2N}}}\right) = 20\log(2^N) = N20\log(2) \approx 6N \text{ dB/octave}$$

This means that in the stopband, the logarithmic gain falls at a rate of $6N$ dB for every frequency doubling (one octave). The above relationship between the order of the filter and the cutoff rate explains the curves of Fig. 2.9b.

Often, instead of giving H_S and Ω_S in the specifications, the cut-off rate is given, from which the order N can be calculated (e.g. given $r = 24$ dB/octave, the order is $N = 24/6 = 4$).

Example 2.1 Determine the Butterworth gain function $G_\alpha(\omega)$ that satisfies the specifications shown in Fig. 2.10a.

We identify the specifications:

$$\omega_C = 600 \text{ rad/s}, \qquad \omega_S = 1800 \text{ rad/s},$$
$$H_o = 2, \qquad H_C = 1.9 \quad \text{and} \quad H_S = 0.1$$

Normalizing with $\omega_C = 600$ rad/s we get the normalized specifications of Fig. 2.10b with $\Omega_C = 1$ and $\Omega_S = 3$. In order to use (2.9), we need a value for β and the order N which will be calculated from (2.13).

$$\beta_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{\left(\frac{2.0}{1.90}\right)^2 - 1} = 0.3286837 \quad \beta_{\max}^2 = 0.1080329$$

$$N_d = \frac{\log(\frac{(H_o/H_S)^2-1}{\beta_{\max}^2})}{2\log\Omega_S} = \frac{\log(\frac{\frac{2.0^2}{0.1^2}-1}{0.1080329})}{2\log(3)} = 3.7385$$

The integer order will be the rounded up N_d, i.e. $N = 4$. Now $\beta_{\min}$ can be calculated:

$$\beta_{\min} = \frac{\sqrt{\frac{H_o^2}{H_S^2} - 1}}{\Omega_S^N} = \frac{\sqrt{\frac{2^2}{0.1^2} - 1}}{3^4} = 0.246605 \quad \text{and} \quad \beta_{\min}^2 = 0.0608$$

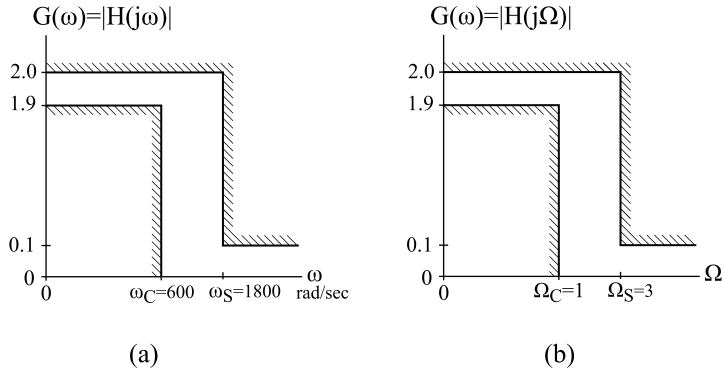


Fig. 2.10 Filter specifications of Example 2.1

For $\beta = \beta_{\max}$ the normalized gain function will be

$$G_{\max}(\Omega) = \frac{2}{\sqrt{1 + 0.108033\Omega^8}}$$

For $\beta = \beta_{\min}$, the normalized gain function will be

$$G_{\min}(\Omega) = \frac{2}{\sqrt{1 + 0.0608\Omega^8}}$$

The corresponding denormalized gain functions for $\omega_C = 600$ rad/s will be

$$G_{1\max}(\omega) = G\left(\frac{\omega}{\omega_C}\right) = \frac{2}{\sqrt{1 + 0.108033(\frac{\omega}{600})^8}}$$

$$G_{1\min}(\omega) = G\left(\frac{\omega}{\omega_C}\right) = \frac{2}{\sqrt{1 + 0.0608(\frac{\omega}{600})^8}}$$

2.2.4 The Normalized Butterworth Lowpass Transfer Function

The gain function $G(\Omega) = |H(s)|_{s=j\Omega}$ of (2.9) is well determined after the calculation of the order N and the selection of the design parameter β. In fact, we now have the following relation:

$$|H(s)|^2_{s=j\Omega} = \frac{H_o^2}{1 + \beta^2\Omega^{2N}} \tag{2.21}$$

From (2.21) and the fact that the denominator of the transfer function $H(s)$ is a Hurwitz polynomial, with no roots in the right hand half-plane, $H(s)$ can be determined from

$$H(s)H(-s) = |H(j\Omega)|^2_{\Omega^2=-s^2} = G(\Omega)^2|_{\Omega^2=-s^2}$$

$$(\Omega^2 = -s^2 \text{ or } \Omega = -js \text{ are used equivalently}) \tag{2.22}$$

On letting $|H(j\Omega)| = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$ in (2.22) we get

$$H(s)H(-s) = \left.\frac{H_o^2}{1+\beta^2\Omega^{2N}}\right|_{\Omega^2=-s^2} = \frac{H_o^2}{1+\beta^2(-s^2)^N} = \frac{\frac{H_o^2}{\beta^2}}{\frac{1}{\beta^2}+(-s^2)^N} \tag{2.23}$$

The denominator of (2.23) is a binomial of the form $x^N + \frac{1}{\beta^2}$ with $x = -s^2$. The roots of this binomial expression are of course the roots of the equation

$$x^N + \left(\frac{1}{\beta^2}\right) = 0 \quad \Leftrightarrow \quad x^N - \frac{1}{\beta^2}e^{j(2k+1)\pi} = 0$$

and henceforth the roots are

$$x_k = \sqrt[N]{\frac{1}{\beta^2}}e^{j\frac{2k+1}{N}\pi} \quad \text{for } k = 0, 1, 2, \ldots, (N-1)$$

Since we have set $x = s^2$, for the roots s_k we have:

$$s_k^2 = -\sqrt[N]{\frac{1}{\beta^2}}e^{j\frac{2k+1}{N}\pi} = \sqrt[N]{\frac{1}{\beta^2}}e^{j(\frac{2k+1}{N}\pi+\pi)} \quad \text{with } k = 0, 1, 2, \ldots, (N-1) \tag{2.24}$$

Finally, the roots of $\frac{1}{\beta^2} + (-s^2)^N = 0$ are given below:

$$\begin{aligned} s_{k+} &= \sqrt[N]{\frac{1}{\beta}}e^{j(\frac{2k+1}{2N}\pi+\frac{\pi}{2})} \quad \text{and} \\ s_{k-} &= \sqrt[N]{\frac{1}{\beta}}e^{j(\frac{2k+1}{2N}\pi-\frac{\pi}{2})} \quad k = 0, 1, 2, \ldots, N-1 \end{aligned} \tag{2.25}$$

The roots s_{k+} and s_{k-} are the poles of $H(s)H(-s)$. It should be noticed from (2.25) that these do not lie on the $j\omega$-axis, i.e. their real part

$$\cos\left(\frac{2k+1}{2N}\pi \pm \frac{\pi}{2}\right)$$

cannot become zero since this would require

$$\frac{2k+1}{2N}\pi \pm \frac{\pi}{2} = \lambda\frac{\pi}{2} \quad \text{with } \lambda \text{ odd}$$

Working out this condition, we find that it requires $2k = N(\lambda \pm 1) - 1$, which is never true since $2k$ is even and $N(\lambda \pm 1) - 1$ is always odd.

$$\begin{aligned} \text{Moreover, real roots exist} \quad &\Leftrightarrow \quad \sin\left(\frac{2k+1}{2N}\pi \pm \frac{\pi}{2}\right) = 0 \\ &\Leftrightarrow \quad \frac{2k+1}{2N}\pi \pm \frac{\pi}{2} = \lambda\pi \text{ where } \lambda \text{ is an integer} \\ &\Leftrightarrow \quad N(2\lambda+1) - 1 \text{ must be even} \end{aligned}$$

$$\Leftrightarrow \quad N(2\lambda+1) \text{ is odd} \quad \Leftrightarrow \quad N \text{ is odd}$$

For odd values of N, there exists a pair of real roots with value $\pm\sqrt[N]{\frac{1}{\beta}} = \pm\beta^{-\frac{1}{N}}$.

The N roots s_{k+} are all located in the left hand half-plane since

$$\frac{\pi}{2} < \frac{2k+1}{2N}\pi + \frac{\pi}{2} < \frac{3\pi}{2} \quad \text{for all } k = 0, 1, 2, \ldots, (N-1)$$

whilst all N roots s_{k-} of (2.25) are located in the right hand half-plane and are one-by-one opposite to s_{k+}.

In view of all of these properties, (2.23) gives:

$$H(s)H(-s) = \frac{\frac{H_o^2}{\beta^2}}{\frac{1}{\beta^2} + (-s^2)^N} = \underbrace{\frac{\frac{H_o}{\beta}}{\prod_{k=0}^{N-1}(s - s_{k+})}}_{\substack{\text{poles} \\ \text{in the LHP}}} \times \underbrace{\frac{\frac{H_o}{\beta}}{\prod_{k=0}^{N-1}(s - s_{k-})}}_{\substack{\text{poles} \\ \text{in the RHP}}}$$

from which $H(s)$ can be identified as:

$$H_{BUT}(s) = \frac{\frac{H_o}{\beta}}{\prod_{k=0}^{N-1}(s - s_{k+})} \tag{2.26}$$

The poles of $H_{BUT}(s)$ are given by

$$s_k = \sqrt[N]{\frac{1}{\beta}} e^{j(\frac{2k+1}{2N}\pi + \frac{\pi}{2})} \quad k = 0, 1, 2, \ldots, (N-1) \tag{2.27}$$

and are located in the s-plane on a circle with radius

$$\sqrt[N]{\frac{1}{\beta}} = \beta^{-\frac{1}{N}} \tag{2.28}$$

It should be noted that the radius given in (2.28) is the magnitude of the poles and is equal to the 3 dB frequency of (2.20). Parameter β in (2.28) has a value given by (2.15), namely

$$\beta_{\min} = \frac{\sqrt{(H_o/H_S)^2 - 1}}{\Omega_S^N} \le \beta \le \sqrt{(H_o/H_C)^2 - 1} = \beta_{\max}$$

For $k = 0$ the first pole is obtained with phase angle $\frac{\pi}{2} + \frac{\pi}{2N}$ and the next poles are at a distance $\frac{\pi}{N}$, as shown in Fig. 2.11. Figure 2.12 shows the poles of the transfer function for $N = 3$ and $N = 4$.

By adjusting the index k in (2.27) so that it varies from $k = 1$ to N, we get

$$H_{BUT}(s) = \frac{\frac{H_o}{\beta}}{\prod_{k=1}^{N}(s - s_k)} \tag{2.29}$$

where

$$s_k = \sqrt[N]{\frac{1}{\beta}} e^{j(\frac{2k+N-1}{2N}\pi)} \quad \text{for } k = 1, 2, \ldots, N \tag{2.30}$$

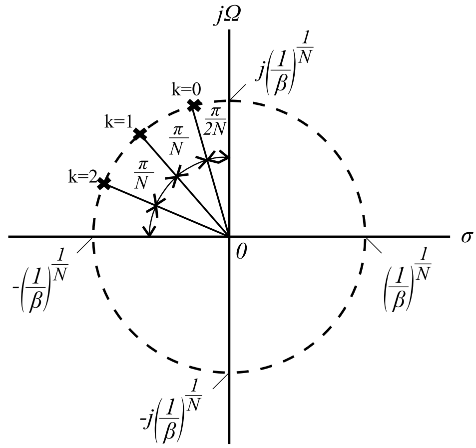


Fig. 2.11 Pole location of normalized Butterworth transfer function

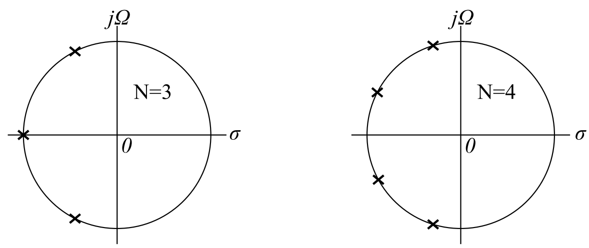


Fig. 2.12 Poles for $N = 3$ and $N = 4$

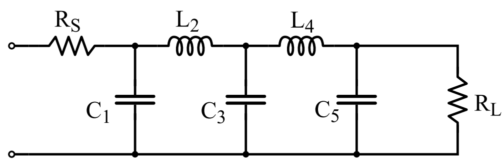


Fig. 2.13 Typical structure of a passive-LC doubly terminated lowpass filter

or

$$s_k = \beta^{-\frac{1}{N}}\left[\cos\left(\frac{2k+N-1}{2N}\pi\right) + j\sin\left(\frac{2k+N-1}{2N}\pi\right)\right]$$
$$\text{for } k = 1, 2, \ldots, N \tag{2.31}$$

The poles of the transfer function of (2.30) for given N, depend only on the design parameter β, and each choice $\beta_{\min} \leq \beta \leq \beta_{\max}$ from (2.15) leads to a different set of poles of the transfer function, the corresponding gain function $|H(j\Omega)|$ of which satisfies the specifications in its own way according to Sect. 2.2.1 and Fig. 2.7.

It will be shown later in this book that the all-pole Butterworth lowpass transfer function can be realized not only with active circuits but also with doubly terminated passive ladder two-port LC circuits like the one in Fig. 2.13. Such a circuit, with the proper calculation of the element values, can realize a Butterworth response with $N = 5$. The total number of lossless elements in this case (inductors and capacitors) equals the order N of the approximation.

Table 2.1 Denominator of the prototype Butterworth lowpass transfer functions

Denominator of the prototype Butterworth lowpass transfer functions
$H_{pBUT}(s) = \frac{H_o}{D_p(s)} |H_{pBUT}(j\Omega)| = \frac{H_o}{\sqrt{1+\beta^2\Omega^{2N}}}$ with $G(1) = \frac{\sqrt{2}}{2} H_o$

N	$D_p(s)$
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s+1)(s^2+s+1) = s^3+2s^2+2s+1$
4	$(s^2+0.765367s+1)(s^2+1.847759s+1) = s^4+2.613126s^3+3.414214s^2+2.613126s+1$
5	$(s+1)(s^2+0.618034s+1)(s^2+1.618034s+1)$ $= s^5+3.236068s^4+5.236068s^3+5.236068s^2+3.236068s+1$
6	$(s^2+0.517638s+1)(s^2+(\sqrt{2}/2)s+1)(s^2+1.931852s+1)$ $= s^6+3.863703s^5+7.464102s^4+9.141620s^3+7.464102s^2+3.863703s+1$
7	$(s+1)(s^2+0.445042s+1)(s^2+1.246980s+1)(s^2+1.801938s+1)$ $= s^7+4.493959s^6+10.097835s^5+14.591794s^4+14.591794s^3$ $+10.097835s^2+4.493959s+1$
8	$(s^2+0.390181s+1)(s^2+1.11114s+1)(s^2+1.662939s+1)(s^2+1.961571s+1)$ $= s^8+5.125831s^7+13.137071s^6+21.846151s^5+25.688356s^4+21.846151s^3$ $+13.137071s^2+5.125831s+1$
9	$(s+1)(s^2+s+1)(s^2+0.347296s+1)(s^2+1.532089s+1)(s^2+1.879385s+1)$ $= s^9+5.758770s^8+16.581719s^7+31.163437s^6+41.986386s^5+41.986386s^4$ $+31.163437s^3+16.581719s^2+5.758770s+1$

2.2.5 Table of Prototype Butterworth Filters

The approximation of prototype specifications of Fig. 1.37 with $G(1) = \frac{1}{\sqrt{2}} H_o$ or $\alpha_{\max} = 3$ dB requires that $\beta = 1$. In this case *the prototype Butterworth lowpass transfer function* is given by (2.32).

$$H_{pBUT}(s) = \frac{H_o}{[s - e^{j(\frac{\pi}{2N}+\frac{\pi}{2})}][s - e^{j(\frac{3\pi}{2N}+\frac{\pi}{2})}]\cdots[s - e^{j(\frac{(2N-1)\pi}{2N}+\frac{\pi}{2})}]} \tag{2.32}$$

Since the only parameter in (2.32) is the order N, a table of prototype Butterworth lowpass transfer functions can be created.

If the normalized lowpass filter specifications require $G(1) = H_C \neq \frac{1}{\sqrt{2}} H_o (a_{\max} \neq 3dB)$, the corresponding normalized transfer functions will simply be frequency scaled with Ω_{3dB} of (2.20), i.e.

$$H_n(s) = H_{pBUT}\left(\frac{s}{\Omega_{3dB}}\right) = \frac{H_o}{D_p(\frac{s}{\Omega_{dB}})} \quad \text{where } \Omega_{3dB} = \left(\frac{1}{\beta}\right)^{\frac{1}{N}} = \beta^{-\frac{1}{N}} \tag{2.33}$$

Example 2.2 Calculate the transfer function of a normalized lowpass Butterworth filter of order $N = 3$ with $G(0) = 4$ and $G(1) = 3.9$.

Obviously, since Ω_S and H_S are not specified, only $\beta_{\max}$ can be calculated. From Table 2.1, we have for the 3rd order prototype Butterworth filter

$$H_{pBUT}(s) = \frac{H_o}{D(s)} = \frac{4}{s^3+2s^2+2s+1}$$

that has $G(1) = 0.7071 \times 4 = 2.8284$ while we require $G(1) = 3.9$.

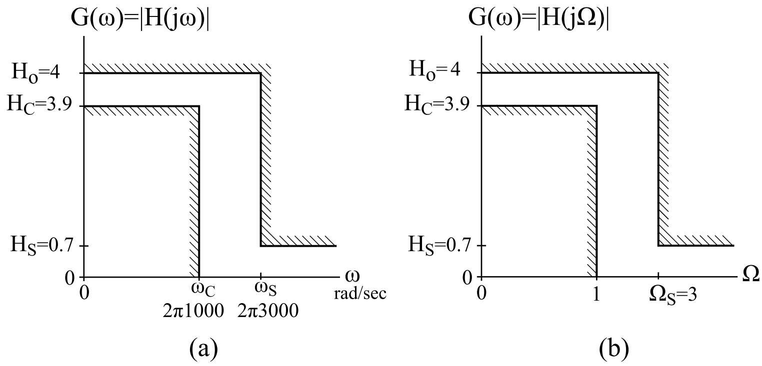


Fig. 2.14 Filter specifications of application Example 2.3

For the specified $H_o = G(0) = 4$ and $H_C = G(1) = 3.9$, the 3 dB frequency can be calculated:

$$\Omega_{3dB} = \left(\frac{1}{\beta_{\max}}\right)^{\frac{1}{N}} = \left(\frac{1}{\sqrt{\frac{G(0)^2}{G(1)^2} - 1}}\right)^{\frac{1}{3}} = 1.63713$$

and the transfer function will be:

$$H(s) = H_{nBUT}\left(\frac{s}{1.63713}\right) = \frac{H_o}{D_p(\frac{s}{1.63713})} = \frac{17.5514}{s^3 + 3.27426s^2 + 5.3604s + 4.3878}$$

Example 2.3 Calculate the transfer function of a lowpass filter with Butterworth response with the specifications given in Fig. 2.14a.

By normalizing the given specifications with $\omega_C = 2\pi 1000$ rad/s, we get the normalized specifications of Fig. 2.14b. The normalized stopband edge frequency is $\Omega_S = \frac{2\pi 3000}{2\pi 1000} = 3$. We next calculate $\beta_{\max}$ from (2.10) or (2.15):

$$\beta_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{\left(\frac{4}{3.9}\right)^2 - 1} = 0.2279$$

From (2.13), the fractional order of the approximation is calculated:

$$N_d = \frac{\log(\frac{\frac{H_o^2}{H_S^2} - 1}{\beta_{\max}^2})}{2\log\Omega_S} = \frac{\log(\frac{\frac{4^2}{0.7^2} - 1}{0.2279^2})}{2\log(3)} = 2.918$$

The integer order of the approximation will be taken $N = 3$. With this value of N, we can now calculate the minimum value of β from (2.15):

$$\beta_{\min} = \frac{\sqrt{\frac{H_o^2}{H_S^2} - 1}}{\Omega_S^N} = \frac{\sqrt{\frac{4^2}{0.7^2} - 1}}{3^3} = 0.208374 \quad \text{and} \quad \beta_{\min}^2 = 0.04342$$

According to (2.29) and (2.30), the transfer function of the filter will be:

$$H_{BUT}(s) = \frac{\frac{H_o}{\beta}}{\prod_{k=1}^{3}(s - s_{k+})} \quad \text{with } s_{k+} = \sqrt[3]{\frac{1}{\beta}} e^{j\left(\frac{2k+3-1}{2\times 3}\pi\right)} \quad \text{for } k = 1, \ldots, 3$$

If for our calculations we use $\beta = \beta_{\min} = 0.208374$, the poles of the transfer function of the normalized filter are found to be -1.686755 and $-0.843377 \pm j1.46077$ and the transfer function of the normalized filter:

$$H_{\min}(s) = \frac{19.196228}{(s+1.686755)(s^2+1.686755s+2.845143)}$$

If $\beta = \beta_{\max} = 0.2279$ is used, the poles will be -1.637132 and $-0.818566 \pm j1.4178$ and for the transfer function of the normalized filter we find:

$$\begin{aligned} H(s) &= \frac{17.5514}{(s+1.63713)(s^2+1.63713s+2.6802)} \\ &= \frac{17.5514}{s^3+3.2743s^2+5.3604s+4.3878} \end{aligned} \tag{2.34}$$

The second form of the transfer function, with the denominator as a formal polynomial, is not particularly useful and the factored form is preferred.

Alternatively, since we know the order N of the approximation, we can take the transfer function of the prototype filter with $N = 3$ from Table 2.1 and use (2.33). From the table with $N = 3$, we get

$$H_{pBUT} = \frac{H_o}{(s+1)(s^2+s+1)} \quad \text{with } H_o = 4$$

Since in the prototype filters $G(1) = |H_{nBUT}(j1)| = \frac{1}{\sqrt{2}}H_o$, the transfer function for our specifications with different value $G(1) = H_C = 3.9$ (i.e. $\beta = \beta_{\max} = 0.2279$), will be

$$H(s) = H_{pBUT}\left(\frac{s}{\Omega_{3dB}}\right) = \frac{H_o}{(\frac{s}{\Omega_{3dB}}+1)[(\frac{s}{\Omega_{3dB}})^2+(\frac{s}{\Omega_{3dB}})+1]}$$

$$\text{where } \Omega_{3dB} = \left(\frac{1}{\beta_{\max}}\right)^{\frac{1}{N}} = \left(\frac{1}{0.2279}\right)^{\frac{1}{3}} = 1.637132$$

Finally, the transfer function of the normalized filter will be:

$$H(s) = \frac{17.5514}{(s+1.63713)(s^2+1.63713s+2.6802)}$$

As expected, the transfer function is identical to the previously calculated (2.34). All transfer functions determined in this example refer to the normalized filter which satisfies the normalized specifications of Fig. 2.14b. The corresponding transfer functions that meet the specifications of Fig. 2.14a will be simply frequency scaled by $\omega_C = 2\pi 1000$, i.e. $H(\frac{s}{\omega_C})$.

Example 2.4 The transfer function calculated in Example 2.3 using $\beta = \beta_{\max}$, can be factored as follows:

$$H(s) = \frac{A_1}{s+1.63713} \times \frac{A_2}{s^2+1.63713s+2.6802} \tag{2.35}$$

where $A_1A_2 = 17.5514$. The first term can be realized using a first order circuit like the one shown in Fig. 2.15a. The circuit is a lossy integrator and the voltage follower (buffer) is used to isolate the circuit from the next section. The second term can be realized with a lowpass Sallen–Key circuit like the one shown in Fig. 2.15b.

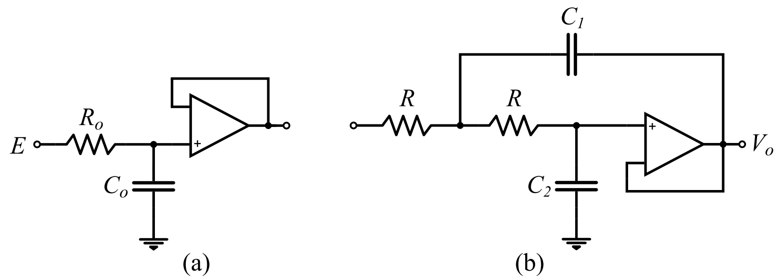


Fig. 2.15 (**a**) Lossy integrator. (**b**) Lowpass Sallen–Key circuit

The transfer function of the first order circuit of Fig. 2.15a is

$$H_1(s) = \frac{\frac{1}{R_oC_o}}{s + \frac{1}{R_oC_o}} \quad \text{with } R_oC_o = \frac{1}{1.63713} = 0.6108 \text{ and } A_1 = 1.63713$$

Choosing $R_o = 1$, we get $C_o = 0.6108$. This choice is not unique and any other would be acceptable as long as it satisfies $R_oC_o = 0.6108$.

The second term $\frac{A_2}{s^2+1.63713s+2.6802}$ of (2.35) is a 2nd order lowpass function of the form $\frac{A_2}{s^2+\frac{\omega_o}{Q}s+\omega_o^2}$ with

$$\omega_o = \sqrt{2.6802} = 1.63713 \quad \text{and} \quad Q = \sqrt{\frac{2.6802}{1.63713}} = 1$$

and can be realized with the unity gain ($k = 1$) Sallen–Key lowpass circuit of Fig. 2.15b (see Chap. 10). This circuit has the following transfer function:

$$H(s) = \frac{\frac{1}{R^2C_1C_2}}{s^2 + s(\frac{2}{RC_1}) + \frac{1}{R^2C_1C_2}}$$

from which, with known Q and ω_o, we find

$$C_1 = \frac{2Q}{\omega_oR} \quad \text{and} \quad C_2 = \frac{1}{2\omega_oRQ}$$

In this circuit, $A_2 = \frac{1}{R^2C_1C_2} = \omega_o^2 = 2.6802$.

We can now choose any value for R, e.g. $R = 1$ (do not forget that we design a normalized filter from a normalized transfer function!), in which case:

$$C_1 = \frac{2Q}{\omega_oR} = \frac{2}{1.63713} = 1.22165 \quad \text{and} \quad C_2 = \frac{1}{2 \cdot 1.63713} = 0.3054$$

When cascading the two circuits, they have

$$A_1A_2 = \frac{\omega_o^2}{R_oC_o} = 4.3878$$

Consequently, if we wish $A_1A_2 = 17.5514$, we must add a unit with gain $A = \frac{17.5514}{4.3878} = 4$, as we have done in Fig. 2.16, by replacing the buffer with a non-inverting voltage amplifier with gain 4.

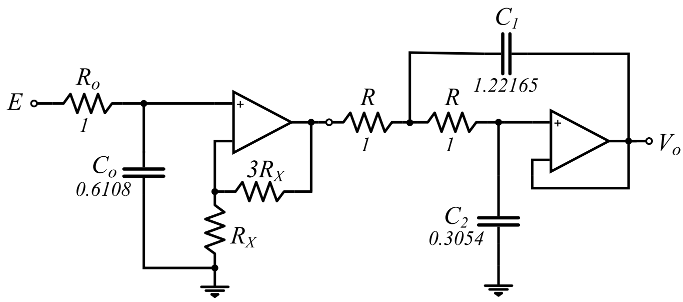


Fig. 2.16 Cascade connection of the two circuits

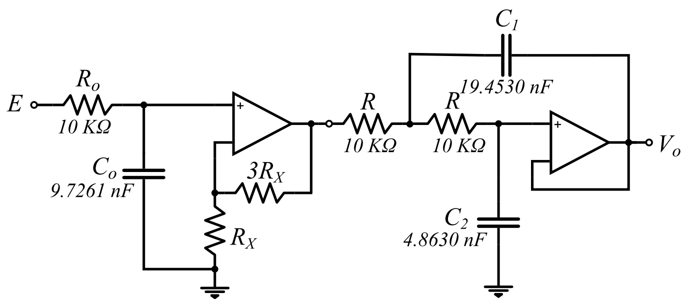


Fig. 2.17 Final denormalized circuit

This filter satisfies the normalized frequency specifications with $\Omega_C = 1$ and can be denormalized simply by dividing the capacitors by the specified $\omega_C = 2\pi \times 1000$. A resistor level adjustment, which does not affect the transfer function and the frequency response, is achieved by multiplying all resistors by R_n and dividing all capacitors by R_n. The value of R_n is so selected as to lead to reasonable resistor values of the order of 10 kOhm. In our case we select $R_n = 10^4$ Ohm.

$$R_o = 1 \times 10^4\ \Omega = 10\ \text{k}\Omega \qquad C_o = \frac{0.6108}{2\pi \times 10^3 \times 10^4} = 9.7261\ \text{nF}$$

$$R = 1 \times 10^4\ \Omega = 10\ \text{k}\Omega \qquad C_1 = \frac{1.22165}{2\pi \times 10^3 \times 10^4} = 19.4530\ \text{nF}$$

$$C_2 = \frac{0.3054}{2\pi \times 10^3 \times 10^4} = 4.8630\ \text{nF}$$

The values of resistors R_x and $3R_x$ are not important, provided that they keep the correct ratio value to realize gain 4. Resistors in active circuits must be kept at high levels, e.g. 10 kOhm, and therefore R_x could be 10 kOhm and $3R_x = 30$ kOhm. Figure 2.17 shows the final filter circuit and Fig. 2.18 the corresponding gain response.

The section ordering is theoretically neither important nor obligatory and the final circuit can be that of Fig. 2.19. However, the circuit of Fig. 2.17 is preferred as it obeys the rule that sections are ordered according to the distance of the pole(s) they realize from the $j\omega$-axis and sections that realize distant poles go first [10, 12].

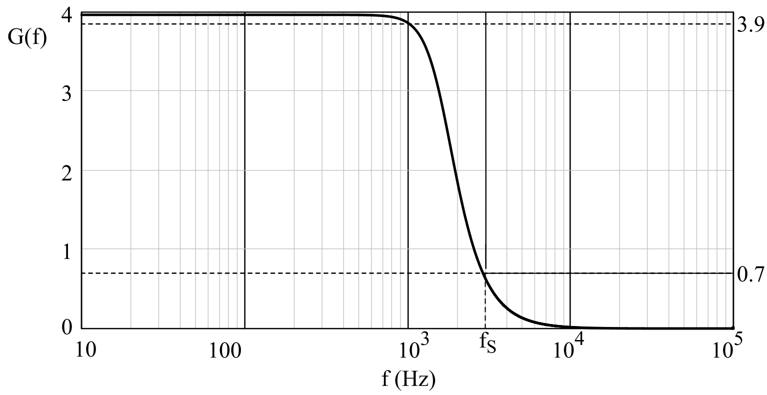


Fig. 2.18 Frequency response

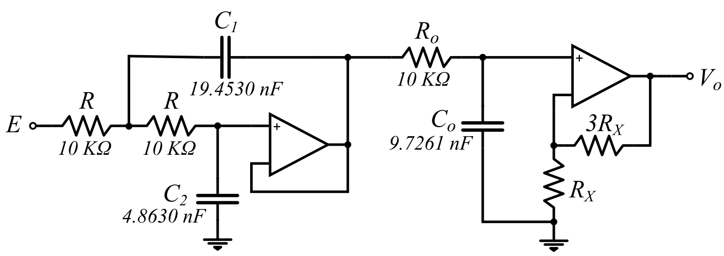


Fig. 2.19 Alternative connection

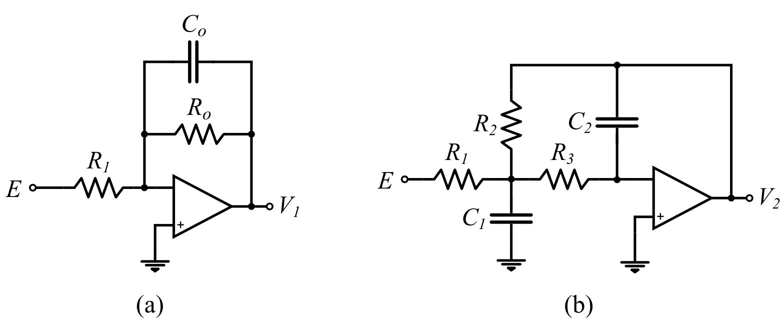


Fig. 2.20 Alternative circuits

A designer's creativity is only limited by the number of the circuits available for the realization of the various sections. The first order circuit, for example, can be realized using the inverting lossy integrator of Fig. 2.20a, and for the second order section a multiple feedback 2nd order lowpass circuit like the one in Fig. 2.20b can be used (see Chap. 10).

2.3 The Chebyshev Approximation

According to Sect. 2.1.1, all approximations use a gain expression of the form of (2.7):

$$G(\Omega) = |H(j\Omega)| = \frac{H_o}{\sqrt{1+\gamma^2 P_N^2(\Omega)}} \tag{2.36}$$

It should be remembered that the approximating polynomial $P_N(\Omega)$ of degree N must be a complete even or odd polynomial in Ω and that in order to achieve lowpass gain characteristics, $P_N(\Omega)$ must assume low values for $0 \le \Omega \le 1$ and high values for $\Omega > 1$.

Due to the form of the gain of (2.36), certain characteristics of the approximating polynomial are inherited to the gain function. If, for example, $P_N(\Omega)$ is monotonically increasing for $\Omega > 1$, then $G(\Omega)$ will be monotonically decreasing in the same interval, as in the Butterworth case with $P(N,\Omega) = \Omega^N$. If $P_N(\Omega)$ assumes zero values at certain frequencies, the related gain $G(\Omega)$ will be maximized at these frequencies, and at the frequencies of any extrema of $P_N(\Omega)$, the gain in (2.36) will be minimized. Such polynomials with roots and extrema in the passband are the Chebyshev polynomials [2] and when used in (2.36) lead to non-monotonic passband gain functions.

2.3.1 Chebyshev Polynomials

The general form of the Chebyshev polynomial of order N is

$$C_N(\Omega) = \cos(N\cos^{-1}(\Omega)) \quad \text{or} \quad C_N(\Omega) = \cosh(N\cosh^{-1}(\Omega)) \tag{2.37}$$

This definition does not indicate that $C_N(\Omega)$ is a polynomial. It will be seen shortly, however, that this is the case. Clearly,

$$C_0(\Omega) = 1 \quad \text{and} \quad C_1(\Omega) = \Omega \tag{2.38}$$

Moreover, by using trigonometric relationships it can be shown that the Chebyshev polynomial of degree N is related to the polynomials of degrees $(N-1)$ and $(N-2)$ by the recursive formula:

$$C_N(\Omega) = 2\Omega C_{N-1}(\Omega) C_{N-2}(\Omega) \tag{2.39}$$

Equation (2.39) provides a way to calculate any Chebyshev polynomial of degree N recursively as follows:

$$C_2(\Omega) = 2\Omega C_1(\Omega) - C_0(\Omega) = 2\Omega^2 - 1$$
$$C_3(\Omega) = 2\Omega C_2(\Omega) - C_1(\Omega) = 2\Omega(2\Omega^2 - 1) - \Omega = 4\Omega^3 - 3\Omega$$

The parameter η, which will be used extensively in this chapter to allow unified expressions, is defined by:

$$\eta = \begin{cases} 0 & \text{for } N \text{ even} \\ 1 & \text{for } N \text{ odd} \end{cases} \tag{2.40}$$

Using η, the general form of the Chebyshev polynomial of degree N will be given by

$$C_N(\Omega) = c_N\Omega^N + c_{N-2}\Omega^{N-2} + \cdots + c_\eta\Omega^\eta$$
$$\text{with: } c_N = 2^{N-1} \text{ and } c_\eta = N^\eta(-1)^{\frac{N+3\eta}{2}} \tag{2.41}$$

Table 2.2 Chebyshev polynomials

Chebyshev Polynomials for $N = 0$–9	
N	$C_N(\Omega) = \cos(N\cos^{-1}(\Omega)) = \cosh(N\cosh^{-1}(\Omega))$
0	1
1	Ω
2	$2\Omega^2 - 1$
3	$4\Omega^3 - 3\Omega$
4	$8\Omega^4 - 8\Omega^2 + 1$
5	$16\Omega^5 - 20\Omega^3 + 5\Omega$
6	$32\Omega^6 - 48\Omega^4 + 18\Omega^2 - 1$
7	$64\Omega^7 - 112\Omega^5 + 56\Omega^3 - 7\Omega$
8	$128\Omega^8 - 256\Omega^6 + 160\Omega^4 - 32\Omega^2 + 1$
9	$256\Omega^9 - 576\Omega^7 + 432\Omega^5 - 120\Omega^3 + 9\Omega$

Table 2.3 Properties of the Chebyshev polynomials

Fundamental properties of Chebyshev polynomials	
N odd	$C_N(0) = 0,\ C_N(1) = 1,\ C_N(-\Omega) = -C_N(\Omega)$
N even	$C_N(0) = \pm 1,\ C_N(1) = 1,\ C_N(-\Omega) = C_N(\Omega)$
All N	$\Omega = [-1, 1] \rightarrow$ ripple between -1 and 1
	For $\Omega > 1$ polynomials increase monotonically.
	$C_N^2(\Omega) = \frac{1}{2}(C_{2N}(\Omega) + 1)$

Table 2.2 has been created using the recursive formula of (2.39).

Some easily demonstrable important and characteristic properties of the Chebyshev polynomials are tabulated in Table 2.3. These properties are visualized in the graphs shown in Fig. 2.21.

2.3.2 The All-Pole Chebyshev Approximation

As a reminder, the parameters

$$\{H_o, H_C, H_S, \Omega_S\} \qquad (\Omega_C = 1)$$

specify the normalized lowpass gain specifications (Fig. 2.22). In logarithmic gain terms, the specifications are of course

$$\{\alpha_o, \alpha_{\max}, \alpha_{\min}, \Omega_S\} \qquad (\Omega_C = 1)$$

where $\alpha_o = 20\log(H_o)$, $\alpha_{\max} = 20\log(H_o/H_C)$ and $\alpha_{\min} = 20\log(H_o/H_S)$.

H_o can be taken equal to 1 (i.e. $\alpha_o = 0$ dB) without loss of generality and therefore the filter requirements are determined by three parameters:

$$\Omega_S \quad \text{and} \quad \{H_C, H_S\} \quad \text{or} \quad \{\alpha_{\max}, \alpha_{\min}\} \tag{2.42}$$

referred to as the normalized lowpass filter specifications.

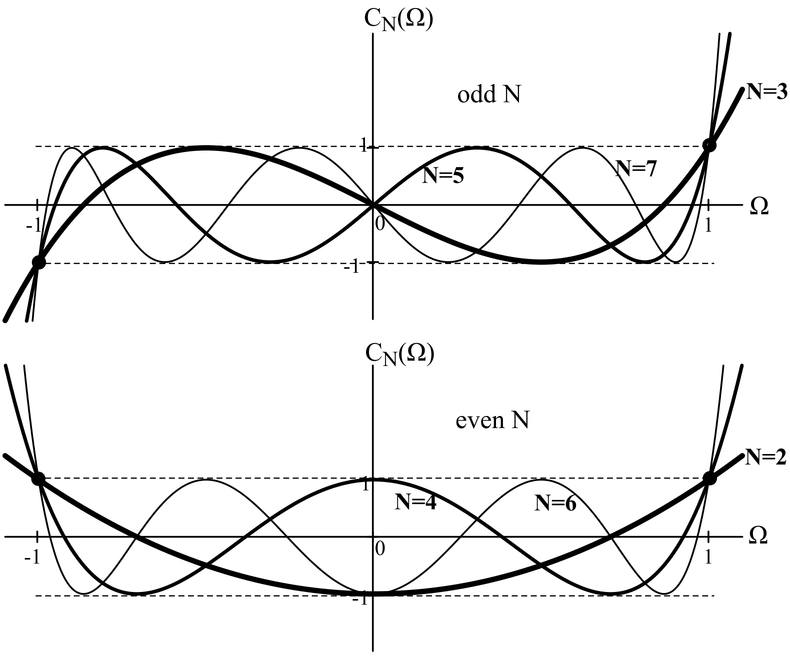


Fig. 2.21 Graphs of odd and even Chebyshev polynomials

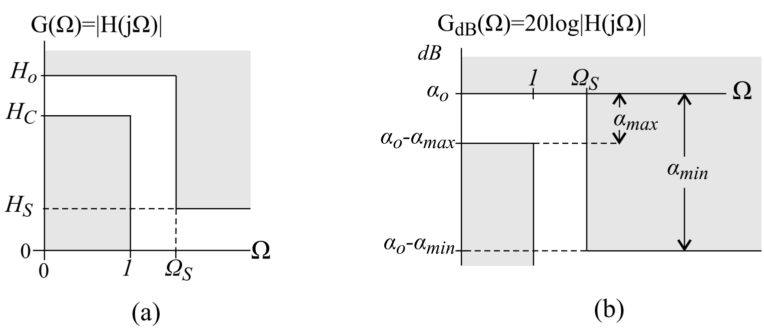


Fig. 2.22 Normalized lowpass specifications. (**a**) Plain gain (**b**). Logarithmic gain

Since Chebyshev polynomials $C_N(\Omega)$ are even or odd, they can be used as approximating functions in (2.7) or (2.36) so as to derive the Chebyshev approximation:

$$G_{CH}(\Omega) = \frac{H_o}{\sqrt{1+\varepsilon^2 C_N^2(\Omega)}} \tag{2.43}$$

The *ripple factor* ε and order N are so chosen to keep the response $G_{CH}(\Omega)$ within the specifications.

In the interval from $\Omega = 0$ to $\Omega = 1$ the function $\Xi_N(\Omega) = 1 + \varepsilon^2 C_N^2(\Omega)$ assumes values which vary from 1 to $1+\varepsilon^2$ and $\Xi_N(1) = 1+\varepsilon^2$ since $C_N(\Omega) = 1$ for odd and even N (Fig. 2.23). Therefore, $\frac{H_o}{\sqrt{\Xi_N(\Omega)}} = \frac{H_o}{\sqrt{1+\varepsilon^2 C_N^2(\Omega)}}$ assumes values from 1 to $\frac{H_o}{\sqrt{1+\varepsilon^2}}$ in the same interval [0, 1] and for $\Omega = 1$ takes the value $\frac{H_o}{\sqrt{1+\varepsilon^2}}$.

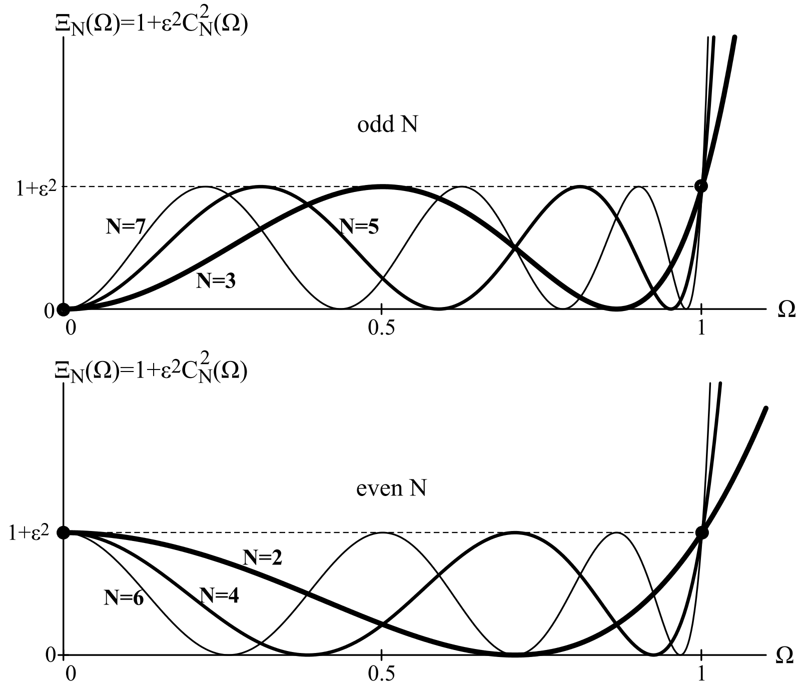


Fig. 2.23 The $\Xi_n(\Omega)$ function

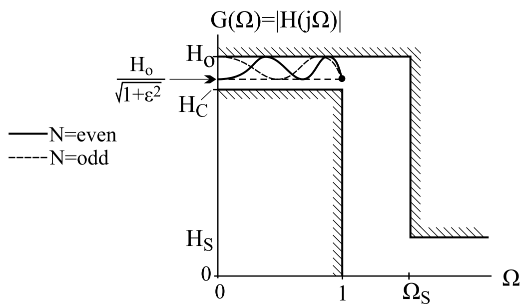


Fig. 2.24 Passband approximation

For $\Omega > 1$, function $\Xi_N(\Omega)$ is positive and monotonically increasing and therefore $\frac{H_o}{\sqrt{\Xi_N(\Omega)}}$ is positive and monotonically decreasing to zero as $\Omega \to \infty$.

The ripple factor ε can be so selected as to ensure that $\frac{H_o}{\sqrt{1+\varepsilon^2}} \geq H_C$, i.e.

$$\varepsilon \leq \sqrt{\frac{H_o^2}{H_C^2} - 1} = \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} = \varepsilon_{\max} \tag{2.44}$$

For values of $\varepsilon \leq \varepsilon_{\max}$, the gain $G_{CH}(\Omega)$ will vary in the passband between H_o and $\frac{H_o}{\sqrt{1+\varepsilon^2}} \geq H_C$ and for $\Omega = 1$ it will assume the value $G_{CH}(1) = \frac{H_o}{\sqrt{1+\varepsilon^2}} \geq H_C$. These properties are shown in Fig. 2.24.

As far as the value $G_{CH}(0)$ is concerned, we note that

$$G_{CH}(0) = H_o \quad \text{for odd } N \quad \text{and} \quad G_{CH}(0) = \frac{H_o}{\sqrt{1+\varepsilon^2}} \quad \text{for even } N \tag{2.45}$$

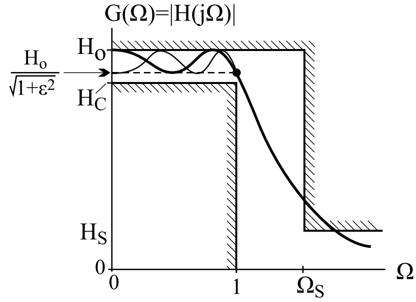


Fig. 2.25 Violation of the specifications in the stopband

Finally, the Chebyshev gain of (2.43) with $\varepsilon \leq \varepsilon_{\max}$ satisfies the passband specifications according to Fig. 2.24 and (2.45) for $G_{CH}(0)$. Keep in mind that for $\varepsilon = \varepsilon_{\max}$, from (2.44) $\rightarrow G_{CH}(1) = H_C$. However, the stopband specifications might not necessarily be satisfied as shown in Fig. 2.25.

Since Chebyshev polynomials are monotonically increasing for $\Omega > 1$, the gain $G_{CH}(\Omega)$ is monotonically decreasing for the same interval and therefore if we ensure that $G_{CH}(\Omega_S) \leq H_S$, the stopband requirements will be met. This is done below.

$$G_{CH}(\Omega_s) = \frac{H_o}{\sqrt{1 + \varepsilon^2 C_N^2(\Omega_S)}} \leq H_S$$

$$\Leftrightarrow \quad C_N^2(\Omega_S) \geq \frac{(H_o/H_S)^2 - 1}{\varepsilon^2}$$

$$\Leftrightarrow \quad N \cosh^{-1}(\Omega_S) \geq \cosh^{-1}\sqrt{\frac{(H_o/H_S)^2 - 1}{\varepsilon^2}}$$

From the above equation, we get for the order N:

$$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2 - 1}{\varepsilon^2}}\right)}{\cosh^{-1}(\Omega_S)} \tag{2.46}$$

The quantity N_d is in general a non-integer number referred to as the *decimal order* of the approximation. As mentioned in the Butterworth approximation, the complexity of a filter is directly proportional to the order of the approximation and this is the reason we try to keep N as low as possible. The decimal order N_d, if seen as a function of ε, assumes its minimum value for $\varepsilon = \varepsilon_{\max}$. Since normally we are looking for the minimum possible N, the maximum value of ε from (2.44) must be used in (2.46) and finally the order equation is given below

$$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2 - 1}{\varepsilon_{\max}^2}}\right)}{\cosh^{-1}(\Omega_S)} = \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2 - 1}{(H_o/H_C)^2 - 1}}\right)}{\cosh^{-1}(\Omega_S)} \tag{2.47}$$

In terms of the logarithmic gain specifications, the order equation (2.47) becomes:

$$N \geq N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{10^{\frac{\alpha_{\min}}{10}} - 1}{10^{\frac{\alpha_{\max}}{10}} - 1}}\right)}{\cosh^{-1}(\Omega_S)} \tag{2.48}$$

The integer order of the approximation will be the minimum integer N that satisfies $N \geq N_d$, i.e. N_d rounded up to the next integer.

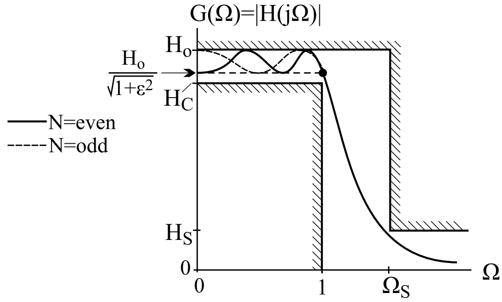


Fig. 2.26 Ripple factor lower than the maximum value

It is apparent from equations (2.47) and (2.48) that the order increases as Ω_S approaches unity and becomes infinite for $\Omega_S = 1$. In other words, the order increases as we try to make the transition band as narrow as possible. In effect, we also say that higher order filters can realize narrower transition bands. The order also increases as the passband tolerance decreases (H_C approaches H_o or $\alpha_{\max}$ approaches zero) and as the maximum allowed stopband gain H_S decreases ($\alpha_{\min}$ increases). Thus, as we try to make the specifications closer and closer to the ideal ones, the order becomes larger and larger, finally becoming infinite. This is another way of saying that the ideal specifications are unrealizable.

Filters designed with the Chebyshev approximation as above are referred to as Chebyshev filters. Chebyshev filters can be implemented using any suitable technology, e.g. as passive, active or digital filters.

Finally, the Chebyshev gain function

$$G_{CH}(\Omega) = \frac{H_o}{\sqrt{1 + \varepsilon^2 C_N^2(\Omega)}} \quad \text{with}$$

$$\varepsilon \le \varepsilon_{\max} = \sqrt{(H_o/H_C)^2 - 1} = \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} \quad \text{and} \quad N \ge N_d \quad \text{(from (2.48))}$$

satisfies the magnitude specifications in the manner shown in Fig. 2.26.

Passband tolerance is reached only when $\varepsilon = \varepsilon_{\max}$, in which case, $G_{CH}(1) = H_C$ and $G_{CH}(\Omega_S) < H_S$. As far as the DC gain is concerned, it is $G_{CH}(0) = H_C$ when N is odd and $G_{CH}(0) = \frac{H_o}{\sqrt{1+\varepsilon^2}}$ when N is even. In both cases, $G_{CH}(1) = \frac{H_o}{\sqrt{1+\varepsilon^2}}$.

It should be noted that the number of extrema in the passband is equal to the integer order N, including the extremum at $\Omega = 0$.

When the order N has been determined from (2.47) or (2.48), the minimum value of the ripple factor ε can be calculated since

$$N \ge \frac{\cosh^{-1}\left(\sqrt{\frac{(H_o/H_S)^2-1}{\varepsilon^2}}\right)}{\cosh^{-1}(\Omega_S)}$$

from which we get

$$\varepsilon \ge \frac{\sqrt{(H_o/H_S)^2 - 1}}{C_N(\Omega_s)} = \varepsilon_{\min} \tag{2.49}$$

The specifications are satisfied for any value of ε

$$\varepsilon_{\min} = \frac{\sqrt{H_o^2/H_S^2 - 1}}{C_N(\Omega_s)} \le \varepsilon \le \sqrt{H_o^2/H_C^2 - 1} = \varepsilon_{\max} \tag{2.50}$$

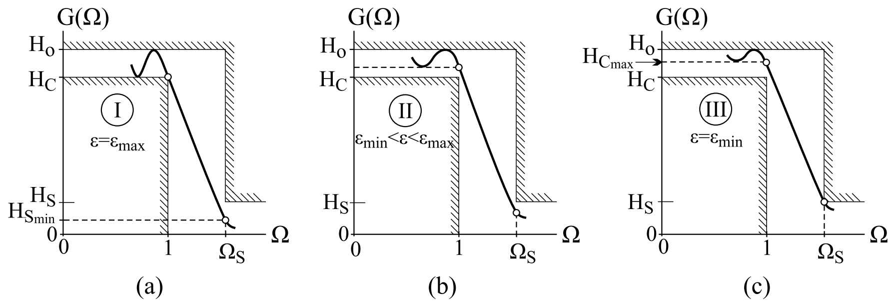


Fig. 2.27 Three choices of the ripple factor ε

or in terms of logarithmic gain

$$\varepsilon_{\min} = \frac{\sqrt{10^{\frac{\alpha_{\min}}{10}} - 1}}{C_N(\Omega_s)} \leq \varepsilon \leq \sqrt{10^{\frac{\alpha_{\max}}{10}} - 1} = \varepsilon_{\max} \tag{2.51}$$

Figure 2.27 shows the manner in which the specifications are met for three choices of the value of the ripple factor.

Case I: $\varepsilon = \varepsilon_{\max}$ When the maximum value of the ripple factor $\varepsilon = \varepsilon_{\max}$ is used, the gain ripple exhausts the passband tolerance with $G_{CH}(1) = H_C$. However, the gain at the stopband edge frequency Ω_S is

$$\begin{aligned} H_{S\min} &= G(\Omega_S) \leq H_S \\ H_{S\min} &= \frac{H_o}{\sqrt{1 + ((H_o/H_C)^2 - 1)C_N^2(\Omega_S)}} = \frac{H_o}{\sqrt{1 + (10^{\frac{\alpha_{\max}}{10}} - 1)C_N^2(\Omega_S)}} \end{aligned} \tag{2.52}$$

This is the minimum gain achieved with a Chebyshev filter of order N at the specified Ω_S and with the specified passband tolerance. Filters designed with $\varepsilon = \varepsilon_{\max}$ are referred to as *stopband edge frequency gain optimized*.

Case II: $\varepsilon_{\min} < \varepsilon < \varepsilon_{\max}$ When the ripple factor ε is not selected to have neither the maximum nor the minimum value, neither passband nor stopband tolerances are exhausted.

Case III: $\varepsilon = \varepsilon_{\min}$ When the minimum value of the ripple factor $\varepsilon = \varepsilon_{\min}$ is used, the gain ripple reaches the stopband tolerance with $G_{CH}(\Omega_S) = H_S$. The gain at $\Omega = 1$ is

$$\begin{aligned} H_{C\max} &= G(1) > H_C \\ H_{C\max} &= \frac{H_o}{\sqrt{1 + \frac{(H_o/H_S)^2 - 1}{C_N^2(\Omega_S)}}} = \frac{H_o}{\sqrt{1 + \frac{10^{\frac{\alpha_{\min}}{10}} - 1}{C_N^2(\Omega_S)}}} > H_C \end{aligned} \tag{2.53}$$

This is the minimum passband ripple that can be achieved with a Chebyshev filter of order N and with the specified stopband tolerance and Ω_S. Filters designed with $\varepsilon = \varepsilon_{\min}$ are referred to as *passband gain optimized* (Fig. 2.27c).

Example 2.5 For $H_o = 1$, $H_C = 0.95$, $H_S = 0.05$ and $\Omega_S = 1.7$, the order can be calculated from (2.47) as $N = 5$. From (2.50), we find $\varepsilon_{\min} = 0.14536$ and $\varepsilon_{\max} = 0.328684$. If $\varepsilon = \varepsilon_{\max} = 0.328684$ is chosen, the ripple will be from $H_o = 1$ to $H_C = 0.95$ and $G_{CH}(1.7) = H_{S\min} = 0.0221$. If $\varepsilon = \varepsilon_{\min} =$

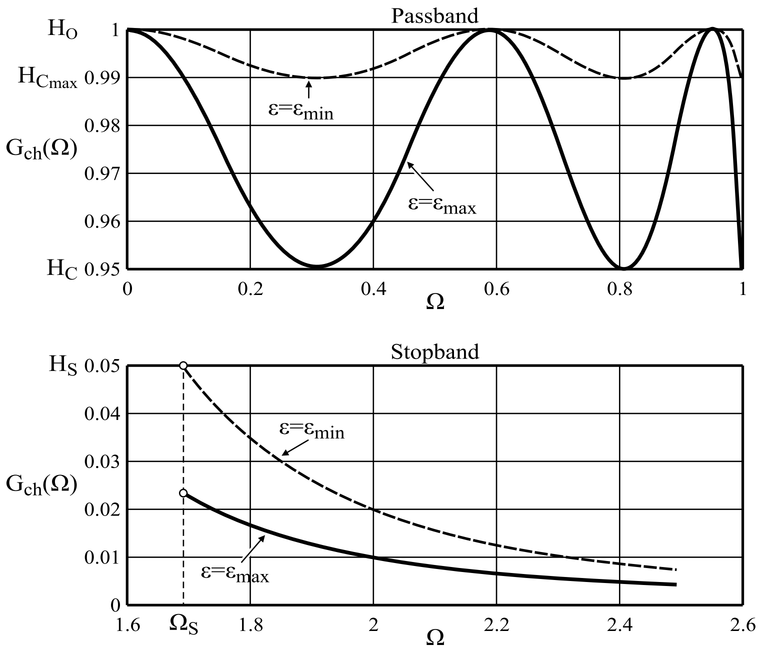


Fig. 2.28 Passband and stopband gain for the maximum and minimum value of the ripple factor

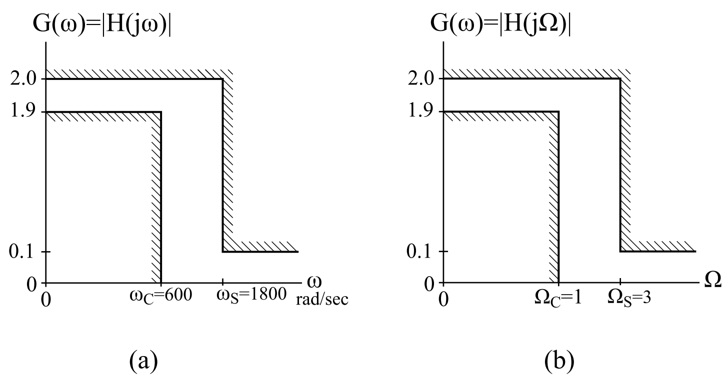


Fig. 2.29 Specifications for Example 2.6a

0.14536 is chosen, the ripple will be from $H_o = 1$ to $H_{C\,\max} = 0.9896$ and $G_{CH}(1.7) = H_S = 0.05$. Figure 2.28 shows these two cases.

Example 2.6 Calculate the stopband edge frequency gain optimized Chebyshev gain function $G_{CH}(\Omega)$ with the specifications given in Fig. 2.29.

First, we identify the specifications: $\omega_C = 600$ rad/s, $\omega_S = 1800$ rad/s, $H_o = 2$, $H_C = 1.9$, and $H_S = 0.1$. Normalizing with $\omega_C = 600$ rad/s, the normalized specifications, shown in Fig. 2.6b will be: $\Omega_C = 1$, $\Omega_S = 3$, $H_o = 2$, $H.C = 1.9$, and $H_S = 0.1$.

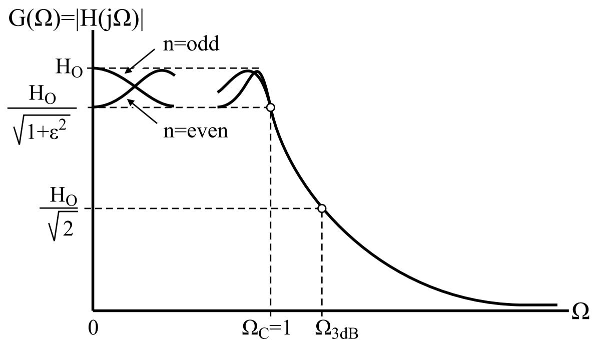


Fig. 2.30 The 3-dB frequency Ω_{3dB}

Since the stopband edge frequency gain optimized filter is required, $\varepsilon_{\max}$ must be used. From (2.50) we have:

$$\varepsilon_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{\left(\frac{2.0}{1.90}\right)^2 - 1} = 0.3286837$$

From (2.47), the decimal order N_d and the integer order N can be calculated:

$$N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{\frac{H_o^2}{H_S^2}-1}{\frac{H_o^2}{H_C^2}-1}}\right)}{\cosh^{-1}(\Omega_S)} = \frac{\cosh^{-1}\left(\sqrt{\frac{\frac{2.0^2}{0.1^2}-1}{\frac{2.0^2}{1.9^2}-1}}\right)}{\cosh^{-1}(3)} = 2.723146$$

Therefore $N = 3$ and

$$G_{CH}(\Omega) = \frac{2}{\sqrt{1 + 0.108033 C_3^2(\Omega)}} = \frac{2}{\sqrt{1 + 0.108033(4\Omega^3 - 3\Omega)^2}}$$

This is the normalized gain, and the gain which satisfies the specifications of Fig. 2.29a with $\omega_C = 600$ rad/s will be

$$G_{CH}(\omega) = \frac{2}{\sqrt{1 + 0.108033(4\frac{\omega^3}{\omega_C^3} - 3\frac{\omega}{\omega_C})^2}}$$

It should be noted that the Butterworth approximation with the same specifications (see Example 2.1) led to $N = 4$. This is not accidental. In general, for a given set of specifications, the Chebyshev approximation leads to a lower degree than all other all-pole approximations.

2.3.3 The 3-dB Frequency and The Cut-off Rate

The frequency Ω_{3dB} at which the gain drops by 3 dB from its maximum value in the passband corresponds to that value at which the gain falls from its maximum value H_o to the value $\frac{1}{\sqrt{2}}H_o$ as shown in Fig. 2.30.

The frequency Ω_{3dB} is determined as follows:

$$G_{CH}(\Omega_{3dB}) = \frac{H_o}{\sqrt{1 + \varepsilon^2 C_N(\Omega_{3dB})}} = \frac{\sqrt{2}}{2} H_o$$

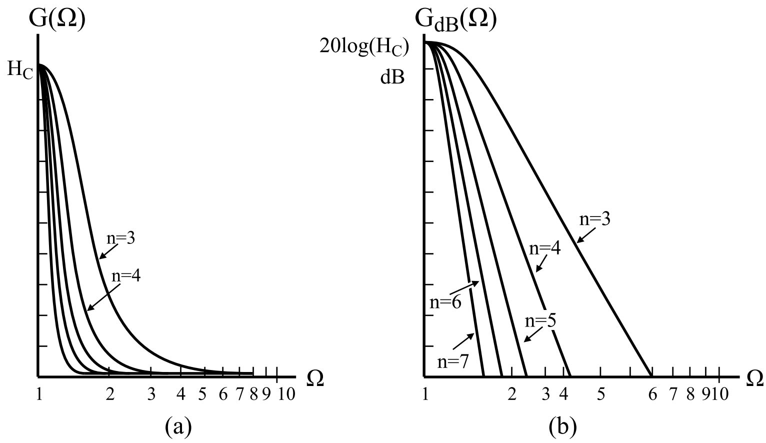


Fig. 2.31 Chebyshev plain and logarithmic gain for $\Omega > 1$

$$\Leftrightarrow \quad \varepsilon C_N(\Omega_{3dB}) = 1$$

$$\Leftrightarrow \quad \cosh(N\cosh^{-1}(\Omega_{3dB})) = \frac{1}{\varepsilon}$$

from which we get

$$\Omega_{3dB} = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \tag{2.54}$$

It is apparent that for given order N, frequency Ω_{3dB} depends only on the choice of the ripple factor ε.

Another characteristic quantity of polynomial filters is the *cut-off rate* at frequencies $\Omega \gg 1$.

Figure 2.31 shows the plain and logarithmic gain of Chebyshev approximation for $\Omega > 1$ and for a range of orders $N = 3$–7. In the logarithmic gain plot versus logarithmic Ω axis, the gain curves become almost straight lines with increasing constant slope for higher N. This slope can be expressed in dB/octave and is referred to as *cut-off rate* and for $\Omega_1 \gg 1$ can be calculated as follows:

$$r = 20\log(G(\Omega_1)) - 20\log(G(2\Omega_1)) = 20\log\left(\frac{G(\Omega_1}{G(2\Omega_1)}\right)$$

$$= 20\log\left(\sqrt{\frac{1+\varepsilon^2 C_N^2(2\Omega_1)}{\sqrt{1+\varepsilon^2 C_N^2(\Omega_1)}}}\right) \approx 20\log\left(\sqrt{\frac{\varepsilon^2 C_N^2(2\Omega_1)}{\sqrt{\varepsilon^2 C_N^2(\Omega_1)}}}\right)$$

$$= 20\log\left(\frac{C_N(2\Omega_1)}{C_N(\Omega_1)}\right) = 20\log\left(\frac{c_N(2\Omega_1)^N + \cdots}{c_N(\Omega_1)^N + \cdots}\right)$$

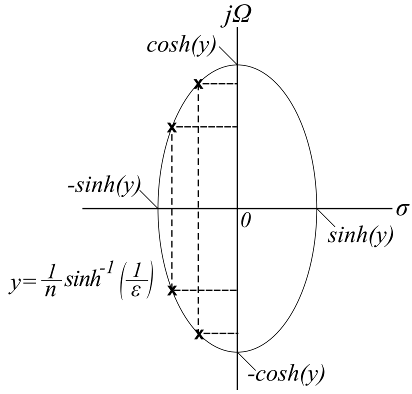
$$\text{which for } \Omega_1 \gg 1 \Rightarrow r \approx 20\log(2^N) \approx 6N \text{ dB/octave}$$

This means that the logarithmic gain in the stopband decreases at a rate $6N$ dB/octave, i.e. $6N$ dB for every frequency doubling. If the cut-off rate r is specified, then the order is $N = \frac{r}{6}$.

2.3.4 The Transfer Function

In order to synthesize the filter, the transfer function itself is required.

Fig. 2.32 Poles location of the transfer function of Chebyshev approximation



Thus far, the approximation process yields $G_{CH}(\Omega) = |H(s)|_{s=j\Omega}$. Now we seek to determine a transfer function $H_{CH}(s) = \frac{N(s)}{D_N(s)}$ for which we know its magnitude G_{CH} and that the denominator polynomial $D_N(s)$ is a Hurwitz polynomial. As we have done in the case of Butterworth filters, from

$$\begin{aligned} H_{CH}(s)H_{CH}(-s) &= \frac{N(s)N(-s)}{D_N(s)D_N(-s)} = |G_{CH}(j\Omega)|^2_{\Omega=-js} \\ &= \frac{H_o^2}{1+\varepsilon^2 C_N^2(\Omega)|_{\Omega=-js}} \end{aligned} \tag{2.55}$$

we set

$$N(s)N(-s) = H_o^2 \quad \Rightarrow \quad N(s) = H_o \tag{2.56}$$

and

$$D_N(s)D_N(-s) = 1+\varepsilon^2 C_N^2(\Omega)|_{\Omega=-js} \tag{2.57}$$

For a specific order N and ripple factor ε, the denominator polynomial $D_N(s)$ has N roots $s_k = \sigma_k + j\Omega_k$ in the left half s-plane given by

$$\begin{aligned} &ChebPoles - 1 = \sin\left(\frac{(2n+2k-1)\pi}{2N}\right)\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \\ &\Omega_k = \cos\left(\frac{(2n+2k-1)\pi}{2N}\right)\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \\ &k = 1, 2, \ldots, N \end{aligned} \tag{2.58}$$

For the values of σ_k and Ω_k given by (2.58), it can be easily shown that:

$$\frac{\sigma_k^2}{\sinh^2[\frac{1}{N}\sinh^{-1}(\frac{1}{\varepsilon})]} + \frac{\Omega_k^2}{\cosh^2[\frac{1}{N}\sinh^{-1}(\frac{1}{\varepsilon})]} = 1 \tag{2.59}$$

This implies that the poles of the Chebyshev transfer function lie on an ellipse with the major axis on the $j\Omega$-axis and the minor on the real axis of the s-plane, as shown in Fig. 2.32. This should be compared to the Butterworth case, in which the poles are all located on a circle.

When N is odd, there is a real pole given by (2.58) for $k_o = \frac{N+1}{2}$:

$$s_{k_o} = -\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] + j0 \tag{2.60}$$

It should be noted that there are no poles on the imaginary axis since this would require $\sin(\frac{(2N+2k-1)\pi}{2N}) = 0$, i.e. the positive integer k needs to be $k = \frac{1}{2} - N$ which is impossible.

Finally, the transfer function is given in (2.61)

$$\left\{\begin{aligned} & H_{CH}(s) = \frac{H_o/(\varepsilon c_N)}{\prod_{k=1}^{N}(s-s_k)} \quad \text{with } s_k = \sigma_k + j\Omega_k \text{ and} \\ & \sigma_k = \sin\left(\frac{(2N+2k-1)\pi}{2N}\right)\sinh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \\ & \Omega_k = \cos\left(\frac{(2N+2k-1)\pi}{2N}\right)\cosh\left[\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right] \\ & k = 1, 2, \ldots, N \end{aligned}\right\} \tag{2.61}$$

The coefficient c_N in (2.61) is the coefficient of the highest order term Ω^N in $C_N(\Omega)$, taken from Table 2.2 or calculated as

$$c_N = 2^{N-1}$$

The all-pole Chebyshev transfer functions can be implemented using active-RC or passive circuits. The complexity of the implementation depends on the order N and this is the main reason why we want it minimized. Since both Butterworth and Chebyshev filters have all-pole transfer functions, they are realized with similar circuits but of element values.

At the end of this chapter a table is given for the denominator polynomial $D_N(s)$ of the transfer function of the prototype Chebyshev filters with $\Omega_{3dB} = 1$. Unfortunately, a separate table is required for each value of the ripple factor. Tables are given for $\alpha_{\max} = 0.1, 0.5, 1.0, 2.0$ and 3.0 dB where

$$\alpha_{\max} = 20\log\left(\frac{H_o}{H_C}\right) = 20\log\sqrt{1+\varepsilon^2} \text{ (dB)}$$

Example 2.7 Calculate the transfer function of the filter of Example 2.6.

In Example 2.6, the order of the filter was found $N = 3$ and the 3rd order Chebyshev polynomial is $C_3(\Omega) = 4\Omega^3 - 3\Omega$. Therefore, $c_N = c_3 = 2^{3-1} = 4$. From the specifications, $H_o = 2$ and $\varepsilon = \varepsilon_{\max} = 0.3286837$. According to (2.61), the transfer function will be

$$H_{CH}(s) = \frac{H_o/(\varepsilon c_N)}{\prod_{k=1}^{N}(s-s_k)} = \frac{2/(0.3286837 \cdot 4)}{\prod_{k=1}^{3}(s-s_k)}$$

with $s_k = \sigma_k + j\Omega_k$ and

$$\sigma_k = \sin\left(\frac{(6+2k-1)\pi}{6}\right)\sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{0.3286837}\right)\right]$$

$$\Omega_k = \cos\left(\frac{(6+2k-1)\pi}{6}\right)\cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{0.3286837}\right)\right]$$

The poles are calculated for $k = 1, \ldots, 3$:

$$s_1 = -0.3246 - j1.0325 \qquad s_2 = -0.6492 \qquad s_3 = -0.3246 + j1.0325$$

The transfer function can then be calculated as

$$H(s) = \frac{2/(4 \cdot 0.32868)}{(s+0.6492)(s^2 + 2 \cdot 0.3246 \cdot s + 0.3246^2 + 1.0325^2)}$$

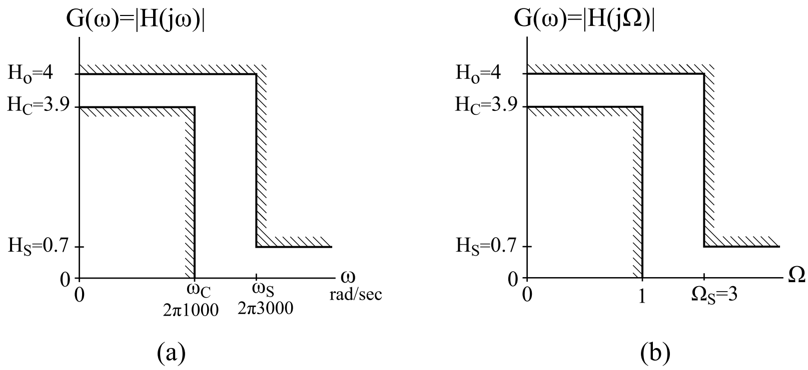


Fig. 2.33 Specifications for Example 2.8

This is the normalized transfer function. The transfer function which corresponds to $\omega_C = 600$ and $\omega_S = 1800$ rad/s will be the following

$$H_o(s) = H(s/600)$$

Example 2.8 Find the Chebyshev transfer function from the specifications of Fig. 2.33a.

First, the specifications are identified and normalized as in Fig. 2.33b. The maximum value of the ripple factor is then calculated:

$$\varepsilon_{\max} = \sqrt{\left(\frac{H_o}{H_C}\right)^2 - 1} = \sqrt{\left(\frac{4}{3.9}\right)^2 - 1} = 0.2279$$

The decimal order N_d is calculated from (2.47)

$$N_d = \frac{\cosh^{-1}\left(\sqrt{\frac{\frac{4^2}{0.7^2}-1}{\frac{4^2}{3.9^2}-1}}\right)}{\cosh^{-1}(3)} = 2.2119$$

The integer order N of the approximation and the filter will of course be $N = 3$. The transfer function will be:

$$H_{CH}(s) = \frac{H_o/(\varepsilon c_3)}{\prod_{k=1}^{3}(s - s_k)} \quad \text{with } s_k = \sigma_k + j\Omega_k$$

where

$$\sigma_k = \sin\left(\frac{(2 \cdot 3 + 2k - 1)\pi}{2 \cdot 3}\right) \sinh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right]$$

$$\Omega_k = \cos\left(\frac{(2 \cdot 3 + 2k - 1)\pi}{2 \cdot 3}\right) \cosh\left[\frac{1}{3}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right]$$

Since the order is $N = 3 \Rightarrow c_N = c_3 = 4$ (from Table 2.2). If the table is not available, c_N can be calculated from $c_N = 2^{N-1}$.

Selecting the maximum ripple factor value $\varepsilon = \varepsilon_{\max} = 0.2279$, the poles can be calculated for $k = 1 \ldots 3$:

$$s_1 = -0.397169 - j1.05997 \qquad s_2 = -0.794339 \qquad s_3 = -0.397169 + j1.05997$$

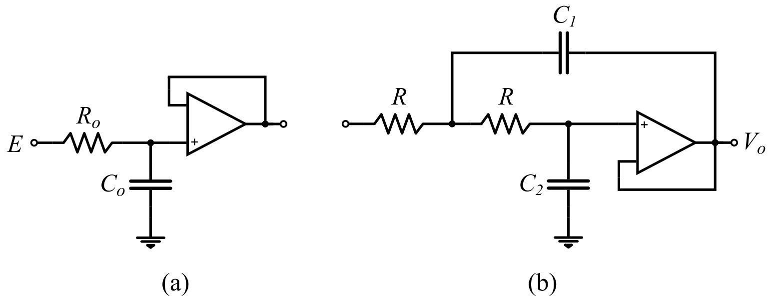


Fig. 2.34 (**a**) Lossy integrator. (**b**) A Sallen-Key 2nd order LP circuit

The transfer function is calculated from (2.61):

$$H_{CH}(s) = \frac{4.38784}{(s+0.794339)(s^2+0.794339s+1.38097375)}$$
$$= \frac{4.38784}{s^3+1.588677s^2+2.011947s+1.09696}$$

Example 2.9 $H_{CH}(s)$ of Example 2.8 can be factorized as follows:

$$H_{CH}(s) = \frac{A_1}{s+0.794339} \times \frac{A_2}{s^2+0.79433s+1.38097375}$$

where $A_1 A_2 = 4.38784$. The first factor $\frac{A_1}{s+0.794339}$ can be realized using a lossy integrator (Fig. 2.34a). The second order factor $\frac{A_2}{s^2+0.79433s+1.38097375}$ can be realized using a Sallen–Key second order low-pass circuit (Fig. 2.34b) with

$$\omega_o = \sqrt{1.38097375} = 1.1751483 \quad \text{and} \quad Q = \sqrt{\frac{1.38097375}{0.794339}} = 1.4794047$$

It is observed that the value of the required quality factor Q is higher than the Q required for the 3rd order Butterworth filter of Example 2.4, which meets the same specifications. However, the Chebyshev filter can meet stricter specifications, for which a higher order Butterworth filter is required.

The element values can be calculated as in Example 2.4:

$$R_o = 1 \qquad C_o = 1.25891 \qquad R = 1 \qquad C_1 = 2.5178178 \qquad C_2 = 0.2876$$

Next, the normalized element values are denormalized using $R_n = 10$ kOhm and $\omega_C = 2\pi 10^3$ rad/s (in order to shift the unity cut-off frequency from 1 to ω_C), i.e. $G(2\pi 10^3) = 3.9$:

$$R_o = R = 10\ \text{kOhm} \qquad C_o = 20.04\ \text{nF} \qquad C_1 = 40\ \text{nF} \qquad C_2 = 4.5770\ \text{nF}$$

The denormalized filter is shown in Fig. 2.35. Figure 2.36 shows the plot of the plain gain of the denormalized filter from its PSpice simulation.

2.4 The Pascal Approximation

According to (2.7), the approximating polynomial $P_N(\Omega)$ must be an even or odd polynomial in order to lead to a realizable transfer function.

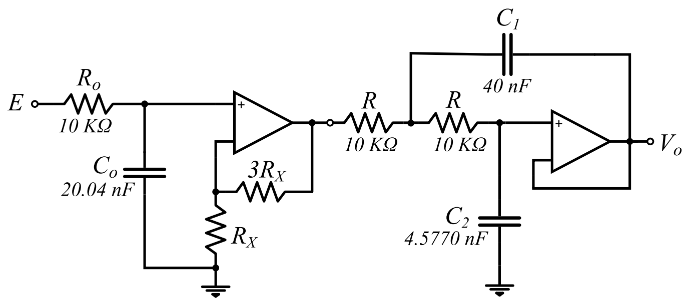


Fig. 2.35 Denormalized circuit

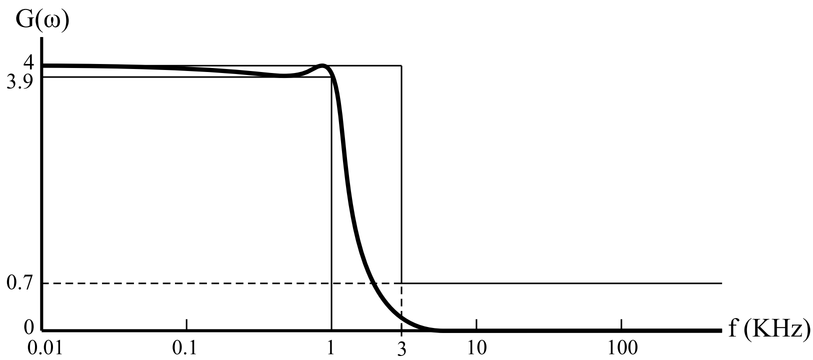


Fig. 2.36 Frequency response of the denormalized circuit

Another set of polynomials are sometimes used in the approximation procedure, the so called Pascal polynomials. The Pascal polynomial [6] of degree N is given by

$$P_o(N,\Omega) = \frac{(-1)^N}{N!}\Omega(\Omega-1)(\Omega-2)\cdots(\Omega-N+1) \tag{2.62}$$

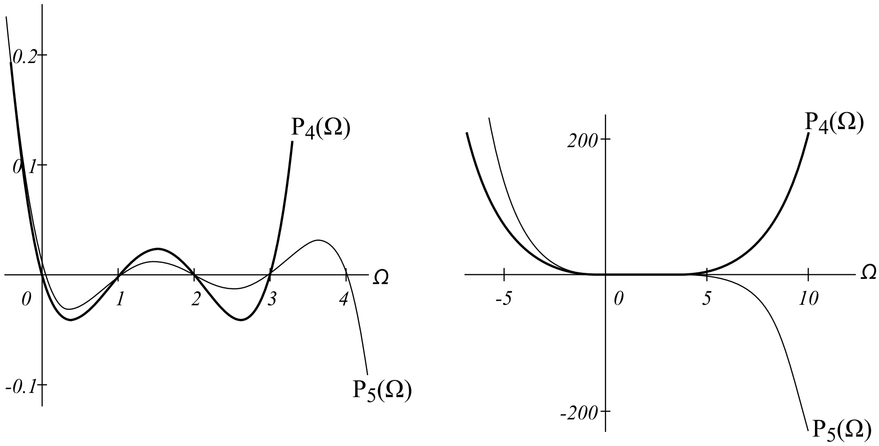
or equivalently

$$P_o(N,\Omega) = \frac{(-1)^N}{N!}\prod_{k=1}^{N}(\Omega-k+1) \tag{2.63}$$

These polynomials, plotted for several orders in Fig. 2.37(a), assume extremely low values for $0 \le \Omega \le N-1$ and very high values for $\Omega \le N-1$. It is clear that they do not satisfy the required symmetry in this form since they are neither odd nor even. However, if they are appropriately shifted this condition can be satisfied.

The Symmetric Shifted Pascal Polynomial $P(N,\Omega)$ The shifted version $P(N,\Omega)$ of the Pascal polynomial $P_o(N,\Omega)$

$$P(N,\Omega) = P_o\left(N,\Omega+\frac{N-1}{2}\right) = \frac{(-1)^N}{N!}\prod_{k=1}^{N}\left(\Omega+\frac{N-1}{2}-k+1\right) \tag{2.64}$$



(a) (b)

Fig. 2.37 Plot of Pascal polynomials

or equivalently

$$P(N,\Omega) = \frac{(-1)^{\eta}}{N!}\cdot\Omega^{\eta}\cdot\prod_{k=1}^{\frac{N-\eta}{2}}\left[\Omega^2-\left(\frac{N+1}{2}-k\right)^2\right]$$
$$\eta = \begin{cases}1 & \text{for } N \text{ odd}\\ 0 & \text{for } N \text{ even}\end{cases} \tag{2.65}$$

is even when the degree N is even and odd when the degree N is odd and therefore $P^2(N,\Omega)$ is an even polynomial. Moreover, it can be shown that this shifted Pascal polynomial possesses the property

$$P\left(N,\frac{N+1}{2}\right) = \pm 1 \quad \Rightarrow \quad P^2\left(N,\frac{N+1}{2}\right) = 1 \tag{2.66}$$

Figure 2.38 shows the plot of $P^2(N,\Omega)$ for $N = 5$–7.

It can be shown analytically that the higher local maximum $P^2_{\max}$ of $P^2(N,\Omega)$ occurs for Ω_m between the higher roots of the symmetric polynomial $P(N,\Omega)$, i.e.:

$$\frac{N-1}{2} - 1 < \Omega_m < \frac{N+1}{2} - 1$$

Since $P^2(N,\Omega)$ is an even function, this absolute maximum $P^2_{\max}$ occurs also at $-\Omega_m$.

The Shifted and Scaled Symmetric Pascal Polynomial $P_a(N,\Omega)$ If we wish to maintain the condition $|P(N,1)| = 1$, the property of (2.66) implies that a frequency scaling by $(N+1)/2$ is required. We denote this symmetric Pascal polynomial by $P_a(N,\Omega)$:

$$P_a(N,\Omega) = P\left(N,\frac{N+1}{2}\Omega\right) = \frac{(-1)^N}{N!}\prod_{k=1}^{N}\left(\frac{N+1}{2}\Omega + \frac{N-1}{2} - k + 1\right) \tag{2.67}$$

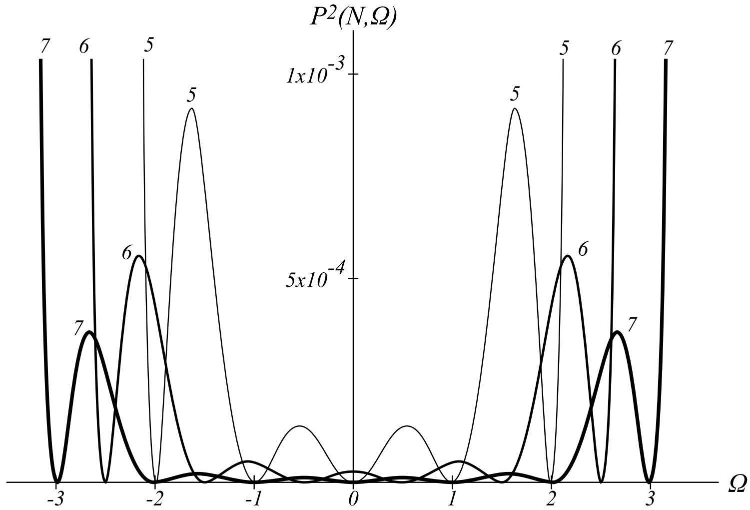


Fig. 2.38 Plot of shifted Pascal polynomials

or

$$P_a(N,\Omega)=\frac{(-1)^\eta}{N!}\cdot\left(\frac{N+1}{2}\Omega\right)^\eta\cdot\prod_{k=1}^{\frac{N-\eta}{2}}\left[\left(\frac{N+1}{2}\Omega\right)^2-\left(\frac{N+1}{2}-k\right)^2\right] \tag{2.68}$$

where $\eta=\begin{cases}1 \text{ for } N \text{ odd}\\ 0 \text{ for } N \text{ even}\end{cases}$.

The highest maximum $P_{\max}$ of $P_a(N,\Omega)$, an indicative plot of which is shown in Fig. 2.39, occurs for $\Omega_{\max}$ between the higher roots of $P_a(N,\Omega)$ i.e.:

$$\frac{N-3}{N+1}<\Omega_{\max}<\frac{N-1}{N+1}$$

Since $P_a^2(N,\Omega)$ is an even polynomial, this highest maximum $P_{\max}^2$ occurs also at $-\Omega_{\max}$. Therefore, the absolute extremum $P_{\max}$ of $P_a(N,\Omega)$ occurs at $\pm\Omega_{\max}$. This extreme value $P_{\max}$ of $P_a(N,\Omega)$ is negative when N is even and positive when it is odd, as shown in Fig. 2.39. It can be shown that there is always a frequency Ω_D in the range $\Omega_{\max}<\Omega_D<1$ at which the polynomial assumes the value $-P_{\max}$, as indicated in Fig. 2.39. If $P_a(N,\Omega)$ is used to approximate the normalized lowpass filter specifications in a manner described by (2.7), the minimum passband gain will occur at $\Omega_{\max}$ and Ω_D.

Due to the nature of Pascal polynomials, $P_{\max}$ and the related frequencies $\Omega_{\max}$ and Ω_D cannot be determined analytically. These characteristic values, calculated using mathematical software, are tabulated in Table 2.4 for $N=2$–15 with reference to Fig. 2.39.

The "Filter-Appropriate" Modified Symmetric Pascal Polynomial $P_D(N,\Omega)$ It is desirable to have the value $-P_{\max}$ at $\Omega=1$ instead of Ω_D. This would ensure that at $\Omega=1$ the gain will be exactly equal to the minimum gain in the passband which is due to the absolute extreme value $-P_{\max}$ of $P_a(N,\Omega)$. To move Ω_D to 1, we simply scale the polynomial $P_a(N,\Omega)$ of (2.67) and (2.68) by $\Omega_D<1$, so defining $P_D(N,\Omega)$:

$$P_D(N,\Omega)=P_a(N,\Omega_D\Omega)=\frac{(-1)^N}{N!}\prod_{k=1}^{N}\left(\frac{N+1}{2}\Omega_D\Omega+\frac{N-1}{2}-k+1\right) \tag{2.69}$$

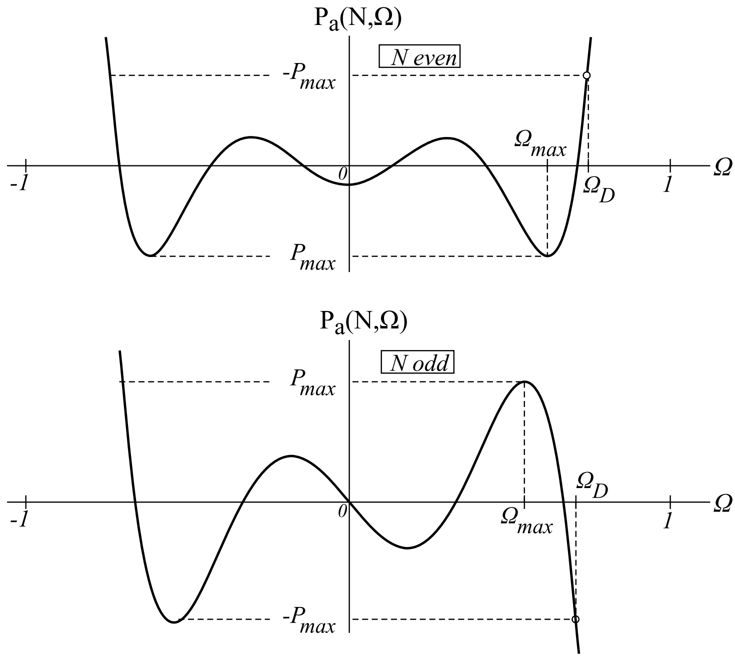


Fig. 2.39 Detailed plot of $P_a(\Omega)$ polynomials

Table 2.4 Characteristic values of $P_a(N, \Omega)$

N	$\Omega_{\max}$	$P_{\max}$	Ω_D
2	0.00000000	−0.12500000	0.47140452
3	0.28867513	+0.06415003	0.57735029
4	0.44721360	−0.04166667	0.65289675
5	0.54814429	+0.03026194	0.70639006
6	0.61812758	−0.02347346	0.74582512
7	0.66950940	+0.01901625	0.77599290
8	0.70882772	−0.01588792	0.79978194
9	0.73987600	+0.01358345	0.81900877
10	0.76500826	−0.01182234	0.83486553
11	0.78576311	+0.01043707	0.84816452
12	0.80318872	−0.00932176	0.85947728
13	0.81802376	+0.00840640	0.86921746
14	0.83080378	−0.00764299	0.87769147
15	0.84192645	+0.00699753	0.88513110

From (2.69), it is clear that $P_D(N, \Omega)$ has $(N-\eta)/2$ positive non-zero roots ρ_k given by

$$\rho_k = \frac{(N+1)-2k}{(N+1)\Omega_D} < 1 \quad \text{for } k = 1, \dots, \frac{N-\eta}{2} \tag{2.70}$$

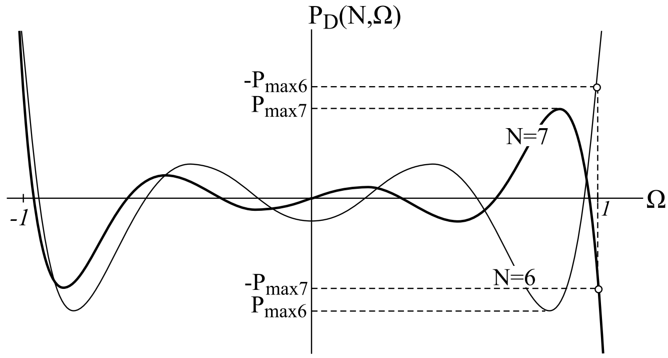


Fig. 2.40 Plot of filter-appropriate $P_D(\Omega)$ polynomials

and $(N-\eta)/2$ negative real roots $-\rho_k$. Parameter Ω_D depends only on N and can be taken from Table 2.4. Equation (2.69) can therefore be written as

$$P_D(\Omega) = \frac{(-1)^\eta}{N!}\left(\frac{N+1}{2}\Omega_D\right)^N \Omega^\eta \prod_{k=1}^{\frac{N-\eta}{2}} [\Omega^2 - \rho_k^2] \tag{2.71}$$

where ρ_k are the positive roots of $P_D(N,\Omega)$ given in (2.70).

The symmetric polynomial $P_D(N,\Omega)$ is related to the original Pascal polynomial $P_o(N,\Omega)$ of (2.63) by the relationship:

$$P_D(N,\Omega) = P_o\left(N, \frac{N+1}{2}\Omega_D\Omega + \frac{N-1}{2}\right) \tag{2.72}$$

The absolute extremum of $P_D(N,\Omega)$ has the same value $P_{\max}$ and occurs at $\Omega_{\max}/\Omega_D < 1$. The property of $P_D(N,\Omega)$ which makes it "filter-appropriate" can be expressed as

$$|P_D(N,1)| = \left|P_D\left(N, \frac{\Omega_{\max}}{\Omega_D}\right)\right| = |P_{\max}| \tag{2.73}$$

The polynomial $P_D(N,\Omega)$ is so constructed as to assume the value $\pm P_{\max}$ at $\Omega = 1$ and to decrease monotonically from $\Omega_{\max}/\Omega_D < 1$. Both characteristic frequencies, as well as $P_{\max}$, can be taken from Table 2.4. An indicative plot of $P_D(N,\Omega)$ for $N = 6$ and $N = 7$ is shown in Fig. 2.40.

Referring to the definition of $P_D(N,\Omega)$ of (2.71), its non-zero roots are given by

$$\pm\frac{(N+1)-2k}{(N+1)\Omega_D} \quad \text{for } k = 1, \ldots, \frac{N-\eta}{2}$$

The $(N-\eta)/2$ positive roots ρ_k of interest are given by

$$\rho_k = \frac{(N+1)-2k}{(N+1)\Omega_D} \quad \text{for } k = 1, \ldots, \frac{N-\eta}{2}, \quad \eta = \begin{cases} 1 & \text{for } N \text{ odd} \\ 0 & \text{for } N \text{ even} \end{cases} \tag{2.74}$$

(Frequency Ω_D depends only on N and can be taken from Table 2.4.)

It is observed from (2.71) that for odd N, in which case $\eta = 1$, there will also exist a root at $\Omega = 0$. Table 2.5 gives the positive roots of $P_D(\Omega)$ for $N = 1$–15.

Table 2.5 Roots of the filter-appropriate Pascal polynomial $P_D(\Omega)$

N	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7
2	0.70710678						
3	0.8660254						
4	0.91898144	0.30632715					
5	0.94376568	0.47188284					
6	0.95771207	0.57462724	0.19154241				
7	0.96650368	0.64433579	0.32216789				
8	0.97248731	0.69463379	0.41678027	0.13892676			
9	0.97679052	0.73259289	0.48839526	0.24419763			
10	0.98001629	0.76223489	0.54445349	0.32667210	0.10889070		
11	0.98251382	0.78601105	0.58950829	0.39300553	0.19650276		
12	0.98449821	0.80549854	0.62649886	0.44749919	0.26849951	0.08949984	
13	0.98610865	0.82175721	0.65740577	0.49305432	0.32870288	0.16435144	
14	0.98743887	0.83552520	0.68361153	0.53169786	0.37978418	0.22787051	0.07595684
15	0.98855417	0.84733215	0.70611012	0.56488810	0.42366607	0.28244405	0.14122202

Table 2.6 Values $P_D(N, 0)$

N	$P_D(N, 0)$
2	$-125 \cdot 10^{-3}$
4	$+23.4375 \cdot 10^{-3}$
6	$-4.882812 \cdot 10^{-3}$
8	$+1.068115 \cdot 10^{-3}$
10	$-240.325928 \cdot 10^{-6}$
12	$+55.074692 \cdot 10^{-6}$
14	$-12.785196 \cdot 10^{-6}$
16	$+2.99653 \cdot 10^{-6}$

While for odd degree polynomials we have $P_D(N, 0) = 0$, even degree polynomials assume a very low value $P_D(N, 0)$ at $\Omega = 0$ given by:

$$P_D(N,0) = \frac{N-1}{2\pi \cdot N!} \sin\left(\frac{N+1}{2}\pi\right) \Gamma\left(\frac{N+1}{2}\right) \Gamma\left(\frac{N-1}{2}\right) \quad \text{or}$$
$$P_D(N,0) = \frac{1}{\pi \cdot N!} \sin\left(\frac{N+1}{2}\pi\right) \Gamma^2\left(\frac{N+1}{2}\right) \tag{2.75}$$

$\Gamma(x)$ is the Gamma function [7]. Table 2.6 gives $P_D(N, 0)$ for the even $N = 2$–16.

The coefficients of the polynomial $P_D(N, \Omega)$ for $N = 2$–7 are tabulated in Table 2.7.

For the coefficients of the highest and lowest order terms of $P_D(N, \Omega)$, analytic formulae can be derived from the definition equations:

$$a_N = \frac{(-1)^N}{N!} \cdot \left(\frac{N+1}{2}\Omega_D\right)^N \tag{2.76}$$

Table 2.7 Polynomial coefficients of $P_D(N, \Omega)$ for $N = 2$–7

$P_D(N,\Omega) = a_N\Omega^N + a_{N-1}\Omega^{N-1} + \cdots + a_1\Omega + a_0$						
$N \rightarrow$	2	3	4	5	6	7
a_0	−0.125000	0	0.023438	0	−0.004882	0
a_1	0	0.192450	0	−0.070639	0	0.022171
a_2	0.250000	0	−0.277522	0	0.153199	0
a_3		−0.256600	0	0.396539	0	−0.2907494
a_4			0.295751	0	−0.564281	0
a_5				−0.356162	0	0.800362
a_6					0.439438	0
a_7						−0.550800

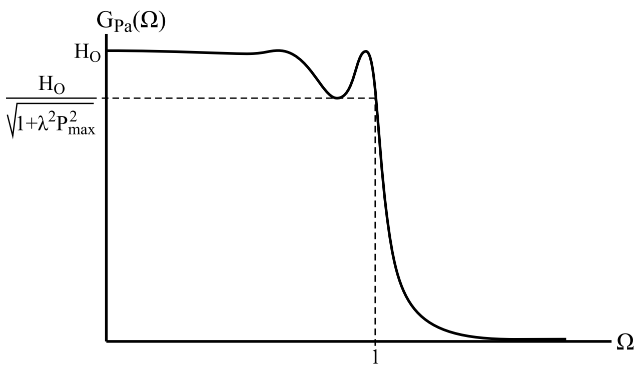


Fig. 2.41 The Pascal approximation gain

$$a_\eta = \frac{(-1)^{\frac{N+\eta}{2}}}{N!}\left(\frac{N+1}{2}\Omega_D\right)^\eta \cdot \prod_{k=1}^{(N-\eta)/2}\left(\frac{N+1}{2} - k\right)^2 \tag{2.77}$$

or

$$a_\eta = \frac{(-1)^{\frac{N+\eta}{2}}}{2^N N!}(N+1)^\eta \Omega_D^\eta \cdot \prod_{k=1}^{(N-\eta)/2} (N+1-2k)^2 \tag{2.78}$$

2.4.1 Optimizing the Pascal Approximation

The polynomial $P_D(N, \Omega)$ is symmetric and therefore can be used for the definition of the Pascal approximation:

$$G_{Pa}(\Omega) = \frac{H_o}{\sqrt{1 + \lambda^2 P_D^2(N, \Omega)}} \tag{2.79}$$

The non-equiripple nature of the Pascal polynomials appears in the gain of the Pascal approximation, as shown in Fig. 2.41. The minimum passband gain depends only on the order N, occurs at $\Omega = \frac{\Omega_{max}}{\Omega_D}$ and its value is $\frac{H_o}{\sqrt{1+P_{max}^2}}$. The gain takes the same value at $\Omega = 1$.

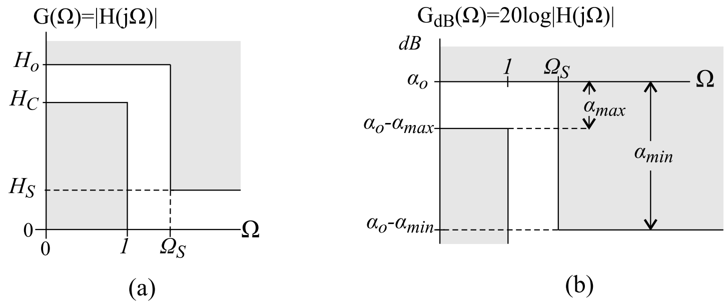


Fig. 2.42 Normalized lowpass specifications: (**a**) Plain gain. (**b**) Logarithmic gain

The constant factor λ, corresponding to the ripple factor ε of the Chebyshev approximation, is calculated next from the normalized lowpass filter specifications Ω_S and $\{H_C, H_S\}$ or $\{\alpha_{\max}, \alpha_{\min}\}$.

The plot of $G_{Pa}(\Omega)$ of (2.79) must not enter the shaded areas of Fig. 2.42a. The gain at $\Omega = 1$, $G_{Pa}(1) = \frac{H_o}{\sqrt{1+\lambda^2 P_D(1)}}$, equals the gain at $(\Omega_{\max}/\Omega_D) < 1$ which is the absolute passband minimum and decreases monotonically for $\Omega > 1$. Therefore, as far as the passband is concerned, the gain will remain in the permitted area by selecting λ to satisfy

$$G_{Pa}(1) = \frac{H_o}{\sqrt{1+\lambda^2 P_D^2(N,1)}} \geq H_C \tag{2.80}$$

The term $P_D^2(N,1) = P_{\max}^2$ and therefore

$$\lambda \leq \lambda_{\max} = \frac{\sqrt{\frac{H_o^2}{H_C^2} - 1}}{|P_{\max}|} = \frac{\sqrt{10^{\frac{\alpha_{\max}}{10}} - 1}}{|P_{\max}|} \tag{2.81}$$

Using the maximum value for the λ parameter we get the stopband edge optimized response:

$$\lambda = \lambda_{\max} \quad \Rightarrow \quad G_{Pa}(1) = G_{Pa}\left(\frac{\Omega_{\max}}{\Omega_D}\right) = H_C$$

and

$$G_{Pa}(\Omega_S) = \frac{H_o}{\sqrt{1 + (10^{\frac{\alpha_{\max}}{10}} - 1)(\frac{P_D(N,\Omega_S)}{P_D(N,1)})^2}} < H_S \tag{2.82}$$

It will be shown that parameter λ has also a lower boundary value $\lambda_{\min}$ derived from the stopband requirement:

$$G_{Pa}(\Omega_S) = \frac{H_o}{\sqrt{1+\lambda^2 P_D^2(N,\Omega_S)}} \leq H_S \tag{2.83}$$

This implies that

$$\lambda \geq \lambda_{\min} = \frac{\sqrt{\frac{H_o^2}{H_S^2} - 1}}{|P_D(N,\Omega_S)|} = \frac{\sqrt{10^{\frac{\alpha_{\min}}{10}} - 1}}{|P_D(N,\Omega_S)|} \tag{2.84}$$

The passband optimized response is derived by selecting

$$\lambda = \lambda_{\min} \quad \Rightarrow \quad G_{Pa}(\Omega_S) = H_S$$

and

$$G_{Pa}(1) = \frac{H_o}{\sqrt{1 + (10^{\frac{\alpha_{\min}}{10}} - 1)(\frac{P_D(N,1)}{P_D(N,\Omega_S)})^2}} > H_C \tag{2.85}$$

Finally, the ripple factor λ can be selected from (2.81) in view of (2.84), or from their combined expression:

$$\frac{\sqrt{10^{\frac{\alpha_{\min}}{10}} - 1}}{|P_D(N,\Omega_S)|} = \lambda_{\min} \le \lambda \le \lambda_{\max} = \frac{\sqrt{10^{\frac{\alpha_{\max}}{10}} - 1}}{|P_D(N,1)|}$$
$$\text{with } |P_D(N,1)| = |P_{\max}| \tag{2.86}$$

2.4.2 Order Calculation

So far, the order N has not been known, but can be calculated from (2.80) by setting $\lambda = \lambda_{\min}$ or from (2.83) by setting $\lambda = \lambda_{\max}$. In both cases, we end up with the order inequality

$$\left|\frac{P_D(N,1)}{P_D(N,\Omega_S)}\right| = \left|\frac{P_{\max}}{P_D(N,\Omega_S)}\right| \le g = \sqrt{\frac{\frac{H_o^2}{H_C^2} - 1}{\frac{H_o^2}{H_S^2} - 1}} = \sqrt{\frac{10^{\frac{\alpha_{\max}}{10}} - 1}{10^{\frac{\alpha_{\min}}{10}} - 1}} \ll 1 \tag{2.87}$$

In this inequality, the only unknown is the order N since the quantity g depends on the gain specifications. Therefore, for the calculation of the order, inequality (2.87) must be solved for N, but due to the nature of the Pascal polynomials, an analytic expression cannot be derived. However, the nomograph shown in Fig. 2.43 can be derived by using a mathematical software package.

The nomograph works as follows: We draw a vertical line from the point of the horizontal axis which corresponds to the specified Ω_S. We then draw a horizontal line from the point of the vertical axis that corresponds to HP:

$$HP = 20 \cdot \log\left(\frac{1}{g}\right) \tag{2.88}$$

The two lines intersect at a point. The order N is the number of the curve that lies above this point.

As an example, let us find the order if the specifications of the filter are $\Omega_S = 1.5$, $\alpha_{\max} = 0.5$ dB and $\alpha_{\min} = 40$ dB. Equation (2.88) gives $HP = 49.14$ and from the nomograph we get $N = 8$, since the point is located above curve 7.

2.4.3 The Transfer Function

The poles and the transfer function can, in theory, be calculated from

$$\begin{aligned} H(s)H(-s) &= G_{Pa}^2(\Omega)|_{\Omega^2=-s^2} \\ &= \left.\frac{H_o^2}{1 + \lambda^2 P_D^2(N,\Omega)}\right|_{\Omega^2=-s^2} = G_{Pa}^2(js) = G_{Pa}^2(-js) \end{aligned} \tag{2.89}$$

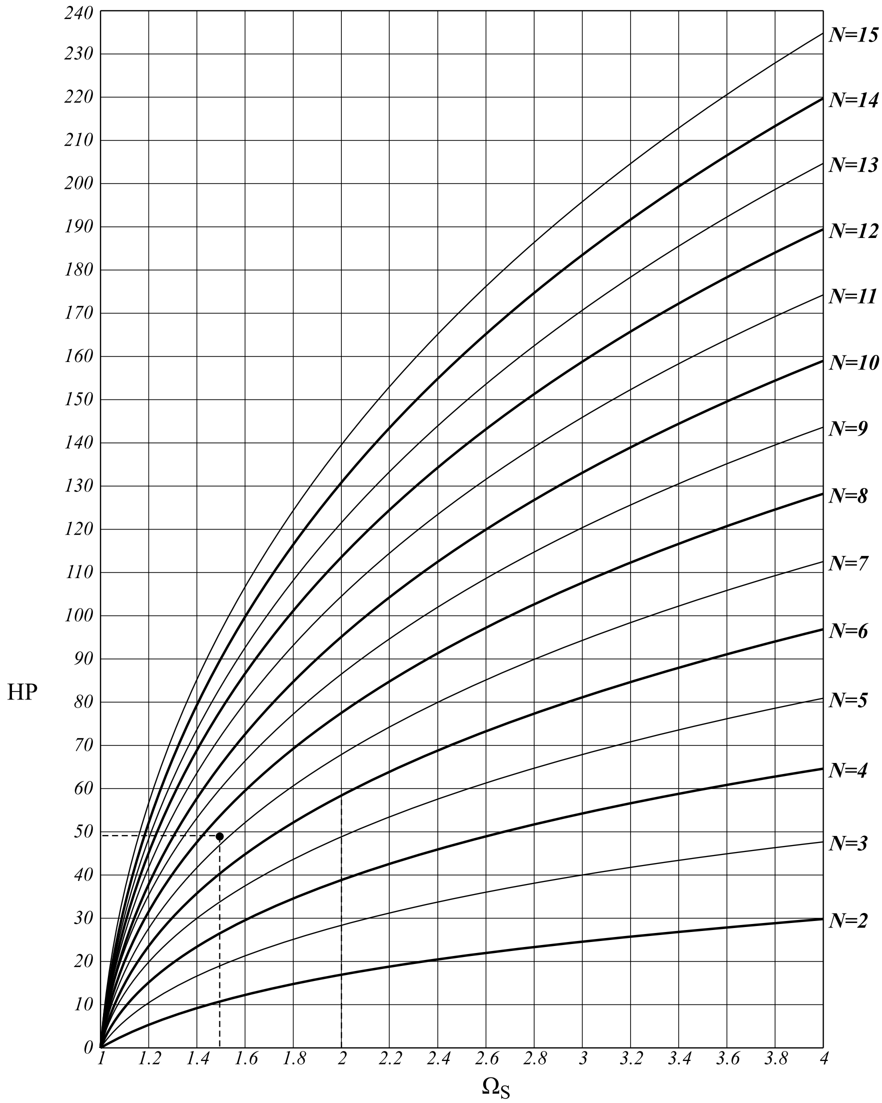


Fig. 2.43 Nomograph for the calculation of the order of the Pascal approximation

Since we have a polynomial approximation, the transfer function is expected to be all-pole of the form:

$$H(s) = \frac{C}{\prod_{k=1}^{N}(s - s_k)} = \frac{C}{(s + s_R)^{\eta} \prod_{k=1}^{\frac{N-\eta}{2}} (s^2 + 2\,\mathrm{Re}[s_k] + |s_k|^2)}$$

$$\text{with } \eta = \begin{cases} 1 & \text{for } N \text{ odd} \\ 0 & \text{for } N \text{ even} \end{cases} \tag{2.90}$$

where $s_k = \sigma_k \pm j\Omega_k$ are the $(N - \eta)/2$ conjugate pole pairs of $H(s)$ and s_R the real pole when N is odd.

The constant C of (2.90) can be calculated by letting

$$P_D(N,\Omega) = a_N\Omega^N + \cdots + a_\eta\Omega^\eta$$

in (2.89), which after some manipulation yields:

$$C = \frac{H_o}{\lambda \cdot |\alpha_N|} \tag{2.91}$$

where α_N is the coefficient of Ω^N in the polynomial $P_D(N,\Omega)$ taken from Table 2.7 or calculated from (2.76).

It seems again that no analytic expressions can be found for the poles and the use of numerical methods is inevitable in order to calculate the poles from equation (2.89). Note that α_N depends only on the order N, and the ripple factor $\lambda_{\max}$ depends on N and the specified $\alpha_{\max}$. In view of (2.89), the poles are therefore functions of N and $\alpha_{\max}$, and a table giving the constant C and the poles can be created for all values of order N and all values of $\alpha_{\max}$. Tabulation for $\lambda = \lambda_{\min}$ would require separate tables for values of Ω_S since $\lambda_{\min}$ depends on Ω_S according to (2.84).

Tables 2.8 at the end of this chapter give the constant C and the poles of the transfer function for $N = 2$–9 and $\alpha_{\max} = 0.01, 0.1, 0.5, 1.0, 1.25$ and 1.5 dB. All tables have been created for the stopband edge optimized case with $\lambda = \lambda_{\max}$. For any other value $\lambda_{\min} < \lambda < \lambda_{\max}$, a mathematical software package must be used.

2.4.4 Design Examples

Given the normalized specifications Ω_S and $\{H_C, H_S\}$ or $\{\alpha_{\max}, \alpha_{\min}\}$, the complete design procedure can be outlined as follows:

1. Calculate HP from (2.88) and find the order N of the filter from the nomograph of Fig. 2.43. If the transfer function of the stopband edge optimized filter ($\lambda = \lambda_{\max}$) is required, skip to step 4.
2. Get $P_{\max}$ and Ω_D from Table 2.4 for the calculated order N.
3. Calculate the limits of the ripple factor λ. If the stopband edge optimized filter is to be designed, $\lambda_{\max}$ can be calculated from (2.86). If the passband optimized filter is required, $L(N,\Omega_S) = P_D(N,\Omega_S)$ can be calculated from (2.69) or (2.71) and $\lambda_{\min}$ from (2.86). At this point, all necessary information for the gain is available.
4. If the stopband edge optimized filter is to be designed with $\lambda = \lambda_{\max}$, the transfer function can be calculated using the tables at the end of the chapter. For any other value $\lambda_{\min} < \lambda < \lambda_{\max}$, the coefficient C and the poles of the transfer function can be calculated from (2.89) using a mathematical software package.

Example 2.10 As an example, a Pascal filter transfer function with specifications $H_o = 1$, $\Omega_S = 2$, $\alpha_{\max} = 1.25$ dB and $\alpha_{\min} = 40$ dB will be calculated and compared to the Butterworth and Chebyshev transfer functions with the same specifications.

From (2.88), we find $HP = 44.77$. Using this value, the order of the filter can be found from the nomograph: $N = 5$ ($N_{But} = 8$, $N_{Che} = 5$). For the stopband edge optimized filter, the ripple factor, although unnecessary, is calculated from (2.86) $\lambda_{\max} = 19.083796$

From Table 2.4 for $N = 5$, we get $P_{\max} = 0.03026194$ and $\Omega_D = 0.70639006$. From Tables 2.8 for $N = 5$ and $\alpha_{\max} = 1.25$, we get $C = 0.147125$, real pole -0.383466 and two conjugate pole pairs:

$$-0.286033 \pm j0.558421 \quad \rightarrow \quad \Omega_1 = 0.62741472, \qquad Q_1 = 1.09675171$$

$$-0.094300 \pm j0.982733 \quad \rightarrow \quad \Omega_2 = 0.98724656, \qquad Q_2 = 5.23459125$$

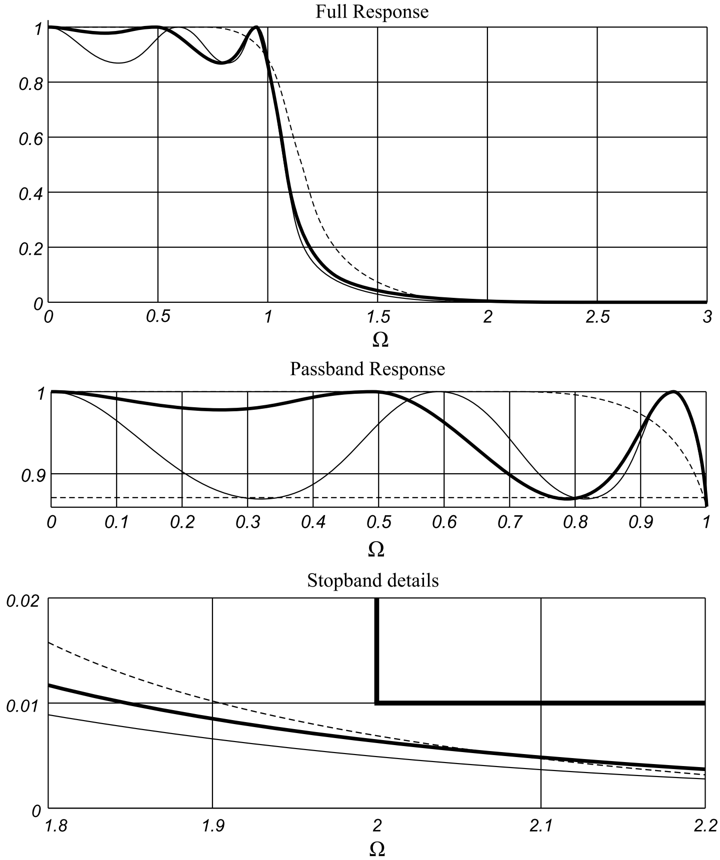


Fig. 2.44 Gain plot of design Example 2.10

Knowing C and the poles, the transfer function is calculated from (2.90):

$$H(s) = \frac{0.147125}{(s+0.383466)(s^2+\frac{\Omega_1}{Q_1}s+\Omega_1^2)(s^2+\frac{\Omega_2}{Q_2}s+\Omega_2^2)}$$

The plot of the gain function $G(\Omega) = |H(j\Omega)|$ is shown in Fig. 2.44 (thick curve) together with the corresponding gain plots for the Butterworth design ($N_{But} = 8$, dotted curve) and Chebyshev design ($N_{Che} = 5$, thin curve).

Example 2.11 As a second example, let us design a Pascal filter with specifications $\Omega_S = 1.5$, $\alpha_{\max} = 0.5$ dB and $\alpha_{\min} = 40$ dB.

From (2.88), $HP = 49.14$. Using this value, the order of the filter can be found from the nomograph: $N = 8$ ($N_{But} = 14$, $N_{Che} = 7$). For the stopband edge optimized filter $\lambda = \lambda_{\max}$, the ripple factor is calculated from (2.86) as $\lambda_{\max} = 21.985965$.

From Table 2.4 for $N = 8$, we get $P_{\max} = -0.01588792$ and $\Omega_D = 0.79978194$. From Tables 2.8 for $N = 8$ and $\alpha_{\max} = 0.5$, we get $C = 0.06514816$ and four conjugate pole pairs:

$$\begin{aligned} -0.43824049 \pm j0.18275053 \quad &\rightarrow \quad \Omega_1 = 0.4748184, \qquad Q_1 = 0.54173266 \\ -0.35480472 \pm j0.52718634 \quad &\rightarrow \quad \Omega_2 = 0.6354619, \qquad Q_2 = 0.89550936 \end{aligned}$$

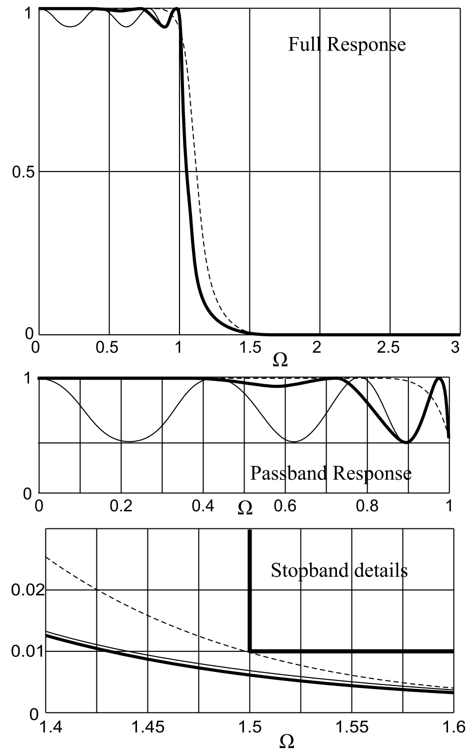


Fig. 2.45 Gain plot of design Example 2.11

$$-0.20803356 \pm j0.81122122 \quad \rightarrow \quad \Omega_3 = 0.8374711, \quad Q_3 = 2.01282686$$
$$-0.06129275 \pm j1.00838152 \quad \rightarrow \quad \Omega_4 = 1.0102419, \quad Q_4 = 8.24261602$$

Therefore, the transfer function will be

$$H(s) = \frac{0.06514816}{\prod_{k=1}^{4}(s^2 + \frac{\Omega_k}{Q_k}s + \Omega_k^2)}$$

Figure 2.45 shows the gain response $G(\Omega) = |H(j\Omega)|$ (thick curve) against the corresponding responses of the Butterworth ($N_{But} = 14$, dotted curve) and Chebyshev ($N_{Che} = 7$, thin curve) responses.

Example 2.12 In order to design the passband optimized filter with the same specifications and $G_{Pa}(\Omega_S) = H_S$, the minimum value $\lambda = \lambda_{\min}$ must be used. Calculation of the value $P_D(N, \Omega_S) = P_D(8, 1.5)$ must firstly be performed from (2.69) or (2.71), and then $\lambda_{\min}$ can be found from (2.86): $\lambda_{\min} = 13.414$. In this case, Tables 2.8 for the transfer function constant C and the poles cannot be used. Using a mathematical software package, we find $C = 0.10677611$ and the four new conjugate pole pairs:

$$-0.4962400 \pm j0.1898488 \quad \rightarrow \quad \Omega_1 = 0.531316, \quad Q_1 = 0.53534173$$
$$-0.4060296 \pm j0.5468592 \quad \rightarrow \quad \Omega_2 = 0.681113, \quad Q_2 = 0.83874807$$
$$-0.2468183 \pm j0.8389962 \quad \rightarrow \quad \Omega_3 = 0.874548, \quad Q_3 = 1.77164396$$
$$-0.0769663 \pm j1.0296597 \quad \rightarrow \quad \Omega_4 = 1.032532, \quad Q_4 = 6.70772459$$

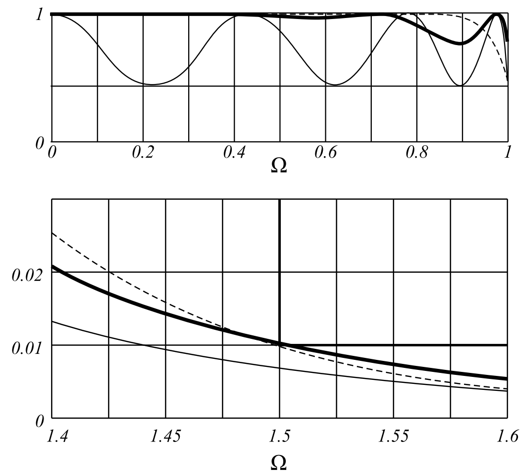


Fig. 2.46 Gain plot of design Example 2.12

The transfer function will in this case be:

$$H(s) = \frac{0.10677611}{\prod_{k=1}^{4}(s^2 + \frac{\omega_k}{Q_k}s + \omega_k^2)}$$

It should be observed that the passband optimized design with $\lambda = \lambda_{\min}$ leads to lower Q values. The gain response of the passband optimized filter is shown in Fig. 2.46 (thick curve).

2.4.5 Comparison to Other Polynomial Approximations

If the order of a Chebyshev filter which meets certain specifications is N_{Cheb}, then the order N of the Pascal filter that meets the same specifications is usually for low orders N_{Cheb}, as in Example 2.10. For higher orders, due to narrow transition bands (Ω_S close to 1), the order of the Pascal filter can be, in the worst case $N_{Cheb}+2$. This statement cannot be proved analytically but can be demonstrated and verified by constructing combined Pascal-Chebyshev order nomographs [6]. It should be noticed that the order of the Butterworth filters in these cases can even be as high as $2N_{Cheb}$, as in Example 2.11.

As far as the pole-Qs are concerned, Pascal filters seem to exhibit lower values, even when the Chebyshev filter that meets the same specifications is of lower order. In Example 2.10, Pascal filter with $N = 5$ has $Q_1 = 1.1$ and $Q_2 = 5.23$. The corresponding Chebyshev filter with $N_{Cheb} = 5$ has $Q_1 = 1.5$ and $Q_2 = 6.0$. In Example 2.11, Pascal filter with $N = 8$ has $Q_1 = 0.54$, $Q_2 = 0.89$, $Q_3 = 2.01$ and $Q_4 = 8.24$. The corresponding Chebyshev filter with $N_{Cheb} = 7$ has $Q_1 = 1.1$, $Q_2 = 2.58$ and $Q_3 = 8.84$. It should be noticed that the passband optimized Pascal filter of Example 2.12 has even lower Q values: $Q_1 = 0.54$, $Q_2 = 0.84$, $Q_3 = 1.77$ and $Q_4 = 6.70$.

Group delay must be compared for filters that meet the same specifications, and not for filters having the same order. Under this principle, Fig. 2.47 shows the group delay plots for the three design examples.

In Fig. 2.47(a), the order of the Pascal and the Chebyshev filter is $N_{Pa} = N_{Cheb} = 5$, and the group delay of the Pascal filter (thick curve) seems to be slightly better and still comparable to the Butterworth delay (dotted).

In Fig. 2.47(b), the group delay of the Pascal filter is smoother and absolutely comparable to that of Chebyshev, although $N_{Pa} = 8$ and $N_{Cheb} = 7$. The group delay of the Butterworth filter (dotted) that meets the same specifications with $N_{But} = 14$ is slightly better.

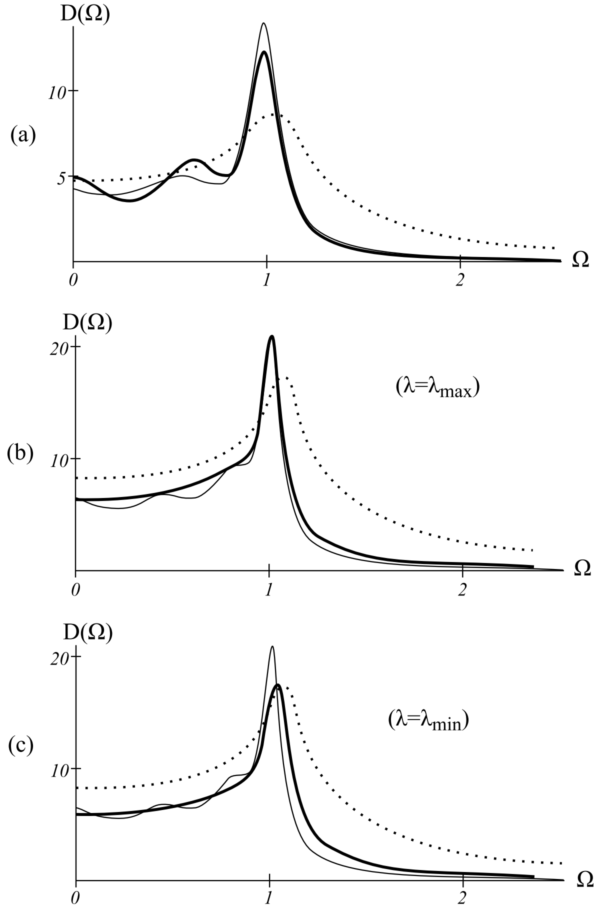


Fig. 2.47 Group delay: (**a**) Design Example 2.10. (**b**) Design Example 2.11. (**c**) Design Example 2.12

In Fig. 2.47(c), the group delay of the passband optimized Pascal filter is better than that of the Chebyshev filter and comparable to that of the Butterworth filter which meets the same specifications with $N_{But} = 14$.

The classical polynomial approximations such as Butterworth and Chebyshev, leading to all-pole transfer functions, are extensively used in analogue and IIR digital filter design. In this chapter, the Pascal approximation was also presented based on a suitably modified symmetric Pascal polynomial, $P_D(N, \Omega)$. The optimization and the application of the Pascal approximation was demonstrated by several design examples. The mathematical complexity involved due to the nature of the Pascal polynomials do not allow the analytical expression of the order equation and the transfer function poles and for this reason an exact nomograph is used for the order calculation and detailed tables for the transfer function constant and poles. The order of the so derived Pascal filters was found to be comparable to that of Chebyshev filters which meet the same specifications with the lowest possible order as compared to all other known polynomial approximations. The Pascal filters were found to require lower pole-Q values than their Chebyshev counterparts and exhibit slightly improved group delay performance comparable, in some cases, to that of the Butterworth filters designed to meet the same specifications.

In all cases, for Butterworth, Chebyshev and Pascal approximations, once the transfer function is available, passive, active, or IIR digital filters can be realized using the classic well-established methods that will be presented in the next chapters. For analogue high frequency integrated filters, a passive LC ladder filter can be initially designed, which can then be properly simulated to derive active structures suitable for integrated realizations.

2.5 Chebyshev and Pascal Design Tables

Design tables for the Chebyshev and Pascal approximations are presented in the following pages.

2.5.1 Table: Chebyshev Approximation

Denominator $D_N(s)$ of $H_{CH}(s) = \frac{H_o/(\varepsilon c_N)}{D_N(s)}$ of *normalized* lowpass Chebyshev filters ($c_N = 2^{N-1}$ is the coefficient of Ω^N in $C_N(\Omega)$)

$$\Omega_C = 1 \qquad G_{CH}(1) = \frac{H_o}{\sqrt{1+\varepsilon^2}} \qquad D_N(s) = \prod_{k=1}^{N}(s - s_k) \quad s_k = \sigma_k + j\Omega_k$$

$$\sigma_k = \sin(x)\sinh(y), \ \Omega_k = \cos(x)\cosh(y), \ x = \frac{(2N+2k-1)\pi}{2N}$$

$$y = \frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

$$G_{CH}(\Omega) = |H_{CH}(j\Omega)| = \frac{H_o}{\sqrt{1+\varepsilon^2 C_N^2(\Omega)}} \qquad \Omega_{3dB} = \cosh\left[\frac{1}{N}\cosh^{-1}\left(\frac{1}{\varepsilon}\right)\right]$$

N	Ripple 0.1 dB ($\varepsilon = 0.152620$)	εc_N	Ω_{3dB}
1	$s + 6.552203$	0.15262	6.552203
2	$s^2 + 2.372356s + 3.314037$	0.305241	1.943219
3	$(s + 0.969406)(s^2 + 0.969406s + 1.689747)$	0.610482	1.388995
4	$(s^2 + 0.528313s + 1.330031)(s^2 + 1.27546s + 0.622925)$	1.220963	1.213099
5	$(s + 0.538914)(s^2 + 0.333067s + 1.194937)(s^2 + 0.871982s + 0.63592)$	2.441927	1.134718
N	Ripple 0.5 dB ($\varepsilon = 0.349311$)	εc_N	Ω_{3dB}
1	$s + 2.862775$	0.349311	2.862775
2	$s^2 + 1.425625s + 1.516203$	0.698623	1.389744
3	$(s + 0.626456)(s^2 + 0.626456s + 1.42448)$	1.397246	1.167485
4	$(s^2 + 0.350706s + 1.063519)(s^2 + 0.84668s + 0.356412)$	2.794491	1.093102
5	$(s + 0.36232)(s^2 + 0.223926s + 1.035784)(s^2 + 0.586245s + 0.476767)$	5.588982	1.059259
N	Ripple 1.0 dB ($\varepsilon = 0.508847$)	εc_N	Ω_{3dB}
1	$s + 1.965227$	0.508847	1.965227
2	$s^2 + 1.097734s + 1.10251$	1.017694	1.217626
3	$(s + 0.494171)(s^2 + 0.494171s + 0.994205)$	2.03589	1.094868
4	$(s^2 + 0.279072s + 0.986505)(s^2 + 0.673739s + 0.279398)$	4.070777	1.053002
5	$(s + 0.289493)(s^2 + 0.178917s + 0.988315)(s^2 + 0.46841s + 0.429298)$	8.141554	1.033815

N	Ripple 2.0 dB ($\varepsilon = 0.764783$)	εc_N	Ω_{3dB}
1	$s + 1.30756$	0.764783	1.30756
2	$s^2 + 0.803816s + 0.82306$	1.529566	1.074142
3	$(s + 0.368911)(s^2 + 0.368911s + 0.886095)$	3.059132	1.032729
4	$(s^2 + 0.209775s + 0.928675)(s^2 + 0.50644s + 0.221568)$	6.118265	1.018367
5	$(s + 0.218308)(s^2 + 0.134922s + 0.952167)(s^2 + 0.35323s + 0.39315)$	12.23653	1.011742
N	Ripple 3.0 dB ($\varepsilon = 0.997628$)	εc_N	Ω_{3dB}
1	$s + 1.002377$	0.997628	1.002377
2	$s^2 + 0.6449s + 0.707948$	1.995257	1.000594
3	$(s + 0.29862)(s^2 + 0.29862s + 0.839174)$	3.990513	1.000264
4	$(s^2 + 0.170341s + 0.903087)(s^2 + 0.411239s + 0.19598)$	7.981027	1.000149
5	$(s + 0.17753)(s^2 + 0.10972s + 0.936025)(s^2 + 0.28725s + 0.377009)$	15.962054	1.000095

Pascal Tables 2.8 (see (2.90) and (2.91)) $\lambda = \lambda_{\max}$

N	C	Real s_R	Pair 1	Pair 2	Pair 3	Pair 4
$\alpha_{\max} = 0.01$						
2	10.413869		-2.227764 $\pm j2.337292$ $Q = 0.724698$			
3	5.206936	-1.589371	-0.794685 $\pm j1.626215$ $Q = 1.138815$			
4	2.934297		-1.044855 $\pm j0.565886$ $Q = 0.568622$	-0.429585 $\pm j1.376373$ $Q = 1.678195$		
5	1.769666	-0.929117	-0.744100 $\pm j0.768710$ $Q = 0.718895$	-0.279541 $\pm j1.259329$ $Q = 2.307319$		
6	1.112555		-0.787056 $\pm j0.311634$ $Q = 0.537767$	-0.564546 $\pm j0.860564$ $Q = 0.911542$	-0.201501 $\pm$j1.193811 $Q = 3.004193$	
7	0.719074	-0.747272	-0.667122 $\pm j0.497552$ $Q = 0.623748$	-0.448356 $\pm j0.908832$ $Q = 1.130140$	-0.154870 $\pm$j1.152746 $Q = 3.755102$	
8	0.473977		-0.685759 $\pm j0.214871$ $Q = 0.523970$	-0.571780 $\pm j0.616491$ $Q = 0.735273$	-0.368266 $\pm j0.936914$ $Q = 1.366801$	-0.124349 $\pm j1.124942$ $Q = 4.550879$
9	0.317021	-0.667452	-0.622821 $\pm j0.367348$ $Q = 0.580491$	-0.496350 $\pm j0.696974$ $Q = 0.861941$	-0.310304 $\pm j0.954500$ $Q = 1.617242$	-0.103048 $\pm j1.105038$ $Q = 5.385007$

Pascal Tables 2.8 (Continued)

N	C	Real s_R	Pair 1	Pair 2	Pair 3	Pair 4
$\alpha_{\max} = 0.1$						
2	3.276101		-1.186178 $\pm j1.380948$ $Q = 0.767359$			
3	1.638051	-0.969406	-0.484703 $\pm j1.206155$ $Q = 1.340928$			
4	0.923101		-0.687948 $\pm j0.457626$ $Q = 0.600520$	-0.278303 $\pm j1.131224$ $Q = 2.092963$		
5	0.556720	-0.652393	-0.513602 $\pm j0.655532$ $Q = 0.810717$	-0.187406 $\pm j1.093326$ $Q = 2.959545$		
6	0.349999		-0.576657 $\pm j0.272841$ $Q = 0.553142$	-0.402354 $\pm j0.758765$ $Q = 1.067273$	-0.138125 $\pm j1.071182$ $Q = 3.909688$	
7	0.226213	-0.569057	-0.503758 $\pm j0.445436$ $Q = 0.667431$	-0.327070 $\pm j0.819700$ $Q = 1.349165$	-0.107841 $\pm j1.056909$ $Q = 4.925736$	
8	0.149108		-0.536721 $\pm j0.194964$ $Q = 0.531966$	-0.441631 $\pm j0.561055$ $Q = 0.808387$	-0.273459 $\pm j0.858899$ $Q = 1.648108$	-0.087613 $\pm j1.047051$ $Q = 5.996295$
9	0.099732	-0.535685	-0.497623 $\pm j0.337401$ $Q = 0.604094$	-0.390185 $\pm j0.642298$ $Q = 0.963038$	-0.233680 $\pm j0.885749$ $Q = 1.960065$	-0.073275 $\pm j1.039887$ $Q = 7.113384$
$\alpha_{\max} = 0.5$						
2	1.431388		-0.712812 $\pm j1.004043$ $Q = 0.863721$			
3	0.715694	-0.626457	-0.313228 $\pm j1.021928$ $Q = 1.706189$			
4	0.403319		-0.473019 $\pm j0.399956$ $Q = 0.654778$	-0.185811 $\pm j1.018171$ $Q = 2.785057$		
5	0.243241	-0.479538	-0.367266 $\pm j0.591875$ $Q = 0.948309$	-0.127497 $\pm j1.014480$ $Q = 4.009746$		
6	0.152921		-0.441502 $\pm j0.250192$ $Q = 0.574702$	-0.295576 $\pm j0.699381$ $Q = 1.284398$	-0.095139 $\pm j1.011785$ $Q = 5.340876$	

Pascal Tables 2.8 (Continued)

N	C	Real s_R	Pair 1	Pair 2	Pair 3	Pair 4
7	0.098837	−0.452860	−0.396560 $\pm j0.414208$ $Q = 0.723012$	−0.245068 $\pm j0.766314$ $Q = 1.641479$	−0.074938 $\pm j1.009831$ $Q = 6.756307$	
8	0.065148		−0.438240 $\pm j0.182751$ $Q = 0.541733$	−0.354805 $\pm j0.527186$ $Q = 0.895509$	−0.208034 $\pm j0.811221$ $Q = 2.012827$	−0.061288 $\pm j1.008378$ $Q = 8.241702$
9	0.043575	−0.447883	−0.413955 $\pm j0.318731$ $Q = 0.631040$	−0.318420 $\pm j0.608385$ $Q = 1.078254$	−0.179934 $\pm j0.843059$ $Q = 2.395455$	−0.051527 $\pm j1.007266$ $Q = 9.786973$
$\alpha_{\max} = 1.0$						
2	0.982613		−0.548867 $\pm j0.895129$ $Q = 0.956520$			
3	0.491307	−0.494171	−0.247085 $\pm j0.965999$ $Q = 2.017720$			
4	0.276869		−0.385603 $\pm j0.377784$ $Q = 0.699974$	−0.148193 $\pm j0.983002$ $Q = 3.354099$		
5	0.166979	−0.407030	−0.305853 $\pm j0.566511$ $Q = 1.052470$	−0.102338 $\pm j0.989590$ $Q = 4.860703$		
6	0.104977		−0.383801 $\pm j0.241065$ $Q = 0.590447$	−0.249813 $\pm j0.675152$ $Q = 1.440849$	−0.076689 $\pm j0.992848$ $Q = 6.492463$	
7	0.067849	−0.402818	−0.350205 $\pm j0.401443$ $Q = 0.760595$	−0.209387 $\pm j0.744156$ $Q = 1.845995$	−0.060590 $\pm j0.994713$ $Q = 8.223764$	
8	0.044723		−0.395505 $\pm j0.177699$ $Q = 0.548148$	−0.316890 $\pm j0.513188$ $Q = 0.951660$	−0.179237 $\pm j0.791172$ $Q = 2.262982$	−0.049668 $\pm j0.995890$ $Q = 10.037888$
9	0.029913	−0.409609	−0.377413 $\pm j0.310934$ $Q = 0.647831$	−0.286839 $\pm j0.594246$ $Q = 1.150213$	−0.156064 $\pm j0.824921$ $Q = 2.689774$	−0.041834 $\pm j0.996687$ $Q = 11.923013$
$\alpha_{\max} = 1.25$						
2	0.865781		−0.499894 $\pm j0.865964$ $Q = 1.000106$			
3	0.432891	−0.453133	−0.226566 $\pm j0.950787$ $Q = 2.157005$			

Pascal Tables 2.8 (Continued)

N	C	Real s_R	Pair 1	Pair 2	Pair 3	Pair 4
4	0.243950		-0.357779 $\pm j0.370834$ $Q = 0.720123$	-0.136308 $\pm j0.973361$ $Q = 3.605295$		
5	0.147125	-0.383466	-0.286033 $\pm j0.558421$ $Q = 1.096752$	-0.094300 $\pm j0.982733$ $Q = 5.234591$		
6	0.092495		-0.364884 $\pm j0.238135$ $Q = 0.597062$	-0.234913 $\pm j0.667338$ $Q = 1.505827$	-0.070751 $\pm j0.987613$ $Q = 6.997370$	
7	0.059782	-0.386340	-0.334920 $\pm j0.397324$ $Q = 0.775785$	-0.197697 $\pm j0.736954$ $Q = 1.929747$	-0.055947 $\pm j0.990524$ $Q = 8.866478$	
8	0.039405		-0.381389 $\pm j0.176063$ $Q = 0.550706$	-0.304335 $\pm j0.508654$ $Q = 0.973840$	-0.169760 $\pm j0.784617$ $Q = 2.364431$	-0.045892 $\pm j0.992424$ $Q = 10.824132$
9	0.026356	-0.396944	-0.365311 $\pm j0.308401$ $Q = 0.654350$	-0.276347 $\pm j0.589650$ $Q = 1.178221$	-0.148181 $\pm j0.818964$ $Q = 2.808264$	-0.038673 $\pm j0.993746$ $Q = 12.857736$
$\alpha_{\max} = 1.5$						
2	0.778464		-0.461089 $\pm j0.844158$ $Q = 1.043049$			
3	0.389232	-0.420112	-0.210056 $\pm j0.939346$ $Q = 2.291162$			
4	0.219346		-0.335093 $\pm j0.365204$ $Q = 0.739560$	-0.126673 $\pm j0.966086$ $Q = 3.845942$		
5	0.132287	-0.364010	-0.269761 $\pm j0.551805$ $Q = 1.138443$	-0.087756 $\pm j0.977549$ $Q = 5.592095$		
6	0.083166		-0.349191 $\pm j0.235725$ $Q = 0.603264$	-0.222628 $\pm j0.660911$ $Q = 1.566289$	-0.065902 $\pm j0.983651$ $Q = 7.479746$	
7	0.053753	-0.372639	-0.322204 $\pm j0.393930$ $Q = 0.789742$	-0.188031 $\pm j0.731006$ $Q = 2.007118$	-0.052147 $\pm j0.987350$ $Q = 9.480258$	
8	0.035431		-0.369634 $\pm j0.174714$ $Q = 0.553040$	-0.293868 $\pm j0.504909$ $Q = 0.993987$	-0.161907 $\pm j0.779188$ $Q = 2.457679$	-0.042796 $\pm j0.989795$ $Q = 11.574823$
9	0.023698	-0.386388	-0.355221 $\pm j0.306307$ $Q = 0.660220$	-0.267586 $\pm j0.585850$ $Q = 1.203477$	-0.141638 $\pm j0.814018$ $Q = 2.916757$	-0.036079 $\pm j0.991514$ $Q = 13.750098$

Fig. 2.48 Filter specifications for Problems 2.3 and 2.4

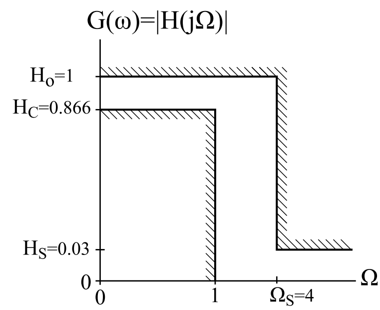
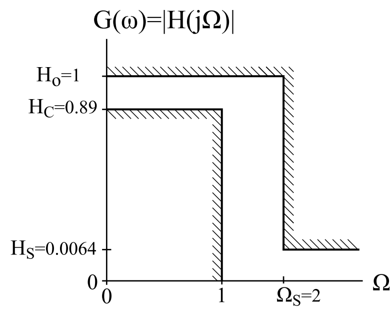


Fig. 2.49 Filter specifications for Problems 2.7–2.9



2.6 Problems

2.1 Show that all $N-1$ derivatives of the Butterworth gain function $G(\Omega)$ of (2.9) assume zero value at $\Omega = 0$.

2.2 Verify that $G_{1\,\max}(\omega)$ and $G_{2\,\max}(\omega)$ of Example 2.1 satisfy the specifications of Fig. 2.10a.

2.3 Calculate the transfer function of the normalized Butterworth lowpass filter with the specifications given in Fig. 2.48 without using Table 2.1.

2.4 Use Table 2.1 to calculate the transfer function of the normalized Butterworth lowpass filter with the specifications given in Fig. 2.48.

2.5 Follow Example 2.4 to realize the normalized transfer function of Problem 2.3 or 2.4.

2.6 Verify the results of Example 2.5.

2.7 Calculate the order of the Butterworth and the Chebyshev filter which satisfy the specifications of Fig. 2.49.

2.8 Calculate the transfer function $H(s)$ of the normalized stopband edge frequency gain optimized Chebyshev lowpass filter with the specifications given in Fig. 2.49 and design an active-RC circuit with voltage transfer function $H(s)$.

2.9 Calculate the transfer function $H(s)$ of the normalized passband optimized Chebyshev lowpass filter with the specifications given in Fig. 2.49 and design an active-RC circuit with voltage transfer function $H(s)$.

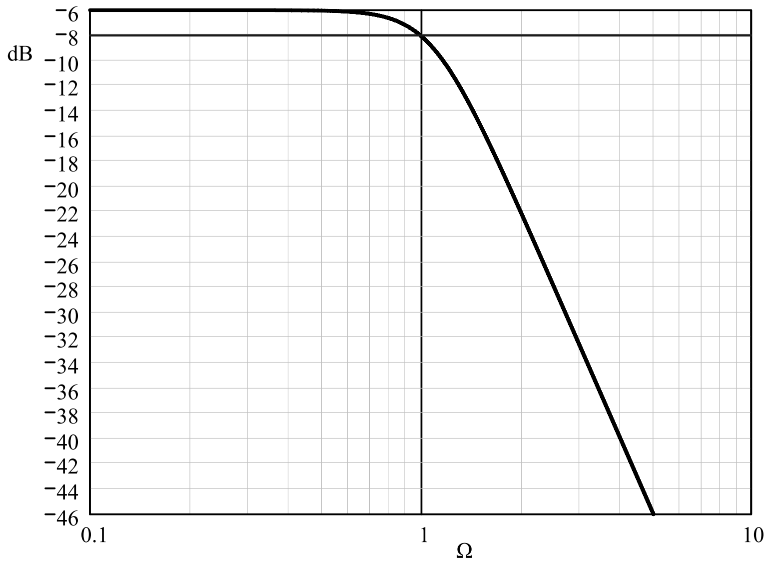


Fig. 2.50 Gain plot for Problem 2.10

Fig. 2.51 Stopband gain plot for Problem 2.11

2.10 Figure 2.50 shows the plot of the logarithmic voltage gain of a normalized lowpass Butterworth filter. Determine the corresponding voltage transfer function.

2.11 Figure 2.51 shows the plot of the stopband logarithmic voltage gain of a normalized lowpass Chebyshev filter with 2 dB passband ripple. Determine the corresponding voltage transfer function.

2.12 Calculate the poles of Pascal design Example 2.10 directly from equation (2.89) using mathematical software, e.g. MATLAB, Mathematica or Mathcad. Use $\lambda = \lambda_{max}$.

2.13 Repeat Pascal design Example 2.10 with $\alpha_{max} = 1.5$ and $\alpha_{min} = 24$ dB. Determine the order of the Pascal filter from the nomograph of Fig. 2.43 and try to calculate the constant C of the transfer function and the poles using a mathematical software package with $\lambda = \lambda_{max}$. Finally, compare your results with the corresponding values from the tables.

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[1] All references, whether or not cited within the text, are recommended for further reading.

http://www.springer.com/978-94-007-2189-0

Analog Electronic Filters
Theory, Design and Synthesis
Dimopoulos, H.G.
2012, XIII, 498 p., Hardcover
ISBN: 978-94-007-2189-0