

Chapter 2

H_2 Performance Criteria

2.1 Introduction

Control performance criteria are a key element in control systems theory. Not only are they fundamental from a conceptual point of view, but this concept also leads to a huge variety of analysis and design methods, which are formulated as optimization problems. This formulation is, in its general form, expressed as the solution of a problem like

$$\min_{\rho} J(\rho) \quad (2.1)$$

where the controller parameter vector ρ is the decision variable. In (2.1) $J(\rho)$ is an objective function which expresses the system's performance in achieving a prescribed control objective, smaller values of the objective function expressing better performance. The most fundamental control objectives, as well as more sophisticated ones, are quite naturally and effectively expressed as the norm of some signal in the control loop. When the 2-norm

$$\|x(t)\|^2 = \frac{1}{N} \sum_{t=1}^N [x(t)]^2$$

is used, this is said to be an H_2 performance criterion.

A large variety of control analysis and design tools have been derived from this general formulation with H_2 performance criteria, since the early days of modern control theory—think of Linear Quadratic Control (LQR/LQG) and Generalized Predictive Control (GPC), for example. These methods can deal with quite sophisticated and ambitious control objectives, but they usually rely on full knowledge of the process and noise, and require the controller's transfer function to be freely chosen. Obtaining a good model for a real process usually demands, among other tasks, collecting data from real system operation. *Data-driven design*, on the other hand, addresses the minimization of the performance criterion (2.1) directly from the data collected from the system, without the intermediate step of deriving a process model

from these data, and for a controller whose structure¹ has been previously fixed—only the parameters in this fixed structure are free to be designed.

In this chapter we will present the H_2 control performance criteria corresponding to the fundamental control objectives: reference tracking, noise rejection and economy of control effort. Most commonly found adaptive control algorithms are concerned with optimizing either reference tracking—like the celebrated Model Reference Adaptive Control—or noise rejection—the equally celebrated Minimum Variance Control. We will dissect each one of these performance criteria, as well as some of its variations and combinations, developing the theoretical framework that will be used in subsequent chapters to analyze the challenges and solutions encountered in their optimization.

2.2 The Different Criteria

2.2.1 Reference Tracking—The Model Reference Control

Reference tracking is concerned with the response of the closed-loop system to the reference alone, disregarding the effect of noise in the output. Let us define this response as

$$y_r(t, \rho) \triangleq T(z, \rho)r(t).$$

A fundamental objective of a control system being to make the process output as close as possible to the reference, the performance from this point of view can be evaluated by the two-norm of the tracking error, that is, by the following performance criterion:

$$J(\rho) = \bar{E}[r(t) - y_r(t, \rho)]^2. \quad (2.2)$$

It is easy to see, and a known fact taught even at the basic levels of control courses, that it is usually impossible to obtain perfect tracking; that is, no controller, whether in the considered class of controllers or not, can make the output to track exactly the reference at all times; the performance criterion in (2.2) can never be made zero. Since perfect tracking is not possible, the tracking objective is usually relaxed into a specification of how close a tracking would be satisfactory for the designer. This is often expressed in terms of control performance measures such as settling time, maximum overshoot, rising time, cutoff frequency, etc. The Model Reference paradigm comes into the scene as an alternative, more detailed, more succinct and analytically treatable description of this relaxation.

In the Model Reference design paradigm, the designer is asked to create a transfer function whose behavior is the one expected from the closed-loop system. This target transfer function is called the *reference model*, and will be henceforth denominated $T_d(z)$ —the subscript d standing for “desired”. The response desired for the

¹The $\bar{C}(z)$ vector introduced in the previous chapter.

closed-loop system under a given reference signal $r(t)$ is then $y_d(t) = T_d(z)r(t)$. The response actually obtained in closed-loop from the reference signal² is $y_r(t, \rho)$ defined above, which should be as close as possible to $y_d(t)$. Then the controller design consists in finding the controller parameters that make these two signals as close as possible to each other. That is, instead of using the performance criterion in (2.2), the following function $J_y(\rho)$ is defined as the reference tracking performance criterion

$$J_y(\rho) \triangleq \bar{E}[y_r(t, \rho) - y_d(t)]^2 = \bar{E}[(T(z, \rho) - T_d(z))r(t)]^2. \quad (2.3)$$

This control design formulation has been also called, in a more general framework, the model matching control. It can be cast as a linear quadratic regulator (LQR) and thus solved by means of tools such as Riccati equations, Linear Matrix Inequalities (LMI's), Bilinear Matrix Inequalities (BMI's), etc. [3]. Provided, of course, that the process model $G(z)$ is known, which is not the case in this book.

The model reference control design paradigm has been around since at least the 1960's. This paradigm has caught more attention within the adaptive control framework, probably because it lends itself naturally to the automatic adjustment of the controller parameters. Before we move on, let us just note a relevant relationship with another control design paradigm of great success in the adaptive control context: the pole assignment design. In pole assignment, as the name says, the designer assigns the poles of the closed-loop system, that is, the denominator of its transfer function; here we assign the whole transfer function—denominator and numerator.

2.2.2 Noise Rejection—The Minimum Variance Control

Another fundamental control objective is to minimize the effect of the noise in the output. The output of the closed-loop system due to the noise alone, that is, disregarding the effect of the reference, is given by

$$y_e(t, \rho) \triangleq S(z, \rho)v(t).$$

The effect of the noise in the output can be measured by the size of this signal, giving rise to the noise rejection performance criterion, which we immediately baptize as $J_e(\rho)$:

$$J_e(\rho) \triangleq \bar{E}[y_e(t)]^2 = \bar{E}[S(z, \rho)v(t)]^2. \quad (2.4)$$

Again, it is clear that no controller in the universe can make the effect of the noise to disappear completely, that is, to make $J_e(\rho) = 0$. Indeed, this would require that $S(z, \rho) = 0 \forall z$ which in turn would demand $C(z, \rho)G(z) \rightarrow \infty \forall z$. Since “perfect performance” is not possible for this performance criterion either, a relaxation can

²That is, neglecting the effect of noise.

also be thought of here, just as in the case of the reference tracking criterion $J_y(\rho)$ and for the same reasons. To do that, define a *desired* sensitivity function $S_e(z)$, and adopt the following criterion:

$$J_e^r(\rho) \triangleq \bar{E}[(S(z, \rho) - S_e(z))v(t)]^2.$$

Even though this relaxation is in all respects dual to the definition of the reference model $T_d(z)$ for the reference tracking criterion, and even though it can lead to better results than using the original $J_e(\rho)$ as the performance criterion, this relaxation is by no means usual and virtually absent from the literature. This fact is probably due to two conceptual differences between the two criteria. First, in tracking the reference, usually the main concern when adjusting the controller's parameters is with the transient performance, as steady-state performance is usually guaranteed by the application of the internal model principle in the choice of the controller's structure.³ And it is rather easy to come up with a transfer function that provides the desired transient performance. For noise rejection, on the other hand, devising a reasonable sensitivity function may not be as intuitive, particularly taking into account that the noise model $H(z)$ is unknown. Second, the input signals in each case— $J_y(\rho)$ and $J_e(\rho)$ —are of different nature. In the noise rejection case, the input signal is the filtered white noise $v(t)$, whereas the reference can in principle be anything and is most likely to be deterministic. The random signal $v(t)$ being not measurable, as opposed to $r(t)$ involved in the tracking performance criterion, complicates the practical computation of the quantities that would be necessary for its optimization—typically the derivatives of the performance criterion. This issue of computation will become clearer in Chap. 7, where we will see computation schemes that yield the derivatives of $J_e(\rho)$ and $J_y(\rho)$, but not of $J_e^r(\rho)$. For these reasons we will adhere to tradition and consider the performance criterion $J_e(\rho)$ instead of the more generic $J_e^r(\rho)$.

2.2.3 The Composite Performance

Each one of the two performance criteria just presented (reference tracking and noise rejection) represents a conceptually different control objective and each one of them has a theoretical and practical relevance of its own. Numerous control design methods, of data-driven, adaptive and model-based natures, have been developed for each one of them independently. Most commonly found adaptive control algorithms are concerned either with minimizing the reference tracking criterion $J_y(\rho)$ (the celebrated Model Reference Adaptive Control) or with the noise rejection criterion $J_e(\rho)$ (Minimum Variance Control).

But equally important is the following *composite performance criterion* $J_T(\rho)$:

$$J_T(\rho) \triangleq \bar{E}[y(t, \rho) - y_d(t)]^2. \quad (2.5)$$

³Including integral action in the controller being by far the most common instance.

Notice the difference between the definitions of $J_T(\rho)$ and $J_y(\rho)$: the definition of $J_y(\rho)$ involves the *tracking error* $y_r(t, \rho) - y_d(t)$, disregarding the effect of noise in the output, whereas in $J_T(\rho)$ the *output error* $y(t, \rho) - y_d(t)$ appears. We call it the composite criterion because it takes into account both external signals acting on the system $r(t)$ and $v(t)$, and as a result it equals the sum of the two previous ones, that is:

$$J_T(\rho) = \bar{E}[(T(z, \rho) - T_d(z))r(t)]^2 + \bar{E}[S(z, \rho)v(t)]^2 = J_y(\rho) + J_e(\rho). \quad (2.6)$$

To prove this equality it suffices to develop the composite objective function from its definition (2.5):

$$\begin{aligned} J_T(\rho) &= \bar{E}[T(z, \rho)r(t) + S(z, \rho)v(t) - T_d(z)r(t)]^2 \\ &= \bar{E}[(T(z, \rho) - T_d(z))r(t) + S(z, \rho)v(t)]^2 \\ &= \bar{E}[(T(z, \rho) - T_d(z))r(t)]^2 + \bar{E}[S(z, \rho)v(t)]^2 \\ &\quad + 2\bar{E}[(T(z, \rho) - T_d(z))r(t)S(z, \rho)v(t)] \end{aligned}$$

and to realize that the last term in this sum is zero because the reference and the noise are independent.

2.2.4 Economy of Control Effort

Many H_2 design methodologies, including some data-driven methodologies, consider an additional control objective in their performance criteria, which is the economy of control energy—or effort

$$J_u(\rho) = \bar{E}[u(t)]^2. \quad (2.7)$$

This performance criterion does not make sense if used alone. Minimizing $J_u(\rho)$ above would always result in $u(t) \equiv 0$ (an open-loop system), and we don't need an optimization procedure to tell us that the best choice regarding economy of control energy is to turn the control off. Using this performance criterion in a control design only makes sense when combined with another performance criterion, such as reference tracking or noise rejection or their combination. The performance criterion to be minimized in such cases will be a weighted sum of the just defined $J_u(\rho)$ and whatever performance criterion $J(\rho)$ is to be minimized ($J_T(\rho)$, $J_e(\rho)$ or $J_y(\rho)$):

$$J_\lambda(\rho) = J(\rho) + \lambda J_u(\rho) \quad (2.8)$$

where $\lambda \in \mathbb{R}$ is a design parameter which weighs the relative importance of control economy versus the performance as specified by $J(\rho)$.

Energy saving is among the strongest social and industrial concerns nowadays, and probably even more in the future, but still we decline to include it explicitly in our analysis almost everywhere in the remaining of this book. This is not because we are not concerned about saving control effort whenever possible—quite

the contrary—but instead because the addition of this term is not necessary to accomplish this end.

In a control design defined by the performance criterion $J_\lambda(\rho)$ in (2.8) there is an additional parameter to be chosen by the designer: the weight λ given to the control effort. It is not clear how to choose λ in order to obtain the desired effect, and most such formulations rely on trial and error to do so. An equivalent effect on control effort's savings can be obtained by properly choosing the reference model $T_d(z)$. Existing constraints in the control action can be taken into account in the choice of the reference model $T_d(z)$, so that we do not require the controller to do any more effort than necessary to provide an appropriate performance. For instance, it may be desirable and possible to achieve a deadbeat response $T_d(z) = \frac{1}{z^n}$ but, if such a performance would require too much control energy, the designer can settle for a slower time response which is also acceptable, and specify a slower reference model—say $T_d(z) = \frac{1}{(z-0.4)^n}$. This slower reference model will most likely require significantly less control effort. So, relaxing the requirements in $T_d(z)$ has a similar effect to adding the term $J_u(\rho)$ in the performance criterion. There seem to be no quantitative guidelines for choosing either one (the reference model and the weight λ) to this end, so it may be wiser to have just one thing to choose—the reference model—instead of two. Accordingly, in almost everything that follows in this book we will not treat the performance criterion $J_u(\rho)$, which also corresponds to set $\lambda = 0$ in (2.8) in all instances.

2.3 Duality with System Identification—The Ideal Controllers

Our aim in this section is to provide a central concept in the framework for the analysis of the properties of data-driven design that will be presented along this book: the “ideal controller”. There are two fundamental objectives to be pursued by the control design: noise rejection and reference tracking. Each control objective is best accomplished individually by a given controller, which is the one that, among all linear time invariant controllers in the universe (and not only within the class, and not even with causality or stability constraints), provides the minimum value for the performance criterion. This is what we call the ideal controller, and there is one ideal controller for each process and for each control objective. We will see in the following that the ideal controller plays in the data-driven control design a very similar role to the one played by the “real system” in system identification, and that this analogy can be used to our benefit.

2.3.1 Reference Tracking

Let us start by analyzing the reference tracking objective function, that is, the performance criterion $J_y(\rho)$ defined previously and reproduced below

$$J_y(\rho) = \bar{E}[(T(z, \rho) - T_d(z))r(t)]^2.$$

To that end, define the ideal controller $C_d(z)$, which is the controller transfer function that would exactly achieve the desired closed-loop transfer function $T_d(z)$:

$$C_d(z) = \frac{T_d(z)}{G(z)(1 - T_d(z))}. \quad (2.9)$$

Should the ideal controller $C_d(z)$ be put in the control loop, the objective function would evaluate to zero—that is, $J_y(\rho) = 0$. The ideal controller may or may not belong to the class of controllers considered. Only when it does the closed-loop system can be made to behave exactly as specified by the reference model by a proper choice of the parameter ρ . Whether this is the case or not is a critical issue in determining the properties of a model reference design. So, let us formalize this assumption.

Assumption B_y (Matched control) $C_d(z) \in \mathcal{C}$ or, equivalently,

$$\exists \rho_d : C(z, \rho_d) = C_d(z). \quad (2.10)$$

This assumption is quite similar to the standard assumption in identification theory that the process model belongs to the model class considered. Though assumptions of this nature are standard in our context [1, 2], they are not weak ones. We can, however, expect them to be violated only moderately in a well formulated design problem. Indeed, it does not make good sense to formulate a problem in which one searches for a performance that is radically different from what can be achieved. Now, it is important to know, for any particular case, whether or not Assumption B_y is satisfied, which usually requires knowledge of a model class for the process. On the other hand, it is possible to arrange things so that this assumption is satisfied. The following example briefly illustrates these ideas, which will be extensively used in this book.

Example 2.1 Consider that a process is controlled by a PID controller whose transfer function is

$$C(z, \rho) = \frac{\varrho_1(z - \varrho_2)(z - \varrho_3)}{z(z - 1)} \quad (2.11)$$

with $\rho = [\varrho_1 \ \varrho_2 \ \varrho_3]^T$ as the parameter vector to be adjusted. The performance criterion to be minimized is $J_y(\rho)$ with the following reference model:

$$T_d(z) = \frac{1 - c}{z - c}$$

where $c \in (0, 1)$ is a given constant.

The controller class is defined as the set of all transfer functions with this particular structure defined in (2.11):

$$\mathcal{C}_{PID} = \left\{ C(z, \rho) = \frac{\varrho_1(z - \varrho_2)(z - \varrho_3)}{z(z - 1)} : \rho \in \mathbb{R}^3 \right\}.$$

The job of setting the PID parameters consists in choosing, among all the controllers in the class \mathcal{C}_{PID} , the best one according to the performance criterion $J_y(\rho)$. If the ideal controller is among them—that is, if $C_d(z) \in \mathcal{C}_{PID}$ —then it is certainly the best choice. Now assume that the process has a second-order BIBO-stable transfer function:

$$G(z) = \frac{kz}{(z-a)(z-b)}$$

with fixed but unknown $a, b \in (-1, 1), k \in \mathbb{R}$. Then, using (2.9), the ideal controller is given by

$$\begin{aligned} C_d(z) &= \frac{\frac{1-c}{z-c}}{\frac{kz}{(z-a)(z-b)} \left(1 - \frac{1-c}{z-c}\right)} \\ &= \frac{\frac{1-c}{z-c}}{\frac{kz}{(z-a)(z-b)} \left(\frac{z-1}{z-c}\right)} \\ &= \frac{(1-c)}{k} \frac{(z-a)(z-b)}{z(z-1)} \end{aligned}$$

which belongs to the class \mathcal{C}_{PID} , with

$$\rho_d = \begin{bmatrix} \frac{1-c}{k} \\ a \\ b \end{bmatrix}.$$

Thus, using only the knowledge of a system class to which the process belongs, it is possible to establish that Assumption \mathbf{B}_y is satisfied.

Now assume that the same process is controlled by a PI controller within the following class:

$$\mathcal{C}_{PI} = \left\{ C(z, \rho) = \frac{q_1(z - q_2)}{z - 1} : \rho \in \mathbb{R}^2 \right\}$$

where the parameter vector has been defined as $\rho = [q_1 \ q_2]^T$. It is evident that $C_d(z) \notin \mathcal{C}_{PI}$; equivalently Assumption \mathbf{B}_y is not satisfied. However, if we really want to work under this assumption, we may choose another reference model which provides a response $y_d(t)$ to be tracked that is similar to the original one. It is straightforward to verify that with the second order reference model

$$T_d(z) = \frac{Kz}{z^2 + (K-1-b)z + b}$$

where K is any real number chosen by the designer, $C_d(z) \in \mathcal{C}_{PI}$ and $\rho_d = [\frac{K}{k} \ a]^T$. And if the parameter K is chosen such that the dominant pole of this new reference model is close to the unique pole c of the original reference model, then both yield similar step responses. Figure 2.1 illustrates for an example: $b = 0.24, c = 0.8$,

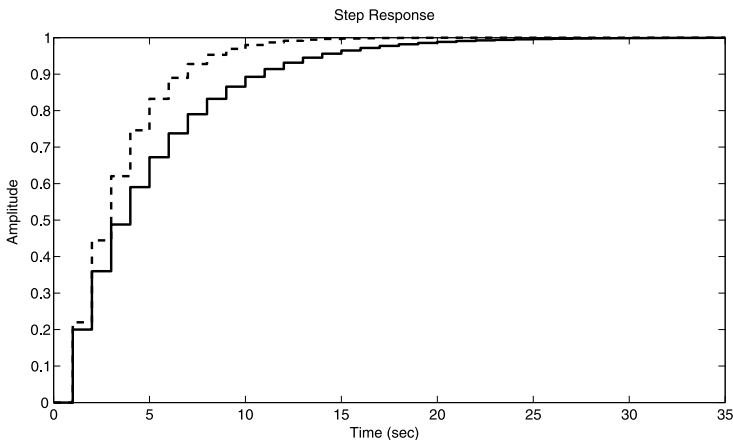


Fig. 2.1 Step responses of the two reference models in Example 2.1: $T_d(z) = \frac{0.2}{z-0.8}$ (continuous line) and $T_d(z) = \frac{0.22z}{z^2-1.02z+0.24}$

$K = 0.22$, and $r(t)$ is a step signal. This new choice of reference model, however, requires the knowledge of the process' pole b .

Assumption B_y is a quite realistic assumption in many cases. Accordingly, we analyze the properties of the solution of the model reference design and derive a number of properties for it under this assumption, which we call the matched control case. We also show in Sect. 2.4 that actually we can—and should—turn the problem around, as in the last part of the example. That is, instead of just picking a reference model and hoping that Assumption B_y is satisfied, we can try and choose the reference model such that it is.

In the mismatched control case, when Assumption B_y is not satisfied, we define the mismatch in terms of the ideal controller. Let $\rho_* = \arg \min J_y(\rho)$; then $C(z, \rho_*)$ is the best controller allowed by the controller class. The mismatch is defined as the difference between this controller and the ideal controller:

$$K(z) \triangleq C_d(z) - C(z, \rho_*). \quad (2.12)$$

This is similar to the bias definition in system identification [4]—another similarity with system identification theory found in this book, and certainly not the last. The norm of the transfer function $K(z)$ can be used as a measure of mismatch. The case where Assumption B_y is not satisfied will also be analyzed, and this analysis will be built upon the results obtained for the matched control case.

When Assumption B_y is satisfied, the global minimum of the performance criterion is ρ_d . On the other hand, when Assumption B_y is not satisfied, the global minimum of $J_y(\rho)$ will be dependent of the reference spectrum. Whether the global minimum is unique or not depends on the richness of the input $r(t)$, in both cases. These properties will be analyzed further ahead in this book.

2.3.2 Noise Rejection

To analyze the noise rejection performance criterion in a similar way to what has just been done for the reference tracking criterion, defining an ideal controller, some preparation is needed. This is due to the fact that, unlike the reference tracking cost, no relaxation has been included in $J_e(\rho)$ and as a result this cost function can not be made zero by any controller.

Applying Parseval's Theorem to (2.4) and using the fact that $v(t) = H(z)e(t)$, and that $e(t)$ is white noise (thus with a flat spectrum) leads to

$$\begin{aligned} J_e(\rho) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |S(e^{j\omega}, \rho)|^2 \Phi_v(e^{j\omega}) d\omega \\ &= \frac{1}{2\pi} \sigma_e^2 \int_{-\pi}^{\pi} |S(e^{j\omega}, \rho)|^2 |H(e^{j\omega})|^2 d\omega. \end{aligned} \quad (2.13)$$

So $J_e(\rho) = 0$ would require $S(e^{j\omega}, \rho) = 0 \forall \omega$, but no controller can provide this because $S(z, \rho) = \frac{1}{1+C(z, \rho)G(z)}$ —an argument already presented a few pages earlier. Hence, in order to check which controller provides the smallest possible value for $J_e(\rho)$, we start by checking which value is this. Manipulating (2.13) yields

$$\begin{aligned} J_e(\rho) &= \sigma_e^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |1 + H(e^{j\omega})S(e^{j\omega}, \rho) - 1|^2 d\omega \\ &= \sigma_e^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + H(e^{j\omega})S(e^{j\omega}, \rho) - 1)(1 + H(e^{j\omega})S(e^{j\omega}, \rho) - 1)^* d\omega \\ &= \sigma_e^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + [H(e^{j\omega})S(e^{j\omega}, \rho) - 1] + [H(e^{j\omega})S(e^{j\omega}, \rho) - 1]^* \\ &\quad + |H(e^{j\omega})S(e^{j\omega}, \rho) - 1|^2) d\omega \\ &= \sigma_e^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + 2\Re\{H(e^{j\omega})S(e^{j\omega}, \rho) - 1\} + |H(e^{j\omega})S(e^{j\omega}, \rho) - 1|^2) d\omega. \end{aligned} \quad (2.14)$$

Now observe that, by hypothesis,

$$\lim_{z \rightarrow \infty} H(z) = 1$$

and

$$\lim_{z \rightarrow \infty} C(z, \rho)G(z) = 0 \quad \forall \rho.$$

So, by construction,

$$\lim_{z \rightarrow \infty} S(z, \rho) = \lim_{z \rightarrow \infty} \frac{1}{1 + C(z, \rho)G(z)} = 1.$$

Hence $[H(z)S(z, \rho) - 1]$ is a strictly proper transfer function, and the integral of the real part of a strictly proper transfer function is zero. As a result, we can write

$$J_e(\rho) = \sigma_e^2 + \sigma_e^2 \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \left| S(e^{j\omega}, \rho) - \frac{1}{H(e^{j\omega})} \right|^2 d\omega. \quad (2.15)$$

Equation (2.15) consists of a constant term and a controller-dependent term which is positive semi-definite. It is seen that the least possible value for the noise rejection term equals σ_e^2 . Moreover, it is also seen that this least value is achieved, among all linear time-invariant controllers, regardless of any additional constraints, when the controller is such that the sensitivity equals the inverse of the noise filter, that is, for $S(z, \rho) = \frac{1}{H(z)}$, because then the second term in (2.15) vanishes. This sensitivity is the desired sensitivity regarding this performance criterion, and the corresponding complementary sensitivity plays here a similar role to $T_d(z)$ in the tracking performance criterion, in the sense that it represents the desired closed-loop transfer function from $r(t)$ to $y(t)$. Let us define this desired transfer function:

$$T_e(z) = 1 - \frac{1}{H(z)}.$$

The controller which achieves this desired closed-loop behavior is given by

$$C_e(z) = \frac{T_e(z)}{G(z)(1 - T_e(z))} = \frac{H(z) - 1}{G(z)}. \quad (2.16)$$

This is the “ideal controller” regarding the noise rejection performance $J_e(\rho)$, which is usually called “minimum variance” controller in the literature. This controller may very well result in an internally unstable closed-loop, which happens if the process is non-minimum phase. We shall study this issue, which is not absent in the tracking performance criterion either, in Sect. 2.4. It is also worthy of note the well known fact that if the system noise is white (that is, $H(z) = 1$) then the minimum variance controller is $C_e(z) = 0$ —closing the loop can only worsen the noise rejection performance in this case.

If and only if the ideal controller $C_e(z)$ lies within the class of controllers considered, then the closed-loop system can be made to behave exactly as desired by a proper choice of the parameter ρ . Let us formalize this assumption.

Assumption B_e $C_e(z) \in \mathcal{C}$ or, equivalently,

$$\exists \rho_e : C(z, \rho_e) = C_e(z). \quad (2.17)$$

Just like in the reference tracking case, whether or not this assumption is satisfied is critical in the study of the solution of the corresponding H_2 minimization problem. But Assumption B_e tends to be more restrictive than Assumption B_y, because $T_e(z)$ is given, whereas $T_d(z)$ can be chosen—a consequence of not relaxing the noise rejection performance criterion.

Example 2.2 Consider the first-order system

$$G(z) = \frac{k}{(z-a)}$$

$$H(z) = \frac{z}{(z-b)}$$

with fixed but unknown $a, b \in (0, 1), k \in \mathbb{R}$. The ideal controller regarding noise rejection is given by

$$C_e(z) = \frac{H(z) - 1}{G(z)} = \frac{\frac{z}{(z-b)} - 1}{\frac{k}{(z-a)}}$$

which, after straightforward manipulation, simplifies to the following lead-lag controller

$$C_e(z) = \frac{b(z-a)}{k(z-b)}.$$

2.3.3 The Composite Criterion

If the same controller happened to optimize the two different control objectives $J_y(\rho)$ and $J_e(\rho)$ at once, that is, if $C_d(z) = C_e(z)$, then the ideal controller for the composite performance criterion $J_T(\rho)$ would also be this same controller. This is, however, unlikely to happen in a real application, for a number of reasons; let us mention one of them. In most applications, one wants to track with zero steady-state error a reference with a given frequency, which requires a controller such that the loop gain tends to infinity at this frequency. In other words, the ideal controller in this case must satisfy:

$$C_d(e^{j\omega_1})G(e^{j\omega_1}) \rightarrow \infty \quad (2.18)$$

where ω_1 is the frequency of the reference to be tracked. If the two ideal controllers are the same, that is $C_d(z) = C_e(z)$, then (2.16) with (2.18) would imply that

$$H(e^{j\omega_1}) \rightarrow \infty \quad (2.19)$$

which violates one of the hypotheses on the noise, which is that the filter $H(z)$ is BIBO-stable. But this is not the only factor preventing the two controllers to be the same, as shown in the following example.

Example 2.3 Consider again the process in Example 2.2, with the following reference model

$$T_d(z) = \frac{1-c}{z-c}.$$

Then the ideal reference tracking controller is given by

$$\begin{aligned} C_d(z) &= \frac{\frac{1-c}{z-c}}{\frac{k}{(z-a)} \left(1 - \frac{1-c}{z-c}\right)} \\ &= \frac{\frac{1-c}{z-c}}{\frac{k}{(z-a)} \left(\frac{z-1}{z-c}\right)} \\ &= \frac{(1-c)}{k} \frac{z-a}{z-1} \end{aligned}$$

which is a PI controller. Hence $C_e(z) = C_d(z)$ would require, according to (2.19) that $H(1) \rightarrow \infty$ —the noise filter would have to be an integrator.

The ideal controller regarding noise rejection was calculated in Example 2.2 as

$$C_e(z) = \frac{b}{k} \frac{z-a}{z-b}$$

which equals $C_d(z)$ if and only if $b = 1$ and $c = 0$. That is, equality of the two ideal controllers would also require a specific reference model: $T_d(z) = \frac{1}{z}$.

So, the ideal reference tracking controller and the ideal noise rejection controller are hardly ever the same. This implies that the minimizer of the composite cost $J_T(\rho)$ depends not only on the process and noise characteristics— $G(z)$, $H(z)$ —but also on the reference signal $r(t)$ and on the noise variance σ_e^2 . This can be seen with a simple mind experiment. Assume that, in a particular problem, both ideal controllers belong to the controller class, that is $C_d(z), C_e(z) \in \mathcal{C}$. If the reference's amplitude is zero, then the composite cost equals the noise rejection cost— $J_T(\rho) = J_e(\rho)$ —and the minimum of this criterion is achieved with $C_e(z)$, resulting in $J_T(\rho) = 0$. If, for the same process, the reference's amplitude is not zero, but the noise variance is, then the composite cost is given by the reference tracking alone— $J_T(\rho) = J_y(\rho)$ —and its minimum becomes $C_d(z)$, again resulting in $J_T(\rho) = 0$. In the midway, as the signal to noise ratio becomes larger starting from zero, the optimal controller drifts away from $C_e(z)$, approaching $C_d(z)$ as the signal-to-noise ratio tends to infinity. Moreover, for any of these “midway” optimal controllers the resulting cost will not be zero— $J_T(\rho_*) \neq 0$. As a consequence, it is not possible for the composite performance criterion $J_T(\rho)$ to obtain a closed-form formula for an “ideal controller”, as obtained in (2.9) and (2.16).

2.4 Beware of What You Ask for—Choosing the Reference Model

When applying a model reference design, the user is asking for the algorithm to find $C_d(z)$, which is the controller that provides the desired input-output relationship $\frac{y(z)}{r(z)} = T_d(z)$. In so doing, what the user has specified is “only” the input-output

behavior of the closed-loop system, and the ideal controller, although it matches the desired input-output behavior, does not guarantee internal stability. While searching for his/her ideal controller, the user may find out that his/her ideal is not the best thing to do—maybe even disastrous. Besides, as we will see further ahead in this book, specifying a model reference that is too far from what can be achieved tends to complicate the optimization so that it becomes less likely that the best controller is ever achieved.

What can be done about it? Precautions can be taken in choosing the reference model to avoid such disastrous quest for an “ideal” that does not provide an appropriate behavior for the closed-loop system. This is what this section is about: these aspects are studied, and guidelines are derived for choosing the reference model safely, without waiving the desired performance. All this can be analyzed starting from $C_d(z)$, so we reproduce its expression here for ease of reference

$$C_d(z) = \frac{T_d(z)}{G(z)(1 - T_d(z))}. \quad (2.20)$$

In the choice of a sensible reference model, the first concern is that the ideal controller must be causal. Causality means that the relative degree of $C_d(z)$ is non-negative. Looking at (2.20) it is easy to verify that the relative degree of $C_d(z)$ equals the difference between the relative degree of the process and that of the reference model. Hence, in order to have a causal ideal controller, the reference model must be chosen according to the guideline below.

Guideline 1 (Causality of the controller) *The relative degree of the reference model $T_d(z)$ can not be smaller than the relative degree of the process $G(z)$.*

In order to be able to follow this guideline, it is necessary the *a priori* knowledge of an overbound for the relative degree of the process. This is the first instance found in this book of an obvious—yet sometimes forgotten—principle: it is impossible to properly design a controller for a given process without knowing anything at all about it. We are dealing with design methods that do not require the knowledge of the process’ transfer function, but this does not mean that we do not know anything about the process. A process model is a rather complete and more or less exact description of the process’ behavior for a wide class of excitations, and as such is often difficult and/or expensive to obtain. But the designer is likely to know basic features of the process with very little or no cost, such as an estimate of its static gain, whether or not it is stable and/or stably invertible, etc. So, what data-driven and adaptive design methods propose is not to design a controller completely in the dark, but instead to proceed with only some basic information about the process. Different methods will require different basic information.

Of course, one could just put an arbitrary controller in the loop, set randomly its parameters and hope that it will work; this does not require any knowledge about the process. Every now and then it will work—even a broken watch is correct at least once a day. But if we do not want to rely on luck to do our job, we are bound to use methods that are guaranteed to work well under reasonable circumstances, and

such methods require knowledge of something about the process to be controlled. There does not exist a universal control/adaptation law, that would work properly for all processes in the world regardless of their static and dynamic properties and the performance requirements; searching for such a panacea is reminiscent of the quest for the *moto perpetuo*.

Let us close this brief philosophical parenthesis and go back to (2.20). There, we observe that the ideal controller is obtained through inversion of the process transfer function $G(z)$, so the transfer function of the ideal controller will have the zeros of the process as its poles and vice-versa. If any of the zeros or poles canceled by the ideal controller is outside the unit circle, then the resulting closed-loop with the ideal controller would be internally unstable, even though its input-output properties are the stable ones that were specified through the reference model. It is of course not acceptable to set the control design problem as the quest for a solution that is not even stabilizing. These unstable cancellations must be prevented at all costs and this can be accomplished by appropriately choosing the reference model. If the unstable singularities of the process are also present in the reference model, then they are canceled in the right hand side of (2.20) and do not appear in the ideal controller $C_d(z)$. We arrive thus at a second guideline.

Guideline 2 (Internal stability with non-minimum phase process) *The non-minimum phase zeros of the process $G(z)$ must be included in the reference model $T_d(z)$.*

Enforcing this second guideline requires the knowledge of the locations of the unstable zeros of the process, if any. Limitations in performance imposed by NMP zeros are well known and not including these zeros in the reference model would be fighting against nature. So, a successful model reference design requires the identification of the non-minimum phase zeros of the process. How to do this within the context of data-driven control design, without requiring an explicit previous identification procedure for the NMP zeros, is the subject of later discussion in Chap. 3.

2.4.1 Too Ambitious Performance

If the reference model $T_d(z)$ specifies a performance that is too different from the best that can be achieved within the given controller class, then the solution may have nothing to do with $T_d(z)$. Start with the following example.

Example 2.4 Consider a process described by the following transfer function

$$G(z) = \frac{z}{z - 0.2}$$

controlled by a purely integral controller

$$C(z, \rho) = \frac{\rho}{z - 1}.$$

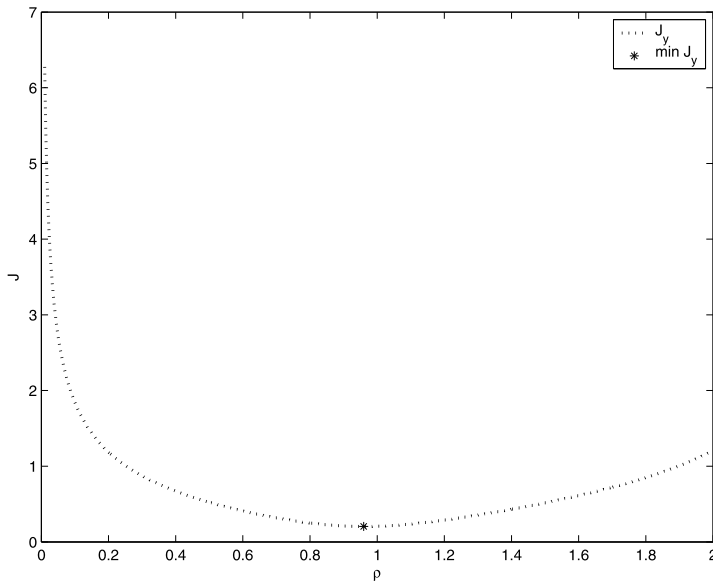


Fig. 2.2 Reference tracking performance criterion as a function of the scalar controller parameter ρ

The performance criterion is reference tracking, with the specification of the deadbeat reference model

$$T_{d1}(z) = \frac{1}{z}.$$

The corresponding ideal controller is given by

$$C_d(z) = \frac{z - 0.2}{z(z - 1)}$$

which does not belong to the controller class considered. Then the minimum of the reference tracking criterion depends on the reference. For a step reference signal, the function $J_y(\rho)$ is as shown in Fig. 2.2, and it is observed that $\rho_* = 0.96$, that is, the optimal controller is $C(z, \rho_*) = \frac{0.96}{z-1}$. The transfer function of the closed-loop system with this controller becomes

$$T_1(z, \rho_*) = \frac{0.96z}{z^2 - 0.24z + 0.2}.$$

Now consider the alternative reference model

$$T_{d2}(z) = \frac{0.3z}{(z - 0.5)(z - 0.4)}$$

for which $C_d(z) = \frac{0.3}{z-1} \in \mathcal{C}$ is the optimal controller.

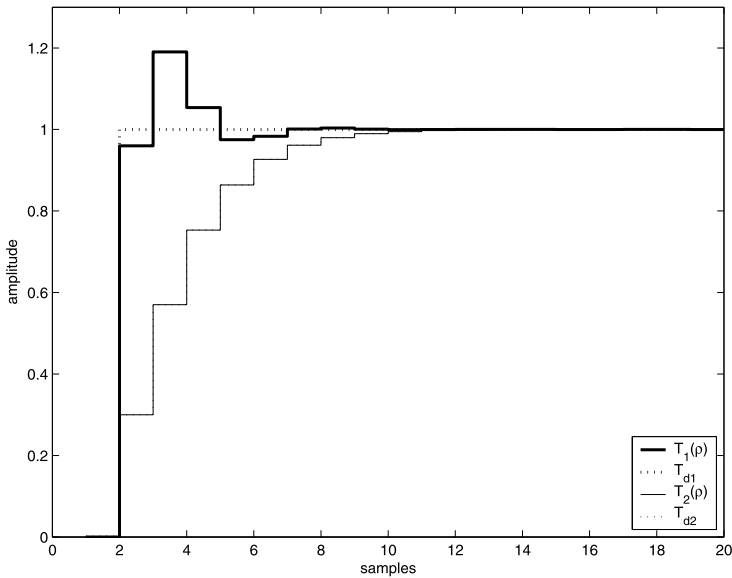


Fig. 2.3 System responses for a step applied at $t = 1$: deadbeat response specified by $T_{d1}(z)$ (dotted line), response achieved minimizing $J_y(\rho)$ with this reference model (thick line), and response specified by the reference model $T_{d2}(z)$, which is achieved exactly by minimizing $J_y(\rho)$ for this reference model

The step responses of the closed-loop system with each one of the optimal controllers found for each reference model are shown in Fig. 2.3. It is observed that the transient performance obtained with the problem formulation using the second reference model $T_{d2}(z)$ is better than the one obtained when the deadbeat reference model $T_{d1}(z)$ was used. This is because the closed-loop system can be made to behave exactly as specified by $T_{d2}(z)$ with the controller structure available, whereas for $T_{d1}(z)$ this is not possible.

To better appreciate the moral behind this example, recall that the original performance criterion that one would like to optimize is

$$\bar{E}[(T(z, \rho) - 1)r(t)]^2$$

which is $J_y(\rho)$ with $T_d(z) = 1$. The primary reason why a different $T_d(z)$ is specified instead of minimizing directly the cost function above is to make it possible to achieve the performance specified. So, it does not make a lot of sense to specify another reference model that is still far from what is achievable; if this were not a concern, we would just keep $T_d(z) = 1$.

Guideline 3 (Realistic ambition) *The reference model should be sufficiently close to what is possible to achieve with the given controller class.*

There is more than just intuition to support this guideline. Analytical results supporting this intuition will be presented in subsequent chapters showing that there exist a good number of properties of data-based methods that can only be proven when Assumption B_y is satisfied. It will also be shown that much can be done to improve these properties in this case, so that it is more likely and easier and safer to obtain optimality when this assumption is satisfied, or at least moderately violated—that is, when $\|K(z)\|$ is small.

2.5 Chapter Conclusions

This chapter presented a framework for the study of H_2 performance criteria, and all that has been said is inherent to this problem formulation. These properties apply to whatever control design based on this formulation. This includes data-driven control design methods, MRAC, and non-adaptive methods as well.

A central concept in this framework is the one of the ideal controller for each performance criterion—reference tracking and noise rejection. And central conditions are Assumptions B_y and B_e ; whether or not these conditions are violated, and to which extent, is key to the properties of the data-driven design, as will be seen in the chapters to follow. A few principles and guidelines have also been established for the choice of the reference model, which must be followed in the problem setting. Verifying Assumptions B_y and/or B_e , and following these guidelines, are tasks that require some basic knowledge about the behavior of the process being controlled. There is no miracle: there is no need for models in data-driven control design, but the designer can not be completely in the dark; *some* (rudimentary) information about the process must be available.

Given the performance criterion, the design is nothing but an optimization problem, to be solved by some numerical algorithm. The H_2 control design thus consists “only” in choosing the reference model and then solving the optimization problem. Data-driven control design usually consists in performing the optimization based on data collected from the system, without the intermediate step of deriving a process model from these data. Data-driven control design methods rely mainly on iterative optimization procedures, mostly gradient descent algorithms. The quantities required in the optimization procedure are the cost function’s gradient and possibly its Hessian, which are estimated pointwise directly from batches of input-output data collected from the closed-loop system. These details will be extensively analyzed along this book.

An alternative to iterative optimization methods is to approximate the cost function by a quadratic function whose minimum is the same. The function to be minimized being quadratic, no iterative algorithm is required, and the minimum can be found by one single least squares calculation. This alternative is presented in the next chapter.

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