

Chapter 2

Theoretical aspects

When studying the asymptotic stability of the basic flow, it is possible to simplify the problem essentially by reducing the consideration of nonlinear equations of motion to the analysis of linearized equations for disturbances. In this chapter, various aspects of the corresponding theory for the so-called parallel shear flows are expounded beginning from formulation of linear hydrodynamic stability problems in time and space for wavy (modal) disturbances. The classical Gaster's transformation between these two approaches is explained. The dual role of viscosity for flow instability is outlined. Then, relevance of oblique waves in instability is discussed. Finally, such special issues, important in the following chapters, as bi-orthogonality of modes and completeness of their set are introduced.

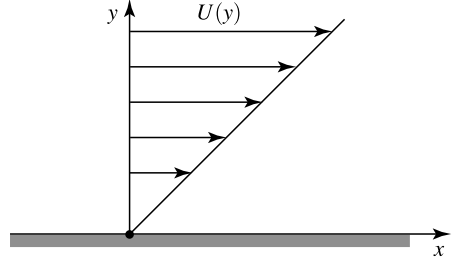
2.1 Formulation of linear hydrodynamic stability problems

Dropping the quadratic terms for perturbations in (1.3a) yields

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{U} \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{U} = -\nabla p + \frac{1}{\text{Re}} \nabla^2 \mathbf{u}. \quad (2.1)$$

These equations, with relevant initial and boundary conditions as well as the continuity equation for disturbances (1.3b), form the *linear stability problem*. A general theoretical study even of these linearized equations is impossible at present time. Therefore, simplified models that reflect real-life phenomena well enough are usually considered.

For this purpose, it is important to choose an appropriate coordinate system, which reflects possible symmetry features of a flow. Frequently, a useful choice is a boundary and streamline-aligned coordinate system. Particularly, when the streamlines are parallel to each other, they should also be parallel to the boundaries, see Fig. 2.1. Then, for plane cases, the Cartesian coordinates x , y , and z ; for axisymmetric cases, the cylindrical coordinates r , θ , and z ; etc. are appropriate. Typical representatives of these so-called parallel shear flows are the Poiseuille flows (e.g.,

Fig. 2.1 A parallel shear flow

flows in a plane channel, round or elliptic pipe, rectangular duct, etc.) as well as some boundary layers (e.g., asymptotic suction profile over a permeable flat plate). Meanwhile, many other flows can be treated in the framework of the linear stability theory (after a proper justification) as locally parallel, in which case the streamline nonparallelism is neglected.

Let us begin with the plane case. The velocity vectors $\mathbf{U}$ and $\mathbf{u}$ in component-wise form in the Cartesian coordinates are

$$\mathbf{U} = (U(y), V = 0, W = 0)^T, \quad (2.2a)$$

$$\mathbf{u} = (u(x, y, z, t), v(x, y, z, t), w(x, y, z, t))^T, \quad (2.2b)$$

where x is the streamwise coordinate, y is the wall-normal coordinate, z is the transverse coordinate normal to x and y , and the superscript ‘ T ’ denotes transpose of the vectors.

Taking into account (2.2), the equations of motion (2.1) and the continuity equation can be written as

$$\frac{\partial u}{\partial t} + U \frac{\partial u}{\partial x} + v \frac{\partial U}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\nabla^2 u}{\text{Re}}, \quad (2.3a)$$

$$\frac{\partial v}{\partial t} + U \frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \frac{\nabla^2 v}{\text{Re}}, \quad (2.3b)$$

$$\frac{\partial w}{\partial t} + U \frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \frac{\nabla^2 w}{\text{Re}}, \quad (2.3c)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (2.3d)$$

At the assumptions made above, these equations together with corresponding boundary and initial conditions describe completely the evolution of an arbitrary infinitesimal disturbance of a given basic flow in space and time. At solid surfaces, which we assume below, the no-slip boundary conditions (vanishing of disturbances) are imposed.

Note that there are six boundary conditions for velocity to close the problem, while the pressure at the boundaries is not specified. Instead, the conditions for the pressure are given implicitly through the continuity equation. Thus, the pressure can be eliminated yielding

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{\nabla^2}{\text{Re}} \right) \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \frac{\partial U}{\partial y} \frac{\partial w}{\partial x} = 0, \quad (2.4a)$$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{\nabla^2}{\text{Re}} \right) \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \frac{\partial U}{\partial y} \frac{\partial v}{\partial z} = 0, \quad (2.4b)$$

$$\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} - \frac{\nabla^2}{\text{Re}} \right) \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) + \frac{\partial U}{\partial y} \frac{\partial w}{\partial z} - \frac{\partial^2 U}{\partial y^2} v = 0, \quad (2.4c)$$

where the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

The equation for pressure can be also obtained by taking the divergence of the momentum equations, summing them up, and eliminating most of the terms with the help of the continuity equation that yields the following Poisson equation:

$$\nabla^2 p = -2 \frac{\partial U}{\partial y} \frac{\partial v}{\partial x}.$$

Further successive elimination of variables makes it possible to form an equation for the single wall-normal velocity:

$$\left[\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x} \right) \nabla^2 - \frac{\partial^2 U}{\partial y^2} \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^4 \right] v = 0. \quad (2.5)$$

This equation, together with equation (2.4a) and the continuity equation (2.3), describes completely the disturbance flowfield.

The classical approach to solving such stability problems is the method of normal modes. Let us assume that the solution is separable, i.e., it can be expressed as a sum of elementary solutions (*normal modes*)¹, which have the form of plane waves

$$(u, v, w, p)^T = (\hat{u}(y), \hat{v}(y), \hat{w}(y), \hat{p}(y))^T e^{i(\kappa \cdot \mathbf{r} - \omega t)}, \quad (2.6)$$

where $\hat{u}$, $\hat{v}$, $\hat{w}$, $\hat{p}$ are the complex amplitude functions of the waves; κ is the *wavevector*, so that its magnitude $|\kappa| = \kappa = \sqrt{\alpha^2 + \beta^2}$, where α and β are the wavenumbers in the streamwise and transverse directions x and z , respectively; ω is the circular frequency (which is generally complex in case of instability in time). The absolute value of an amplitude function of the wave can be considered as its ‘shape’ or amplitude profile along the coordinate y , whereas the phase ψ , which also depends on y , denotes time shifts of the oscillations, so that, e.g., $\hat{u}(y) = |\hat{u}(y)| e^{i\psi y}$.

Some further kinematic definitions are useful. The quantity ω_i is the *growth rate* of the wave. The angle between the wavevector κ and the streamwise axis x is $\gamma = \arctan(\beta/\alpha)$. If $\gamma = 0$, the normal mode and the corresponding plane wave are two-dimensional (independent of z); otherwise the normal mode is three-dimensional, i.e., the wave is *oblique*. The complex vector of the *phase velocity* (the velocity of the wave crest in the direction of the wavevector) is determined by the expression

¹ Hence, in such reducing the initial-boundary-value problem, it is necessary to check the completeness of the obtained set of the modes (see Sect. 2.6 for further details).

$$\mathbf{c} = \mathbf{c}_r + i\mathbf{c}_i = \frac{\kappa}{\varkappa} \frac{\omega}{\varkappa}, \quad (2.7)$$

where the unit vector $\kappa/\varkappa$ determines the direction of the vector $\mathbf{c}$ and the scalar values of the real and imaginary parts of $\omega/\varkappa$ determine the real phase velocity and the reduced growth rate, respectively. Hence, the projections of the phase velocity vector to the x and z axes are

$$c_x = \frac{\alpha\omega}{\varkappa^2}, \quad c_z = \frac{\beta\omega}{\varkappa^2}.$$

The decomposition (2.6) is equivalent to the transformation from the physical space of the Cartesian coordinates and time to the Fourier–Laplace spectral space of wavenumbers and frequencies. In certain configurations, the problem can be further simplified to consider separately symmetrical or antisymmetrical modes if the flow has a symmetry in the wall-normal direction as, e.g., in the plane channel flows.

Solving the corresponding set of homogeneous equations with homogeneous boundary conditions constitutes an *eigenvalue problem*, that means finding such values of α , β , and ω (with the Reynolds number Re and flow velocity U as basic parameters) at which there is a nontrivial solution $\hat{u}$, $\hat{v}$, $\hat{w}$, and $\hat{p}$. Such an implicit relation between α , β and ω can be characterized by the so-called characteristic (*dispersion*) relation written in general form as

$$\mathcal{F}(\alpha, \beta, \omega) = 0. \quad (2.8)$$

The elementary solutions (the normal modes) are the *eigenfunctions* of the corresponding spectral problem. From the viewpoint of physics, they can be interpreted as free hydrodynamic waves inside the shear flow appearing as a ‘hydrodynamic resonator.’

Substituting (2.6) in (2.5) yields the famous *Orr–Sommerfeld equation*

$$\left[i(\alpha U - \omega)(\mathcal{D}^2 - \varkappa^2) - i\alpha U_{yy} - \frac{1}{Re}(\mathcal{D}^2 - \varkappa^2)^2 \right] \hat{v} = 0, \quad (2.9)$$

where the symbol $\mathcal{D}$ and subscript ‘y’ denote the first derivative with respect to y . The Orr–Sommerfeld equation at homogeneous boundary conditions² $\hat{v} = \hat{v}_y = 0$ plays the role of the dispersion relation that constitutes the eigenvalue problem for the wall-normal velocity of the disturbance $\hat{v}$. Finally, by solving the Orr–Sommerfeld equation, we can make a step back and find, e.g., the transverse velocity $\hat{w}$ from

$$\left[i(\alpha U - \omega) - \frac{1}{Re}(\mathcal{D}^2 - \varkappa^2) \right] (\mathcal{D}\hat{w} - i\beta\hat{v}) + i\alpha U_y \hat{w} = 0, \quad (2.10)$$

and then the streamwise velocity $\hat{u}$ from the continuity equation

$$i\alpha\hat{u} + \mathcal{D}\hat{v} + i\beta\hat{w} = 0. \quad (2.11)$$

Taking into account that we do not need to integrate to find $\hat{u}$, we can conclude that only two velocity components are required to represent the disturbance flowfield

² The condition $\hat{v}_y = 0$ following from the continuity equation is introduced to make the problem complete.

completely. For this purpose, instead of $\hat{w}$ or $\hat{u}$, sometimes their combination in the form of the normal vorticity

$$\tilde{\eta} = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

or

$$\hat{\eta} = i(\beta \hat{u} - \alpha \hat{w})$$

is used. Comparing terms in (2.4b) with these definitions, we find that

$$\left[i(\alpha U - \omega) - \frac{1}{\text{Re}} (\mathcal{D}^2 - \kappa^2) \right] \hat{\eta} = -i\beta U_y \hat{v}. \quad (2.12)$$

This equation is known as the normal vorticity or the *Squire equation*.

2.1.1 Spectral formulation of stability

Let us assume that an elementary disturbance (a mode) develops in time, i.e., α and β are real, and $\omega = \omega_r + i\omega_i$ is complex. If there are complex eigenvalues $\omega = \omega_r + i\omega_i$ of the characteristic equation (2.8) such that $\omega_i > 0$ at certain real α and β , the basic state is (linearly) unstable. If $\omega_i < 0$ for all α and β , it is stable to small disturbances. In the case when $\omega_i = 0$ at certain α and β , it is said that the system is *neutrally (indefinitely) stable*. The neutral disturbance can be of two types: if $\omega_r = 0$, it is stationary; if $\omega_r \neq 0$, it is periodic in time and represents a traveling wave.

The value of Re_L minimized over all frequencies at which the instability occurs is called the (first) critical Reynolds number of the spectral problem. If Re_L is finite, the equation $\omega_i(\alpha, \beta, \text{Re}) = 0$ parametrically determines a surface or several surfaces in space $(\alpha, \beta, \text{Re})$ separating the region where the flow is stable from an instability region; it is called a boundary or *surface of neutral stability* (Fig. 2.2). As the neutral stability surface and the critical Reynolds number bound the parameter regions where the basic laminar flow is stable or unstable to small disturbances, their determination is the primary problem of the stability theory. Certainly, it is possible to formulate other problems, for example, to find the parameters of the dispersion relation at which the growth rate is maximal, etc.

Note that $\hat{v}$ plays the role of a forcing term in both (2.10) and (2.12). The homogeneous parts of these equations (i.e., assuming $\hat{v} = 0$) together with the corresponding homogeneous boundary conditions constitute eigenvalue problems which should be considered separately. For example, the solution of the homogeneous equation (2.12) gives a set of eigenvalues and eigenfunctions of the normal vorticity waves for the flow of interest. However, these modes are always stable. Multiplication of the homogeneous normal vorticity equation by its complex conjugate solution $\hat{\eta}^*$, integration over y , and subsequent separation of imaginary and real parts yield

$$\int_{y_1}^{y_2} \left[|\mathcal{D}\hat{\eta}|^2 + (\kappa^2 + \omega_i \text{Re}) |\hat{\eta}|^2 \right] dy = 0,$$

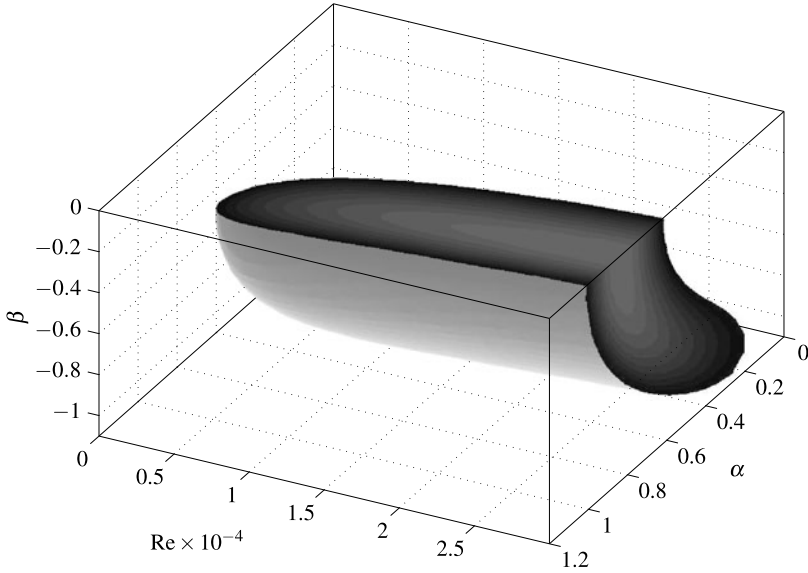


Fig. 2.2 Neutral stability surface with isocaps, which show cross-sectional views of the interior of the isosurfaces $\omega_i > 0$, for the plane Poiseuille flow (calculated by A. V. Boiko)

$$\int_{y_1}^{y_2} \left(U - \frac{\omega_r}{\alpha} \right) |\hat{\eta}|^2 dy = 0.$$

The first of these equations can be satisfied only if the waves decay, i.e., $\omega_i < 0$. From the second equation, it follows directly that the propagation velocity of a vorticity wave lies between the minimum and maximum velocities of the basic flow.

Therefore, it is possible to study the asymptotic linear stability of a parallel flow by considering only the Orr–Sommerfeld equation³. Elementary solutions of the eigenvalue problem for the Orr–Sommerfeld equation with $\omega \neq 0$, i.e., traveling normal modes of the shear layer, are usually called *Tollmien–Schlichting waves* in honor of the pioneers in the boundary-layer stability W. Tollmien and H. Schlichting.

2.1.2 Inviscid instability mechanism

Difficulties with the general theoretical analysis of the Orr–Sommerfeld equation associated with the presence of the small parameter $(\alpha \text{Re})^{-1}$ at the highest deriva-

³ The streamwise and transverse velocities of the modes, however, are still significant in considering the initial-value problem of disturbance propagation.

tive can be reduced by considering the linear stability of a parallel shear flow at high Reynolds numbers. Then it seems natural to drop the viscous terms in (2.9). By putting the transverse wavenumber $\beta = 0$, we obtain the following Rayleigh equation for two-dimensional waves

$$\mathcal{D}^2 \hat{v} - \alpha^2 \hat{v} - \frac{U_{yy}}{U - c} \hat{v} = 0, \quad (2.13)$$

where $c = \omega/\alpha$ is the value of phase velocity, cf. (2.7). Since it is an ordinary differential equation of the second order, it is possible to satisfy only two of the four boundary conditions of the viscous problem—keeping those of zero wall-normal velocity at the boundaries and omitting the condition $\hat{v}_y = 0$. As practice shows, such simplification at high Reynolds numbers is justified, in particular, for free or separated shear flows when solid boundaries are absent or far from the shear layer of interest. At low Reynolds numbers, however, the influence of the boundaries is essential and the role of viscosity cannot be neglected, see, e.g., Chap. 7.

Note first of all that all coefficients in the Rayleigh equation are real; hence, its eigenvalues either are real (in which case the waves are neutrally stable) or occur in complex conjugate pairs. The solutions with $c_i > 0$ correspond to the instability waves, and, hence, are called Rayleigh or inviscid instability waves.

Considering the inviscid stability problem in time for two-dimensional waves, Lord Rayleigh (1880) proved some important general theorems. The first one (or the *inflection point criterion*) states that the necessary condition for inviscid instability is the presence of an inflection point in the basic flow velocity distribution $U(y)$. The *inflection point* is a point y_s such that $U_{yy}(y_s) = 0$. Later Tollmien (1935) proved that the condition is also sufficient for symmetric velocity distributions in channels and velocity distributions of the boundary-layer type.

The *second Rayleigh theorem* states that the velocity of propagation of neutral disturbances (i.e., when $c_i = 0$) in a shear layer is between the minimum $U_{\min}$ and maximum $U_{\max}$ flow velocities, i.e., $U_{\min} < c_r < U_{\max}$. The slice of the basic shear flow where the local velocity is equal to the phase velocity of the wave, i.e., $U(y_c) = c_r$, is called the *critical layer*, as it corresponds to a singularity in the Rayleigh equation (2.13) for the neutral disturbances, unless $y_s = y_c$. Hence, to get rid of the singularity in a general case, viscosity has to be taken into account in the critical layer.

Based on the inflection point criterion, nothing can be said about, e.g., the linear stability of a plane Couette flow with $U(y) = y$, where $U_{yy} = 0$ for all y . An additional necessary condition for instability of monotonous basic flow velocity profiles, the *criterion of the maximum vorticity* (i.e., the gradient U_y), was found later by Fjørtoft (1950), see Fig. 2.3.

Howard (1961) proved the so-called *semicircle theorem* for the bound of the growth rates for unstable Rayleigh waves. It appears that

$$\left[c_r - \frac{1}{2}(U_{\max} + U_{\min}) \right]^2 + c_i^2 \leq \left[\frac{1}{2}(U_{\max} - U_{\min}) \right]^2.$$

Hence, in the phase-velocity plane, all unstable modes lie inside a semicircle.

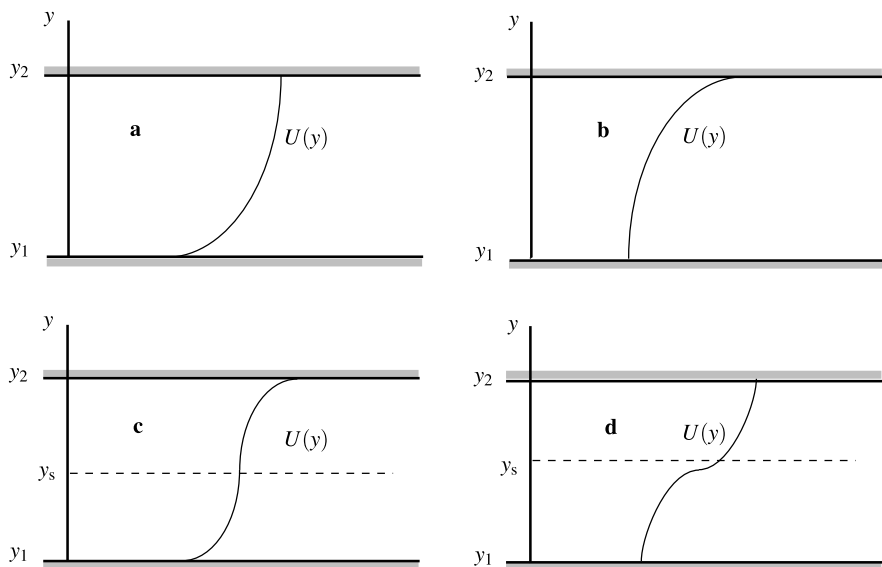
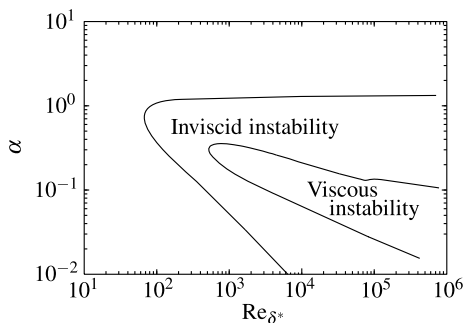


Fig. 2.3 Stability criteria: stable, $U_{yy} < 0$ (a); stable, $U_{yy} > 0$ (b); stable, $U_{s,yy} = 0$ in y_s , but $U_{yy}(U - U_s) \geq 0$ (c); probably unstable, $U_{s,yy} = 0$ and $U_{yy}(U - U_s) \leq 0$ (d)

2.1.3 Viscous instability mechanism

The equation (2.9) that was later called the Orr–Sommerfeld equation was derived independently by Orr (1907a,b) and Sommerfeld (1908). One of the first attempts to solve it belongs to Prandtl (1921, 1922), who studied stability of an idealized flow with a piecewise linear velocity distribution and came to an unexpected conclusion that viscosity can destabilize the flow. Now we know that many shear flows with velocity profiles without inflection are linearly unstable. Such instability is called *viscous instability*. Obviously, it can manifest itself only at finite Reynolds numbers. For example, Fig. 2.4 illustrates that the asymptotic behavior of the neutral

Fig. 2.4 Typical viscous (for Blasius boundary-layer profile) and inviscid (for pre-separation Hartree boundary-layer profile) neutral stability curves which are the slices of the neutral stability surfaces at $\beta = 0$ (calculated by A. V. Boiko)



stability curves for a boundary layer without the inflection point in the velocity profile interior differs from that with an inflection point, as Re approaches infinity. In the former case, it tends to zero; in the latter case, it becomes a constant value.

The physical mechanism of viscous instability can be illustrated by the following two-dimensional considerations. In a case of a two-dimensional normal mode, Eq. (1.4) can be reduced to the form

$$\frac{dE}{dt} = \int \tau_{12} U_y dy - \frac{1}{Re} \iint \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)^2 dx dy ,$$

where $\tau_{12} = (\hat{v}\hat{u}^* - \hat{u}\hat{v}^*)/2$ is the Reynolds stress averaged in the streamwise direction x over one wavelength of the traveling wave. As can be seen, the disturbance kinetic energy increases only if the phase shift between $\hat{u}$ and $\hat{v}$ changes the sign of τ_{12} . The Orr–Sommerfeld equation shows that viscosity (as a part of the imaginary multiplier containing Re at the higher derivatives) introduces a phase shift compared to the inviscid case. The phases of $\hat{u}$ and $\hat{v}$ depending on the wall-normal distance are different, since the viscous terms for these velocities are not the same and $\hat{v}$ should satisfy the additional *viscous boundary condition* $\hat{v}_y = 0$. At a given velocity distribution $U(y)$, certain scalar parameters ω , α , β , and some finite values of Re , this effect can appear to be destabilizing and exceed the effect of energy dissipation due to viscosity.

2.2 Instability in space

In the framework of the eigenvalue approach, there is no means to guess a priori, which formulation—spatial or temporal—corresponds to a particular problem of interest in a given reference system. To identify how the disturbance develops, it is necessary to consider the impulse response of the system as illustrated in Fig. 1.2. This problem can be attacked analytically or numerically by considering the wave-packet group velocity (see Chap. 8).

Meanwhile, it is known from laboratory experiments that small-amplitude disturbances do grow downstream in many open near-wall shear layers (e.g., in attached boundary layers, channel flows, etc.) at least at the Reynolds numbers which are important in practice. Hence, we may assume even without a rigorous (and quite delicate) theoretical justification that it is more appropriate for such basic flows to consider the stability of waves with real frequencies ω and complex wavenumbers α and β . However, spatial instability is a process that differs significantly from instability in time. The initial-value problem for the wall-normal velocity disturbance as $v(x, y, z, t) = \tilde{v}(y, t)e^{i(\alpha x + \beta z)}$ obtained with the help of (2.3) is given by the parabolic partial differential equation

$$\left[\left(\frac{\partial}{\partial t} + i\alpha U \right) (D^2 - \kappa^2) - i\alpha U_{yy} \frac{1}{Re} (D^2 - \kappa^2)^2 \right] \tilde{v}(y, t) = 0 .$$

The spatial evolution problem for $v(x, y, z, t) = \bar{v}(x, y, z)e^{-i\omega t}$, on the contrary, is described by the elliptic equation

$$\left[\left(-i\omega + U \frac{\partial}{\partial x} \right) \nabla^2 - U_{yy} \frac{\partial}{\partial x} - \frac{1}{\text{Re}} \nabla^4 \right] \bar{v}(x, y, z) = 0.$$

The corresponding boundary-value problem requires the conditions at the downstream boundary (e.g., at the trailing edge of a flat plate or channel exit), i.e., there exist disturbances that propagate upstream. It is known from experimental and numerical studies, however, that such disturbances in parallel flows decay and can be neglected. Hence, one can parabolize the problem by imposing additional constraints to exclude all upstream-propagating solutions.

In mathematical studies of convective instability, the Reynolds number Re , the disturbance frequency ω , and one of the wavenumbers, usually β , are given, while α is obtained by solving the characteristic equation (2.8), which parametrically determines the eigenvalues of the spatial problem. For simple two-dimensional shear layers, the flow homogeneity with respect to the z coordinate ensures that the transverse wavenumbers β are real.

As in the case of instability in time, it is possible to introduce a real wavevector κ_r such that its magnitude and the angle between the direction κ_r and the axis x are

$$\kappa_r = \sqrt{\alpha_r^2 + \beta_r^2} \quad \text{and} \quad \gamma_r = \arctan \left(\frac{\beta_r}{\alpha_r} \right),$$

respectively, so that $\kappa_r = \kappa_r \exp(i\gamma_r)$. The phase velocity is then expressed by the formula

$$\mathbf{c} = \frac{\kappa_r}{\kappa_r} \frac{\omega}{\kappa_r}.$$

However, in contrast to the temporal problem, it is also possible to define the vector $\kappa_i = \kappa_i \exp(i\gamma_i)$, where

$$\kappa_i = \sqrt{\alpha_i^2 + \beta_i^2} \quad \text{and} \quad \gamma_i = \arctan \left(\frac{\beta_i}{\alpha_i} \right).$$

If the direction along the vector κ_i is denoted as $\mathbf{r}_i$ and the direction along the vector κ_r as $\mathbf{r}_r$, then the expression for the wave reads

$$\bar{v} = \hat{v}(y) \exp(-\kappa_i \cdot \mathbf{r}_i) \exp[i(\kappa_r \cdot \mathbf{r}_r - \omega t)].$$

Hence, $-\kappa_i$ is the spatial amplification rate along the direction $\mathbf{r}_i$. In particular, if $\beta_i = 0$ and $\beta_r \neq 0$, the wave propagates at an angle $\gamma_r \neq 0$, but amplifies strictly downstream, because $\gamma_i = 0$.

Assuming that β is real, the condition $\alpha_i = 0$ provides the neutral stability curves $\omega = \omega(\beta, \text{Re})$ for different β (Fig. 2.4) constituting together the neutral stability surface separating the regions of flow stability at $\alpha_i > 0$ and flow instability at $\alpha_i < 0$.

Due to differences in the mathematics of linear equations for eigenvalue problems, some of the general theorems of the inviscid temporal problem related to the real wavenumbers and the complex frequency given in Sect. 2.1.2 are not applicable to the problem of spatial amplification.

2.3 Gaster's transformation

Compared to the temporal instability case, the spatial instability problem is complicated as α enters the Orr–Sommerfeld equation nonlinearly which leads to a non-linear (polynomial) eigenvalue problem. However, Gaster's transformation (Gaster, 1962) outlined below provides a useful approximate relation between the temporal and spatial growth rates for two-dimensional waves. This makes it possible to estimate the spatial growth rates from the temporal ones with minimized efforts.

Let us consider the dispersion relation in the form

$$\mathcal{F}(\alpha, \omega) = 0.$$

Its total differential is

$$d\mathcal{F} = \frac{\partial \mathcal{F}}{\partial \alpha} d\alpha + \frac{\partial \mathcal{F}}{\partial \omega} d\omega + \frac{\partial \mathcal{F}}{\partial \text{Re}} d\text{Re} = 0.$$

By denoting the neutral stability curve parameters as α_0 , ω_0 , and Re_0 , we obtain

$$\begin{aligned} d\omega|_{\alpha_0} &= -d\text{Re} \frac{\partial \mathcal{F}}{\partial \text{Re}} / \frac{\partial \mathcal{F}}{\partial \omega}, \\ d\alpha|_{\omega_0} &= -d\text{Re} \frac{\partial \mathcal{F}}{\partial \text{Re}} / \frac{\partial \mathcal{F}}{\partial \alpha}, \\ d\omega|_{\text{Re}_0} &= -d\alpha \frac{\partial \mathcal{F}}{\partial \alpha} / \frac{\partial \mathcal{F}}{\partial \omega} = \frac{\partial \omega}{\partial \alpha} \Big|_{\text{Re}_0} d\alpha. \end{aligned}$$

It follows that

$$d\omega|_{\alpha_0} = d\alpha|_{\omega_0} \frac{\partial \mathcal{F} / \partial \alpha}{\partial \mathcal{F} / \partial \omega} = - \left(\frac{\partial \omega}{\partial \alpha} \Big|_{\text{Re}_0} \right) d\alpha|_{\omega_0} = -c_g d\alpha|_{\omega_0},$$

where c_g is called the group velocity of disturbances, which is a complex number. To perform the desired spatial-to-temporal growth rate transformation, we put $\Delta\alpha = -i\alpha_i$ and obtain the relation

$$\Delta\omega = -ic_g \alpha_i.$$

At the opposite transformation, we put $\Delta\omega = i\omega$, which yields

$$\Delta\alpha = i\omega/c_g.$$

The expression for c_g can be found, e.g., by finite differences, performing two numerical calculations for close values of α . A more rigorous approach described in Sect. 8.2, however, utilizes adjoint functions, which are considered in Sect. 2.5 and are important in some other aspects of linear stability.

2.4 Squire theorem

When $\beta = 0$, the Orr–Sommerfeld equation (2.9) reduces to describe only two-dimensional waves. Written in the form

$$\left[\left(U - \frac{\omega_{2D}}{\alpha_{2D}} \right) (\mathcal{D}^2 - \alpha_{2D}^2) - U_{yy} + \frac{i}{\alpha_{2D} \text{Re}_{2D}} (\mathcal{D}^2 - \alpha_{2D}^2)^2 \right] \hat{v} = 0 ,$$

it clearly shows equivalence to (2.9) written in the form

$$\left[\left(U - \frac{\omega}{\alpha} \right) (\mathcal{D}^2 - \kappa^2) - U_{yy} + \frac{i}{\alpha \text{Re}} (\mathcal{D}^2 - \kappa^2)^2 \right] \hat{v} = 0$$

if to put

$$\begin{aligned} \frac{\omega_{2D}}{\alpha_{2D}} &= \frac{\omega}{\alpha} , \\ \alpha_{2D}^2 &= \kappa^2 , \\ \text{Re}_{2D} &= \text{Re} \frac{\alpha}{\alpha_{2D}} = \text{Re} \frac{\alpha}{\kappa} . \end{aligned}$$

Hence, in the case of the real wavenumbers α and β , the three-dimensional modes have effective Reynolds numbers that are smaller compared to the corresponding two-dimensional waves. Accordingly, if a two-dimensional parallel shear flow loses linear stability at a certain Reynolds number to a two-dimensional normal mode with real streamwise wavenumber α_{2D} , then it loses linear stability to a three-dimensional mode with the same wavevector magnitude κ at higher Reynolds number; see, e.g., Fig. 2.2. Hence, to study linear stability of a parallel flow in time, it is sufficient to confine attention to the two-dimensional waves. This statement constitutes the *Squire theorem*. In other words, to find Re_L , it is possible to consider only two-dimensional waves and to build the *neutral stability curve* formed by values $\omega_i(\alpha, \beta = 0, \text{Re}) = 0$ instead of the neutral stability surface as shown in Fig. 2.4.

Note that the result of the Squire theorem with respect to the critical Reynolds number Re_L is independent of either temporal or spatial stability of the basic flow considered, because both α_i and ω_i for neutral disturbances are zero by definition. However, cases when the oblique waves amplify faster than their two-dimensional counterparts are not excluded: under certain conditions they can be the main source for the transition to turbulence. For example, situations in boundary layers, considered in Chap. 4, with the downstream-growing Reynolds number are possible where the two-dimensional wave has already left the instability region, but some oblique waves still amplify.

Finally, for solving the initial-value problem, e.g., in studying the development of wave packets and transient disturbances (see Chaps. 8 and 9), the solution of the linear problem is only a starting point, since it is necessary to know not only the eigenvalues but also the eigenfunctions $\hat{u} \hat{v}$, and $\hat{w}$, which are different for the cases of two-dimensional and oblique waves. Then, to describe the temporal development of an arbitrary initial disturbance under homogeneous boundary conditions, it is necessary to know a complete set of the waves constituting it. Moreover, in this case, the attempt to reduce the initial-value problem *only* to searching for the eigenvalues and eigenfunctions $\hat{v}$ is not physically justified, as the Orr–Sommerfeld operator is not self-adjoint, and its eigenfunctions are not orthogonal.

2.5 Adjoint problem and bi-orthogonality of normal modes

Let us consider the homogeneous linearized Navier–Stokes momentum equations and the continuity equation describing the wave dynamics in a shear flow (2.3) written in the form:

$$\begin{aligned}\frac{\partial \mathbf{u}}{\partial t} + \mathcal{L}\mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0,\end{aligned}$$

with homogeneous boundary conditions, where $\mathcal{L}$ is the linearized Navier–Stokes operator, which is expressed in the Cartesian coordinates $x \equiv x_1, y \equiv x_2, z \equiv x_3$ as

$$(\mathcal{L}\mathbf{u})_i = U_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial U_i}{\partial x_j} - \frac{1}{\text{Re}} \frac{\partial^2 u_i}{\partial x_j^2}, \quad i = 1, 2, 3,$$

where for brevity the summation convention is utilized. Using the Lagrange identity, one of the fundamental identities in the theory of ordinary differential equations, we can now define the adjoint linearized Navier–Stokes operator $\mathcal{L}^\dagger$ through the scalar products defined as

$$\mathbf{p} \cdot \mathbf{s} = \int_{y_1}^{y_2} \mathbf{p} \mathbf{s} dy,$$

where the values y_1 and y_2 bound the domain of the vectors $\mathbf{p}$ and $\mathbf{s}$:

$$\begin{pmatrix} \mathbf{u}^\dagger \\ p^\dagger \end{pmatrix} \cdot \left[\begin{pmatrix} \frac{\partial}{\partial t} + \mathcal{L} & \nabla \\ \nabla \cdot & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \right] + \begin{pmatrix} \mathbf{u} \\ p \end{pmatrix} \cdot \left[\begin{pmatrix} \frac{\partial}{\partial t} + \mathcal{L}^\dagger & \nabla \\ \nabla \cdot & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}^\dagger \\ p^\dagger \end{pmatrix} \right] = \frac{\partial (\mathbf{u}^\dagger \cdot \mathbf{u})}{\partial t} + \nabla \cdot \mathbf{j},$$

where $\mathbf{q} = (\mathbf{u}, p)^T$ and $\mathbf{q}^\dagger = (\mathbf{u}^\dagger, p^\dagger)^T$ are suitably differentiable fields, which do not necessarily satisfy (2.3),

$$\mathcal{L}^\dagger \mathbf{u}^\dagger = U_j \frac{\partial u_i^\dagger}{\partial x_j} - u_j^\dagger \frac{\partial U_i}{\partial x_j} + \frac{1}{\text{Re}} \frac{\partial^2 u_i^\dagger}{\partial x_j^2}, \quad i = 1, 2, 3,$$

(which can be obtained using integration by parts) and the vector $\mathbf{j}$ is the bilinear concomitant depending on both $\mathbf{q}$ and $\mathbf{q}^\dagger$ so that

$$\mathbf{j}_j(\mathbf{q}^\dagger, \mathbf{q}) = \mathcal{S}_{ij}^\dagger u_i + \mathcal{S}_{ij} u_i^\dagger, \quad i, j = 1, 2, 3, \quad (2.14)$$

where

$$\mathcal{S}_{ij} = \left(U_j - \frac{1}{\text{Re}} \frac{\partial}{\partial x_j} \right) u_i + p \delta_{ij}, \quad \mathcal{S}_{ij}^\dagger = \frac{1}{\text{Re}} \frac{\partial u_i^\dagger}{\partial x_j} + p^\dagger \delta_{ij}.$$

The corresponding adjoint equations by definition are related to the part in square brackets of the second term in the Lagrange identity

$$\begin{aligned}\frac{\partial \mathbf{u}^\dagger}{\partial t} + \mathcal{L}^\dagger \mathbf{u}^\dagger + \nabla p^\dagger &= 0, \\ \nabla \cdot \mathbf{u}^\dagger &= 0,\end{aligned}$$

where $\mathbf{u}^\dagger = (u^\dagger, v^\dagger, w^\dagger)^T$, and $p^\dagger$ are the adjoint velocity and pressure, respectively, and homogeneous boundary conditions are assumed to zero the concomitant term.

Next, let us consider the normal modes, whose velocities and pressure are proportional to $\exp[i(\alpha x - \omega t)]$, and their adjoints are proportional to $\exp[-i(\alpha^\dagger x - \omega^\dagger t)]$ and assume that their set is complete to form a Hilbert space. Substituting the corresponding expressions into the Lagrange identity, assuming $\mathbf{U} = (U, 0, 0)^T$, and noticing that the left-hand side is identically zero, we obtain the following bi-orthogonality condition

$$(\alpha - \alpha^\dagger) \mathbf{e}_x \cdot \mathbf{j} - (\omega - \omega^\dagger) \mathbf{u}^\dagger \cdot \mathbf{u} = 0 ,$$

where $\mathbf{e}_x$ is the unit vector codirectional with the x -axis. These are the required bi-orthogonality conditions. When considering, for example, the spatial stability of a shear flow, $\omega = \omega^\dagger$ is given. Hence, the equality is possible either if $\alpha = \alpha^\dagger$, in which case we can put $\mathbf{e}_x \cdot \mathbf{j} = 1$ without loss of generality, or if $\alpha \neq \alpha^\dagger$, in which case $\mathbf{e}_x \cdot \mathbf{j} = 0$.

In a similar way one can obtain the adjoint Orr–Sommerfeld equation for adjoint disturbances $\hat{v}^\dagger$, which takes the form

$$\mathcal{L}_{\text{OS}}^\dagger \hat{v}^\dagger = 0 ,$$

with the boundary conditions $\hat{v}^\dagger = \hat{v}_y^\dagger = 0$. Here

$$\mathcal{L}_{\text{OS}}^\dagger = -i(\alpha U - \omega) (\mathcal{D}^2 - \varkappa^2) - 2i\alpha U_y \mathcal{D} - \frac{1}{\text{Re}} (\mathcal{D}^2 - \varkappa^2)^2 .$$

The orthogonality relation between the modes $\hat{v}_i$ and $\hat{v}_j^\dagger$ can be now obtained directly from the corresponding Lagrange identity:

$$\left(\hat{v}_j^\dagger, \mathcal{L}_{\text{OS}} \hat{v}_i \right) - \left(\mathcal{L}_{\text{OS}}^\dagger \hat{v}_j^\dagger, \hat{v}_i \right) = 0 .$$

Integrating by parts yields the following relation for the modes developing in time⁴:

$$i(\omega_i - \omega_j^\dagger) \int_{y_1}^{y_2} \hat{v}_j^\dagger (\mathcal{D}^2 - \varkappa^2) \hat{v}_i dy = 0 .$$

Hence, finally,

$$\int_{y_1}^{y_2} \hat{v}_j^\dagger (\mathcal{D}^2 - \varkappa^2) \hat{v}_i dy = C \delta_{ij} , \quad (2.15)$$

where δ_{ij} is the Kronecker delta and C is an arbitrary constant.

2.6 Completeness of solutions for the Orr–Sommerfeld and Squire equations

As has already been mentioned in Sect. 2.1, to describe the development of an arbitrary disturbance in a particular flow, it is necessary to know the complete set of the

⁴ Using a similar procedure, one can obtain the orthogonality relation for normal modes developing in space.

waves constituting the disturbance. The completeness theorem, i.e., the existence of the complete set of *discrete* eigenvalues with corresponding eigenfunctions for the linearized Navier–Stokes equations, was proved for the plane Poiseuille flow by [Schensted \(1960\)](#), and later for *any* parallel internal flow by [Yudovich \(1965\)](#) and independently by [DiPrima and Habetler \(1969\)](#). For external parallel flows, the completeness theorem in such a formulation cannot be justified. In that case, to obtain a complete set of eigenfunctions, as it has been shown by [Salwen and Grosch \(1981\)](#) and [Zhigulev and Tumin \(1987\)](#), it is necessary to supplement the set of discrete modes by a *continuous* spectrum: to consider the so-called generalized eigenfunctions, i.e., the eigenfunctions corresponding to the Orr–Sommerfeld and Squire equations, which do not decay at infinity, but rather are only bounded. Physically, these generalized eigenfunctions are required to describe the behavior of disturbances in the free stream (including infinity).

Let us specify some general features of the modes in a semibounded flow. The Orr–Sommerfeld equation has four linearly independent fundamental solutions. It is easy to establish their asymptotic behavior at $y \rightarrow \infty$. First, we present the Orr–Sommerfeld equation (2.9) in the form

$$\left[(\kappa \bar{U} - \omega) (\mathcal{D}^2 - \kappa^2) - \kappa \bar{U}_{yy} + \frac{i}{\text{Re}} (\mathcal{D}^2 - \kappa^2)^2 \right] \hat{v} = 0 ,$$

where $\bar{U} = \alpha / \kappa U$ is the basic flow velocity projection to the direction of the wavevector κ . In this form it is equivalent to the Orr–Sommerfeld equation for two-dimensional waves with the effective basic flow velocity $\bar{U}$ and the streamwise wavenumber κ .

For simplicity, we assume that $\bar{U} \rightarrow 1$ beyond a narrow shear layer at the wall. The Orr–Sommerfeld equation is reduced at $y \rightarrow \infty$ to

$$\left[(\kappa - \omega) (\mathcal{D}^2 - \kappa^2) + \frac{i}{\text{Re}} (\mathcal{D}^2 - \kappa^2)^2 \right] \hat{v} = 0 . \quad (2.16)$$

After the standard substitution $\hat{v} = e^{qy}$, it becomes a biquadratic algebraic equation

$$(\kappa - \omega) (q^2 - \kappa^2) + \frac{i}{\text{Re}} (q^2 - \kappa^2)^2 = 0 .$$

This equation has four roots, so that the general solution of (2.16) is

$$\hat{v} = C_1 \varphi_1 + C_2 \varphi_2 + C_3 \varphi_3 + C_4 \varphi_4 , \quad (2.17)$$

where

$$\varphi_1 = e^{\kappa y}, \quad \varphi_2 = e^{-\kappa y}, \quad \varphi_3 = e^{\varpi y}, \quad \varphi_4 = e^{-\varpi y},$$

and $\varpi = \sqrt{\kappa^2 + i \text{Re}(\kappa - \omega)}$.

Let us first consider the case where κ and ϖ are complex and $\text{Real}(\kappa) < 0$, $\text{Real}(\varpi) < 0$. Then the only allowable bounded solution

$$\hat{v} = C_1 \varphi_1 + C_3 \varphi_3$$

exhibits $\hat{v} = 0$ as $y \rightarrow \infty$. The boundary conditions at $y = 0$ yield

$$\begin{aligned} C_1 \varphi_1(0) + C_3 \varphi_3(0) &= 0, \\ C_1 \varphi_{1,y}(0) + C_3 \varphi_{3,y}(0) &= 0. \end{aligned}$$

For the existence of nonzero C_1 and C_3 , it is necessary that the determinant of the system be zero:

$$\varphi_1(0)\varphi_{3,y}(0) - \varphi_3(0)\varphi_{1,y}(0) = 0. \quad (2.18)$$

Thus, the spectrum of these waves is discrete; i.e., at a fixed ω and Re there is a number of $\varkappa_n$ ($n = 1, 2, 3, \dots$) satisfying (2.18). They can both grow or decay in the direction of wave propagation as well as be neutral. The waves are near-wall waves corresponding to that discussed in Sect. 2.1: their amplitudes approach zero at large y , i.e., they have a maximum (or maxima) near the wall (usually in the shear layer). The requirement of disappearance of the disturbances at infinity guarantees that the flow relaxes to its undisturbed state in the free stream.

It is seen that finite, but nonzero disturbances exist at $y \rightarrow \infty$ when either $\varkappa$ or ϖ is purely imaginary. Let $\varkappa = iq$, $q > 0$ (note that this is possible only when disturbances developing in space are considered). If $\text{Real}(\varpi) < 0$, the solutions limited at $y \rightarrow \infty$ take the form

$$\hat{v} = C_1 \varphi_1 + C_2 \varphi_2 + C_3 \varphi_3.$$

The boundary conditions $\hat{v}(0) = \hat{v}_y(0) = 0$ result in the equations

$$\begin{aligned} C_1 \varphi_1(0) + C_2 \varphi_2(0) + C_3 \varphi_3(0) &= 0, \\ C_1 \varphi_{1,y}(0) + C_2 \varphi_{2,y}(0) + C_3 \varphi_{3,y}(0) &= 0. \end{aligned}$$

Based on them, two constants, e.g., C_2 and C_3 , can be expressed through the third one which is an arbitrary constant. To be specific, let us choose $C_1 = 1$, then

$$\hat{v} = \varphi_1 + C_2 \varphi_2 + C_3 \varphi_3.$$

A particular form of C_2 and C_3 for the other y coordinates is determined by the basic flow velocity distribution through the Orr–Sommerfeld equation and the boundary conditions. The solution exists at any q and, consequently, corresponds to the continuous spectrum. In the case under consideration, we have

$$\lim_{y \rightarrow \infty} \hat{v} = e^{iqy} + C_2 e^{-iqy}.$$

It is possible to show that, as $y \rightarrow \infty$, the wave appears as a disturbance of vorticity equal to zero and a finite disturbance of pressure. Therefore, such waves are called *pressure waves*. Furthermore, these waves are standing, as their phase velocities are purely imaginary. The case $\varkappa = iq$, $q < 0$ can be considered similarly. Depending on the sign at q , the wave amplitudes decrease or grow in the direction of the wavevector κ .

Let now $\varpi = \pm iq$, $q > 0$. Then the equation for the wavenumbers α is

$$\varkappa^2 + i\varkappa \text{Re} + q^2 - i\omega \text{Re} = 0,$$

whence, under the assumption that $\text{Re} \gg 1$,

$$\varkappa_1 \approx \omega + iq^2/\text{Re} \quad \text{and} \quad \varkappa_2 \approx -\omega - i\text{Re}.$$

The first wave has the real part of the phase velocity coinciding with the effective free-stream velocity $\overline{U}$

$$c_r = \text{Real} \left(\frac{\omega}{\varkappa} \right) = 1$$

and is damped weakly with x . The second wave propagates with the same velocity, but upstream ($c_r = -1$), and strongly decays in the direction of its propagation. Their amplitude functions are

$$\hat{v} = C_2 \varphi_2 + C_3 \varphi_3 + C_4 \varphi_4, \quad \hat{v} = C_1 \varphi_1 + C_3 \varphi_3 + C_4 \varphi_4.$$

As in the previous case, two homogeneous boundary conditions at the wall determine two arbitrary constants through the third one. Thus, the considered waves exist at any q and, consequently, also constitute a part of the continuous spectrum. It could be shown that they are *vorticity waves*.

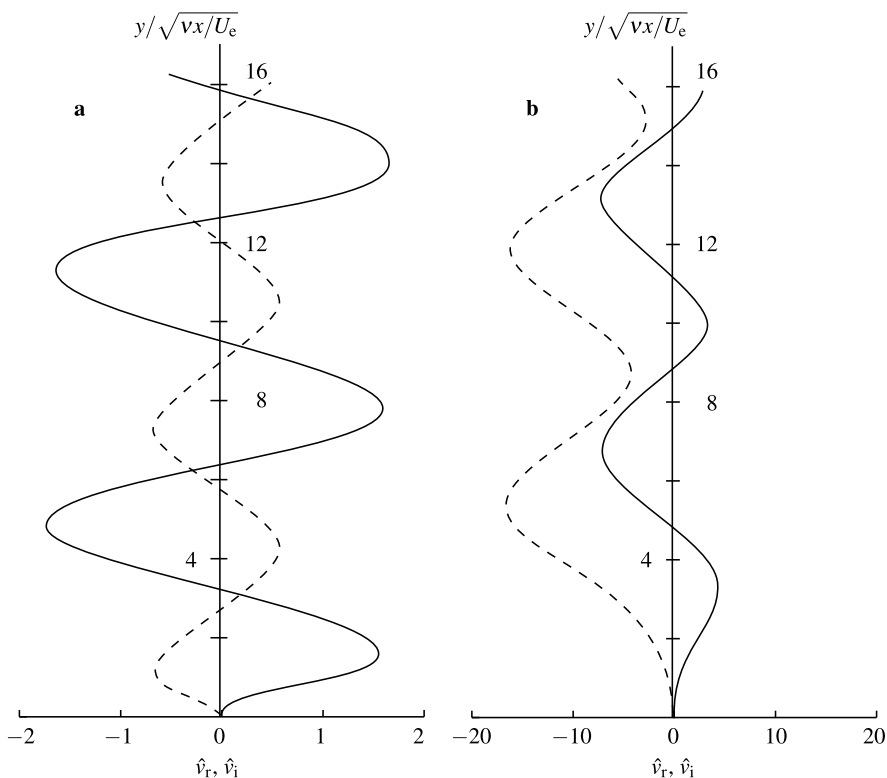


Fig. 2.5 Modes of the continuous spectrum for the Blasius boundary layer: pressure wave (a); vorticity wave (b). $\text{Re}_x = 9 \times 10^4$, $\omega = 0.006$, $\beta = 0$, $q = 1$; (Zhigulev and Tumin, 1987)

The waves of the indicated classes constitute the complete set of solutions of the spatial Orr–Sommerfeld equation. In the temporal stability formulation, pressure

waves are obviously absent, and the continuous spectrum consists only of vorticity waves. Figure 2.5 shows some examples of the real $\hat{v}_r$ and imaginary $\hat{v}_i$ parts of the eigenfunctions for pressure and vorticity waves obtained numerically.

Finally, the analysis of the homogeneous Squire equation shows that it also has a continuous spectrum of vorticity waves, the discrete spectrum being empty. Note that at $\alpha = 0$ the Orr–Sommerfeld equation reduces to

$$(\mathcal{D}^2 - \beta^2 - i\omega\text{Re})(\mathcal{D}^2 - \beta^2)\hat{v} = 0,$$

which also has no discrete solutions.

The observation that parallel flows are always stable to continuous spectrum components must be interpreted with caution. In particular, among these modes, there are weakly decaying low-frequency waves. Though such two-dimensional waves cannot penetrate deeply into the boundary layer, they can produce small quasi-stationary changes of the basic flow velocity distribution close to the shear-layer edge, where U_{yy} is small enough. This results in quasi-stationary inflections in the basic flow velocity profile capable of affecting the flow stability. Moreover, large-amplitude free-stream disturbances and the boundary-layer waves can interact nonlinearly. Additionally, as will be shown in Chap. 9, three-dimensional modes of the continuous spectrum can play a key role in the algebraic instability phenomenon in external flows.

Additional material for this chapter is given in the multimedia file No. 2, which supplements the book, titled ‘Formulation of linear hydrodynamic stability problems.’

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