Chapter 2
Comments on Signals and Systems

Abstract To prepare the mechatronic foundation of active noise control, this chapter contains some elements of system theory. Since we will be talking about active noise control *systems* and *signals*, it is necessary to define these terms. Furthermore, it is necessary to introduce values and functions that can be used to characterize signals and systems in time domain but also in frequency domain. However, it is not intended to present a compact summary of system theory that is in great detail presented in textbooks, such as in (Cadzov and van Landingham in Signals, systems, and transforms, Prentice Hall, New Jersey, 1985), (Fliege in Systemtheorie, Teubner, Stuttgart, 1991), (Girod et al. in Einführung in die Systemtheorie, Teubner, Stuttgart, 2005), (Johnson in Digitale Signalverarbeitung, Hanser, München in Cooperation with Prentice Hall International, London, 1991), (Oppenheim and Willsky in Signale und Systeme—Lehrbuch, VCH Verlagsgesellschaft, Weinheim, 1989), (Sundararajan in A practical approach to signals and systems, Wiley Eastern, Singapore, 2008), (Ziemer et al. in Signals and systems, continuous and discrete, Macmillan, New York, 1983). A description of stochastic signals and random vibrations of both linear and non-linear mechanical systems is to be found in (Lajos in Zufallsschwingungen und ihre Behandlung, Springer, Berlin, 1973), whereas digital audio signal processing is discussed in (Zölzer in Digital audio signal processing, Wiley, Chichester, 2008).

2.1 Comments on Signals

The concept of *signal* is central to the system theory philosophy. According to (Cadzov and van Landingham 1985) a signal connotes the process of conveying information in some format. In the present work, the expression signal is used to denote a measurement or observation that contains information describing some acoustic or electric phenomenon.

To define a mathematical structure, we designate signals by means of symbols such as x or y and refer them as *the signals* x and y. A signal in which the information characteristics fluctuate depends on other variables such as time, distance or frequency. For the purpose of this chapter, however, it is sufficient to use time as an independent variable.

T. Kletschkowski, *Adaptive Feed-Forward Control of Low Frequency Interior Noise*, Intelligent Systems, Control and Automation: Science and Engineering 56,
DOI 10.1007/978-94-007-2537-9_2,

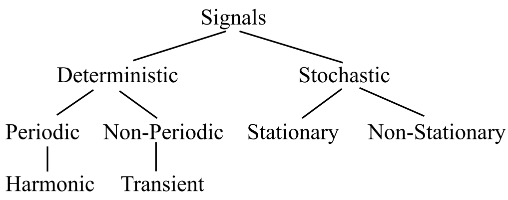

Fig. 2.1 Time-history classification of signals

2.1.1 Classification

A signal that can change at any instant of time is called *continuous-time signal* and will be denoted by $x(t)$ or $y(t)$. On the other hand, if a signal can only change its value at specific instants of time, it is called *discrete-time signal*. Such a signal can be denoted by $x(t_n)$ or $y(t_n)$, where $t_n = nT$ for $n = 0, \pm 1, \pm 2, \ldots$. T is a fixed time interval (e.g. the sample time) and n is the number of the time interval (e.g. the sample number). In the present work a discrete-time signal will be denoted by $x(n)$ or $y(n)$.

As shown in (DIN 1311-1 2000), signals can also be classified according to time history. An overview on time-history classification of signals discussed in this section is given in Fig. 2.1.

A signal is called *deterministic*, if the time history can be described by a function, e.g. $x(t) = f(t)$, that links the time t to the instantaneous value $x(t)$. A deterministic signal with a time history that is repeated after some period T such as

$$x(t) = x(t+T) \quad \forall t \tag{2.1}$$

is a *periodic signal*. Equation (2.1) also holds for $x(t) = x(t+nT)$ $\forall t$ with $n = 0, 1, 2, \ldots$. If a periodic signal can be described by one sine function (or one cosine function) such as

$$x(t) = \hat{x} \sin(\omega t + \varphi_0) \tag{2.2}$$

it is called a *harmonic signal*. Here $\omega = 2\pi/T$ is known as the angular frequency, φ_0 is known as the *zero phase angle*, and $\hat{x} > 0$ is known as the *amplitude*. Deterministic signals with time history

$$x(t) \neq x(t+T) \quad \forall t \tag{2.3}$$

are *non-periodic* signals. An important subclass of non-periodic signals are *transient signals* that describe the crossover from one condition to another one. Typical examples, see (DIN 1311-1 2000), are *quasi-harmonic signals* such as

$$x(t) = \hat{x} e^{-\sigma t} \cos(\omega t + \varphi_0) \quad \text{with decay coefficient } \sigma > 0 \tag{2.4}$$

and *sweep sine* signals such as

$$x(t) = \hat{x} \sin\big[\varphi(t)\big] \quad \text{with } \omega(t) = \frac{d\varphi(t)}{dt}. \tag{2.5}$$

A signal is called *stochastic* or *random*, if the time history cannot be described by a function. This means that predetermination of x is impossible for any point in time t. A stochastic signal can be seen as one realization of a stochastic process ξ that consists of an ensemble of realizations $x_i(t)$. The probability P_w that the realization $x_i(t)$ has a value between a and b at $t = t_j$ is obtained by integrating the probability density function of the stochastic process $f_{x(t)}(\xi)$ over the process such as

$$P_w\{a \leqslant x_i(t) \leqslant b\} = \int_{\xi=a}^{\xi=b} f_{x(t=t_j)}(\xi)d\xi. \tag{2.6}$$

If the probability P_w is independent of the observation time t_j for all values of a and b, the process is *stationary*, otherwise the process is *non-stationary*. Stochastic signals are characterized by their moments or *expectation values*. The most important moments are the *mean*

$$\bar{x} = E[x(t)] = \int_{\xi=-\infty}^{\xi=\infty} \xi \cdot f_x(\xi)d\xi \tag{2.7}$$

and the *variance*

$$\sigma_x^2 = E[x(t) - \bar{x}] = \int_{\xi=-\infty}^{\xi=\infty} (\xi - \bar{x})^2 \cdot f_x(\xi)d\xi. \tag{2.8}$$

If $\bar{x} = 0$ and σ_x^2 are independent of t_j, the signal is a weak (or wide-sense) *stationary* stochastic signal, otherwise the signal is a *non-stationary* stochastic signal. If the stochastic process is an *ergodic process*, the moments defined by (2.7) and (2.8) are identical with time averaged values obtained from analyzing one realization $x_i(t)$ of the stochastic process.

2.1.2 Characteristic Values and Functions

A signal x can be characterized by characteristic values and by characteristic functions. These values and functions will be introduced according to (Ziemer et al. 1983) and (Fliege 1991) but also considering (DIN 1311-1 2000).

Characteristic values that can directly be obtained from the time history of x measured in the time interval $t \in [t_1, t_2]$ are summarized in Table 2.1. In addition to these values other characteristic values are obtained by time averaging of continuous-time and discrete-time signals. Strictly speaking, time averaging of stochastic signals has to be performed on the i-th realization x_i of the stochastic process ξ. However, in what follows the i-th realization is identified with the signal x for convenience.

Continuous-Time Signals

If the continuous-time signal $x(t)$ is observed in the time interval $T = t_2 - t_1$, the *arithmetic mean* of $x(t)$ is given by

Table 2.1 Characteristic values directly obtained from time history of signal

Name and description	Definition
Maximum value: Maximum value of signal for $t \in [t_1, t_2]$.	$x_{\max} := \max(x)$
Minimum value: Minimum value of a signal for $t \in [t_1, t_2]$.	$x_{\min} := \min(x)$
Maximum absolute value: Minimum absolute value of a signal for $t \in [t_1, t_2]$.	$\lvert x_{\max}\rvert := \max(\lvert x(t)\rvert)$
Signal range: Difference between maximum and minimum value.	$x_h := x_{\max} - x_{\min}$

$$\bar{x} = E(x) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)dt. \tag{2.9}$$

The *variance* of $x(t)$ is defined by

$$\sigma_x^2 = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big(x(t) - \bar{x}\big)^2 dt. \tag{2.10}$$

The *mean signal power*[1] of $x(t)$ can be defined by

$$\Pi_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big|x(t)\big|^2 dt. \tag{2.11}$$

It is obvious that the Π_x is given by σ_x^2, if the arithmetic mean is zero, i.e. $\bar{x} = 0$. The RMS[2] value of $x(t)$ is defined as the square root of Π_x

$$x_{RMS} = \sqrt{\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \big|x(t)\big|^2 dt}. \tag{2.12}$$

Furthermore, it is possible to define the *mean signal energy* for $x(t)$ by

$$E_x = \int_{-\infty}^{\infty} \big|x(t)\big|^2 dt. \tag{2.13}$$

The similarity between two continuous-time signals $x(t)$ and $y(t)$ can be characterized by the *cross correlation function* (CCF)

$$r_{xy}(\tau) = \overline{x(t)y(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)y(t+\tau)dt. \tag{2.14}$$

Consequently the *auto correlation function* (ACF) can be understood as a function that describes the change in time history of the signal $x(t)$. The ACF is defined by

[1] In system theory it is assumed that the signal x is normalized and dimensionless, see (Ziemer et al. 1983) and (Fliege 1991). Furthermore, it is assumed that the associated non-normalized signal has the dimension Volt and its electric energy is measured using a 1 Ohm resistance.

[2] The RMS of x is also used to define a logarithmic measure such as $L_x = 10\log_{10}(x_{RMS}^2/x_0^2)$, where x_0 is a normalization factor.

$$r_{xx}(\tau) = \overline{x(t)x(t+\tau)} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t+\tau)dt. \tag{2.15}$$

It is also possible to characterize continuous-time signals in frequency domain. The frequency domain representation of a continuous-time signal is obtained by applying the (continuous) *Fourier transform* which (according to Ziemer et al. 1983 and Fliege 1991) is defined by

$$x(t) \circ\!\!-\!\!\bullet X(j\omega) = \mathfrak{F}\big[x(t)\big] := \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \tag{2.16}$$

Conditions for the existence of the Fourier transform are complicated to state in general (Champeney 1987), but it is sufficient for $x(t)$ to be *absolutely integrable*, i.e.,

$$\int_{-\infty}^{\infty} x(t)dt < \infty. \tag{2.17}$$

The inverse operation is defined by

$$X(j\omega) \bullet\!\!-\!\!\circ x(t) = \mathfrak{F}^{-1}\big[X(j\omega)\big] := \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega. \tag{2.18}$$

Using (2.16) it is possible to introduce the frequency domain counterparts of the CCF and the ACF, known as the *cross spectral density* (CSD)

$$S_{xy}(j\omega) = \mathfrak{F}\big[r_{xy}(\tau)\big] \tag{2.19}$$

and the *auto spectral density* (ASD)

$$S_{xx}(j\omega) = \mathfrak{F}\big[r_{xx}(\tau)\big]. \tag{2.20}$$

For some applications it is advantageous to use the single-sided CSD or the single-sided ASD that are, according to (DIN 1311-1 2000), introduced as

$$G_{xy}(j\omega) = 2S_{xy}(j\omega) \quad \text{with } 0 \le \omega < \infty \tag{2.21}$$

and

$$G_{xx}(j\omega) = 2S_{xx}(j\omega) \quad \text{with } 0 \le \omega < \infty. \tag{2.22}$$

In general the CSD is a complex function of frequency, whereas the ASD is a real valued quantity. The complex nature of the CSD can be seen, if (2.19) is evaluated for two signals $x_T(t)$ and $y_T(t)$ that are observed in the time interval T. Using (2.14) the evaluation of (2.19) results in

$$\begin{aligned} S_{x_T y_T}(j\omega) &= \mathfrak{F}\big[r_{x_T y_T}(\tau)\big] \\ &= \int_{-\infty}^{\infty} \overline{x_T(t)y_T(t+\tau)}e^{-j\omega\tau}d\tau \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{-T/2} x_T(t)e^{j\omega t}\left(\int_{-\infty}^{\infty} y_T(t+\tau)e^{-j\omega(t+\tau)}d\tau\right)dt \end{aligned}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{-T/2} x_T(t) e^{j\omega t} Y_T(j\omega) dt$$
$$= \lim_{T\to\infty} \frac{1}{T} \left[X_T^*(j\omega) Y_T(j\omega) \right]. \tag{2.23}$$

Equation (2.23) also proves that $S_{x_T x_T}(j\omega) = \lim_{T\to\infty} \frac{1}{T} |X_T(j\omega)|^2$ is a real valued function.

Discrete-Time Signals

Discrete-time signals can be characterized similarly to continuous-time signals. However, instead of time integration, summation over all time steps is required for time averaging and the computing of both ACF and CCF.

The *arithmetic mean* of the discrete-time signal $x(n)$ is given by

$$\bar{x} = E(x) = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n). \tag{2.24}$$

The *variance* of $x(n)$ is defined by

$$\sigma_x^2 = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} \left(x(n) - \bar{x} \right)^2. \tag{2.25}$$

The *mean signal power* of $x(n)$ can be defined by

$$\Pi_x = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} \left| x(n) \right|^2. \tag{2.26}$$

The RMS value of $x(n)$ is defined as the square root of Π_x

$$x_{RMS} = \sqrt{\lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} \left| x(n) \right|^2}. \tag{2.27}$$

The *Mean signal energy* of a discrete-time signal is calculated as

$$E_x = \sum_{-\infty}^{\infty} \left| x(n) \right|^2 (t_{n+1} - t_n). \tag{2.28}$$

The similarity between two discrete-time signals $x(n)$ and $y(n)$ can also be characterized by the CCF that is now defined as

$$r_{xy}(m) = \overline{x(n)y(n+m)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n)y(n+m). \tag{2.29}$$

Consequently the discrete version of the ACF can be understood as a function that describes the change in time history of the signal $x(n)$. The ACF for the discrete-time signal $x(n)$ is defined as

$$r_{xx}(m) = \overline{x(n)x(n+m)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{n=N} x(n)x(n+m). \tag{2.30}$$

As with continuous-time signals it is also possible to characterize discrete-time signals in frequency domain. The (discrete) frequency domain representation of a discrete-time signal is obtained by applying the *Fourier transform of sampled signals* that is (according to Fliege 1991 and Nelson and Elliott 1992) defined by

$$x(n) \circ\!\!-\!\!\bullet X\left(e^{j\omega T}\right) = \mathfrak{F}_d\left[x(n)\right] := \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega nT}. \tag{2.31}$$

In direct analogy to (2.17) the existence of the Fourier transform for sampled signals is ensured, if

$$\sum_{-\infty}^{\infty} \left|x(n)\right| < \infty \tag{2.32}$$

holds for $x(n)$. The inverse operation is (according to Fliege 1991) defined by

$$X\left(e^{j\omega T}\right) \bullet\!\!-\!\!\circ x(n) = \mathfrak{F}_d^{-1}\left[X\left(e^{j\omega T}\right)\right] := \frac{1}{2\pi} \int_{-\pi}^{+\pi} X\left(e^{j\omega T}\right)e^{j\omega nT} d(\omega T). \tag{2.33}$$

Using (2.31) it is possible to introduce the frequency domain counterparts of the CCR and the ACF for discrete-time signals. These are the CSD (of discrete-time signals)

$$S_{xy}\left(e^{j\omega T}\right) = \mathfrak{F}_d\left[r_{xy}(m)\right] \tag{2.34}$$

and the ASD (of discrete-time signals)

$$S_{xx}\left(e^{j\omega T}\right) = \mathfrak{F}_d\left[r_{xx}(m)\right]. \tag{2.35}$$

The *discrete Fourier transform* is used to calculate an approximate solution of the Fourier integral (2.16). According to (Fliege 1991) and (Nelson and Elliott 1992) it is defined by

$$x(n) \circ\!\!-\!\!\bullet X(k) = \mathfrak{F}_{DFT}\left[x(n)\right] := \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi nk}{N}}. \tag{2.36}$$

The associated inverse operation is defined by

$$X(k) \bullet\!\!-\!\!\circ x(n) = \mathfrak{F}_{DFT}^{-1}\left[X(k)\right] := \frac{1}{N} \sum_{n=0}^{N-1} X(k)e^{j\frac{2\pi nk}{N}}, \quad \text{with } 0 \leqslant n \leqslant N-1, \tag{2.37}$$

where N is the number of time samples used to split the frequency range between zero and the sampling frequency into N values spaced at a frequency increment of $2\pi(NT)^{-1}$, and k is the index associated with the k-th discrete frequency line.

2.2 Comments on Systems

2.2.1 Definitions

In system theory the concept of a *system* is used alongside to that of signal. From the mathematical point of view a (single input/single output) system represents a transformation $\mathfrak{T}$ by which the input signal $x(t)$ is changed into the output signal $y(t)$, see (Cadzov and van Landingham 1985) and (Fliege 1991). A block diagram representing a system is shown in Fig. 2.2.

A system is called *linear*, if and only if

$$\mathfrak{T}\big[k_1x_1(t)+k_2x_2(t)\big]=k_1\mathfrak{T}\big[x_1(t)\big]+k_2\big[x_2(t)\big] \tag{2.38}$$

is fulfilled for arbitrary scalars k_1 and k_2 and arbitrary pairs of transformations $y_1(t)=k_1\mathfrak{T}[x_1(t)]$ and $y_2(t)=k_2\mathfrak{T}[x_2(t)]$.

An important subclass of linear systems are *linear time invariant systems* (LTI-systems). A (continuous) linear system with impulse response $h(t)$ caused by a Dirac impulse $\delta(t)$, compare Fig. 2.3, is a LTI-system if, and only if an arbitrary time shift τ in the input signal $\delta(t-\tau)$ causes the same time shift in the output signal such as

$$h(t-\tau)=\mathfrak{T}\big[\delta(t-\tau)\big]. \tag{2.39}$$

A (discrete) linear system with impulse response $h(n)$ caused by a Dirac impulse $\delta(n)$ is a LTI-system if, and only if an arbitrary time shift k in the input signal $\delta(n-k)$ causes the same time shift in the output signal such as

$$h(n-k)=\mathfrak{T}\big[\delta(n-k)\big]. \tag{2.40}$$

2.2.2 Transfer Behavior of LTI-Systems

The transfer behavior of a LTI-system can be described in both time domain and frequency domain. In time domain, the transformation or transfer behavior is described by the *impulse response* h, whereas in frequency domain the system behavior is described by its *transfer function* (or frequency response function (FRF)).

The transfer behavior of a continuous LTI-system is shown in Fig. 2.4. If the input signal $x(t)$ is a deterministic continuous-time signal, the output signal $y(t)$ can directly be obtained from the input by solving

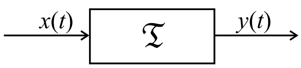

Fig. 2.2 Block diagram representing a system

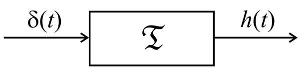

Fig. 2.3 Linear system with Dirac impulse $\delta(t)$ as input and impulse response $h(t)$ as output

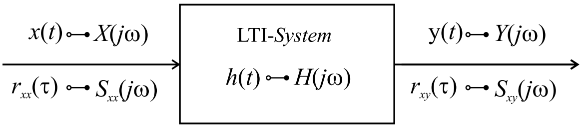

Fig. 2.4 Transmission of deterministic and stochastic signals through a continuous LTI-system

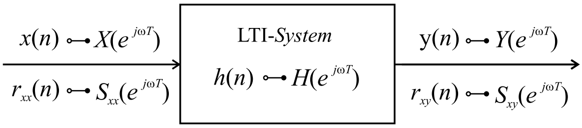

Fig. 2.5 Transmission of deterministic and stochastic signals through a discrete LTI-system

$$y(t) = x(t) * h(t) := \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \tag{2.41}$$

If the calculation is performed in frequency domain using $X(j\omega) = \mathfrak{F}[x(t)]$ and $Y(j\omega) = \mathfrak{F}[y(t)]$, the convolution integral in (2.41) is replaced by a simple multiplication

$$Y(j\omega) = H(j\omega)X(j\omega). \tag{2.42}$$

Direct computation of the output signal is also possible for discrete LTI-systems with deterministic input

$$y(n) = x(n) * h(n) := \sum_{k=-\infty}^{+\infty} x(k)h(n-k). \tag{2.43}$$

The associated equation in frequency domain reads

$$Y\left(e^{j\omega T}\right) = H\left(e^{j\omega T}\right)X\left(e^{j\omega T}\right). \tag{2.44}$$

For stochastic input signals, see Fig. 2.5, it is only possible to analyze the similarity between the input signal $x(t)$ and the output signal $y(t)$ using the CCF (2.14) that is, for stationary continuous-time signals and continuous LTI-systems, given by

$$\begin{aligned} r_{xy}(\tau) &= \overline{x(t)y(t+\tau)} \\ &= \int_{-\infty}^{\infty} \overline{x(t)x(\lambda)}h(t+\tau-\lambda)d\lambda \\ &= \int_{-\infty}^{\infty} r_{xx}(\lambda-t)h(t+\tau-\lambda)d\lambda \\ &\quad \text{with } \lambda - t \to k \text{ and } d\lambda \to dk \\ &= \int_{-\infty}^{\infty} r_{xx}(k)h(\tau-k)dk =: r_{xx}(\tau) * h(\tau). \end{aligned} \tag{2.45}$$

The CCF (2.29) between a stationary discrete-time input signal $x(n)$ that is transformed into the stationary discrete-time output signal $y(n)$ by a discrete LTI-system is given by

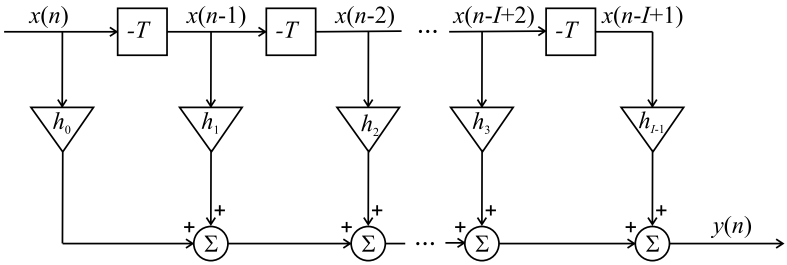

Fig. 2.6 Schematic diagram of filter operation using a finite impulse response filter of order I

$$\begin{aligned}
r_{xy}(n) &= \overline{x(k)y(k+n)} \\
&= \sum_{m=-\infty}^{\infty} \overline{x(k)x(m)}h(k+n-m) \\
&= \sum_{m=-\infty}^{\infty} r_{xx}(m-k)h(k+n-m) \\
&\quad \text{with } m-k \to l \\
&= \sum_{l=-\infty}^{\infty} r_{xx}(l)h(n-l) =: r_{xx}(n) * h(n).
\end{aligned} \tag{2.46}$$

The associated frequency domain equations are

$$S_{xy}(j\omega) = H(j\omega)S_{xx}(j\omega) \tag{2.47}$$

and

$$S_{xy}\big(e^{j\omega T}\big) = H\big(e^{j\omega T}\big)S_{xx}\big(e^{j\omega T}\big). \tag{2.48}$$

A convolution defined by (2.43) or (2.46) can be approximated by the inner product between a finite set of I coefficients $h_i(n)$ and a finite set of I input signals $x(n-i)$ such as

$$y(n) = x(n) * h(n) \approx \sum_{i=0}^{I-1} h_i(n)x(n-i) =: \mathbf{h}(n)^T \mathbf{x}(n), \tag{2.49}$$

where the $(I \times 1)$ column matrix $\mathbf{h}(n)$ is a *finite impulse response* (FIR) filter[3] used to approximate the impulse response $h(n)$ of the LTI-system at discrete time n.

The realization of the FIR filter operation defined by (2.49) is illustrated by Fig. 2.6 in which the I coefficients of $\mathbf{h}(n)$ can be interpreted as weights used to modify the actual as well as the previous values of the input signal x. The latter are obtained by successive delaying x with T. To get the output signal y, all weighted inputs are summed up, as also shown in Fig. 2.6.

[3]For details on FIR filters see for example (Kuo and Morgan 1996) or (Moschytz and Hofbauer 2000).

http://www.springer.com/978-94-007-2536-2

Adaptive Feed-Forward Control of Low Frequency Interior Noise
Kletschkowski, Th.
2012, XXXVI, 330 p., Hardcover
ISBN: 978-94-007-2536-2