Chapter 2
Individual Motion of a Charged Particle in Electric and Magnetic Fields

There are three distinct levels of modelling of the action of $\boldsymbol{E}$ and $\boldsymbol{B}$ fields on the charged particles in a plasma. Starting with the simplest and moving to the most complicated, we have:

The single trajectory model

In this description, the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are given, imposed from the exterior: no account is taken of the fields created by the motion of the particles. Further, collisions are completely neglected, including Coulomb interactions: this model only describes the motion of an isolated particle.

The hydrodynamic model

In this case, the plasma consists either of two fluids (that of the electrons and that of the ions), or of a single fluid (for instance, that of the electrons, the ions remaining at rest and forming a continuous background, providing an effective viscosity to the electron motion). The motion of each fluid is characterised locally by an average velocity $\boldsymbol{v}$ whose value results from an integration of the velocity distribution of the particles contained in the volume element considered (Sect. 3.3). The motion of the charged particles creates the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ (for which the average local value is retained (macroscopic fields)) which are included in a self-consistent manner in the equations of motion[50]. In addition, the model includes collisions, which modify the pre-determined motion defined by the superposition of the external and induced fields.

In order to establish self-consistency between the charged particle motion and the fields they produce, we need to consider first the velocity of the

[50] The coupling of the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields with the charged particles is said to be self-consistent because the motion of the particles creating the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ is itself influenced by the fields that it produces.

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fluid elements. This is obtained from the equation of motion, in which the Lorentz' force (Sect. 2.1) is included, assuming values for the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields for the first iteration. Once $\boldsymbol{v}$ has been determined, we can calculate the total current density $\boldsymbol{J}$ from the component fluids involved ($\boldsymbol{J} = \sum_\alpha n_\alpha q_\alpha \boldsymbol{v}_\alpha$). We can then complete the loop in two ways to obtain iterated values for $\boldsymbol{E}$ and $\boldsymbol{B}$:

- from $\boldsymbol{J}$, recover $\boldsymbol{E}$ from the electromagnetic relation:

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ , \tag{2.1}$$

  where σ is the electrical conductivity from the fluids involved, and from the known value of $\boldsymbol{E}$, calculate $\boldsymbol{B}$ by one or other of Maxwell's curl equations:

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \ , \tag{2.2}$$

$$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} \ , \tag{2.3}$$

- from the density $\boldsymbol{J}$, calculate the charge density ρ from the continuity equation (for example $\partial\rho/\partial t + \nabla \cdot \boldsymbol{J} = 0$) and obtain $\boldsymbol{E}$ from Poisson's equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \rho/\epsilon_0 \ , \tag{1.1}$$

  then, determine $\boldsymbol{B}$ through (2.2) or (2.3).

Remark: Note that the conductivity σ, which relates $\boldsymbol{J}$ and $\boldsymbol{E}$, plays a key role in obtaining field-particle self-consistence: we shall calculate the expression for σ in the framework of various models.

The kinetic or microscopic model

This is the description with the highest resolution. It uses the individual velocity distributions of the particles: this allows us to include certain phenomena that escape the hydrodynamic model, such as, for example, the Landau damping (resonance effect between a wave propagating in the plasma and particles with velocities within a certain interval). This model includes the fields and collisions self consistently, this time on the microscopic scale (individual particles), a more refined approach than that provided by the macroscopic values (average values over the velocity distribution of the particles).

The present chapter is devoted to the study of the individual motion of a charged particle in given $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. This model gives a first glimpse of the complex phenomena taking place at the heart of a plasma, with the assumption that there are no collisions in the body of the plasma or at the walls. In the first place, we will examine the solution of the equation of motion through a series of particular cases, to finally determine the general solution[51].

2.1 The general equation of motion of a charged particle in $\boldsymbol{E}$ and $\boldsymbol{B}$ fields and properties of that equation

Suppose q_a is the charge of a particle of mass m_a, moving with a velocity $\boldsymbol{w} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$ and suppose $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$ are the external fields: the particle is subject to the action of the Lorentz' *force* that, in the non-relativistic case, takes the form[52]:

$$\boldsymbol{F} \equiv q_a \left[\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{w} \wedge \boldsymbol{B}(\boldsymbol{r}, t)\right] \ . \tag{2.4}$$

This equation is the result of observation. It is valid if the particle is sufficiently small to be taken as a point (this therefore avoids the need to consider the problem of repartition of charges in the particle volume).

2.1.1 The equation of motion

From (2.4), we can write:

$$m_\alpha \frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = q_\alpha \left[\boldsymbol{E}(\boldsymbol{r}, t) + \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} \wedge \boldsymbol{B}(\boldsymbol{r}, t)\right] \ . \tag{2.5}$$

This equation leads to a second order differential equation for each axial component of the coordinate system. For example, in Cartesian coordinates:

$$m_\alpha \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = q_\alpha \left[E_x + \left(B_z \frac{\mathrm{d}y}{\mathrm{d}t} - B_y \frac{\mathrm{d}z}{\mathrm{d}t}\right)\right] \ , \tag{2.6}$$

$$m_\alpha \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = q_\alpha \left[E_y + \left(B_x \frac{\mathrm{d}z}{\mathrm{d}t} - B_z \frac{\mathrm{d}x}{\mathrm{d}t}\right)\right] \ , \tag{2.7}$$

$$m_\alpha \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = q_\alpha \left[E_z + \left(B_y \frac{\mathrm{d}x}{\mathrm{d}t} - B_x \frac{\mathrm{d}y}{\mathrm{d}t}\right)\right] \ . \tag{2.8}$$

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[51] The principal reference for this section is Electrodynamics of Plasmas by Jancel and Kahan, Chap. 4. See also Delcroix, Physique des plasmas, Vol. I, Sect. 12.3, Delcroix and Bers, Physique des plasmas, Vol. I, Sect. 2.3, and Allis, Motions of Ions and Electrons [2].

[52] The relativistic equation is:

$$m_a \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = q_a \left(1 - \frac{w^2}{c^2}\right)^{\frac{1}{2}} \left[\boldsymbol{E}(\boldsymbol{r}, t) + \boldsymbol{w} \wedge \boldsymbol{B}(\boldsymbol{r}, t) - \frac{w^2}{c^2}(\boldsymbol{w} \cdot \boldsymbol{E})\right] \ ,$$

where c is the speed of light in vacuum.

2.1.2 The kinetic energy equation

Taking the scalar product of (2.5) with $\boldsymbol{w} = \mathrm{d}\boldsymbol{r}/\mathrm{d}t$, we obtain the *kinetic energy equation*:

$$\frac{m_\alpha}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2 = q_\alpha \boldsymbol{E}(\boldsymbol{r},t)\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + q_\alpha\left(\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\wedge\boldsymbol{B}(\boldsymbol{r},t)\right)\cdot\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\ , \tag{2.9}$$

where the second term on the RHS vanishes, since $(\boldsymbol{A}\wedge\boldsymbol{B})\cdot\boldsymbol{A} = \boldsymbol{0}$: the resulting equation is in scalar form and constitutes an invariant in any frame of reference. After integration of the equation over time t from t_0 to t (in position, from $\boldsymbol{r}_0$ to $\boldsymbol{r}$), we have:

$$\frac{m_\alpha}{2}\left[\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2_{\boldsymbol{r}} - \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2_{\boldsymbol{r}_0}\right] = q_\alpha\int_{t_0}^{t}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{r}\ , \tag{2.10}$$

where the RHS of the equation represents the *work* done on the particle *by the electric field.* From this, we can draw the following important conclusions:

1. *The magnetic field* does “no work” because the force it exerts on the particle is perpendicular to its velocity[53]. It follows that the magnitude of the velocity of a charged particle is not affected by the presence of a magnetic field. However, the magnitudes of the velocity components perpendicular to $\boldsymbol{B}$ can vary, as we will show for the cyclotron motion (Sect. 2.2.2). Supposing that $\boldsymbol{B}$ is directed along Ox, this implies that:

$$w_\perp^2 = w_{y0}^2 + w_{z0}^2 = w_y^2(t) + w_z^2(t)\ , \tag{2.11}$$

   where the subscript 0 denotes the velocity at $t = 0$: in other words, a magnetic field can only change the direction of the velocity, not its magnitude. However, the application of a magnetic field to a plasma makes it possible, among other things, to conserve the energy of the system by reducing the diffusion losses of the charged particles to the walls, as we shall see (Sect. 3.8).
2. Only the electric field can “heat” the charged particles, i.e., give them energy.

2.2 Analysis of particular cases of $\boldsymbol{E}$ and $\boldsymbol{B}$

We will successively treat the following cases: only an electric field acting on the particle (Sect. 2.2.1); the particle is subjected to a constant, uniform

[53] Heating by magnetic pumping, where $\boldsymbol{B}$ varies periodically, can be considered as resulting from the action of the $\boldsymbol{E}$ field through the Maxwell equation $\nabla\wedge\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t$.

magnetic field, with or without an electric field $\boldsymbol{E}$ (Sect. 2.2.2); and finally, the most complex situation, the particle moves in a magnetic field that is (slightly) non uniform or (slowly) varying in time (Sect. 2.2.3). We will see that the different solutions obtained for the particular cases can be included in a general equation describing the particle motion in such $\boldsymbol{E}$ and $\boldsymbol{B}$ fields.

2.2.1 Electric field only ($\boldsymbol{B} = 0$)

From (2.6), (2.7) and (2.8), we obtain:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = \frac{q_a}{m_a} E_x(\boldsymbol{r}, t) \ , \quad \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{q_a}{m_a} E_y(\boldsymbol{r}, t) \ , \quad \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = \frac{q_a}{m_a} E_z(\boldsymbol{r}, t) \ . \tag{2.12}$$

We can now treat the following cases.

Constant and uniform electric field $\boldsymbol{E}$

By direct integration of (2.12) in vectorial form, we deduce:

$$\boldsymbol{w} = \frac{q_\alpha}{m_\alpha} \boldsymbol{E} t + \boldsymbol{w}_0 \ , \tag{2.13}$$

$$\boldsymbol{r} = \frac{q_\alpha}{m_\alpha} \boldsymbol{E} \frac{t^2}{2} + \boldsymbol{w}_0 t + \boldsymbol{r}_0 \ , \tag{2.14}$$

which describe uniformly accelerated motion.

Remarks:

1. From (2.13), one can see that the component of motion along a direction perpendicular to $\boldsymbol{E}$ is not affected by the presence of this field; this can be shown by decomposing $\boldsymbol{w}$ in directions parallel and perpendicular to $\boldsymbol{E}$. The situation is completely different with $\boldsymbol{B}$, because the corresponding force acts perpendicularly to $\boldsymbol{B}$ (and to $\boldsymbol{w}$) (2.4).
2. Since the field $\boldsymbol{E}$ selectively accelerates the component of velocity parallel to it, we could say that it tends, if not to confine, at least to orient the particle in this direction.
3. From (2.13) and (2.14), we can conclude that the velocity, as well as the distance travelled by an ion of mass m_i under the effect of a field $\boldsymbol{E}$ during a given time is m_e/m_i times smaller than that of an electron of mass m_e in the same field, which justifies the commonly used assumption that the ion is at rest with respect to the electron.

Conservative field $\boldsymbol{E}(\boldsymbol{r}, t)$

Since the electric field is conservative, we can write:

$$\boldsymbol{E} = -\boldsymbol{\nabla}\phi(\boldsymbol{r}, t) , \tag{2.15}$$

where ϕ is the potential acting on the particle. The vectorial equation of motion:

$$m_\alpha \frac{\mathrm{d}^2\boldsymbol{r}}{\mathrm{d}t^2} = -q_\alpha \boldsymbol{\nabla}\phi , \tag{2.16}$$

scalar multiplied by $\mathrm{d}\boldsymbol{r}/dt$ shows, after integration over time t, that the variation of kinetic energy is equal to the (negative) variation of the potential energy, such that the total energy is, of course, conserved:

$$\frac{m_\alpha}{2}\left[\left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2_{\boldsymbol{r}} - \left|\frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t}\right|^2_{\boldsymbol{r}_0}\right] = -q_\alpha[\phi(\boldsymbol{r}, t) - \phi(\boldsymbol{r}_0, t_0)] . \tag{2.17}$$

Equation (2.17) is a variant of (2.10).

Application to the case where ϕ is time independent

The motion of an electron in an electrostatic potential is similar to the propagation of a luminous wave in a medium of refractive index n_r, as shown below.

Consider the case of two media where ϕ, moreover, does not depend on $\boldsymbol{r}$, thus $\boldsymbol{E}$ is zero (2.15). The crossing of a discontinuity in potential ($\phi_1 \neq \phi_2$, Fig. 2.1) determines the existence of a field $\boldsymbol{E}$ (at the interface only) and, as a result, the particle experiences an instantaneous acceleration (or deceleration), the velocity thus changing from $\boldsymbol{w}_1$ to $\boldsymbol{w}_2$.

However, the components of the velocities parallel to the interface between the two media remain the same from one side to the other, because the electric field $\boldsymbol{E}$ is perpendicular to this interface (Remark 1 above) from which, noting $\boldsymbol{p} = m_e\boldsymbol{w}$:

$$|\boldsymbol{p}_1| \sin\theta_1 = |\boldsymbol{p}_2| \sin\theta_2 , \tag{2.18}$$

which, when written in the form:

$$\frac{|\boldsymbol{p}_1|}{|\boldsymbol{p}_2|} = \frac{\sin\theta_2}{\sin\theta_1} , \tag{2.19}$$

appears as the well known geometrical optics law of Descartes, if θ_1 and θ_2 are considered as the angle of incidence and refraction respectively, and where the momentum p_i of the particle is proportional to the index of the medium[54].

[54] Doing this, one finds that $n_r = A\sqrt{\mathcal{E} - q_\alpha\phi}$, where A is a constant and $\mathcal{E}$ the total energy of the particle.

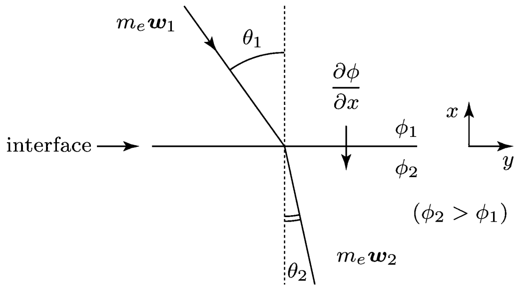

Fig. 2.1 Description of the refraction path in optical electronics.

The field $\boldsymbol{E}$ is uniform, but oscillates periodically as a function of time

This case corresponds to that in which the charged particles are present either in a plasma created by a high frequency field (HF), or in a plasma produced by other means (for example, a continuous current discharge) onto which a significant HF field has been superimposed.

The equation of motion is, in this case:

$$\frac{\mathrm{d}^2 \boldsymbol{r}}{\mathrm{d}t^2} = \frac{q_\alpha}{m_\alpha} \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} \tag{2.20}$$

and, after successive integrations from $t = 0$ to t, and supposing that the initial velocity of the particle is $\boldsymbol{w}_0$ (taking $w_0 \neq 0$, to remain completely general), we obtain:

$$\boldsymbol{w} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} = \frac{1}{\mathrm{i}\omega} \left[\frac{q_\alpha \boldsymbol{E}_0}{m_\alpha} \mathrm{e}^{\mathrm{i}\omega t} - \frac{q_\alpha \boldsymbol{E}_0}{m_\alpha} \right] + \boldsymbol{w}_0 \ , \tag{2.21}$$

or:

$$\boldsymbol{w} = \frac{q_\alpha \boldsymbol{E}_0}{\mathrm{i} m_\alpha \omega} \mathrm{e}^{\mathrm{i}\omega t} + \left(\boldsymbol{w}_0 - \frac{q_\alpha \boldsymbol{E}_0}{\mathrm{i}\omega m_\alpha} \right) \ , \tag{2.22}$$

and:

$$\boldsymbol{r} = -\frac{q_\alpha \boldsymbol{E}_0}{m_\alpha \omega^2} \mathrm{e}^{\mathrm{i}\omega t} + \left(\boldsymbol{w}_0 - \frac{q_\alpha \boldsymbol{E}_0}{\mathrm{i}\omega m_\alpha} \right) t + \boldsymbol{r}_c \ , \tag{2.23}$$

where $\boldsymbol{r}_c$ is a constant of integration, the initial position of the particle being

$$\boldsymbol{r}_0 = -\frac{q_\alpha \boldsymbol{E}_0}{m_\alpha \omega^2} + \boldsymbol{r}_c \ . \tag{2.24}$$

Examination of the relative phases of $\boldsymbol{E}$, $\boldsymbol{w}$ and $\boldsymbol{r}$

We consider a charged particle (taken to be a positive ion), with zero initial velocity, in an electric field $E_0 \cos \omega t$ of period $\mathcal{T}$, and examine the detailed behaviour of its velocity and trajectory[55] as a function of time, with the aid of Fig. 2.2. To simplify this presentation, we ignore the non-periodic term in velocity in (2.22).

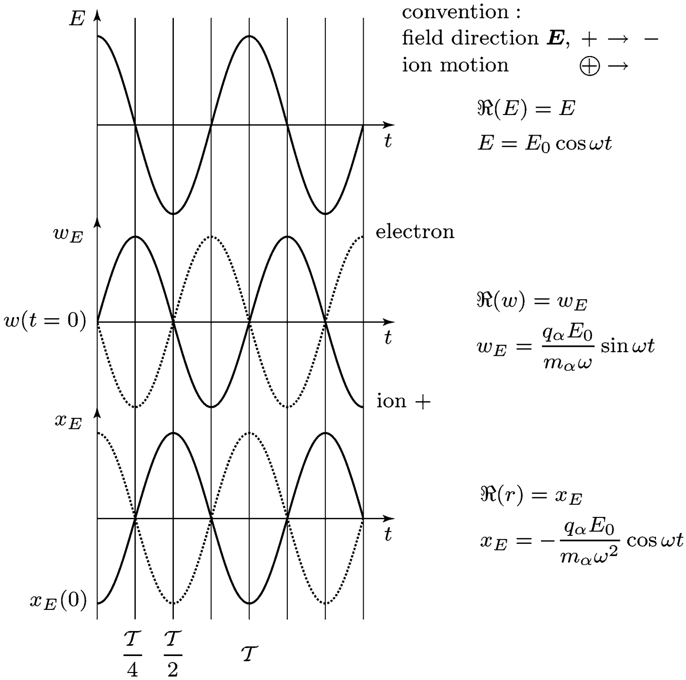

Fig. 2.2 Velocity and trajectory of a positive ion (full curve) and of an electron (dotted) in an alternating electric field of period $\mathcal{T}$.

1. Velocity: the velocity of the charged particle is in phase with the field $\boldsymbol{E}$. From $t = 0$ to $t = \mathcal{T}/4$, the positive ion is accelerated in the positive direction of the field: its velocity increases during the entire quarter period and reaches its maximum value at $t = \mathcal{T}/4$, when the electric field passes through zero.
   Between $t = \mathcal{T}/4$ and $\mathcal{T}/2$, the field $\boldsymbol{E}$ is in the opposite direction to the velocity of the positive ion, so it can only be retarded. The velocity passes through zero at the same time as the electric field reaches its maximum, the situation being symmetric to $t = 0$: in order to return to zero velocity, a field of the same amplitude but in the opposite direction is required.
   Between $t = \mathcal{T}/2$ and $3\mathcal{T}/4$, by symmetry, the velocity of the particle reaches its maximum opposite to its initial direction at the same time as

[55] By convention, the electric field existing between a positive charge and a negative charge is directed towards the negative charge. As a result, a positive ion is accelerated in the direction of the field (see Fig. 2.2).

the electric field passes through zero, just before changing sign, and so on. The velocity of the ion is then $\pi/2$ behind the phase of the field $\boldsymbol{E}$. This de-phasing with respect to the field $\boldsymbol{E}$, as we shall see, is such that the transfer of energy from the field to the charged particle is zero over a complete period.

2. Trajectory: in the case of a positive ion, the phase of the trajectory lags by π behind that of the electric field (in opposite phase), while an electron is in phase with the field. The amplitude of motion *of a charged particle* in a HF field $\boldsymbol{E}$ is referred to as the *extension of the periodic motion* and denoted by x_E.
   For a positive ion (initial position $x_E(0)$ in Fig. 2.2), since the initial velocity is assumed to be zero, the direction of motion, according to our convention, is in the direction of the field and only changes direction when the velocity $\boldsymbol{w}_E$ passes through zero (at $t = \mathcal{T}/2$): at this time, the field has its maximum in the opposite direction: there is clearly a lag in phase of π in the motion of the ion in the field.
   In contrast, the spatial oscillation of the electron motion is in phase with the HF field (following the convention of the direction of the field that we have adopted).

Transfer of energy from an oscillating electric field $\boldsymbol{E}$ to a charged particle

The kinetic energy resulting from the work done by an electric field $\boldsymbol{E}$ on the charge, in the time interval t_0 to t can be written (see (2.10)):

$$W \equiv \frac{m_\alpha}{2} w^2 \Big|_{r_0}^{r} = q_\alpha \int_{r_0}^{r} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{r} = q_\alpha \int_{t_0}^{t} \boldsymbol{E} \cdot \boldsymbol{w} \mathrm{d}t \,, \tag{2.25}$$

and, following (2.22):

$$\begin{aligned} W &= \Re \left[q_\alpha \int_{t_0}^{t} \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} \cdot \left(\frac{q_\alpha \boldsymbol{E}_0}{m_\alpha \mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} + \boldsymbol{w}_0 - \frac{q_\alpha \boldsymbol{E}_0^2}{\mathrm{i}\omega m_\alpha} \right) \mathrm{d}t \right] , \\ &= \Re \left[\frac{q_\alpha^2 E_0^2}{\mathrm{i} m_\alpha \omega} \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}2\omega t} \mathrm{d}t + \left(q_\alpha \boldsymbol{E}_0 \cdot \boldsymbol{w}_0 - \frac{q_\alpha^2 E_0^2}{\mathrm{i} m_\alpha \omega} \right) \int_{t_0}^{t} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t \right] , \\ &= \Re \left[- \frac{q_\alpha^2 E_0^2}{2 m_\alpha \omega^2} \mathrm{e}^{\mathrm{i}2\omega t} \Big|_{t_0}^{t} + \left(\frac{q_\alpha \boldsymbol{E}_0 \cdot \boldsymbol{w}_0}{\mathrm{i}\omega} + \frac{q_\alpha^2 E_0^2}{m_\alpha \omega^2} \right) \mathrm{e}^{\mathrm{i}\omega t} \Big|_{t_0}^{t} \right] , \\ &= - \frac{q_\alpha^2 E_0^2}{2 m_\alpha \omega^2} \cos 2\omega t \Big|_{t_0}^{t} + \frac{q_\alpha^2 E_0^2}{m_\alpha \omega^2} \cos \omega t \Big|_{t_0}^{t} + \frac{q_\alpha \boldsymbol{E}_0 \cdot \boldsymbol{w}_0}{\omega} \sin \omega t \Big|_{t_0}^{t} , \end{aligned} \tag{2.26}$$

where $\Re(A)$ denotes the real part of a complex quantity A. In the scalar product under the integral, $\boldsymbol{w}$ reduces to $\boldsymbol{w}_E$ (2.22), the component of the velocity parallel to $\boldsymbol{E}$ (there is no work done in the direction perpendicular to $\boldsymbol{E}$).

The value of the integral (2.26) over a period $\mathcal{T} = 2\pi/\omega$, i.e. between the times t_0 and $t_0 + 2\pi/\omega$, is zero. The total kinetic energy acquired during a period is actually zero, because during the first half-period the work is done in one direction and in the opposite direction during the second half-period.

However, if the oscillatory motion of the particle is interrupted by a collision before the repetition of a complete period starting from t_0, when the field has been applied, the integral (2.26) is non-zero and the corresponding energy taken from the field will be acquired by the particle[56]. In order to demonstrate this, we must leave the very simplified model of individual trajectories (collisionless plasma model) for a moment and consider the hydrodynamic model including collisions.

Transfer of energy from an oscillating field $\boldsymbol{E}$ to electrons via collisions: power absorbed by the electrons and plasma permittivity (a digression from individual trajectories)

Consider an electron fluid, coupled to ions and neutrals via collisions. Assuming that the thermal motion of electrons is negligible compared to their motion resulting from the field $\boldsymbol{E}$ ($v_{th} \ll v_E$, *cold plasma approximation*), the corresponding hydrodynamic equation for momentum transport (Sect. 3.7) can then be written:

$$m_e \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = -e\boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} - m_e \nu \boldsymbol{v} \ , \tag{2.27}$$

where $\boldsymbol{v}$ is the (macroscopic) velocity of electrons and ν the average electron-neutral momentum transfer collision frequency. The physical meaning of this equation has already been discussed (1.147).

In fact, we are not very far from the context of individual trajectories in the sense that we can consider that (2.27) describes the motion of a single particle in a medium where it is subject to a friction force.

In the cold plasma approximation, the electron velocity is purely periodic, such that:

$$\boldsymbol{v} = \boldsymbol{v}_0 \mathrm{e}^{\mathrm{i}\omega t} \ , \tag{2.28}$$

and, substituting $\boldsymbol{v}$ in (2.27), we obtain:

[56] The particle "acquires" this energy at the moment of collision, this energy being totally or partially shared with the particle with which it interacts. Recall that in the case of an electron-neutral collision, the electron only partially transfers its energy; more exactly, a fraction of the order of m_e/M of that energy (Sect. 1.7.2).

$$\boldsymbol{v} = -\frac{e\boldsymbol{E}(t)}{m_e(\nu + \mathrm{i}\omega)} , \tag{2.29}$$

which determines $\boldsymbol{v}_0$.
Since $\mathrm{d}\boldsymbol{r}/\mathrm{d}t \equiv \boldsymbol{v}$, again neglecting thermal motion, we have:

$$\boldsymbol{r} = \frac{\boldsymbol{v}}{\mathrm{i}\omega} , \tag{2.30}$$

that is:

$$\boldsymbol{r} = \frac{e\boldsymbol{E}(t)}{m_e\omega(\omega - \mathrm{i}\nu)} . \tag{2.31}$$

- Average HF power absorbed per electron
  The work per unit time and per electron in the field $\boldsymbol{E}$ can be written:

$$-\, e\boldsymbol{E}\cdot\boldsymbol{v} , \tag{2.32}$$

  which thus represents the instantaneous power taken from the field. The average value of the product of two complex variables A and B over a period, each varying sinusoidally with the same frequency, is $\Re(AB^*)/2$ (B^* is the complex conjugate of B). The power taken from the field over a period, or the average power, per electron, is then:

$$\theta_a \equiv \Re\left(\frac{-e\boldsymbol{E}\cdot\boldsymbol{v}^*}{2}\right) = \Re\left[\frac{e^2E_0^2}{2m_e}\frac{1}{(\nu - \mathrm{i}\omega)}\right] = \frac{e^2}{m_e}\frac{\nu}{\nu^2+\omega^2}\overline{E^2} , \tag{2.33}$$

  where $\sqrt{\overline{E^2}} = E_0/\sqrt{2}$ is the mean squared value of the electric field.
  If $\nu/\omega \ll 1$ (HF discharge approximation), we have (2.33):

$$\theta_a \approx \frac{e^2}{m_e}\frac{\nu}{\omega^2}\overline{E^2} , \tag{2.34}$$

  and we can verify that, for $\nu = 0$, the transfer of energy from the field $\boldsymbol{E}$ is zero, $\theta_a = 0$, conforming to the result we have already obtained above in the case of individual trajectories.
  In the opposite case of $\nu/\omega \gg 1$ (low-frequency discharge approximation), we obtain:

$$\theta_a \approx \frac{e^2}{m_e}\frac{\overline{E^2}}{\nu} . \tag{2.35}$$

  Expressions (2.34) and (2.35) are essential to the understanding of HF plasmas (Sect. 4.2).
- Electrical conductivity and permittivity in the presence of collisions
  The motion of charged particles in the field $\boldsymbol{E}$ creates a *current*, called the *conduction current*. For an electron density n_e, the current density can be written:

$$\boldsymbol{J} = -n_e e\boldsymbol{v} \tag{2.36}$$

and in complex notation, following (2.29):

$$\boldsymbol{J} = \frac{n_e e^2}{m_e(\nu + \mathrm{i}\omega)} \boldsymbol{E}(t) \ . \tag{2.37}$$

Since from electromagnetism:

$$\boldsymbol{J} = \sigma \boldsymbol{E} \ , \tag{2.38}$$

where σ is the (scalar) electrical conductivity of electrons, we find by identification from (2.37) and (2.38):

$$\sigma = \frac{n_e e^2}{m_e(\nu + \mathrm{i}\omega)} \ . \tag{2.39}$$

Note that in the case where there are no collisions ($\nu = 0$), σ is purely imaginary and the plasma then behaves as a **perfect dielectric**.
The permittivity ϵ_p of the plasma relative to vacuum in a field $E_0 \mathrm{e}^{\mathrm{i}\omega t}$ is related to the conductivity σ (demonstrated in Remark 2 below):

$$\epsilon_p = 1 + \frac{\sigma}{\mathrm{i}\omega\epsilon_0} \ , \tag{2.40}$$

where ϵ_0 is the permittivity of vacuum. Substituting σ from (2.39), we find:

$$\epsilon_p = 1 - \frac{\omega_{pe}^2}{\omega(\omega - \mathrm{i}\nu)} \ , \tag{2.41}$$

which, in the absence of collisions, reduces to:

$$\epsilon_p = 1 - \frac{\omega_{pe}^2}{\omega^2} \ , \tag{2.42}$$

an expression which shows that the exact case where $\omega = \omega_{pe}$ represents a singular value for the propagation of a wave, since $\epsilon_p = 0$.

Remarks:

1. Note that the value of θ_a (2.33) is inversely proportional to the mass of the particles, which means that we can usually neglect the power transferred to the ions in assessing the HF-particle power balance. We can also verify that for constant ω, θ_a passes through a maximum when $\nu = \omega$[57]; this is the case in which the transfer of energy is the most efficient.
2. The use of the conductivity σ in the preceding pages corresponds to the representation of *charges in vacuum*, as distinct from the *dielectric description* expressed by ϵ_p where, from the beginning, we prefer to consider the

[57] Recall that the collision frequency ν depends on the average energy of the electrons (and on the energy distribution function) and gas pressure (Sect. 1.7.8).

displacement current rather than the conduction current to describe the motion of charged particles in a HF field.

In effect, in the case of a purely dielectric description of the plasma, (2.3) can be expressed in the form:

$$\boldsymbol{\nabla} \wedge \boldsymbol{B} = \mu_0 \frac{\partial \boldsymbol{D}}{\partial t} \equiv \mu_0 \epsilon_0 \epsilon_p \frac{\partial \boldsymbol{E}}{\partial t} \ . \tag{2.43}$$

Assuming a periodic variation $\mathrm{e}^{\mathrm{i}\omega t}$ in the electromagnetic field with angular frequency ω, we obtain the terms on the RHS of (2.3) and (2.43) respectively:

$$\mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} = \mu_0 \epsilon_0 \mathrm{i}\omega \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} + \mu_0 \sigma \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} \ , \tag{2.44}$$

$$\mu_0 \epsilon_0 \epsilon_p \frac{\partial \boldsymbol{E}}{\partial t} = \mu_0 \epsilon_0 \epsilon_p \mathrm{i}\omega \boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} \ , \tag{2.45}$$

which, by identification, leads to:

$$\mathrm{i}\omega \epsilon_0 \epsilon_p = \mathrm{i}\omega \epsilon_0 + \sigma \ , \tag{2.46}$$

from which we obtain the complex relative permittivity of the plasma given by (2.40).

2.2.2 Uniform static magnetic field

MAGNETIC FIELD ONLY ($\boldsymbol{E} = \boldsymbol{0}$)

The study of this simple case will allow us to introduce the concepts of cyclotron gyration and helical motion. Cyclotron motion of particles produces a magnetic field $\boldsymbol{B}'$, in the opposite direction to the externally applied field $\boldsymbol{B}$, giving the plasma a diamagnetic character.

We will use Cartesian coordinates, such that Ox is oriented in the direction of $\boldsymbol{B}$. From the general equations of motion (2.6) and (2.8), setting $\boldsymbol{E} = (0, 0, 0)$ and $\boldsymbol{B} = (B, 0, 0)$, we obtain:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 0 \ , \tag{2.47}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{q_\alpha B}{m_\alpha} \frac{\mathrm{d}z}{\mathrm{d}t} \ , \tag{2.48}$$

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{q_\alpha B}{m_\alpha} \frac{\mathrm{d}y}{\mathrm{d}t} \ . \tag{2.49}$$

These equations can be rewritten by introducing the *cyclotron* (angular) *frequency*:

$$\omega_{c\alpha} = -\frac{q_\alpha B}{m_\alpha} , \tag{2.50}$$

the sign convention being such that $\omega_{c\alpha}$ is positive for electrons[58].
Ignoring the subscript α for simplicity, (2.47) to (2.49) take the form:

$$\ddot{x} = 0 , \tag{2.51}$$

$$\ddot{y} = -\omega_c \dot{z} , \tag{2.52}$$

$$\ddot{z} = \omega_c \dot{y} . \tag{2.53}$$

We will solve these equations, using the initial conditions ($t = 0$): $x = y = z = 0$ (the particle is initially at the origin of the coordinate system), $\dot{x} = w_{x0} = w_{\|0}$, $\dot{y} = w_{y0}$ and $\dot{z} = w_{z0}$: for complete generality, the components of the initial velocity parallel and perpendicular to $\boldsymbol{B}$ are non zero.
Integrating (2.53), we obtain:

$$\dot{z} = \omega_c y + C_1 = \omega_c y + w_{z0} , \tag{2.54}$$

where the constant of integration C_1, in view of our initial conditions, is equal to w_{z0}. Introducing this value of $\dot{z}$ into (2.52) for $\ddot{y}$;

$$\ddot{y} = -\omega_c^2 y - \omega_c w_{z0} . \tag{2.55}$$

This equation can be rearranged such that the LHS is homogeneous in y:

$$\ddot{y} + \omega_c^2 y = -\omega_c w_{z0} , \tag{2.56}$$

which has the form of a "forced" harmonic oscillator. The solution to this equation is given by the sum of the general solution without the RHS, and a particular solution of the differential equation including the RHS, thus:

$$y = A_1 \cos \omega_c t + A_2 \sin \omega_c t - \frac{w_{z0}}{\omega_c} . \tag{2.57}$$

We will now determine the constants A_1 and A_2 in (2.57):

$$y(t=0) \equiv A_1 - \frac{w_{z0}}{\omega_c} = 0 \text{ from which } A_1 = \frac{w_{z0}}{\omega_c} , \tag{2.58}$$

$$\dot{y}(t=0) \equiv w_{y0} = A_2 \omega_c \text{ from which } A_2 = \frac{w_{y0}}{\omega_c} . \tag{2.59}$$

We now need to calculate $z(t)$: from (2.54) with (2.57)–(2.59),

---

[58] Some authors prefer to write $\omega_{c\alpha} = |q_\alpha| B/m_\alpha$, but it is still necessary to define the direction in which the respective positively and negatively charged particles rotate around a line of force of the field $\boldsymbol{B}$.

$$\dot{z} = \omega_c \left[\frac{w_{z0}}{\omega_c} \cos \omega_c t + \frac{w_{y0}}{\omega_c} \sin \omega_c t - \frac{w_{z0}}{\omega_c} \right] + w_{z0} \ , \tag{2.60}$$

and, after integrating over t:

$$z = \frac{w_{z0}}{\omega_c} \sin \omega_c t - \frac{w_{y0}}{\omega_c} \cos \omega_c t + C_2 \ , \tag{2.61}$$

and since $z(t = 0) = 0$, we find $C_2 = w_{y0}/\omega_c$.
The three equations describing the orbit of a charged particle can finally be written:

$$x = w_{x0} t = w_{\|0} t \ , \tag{2.62}$$

$$y = \frac{w_{z0}}{\omega_c} \cos \omega_c t + \frac{w_{y0}}{\omega_c} \sin \omega_c t - \frac{w_{z0}}{\omega_c} \ , \tag{2.63}$$

$$z = \frac{w_{z0}}{\omega_c} \sin \omega_c t - \frac{w_{y0}}{\omega_c} \cos \omega_c t + \frac{w_{y0}}{\omega_c} \ . \tag{2.64}$$

In the yOz plane, the particle motion describes a circle[59], for which the centre is fixed by the constants of integration, in this case $Y, Z = -w_{z0}/\omega_c, -w_{y0}/\omega_c$. To demonstrate this, we will write the equation of the corresponding circular trajectory:

$$\begin{aligned}(y - Y)^2 + (z - Z)^2 &\equiv \left(y + \frac{w_{z0}}{\omega_c} \right)^2 + \left(z - \frac{w_{y0}}{\omega_c} \right)^2 \\ &= \frac{w_{z0}^2}{\omega_c^2} \cos^2 \omega_c t + \frac{w_{y0}^2}{\omega_c^2} \sin^2 \omega_c t + \frac{2 w_{z0} w_{y0}}{\omega_c^2} \cos \omega_c t \sin \omega_c t \\ &+ \frac{w_{z0}^2}{\omega_c^2} \sin^2 \omega_c t + \frac{w_{y0}^2}{\omega_c^2} \cos^2 \omega_c t - \frac{2 w_{z0} w_{y0}}{\omega_c^2} \cos \omega_c t \sin \omega_c t \\ &= \frac{w_{z0}^2 + w_{y0}^2}{\omega_c^2} \equiv \frac{w_{\perp 0}^2}{\omega_c^2} = r_B^2 \ , \end{aligned} \tag{2.65}$$

from which we can define a radius whose value is:

$$r_B = \frac{w_{\perp 0}}{\omega_c} = \frac{m_e}{eB} w_{\perp 0} \ . \tag{2.66}$$

In summary, in the plane perpendicular to $\boldsymbol{B}$, we observe a periodic circular motion with an angular frequency ω_c, the cyclotron frequency[60], whose

[59] The relations (2.63) and (2.64) which describe a periodic motion have the same amplitude and the same frequency, with a difference of phase $\pi/2$. In the framework of Lissajous curves, this gives rise to a circle. Note that, in English, the distinction between frequency and angular frequency is often ignored.

[60] Equivalently, the gyro-frequency of particles α ($\alpha = e, i$).

radius r_B is called the *Larmor radius*[61], $w_{\perp 0}$ being the initial speed of the particle in the yOz plane. To determine the direction of rotation of particles of mass m_α and of charge q_α, we ignore the constant, initial velocity of the particle in the yOz plane. For the electron, since by convention $\omega_{ce} > 0$, we see from (2.63) and (2.64) that for $\omega_c t = 0$, $y = w_{z0}/\omega_c$ and $z = -w_{y0}/\omega_c$, while for $\omega_c t = \pi/2$ ($t = \mathcal{T}_c/4$, where $\mathcal{T}_c$ is the cyclotron period), $y = w_{y0}/\omega_c$ and $z = w_{z0}/\omega_c$. It follows that, for a field $\boldsymbol{B}$ away from the reader, the gyration of the electron is in the clockwise direction (towards the right), as is shown in Fig. 2.3a, while the positive ion rotates in the anti-clockwise direction (towards the left). In the direction parallel to $\boldsymbol{B}$, the velocity is constant, equal to $w_{\parallel 0}$, and the motion is uniform, since this velocity is not modified by $\boldsymbol{B}$. The combination of the cyclotron motion and uniform motion gives rise to a trajectory in the form of a helix (Fig. 2.3b), which rotates around the magnetic field line (referred to as the *guiding centre*).

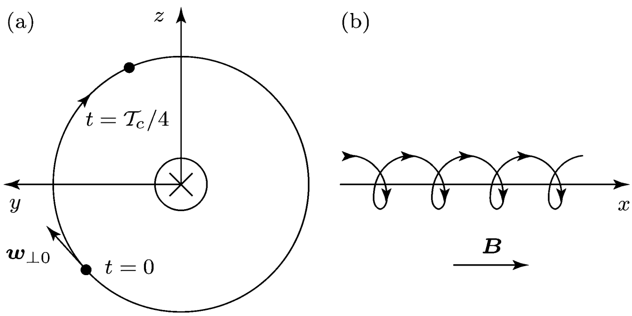

Fig. 2.3 **a** Cyclotron motion of an electron in the plane perpendicular to $\boldsymbol{B}$, the field directed along the Ox axis, away from the reader. The points on the circle show the position of the electron at $t = 0$ and $t = \mathcal{T}_c/4$. **b** Helical motion of the electron along the B field.

Interesting particular cases:

- If $w_{\parallel 0} = 0$, the helical trajectory degenerates into a circular orbit. The radius of the orbit is then dependent on the total velocity w_0 of the particle, and $r_B = w_0 m_e/eB$.
- If $w_{\perp 0} = 0$, the trajectory is rectilinear and parallel to $\boldsymbol{B}$.

Remarks:

1. The decrease in the diameter of the helix with increasing $\boldsymbol{B}$ results in a confinement of charged particles in the direction perpendicular to $\boldsymbol{B}$. In fact, as $\boldsymbol{B}$ tends to infinity, $r_B \to 0$, such that transverse motion is not

[61] Equivalently, the cyclotron radius or radius of gyration.

possible: we will see in Sect. 3.8 that this effect reduces the particle diffusion perpendicular to $\boldsymbol{B}$, towards the walls.

2. A uniform field $\boldsymbol{B}$ cannot affect $w_\parallel$ so $w_\parallel(t) = w_{\parallel 0}$ where the subscript zero corresponds to the time $t = 0$: this is a property of the Lorentz force in the case $\boldsymbol{E} = 0$. If $\boldsymbol{E} = 0$, from conservation of kinetic energy: $w_\perp^2(t) + w_\parallel^2(t) \equiv w^2(t) = w_0^2$. Since we have just seen that $w_\parallel = w_{\parallel 0}$, then $w_\perp^2 = w_{\perp 0}^2$ and, thus $w_\perp^2(t) \equiv w_y^2(t) + w_z^2(t) = w_{\perp 0}^2$. Thus, in a field $\boldsymbol{B}$, the components w_y and w_z can vary, as was mentioned in Sect. 2.1 (Remark 1).
3. The *pitch of the helix* is obtained by calculating the axial distance travelled during one revolution. If this pitch is p_h, and $\mathcal{T}_c$ is the cyclotron period, then $p_h = w_{\parallel 0}\mathcal{T}_c = w_{\parallel 0}/f_c = 2\pi w_{\parallel 0}/\omega_c$, and we obtain:

$$p_h = 2\pi \left(\frac{w_{\parallel 0}}{w_{\perp 0}} \right) r_B \ . \tag{2.67}$$

4. A useful way to represent the helical motion is:

$$\boldsymbol{w} = \boldsymbol{w}_{\parallel 0} + \boldsymbol{\omega}_c \wedge \boldsymbol{r}_B \ , \tag{2.68}$$

where $\boldsymbol{w}_{\parallel 0}$ describes the motion of the guiding centre and the second term, the circular cyclotron motion of the particle; the vector $\boldsymbol{\omega}_c$ is directed along $\boldsymbol{B}$ and defines the axis of rotation and its direction; the vector $\boldsymbol{r}_B$, the orbit radius, has its origin at the guiding centre.
5. Since the Larmor radius is proportional to the mass of the particles, (see (2.66)), it follows that for ions of mass m_i, $r_{Bi} = r_{Be} m_i/m_e$, i.e. $r_{Bi} \gg r_{Be}$.
6. The cyclotron frequency (2.50) or gyration frequency does not depend on the velocity of the particles, but only on their mass and charge. This property allows energy to be given uniquely to particles of a given mass and charge by means of an electric field oscillating at $\omega = \omega_{c\alpha}$, independently of their velocity distribution: we can therefore obtain a form of selective heating by means of cyclotron resonance, which will be treated in detail later (2.146).
7. A usefull numerical relation to calculate the cyclotron frequency for electrons is:

$$f_{ce}(\mathrm{Hz}) = 2.799 \times 10^{10} B\ (\mathrm{tesla}) \ . \tag{2.69}$$

Thus for $B = 0.1\,\mathrm{T}$ (10^3 gauss), $f_{ce} = 2.8\,\mathrm{GHz}$. The corresponding frequency for ions of mass m_i is m_i/m_e times smaller.
8. The *diamagnetic field* created by the circulating cyclotron current is given by the Biot-Savart Law (Lorrain et al):

$$\boldsymbol{B}' = \frac{\mu_0}{4\pi} \int_V \frac{\boldsymbol{J} \wedge \boldsymbol{r}}{r^3}\, \mathrm{d}V \ . \tag{2.70}$$

In this expression, $\boldsymbol{r}$ points from the source (charge) towards the guiding centre axis (Fig. 2.4). Note that $\boldsymbol{B}$ and $\boldsymbol{B}'$ are calculated at the same $\boldsymbol{r}$

position for comparison purposes. The diamagnetic field $\boldsymbol{B}'$ points in the same direction for electrons and ions: particles of opposite charge revolve in opposite directions around $\boldsymbol{B}$, such that their respective currents rotate in the same direction, as is shown in Fig. 2.4. The vectorial product $\boldsymbol{J} \wedge \boldsymbol{r}$ from (2.70) indicates that $\boldsymbol{B}'$ is in the opposite direction to the field $\boldsymbol{B}$ responsible for the cyclotron motion (this cannot be otherwise!). The magnetic field in the plasma is given by the vectorial sum of $\boldsymbol{B}$ and $\boldsymbol{B}'$ (see exercise 2.2).

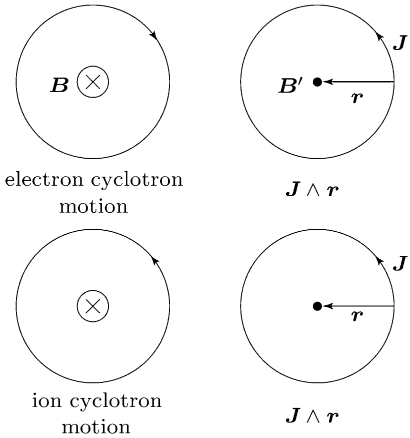

Fig. 2.4 Determining the orientation of the diamagnetic field $\boldsymbol{B}'$ created by the cyclotron motion in a field $\boldsymbol{B}$ imposed into the page: $\boldsymbol{B}'$ comes out of the page towards the reader.

STATIC UNIFORM ELECTRIC AND MAGNETIC FIELDS

In this section, we will show that the effect of uniform, constant fields $\boldsymbol{E}$ and $\boldsymbol{B}$ leads to a motion, called the electric field drift (also known as the $\boldsymbol{E} \wedge \boldsymbol{B}$ drift), of ions and electrons in the plasma, perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{B}$. After this, we will derive an equation that incorporates all the fundamental motions studied to date. As a further application, we will calculate the electric conductivity, for the same $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, and show that it is a tensor.

In the first case, the superposition of electric and magnetic fields modifies the magnitude of the velocity, such that the part of the Lorentz force tied to the magnetic field, $q_\alpha \boldsymbol{w} \wedge \boldsymbol{B}$, is continuously varying. It is noteworthy that, since the two fields are uniform and constant in time, the orbits can be calculated analytically and are easily represented graphically.

The case where $\boldsymbol{E}$ and $\boldsymbol{B}$ are arbitrarily oriented (with $\boldsymbol{w_0} = 0$)

The Cartesian frame is once again constructed such that $\boldsymbol{B}$ is directed along the Ox axis. Since the orientation of $\boldsymbol{E}$ in this frame is independent of $\boldsymbol{B}$, the $\boldsymbol{E}$ field has a component along each of the axes. We suppose that the charged particle, at $t = 0$, is situated at the origin of the frame $x = y = z = 0$, and, in contrast to the previous case ($\boldsymbol{B}$ only), at rest $\dot{x} = \dot{y} = \dot{z} = 0$. This last condition implies that $w_{\perp 0} = 0$, removing the contribution of the cyclotron motion to the particle trajectory entirely, allowing us to examine the effect of the electric field drift alone (the case $w_{\perp 0} \neq 0$ is treated further in the text, for $\boldsymbol{E}$ perpendicular and parallel to $\boldsymbol{B}$.)

1. The equations of motion
   From (2.6)–(2.8), we obtain:

$$\ddot{x} = \frac{q_\alpha}{m_\alpha} E_x \ , \tag{2.71}$$

$$\ddot{y} = \frac{q_\alpha}{m_\alpha} E_y - \omega_c \dot{z} \ , \tag{2.72}$$

$$\ddot{z} = \frac{q_\alpha}{m_\alpha} E_z + \omega_c \dot{y} \ . \tag{2.73}$$

2. Calculation of the trajectories
   The equations of motion are integrated analogously to the previous case.
   Calculation of y: Integration of (2.73) gives:

$$\dot{z} = \frac{q_\alpha}{m_\alpha} E_z t + \omega_c y \ . \tag{2.74}$$

   Substituting $\dot{z}$ in (2.72):

$$\ddot{y} = \frac{q_\alpha}{m_\alpha} E_y - \omega_c \left[\frac{q_\alpha}{m_\alpha} E_z t + \omega_c y \right] \ . \tag{2.75}$$

   This equation can be rearranged such that the LHS is homogeneous:

$$\ddot{y} + \omega_c^2 y = -\frac{\omega_c q_\alpha}{m_\alpha} E_z t + \frac{q_\alpha}{m_\alpha} E_y \ , \tag{2.76}$$

   for which the solution is:

$$y = A_1 \cos \omega_c t + A_2 \sin \omega_c t - \frac{q_\alpha}{m_\alpha \omega_c} E_z t + \frac{q_\alpha}{m_\alpha \omega_c^2} E_y \ . \tag{2.77}$$

   The constants A_1 and A_2 are fixed by the initial conditions.
   Since $y(t = 0) = 0$, (2.77) yields:

$$A_1 + \frac{q_\alpha}{m_\alpha \omega_c^2} E_y = 0 \ , \tag{2.78}$$

from which:

$$A_1 = -\frac{q_\alpha}{m_\alpha \omega_c^2} E_y \tag{2.79}$$

and since $\dot{y}(t=0) = 0$, $A_2\omega_c - (q_\alpha/m_\alpha\omega_c)E_z = 0$, such that:

$$A_2 = \frac{q_\alpha}{m_\alpha \omega_c^2} E_z \ . \tag{2.80}$$

Calculation of z. Substituting the value of y obtained from (2.77), together with (2.79) and (2.80), in (2.74):

$$\dot{z} = \frac{q_\alpha E_z t}{m_\alpha} + \omega_c \left[-\frac{q_\alpha E_y}{m_\alpha \omega_c^2} \cos\omega_c t + \frac{q_\alpha E_z}{m_\alpha \omega_c^2} \sin\omega_c t - \frac{q_\alpha E_z t}{m_\alpha \omega_c} + \frac{q_\alpha E_y}{m_\alpha \omega_c^2} \right] , \tag{2.81}$$

and, after integrating:

$$z = -\frac{q_\alpha E_y}{\omega_c^2 m_\alpha} \sin\omega_c t - \frac{q_\alpha E_z}{\omega_c^2 m_\alpha} \cos\omega_c t + \frac{q_\alpha E_y t}{\omega_c m_\alpha} + C_3 \ . \tag{2.82}$$

Since $z(t=0) = 0 = -(q_\alpha/\omega_c^2 m_\alpha)E_z + C_3$:

$$C_3 = \frac{q_\alpha E_z}{m_\alpha \omega_c^2} \ . \tag{2.83}$$

Calculation of x. Two successive integrations of (2.71) lead to:

$$x = \frac{q_\alpha}{m_\alpha} E_x \frac{t^2}{2} \ . \tag{2.84}$$

Finally, the equations for the trajectory as a function of time (for $\boldsymbol{B} \parallel \hat{\mathbf{e}}_x$) can be written:

$$x = \frac{q_\alpha}{m_\alpha} E_x \frac{t^2}{2} \ , \tag{2.85}$$

$$y = -\frac{q_\alpha}{\omega_c^2 m_\alpha} E_y \cos\omega_c t + \frac{q_\alpha}{\omega_c^2 m_\alpha} E_z \sin\omega_c t - \frac{q_\alpha}{\omega_c m_\alpha} E_z t + \frac{q_\alpha}{\omega_c^2 m_\alpha} E_y \ , \tag{2.86}$$

$$z = -\frac{q_\alpha}{\omega_c^2 m_\alpha} E_y \sin\omega_c t - \frac{q_\alpha}{\omega_c^2 m_\alpha} E_z \cos\omega_c t + \frac{q_\alpha}{\omega_c m_\alpha} E_y t + \frac{q_\alpha}{\omega_c^2 m_\alpha} E_z \ . \tag{2.87}$$

3. Study of the motion described by (2.85) to (2.87)
The presence of the uniform and constant fields $\boldsymbol{E}$ and $\boldsymbol{B}$ results in a *drift motion* (called the *electric field drift*) of the charged particle perpendicular to $\boldsymbol{B}$ and $\boldsymbol{E}_\perp$, the component of $\boldsymbol{E}$ perpendicular to $\boldsymbol{B}$. In fact, if $w_0 = 0$, as is the case here, the non-periodic part of the motion in the plane $y\mathrm{O}z$ is as follows: the particle initially moves in the direction of $\boldsymbol{E}_\perp$ (for a positive

ion, Fig. 2.5) or in the opposite direction (electron). Due to the velocity $\boldsymbol{w}_\perp$ thus acquired, the magnetic part of the Lorentz' force $\boldsymbol{F}_{Lm}$ produces a motion perpendicular to $\boldsymbol{E}_\perp$ and $\boldsymbol{B}$, precisely in the direction of the drift motion, since $\boldsymbol{F}_{Lm} = q_\alpha \boldsymbol{w}_\perp \wedge \boldsymbol{B}$.

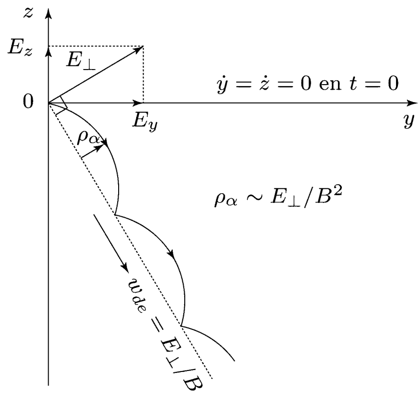

Fig. 2.5 Cycloidal motion of the drift for a positive ion (the field $\boldsymbol{B}$ is out of the page). The ion is initially ($t = 0$) at the origin of the frame and at rest, then it moves, on average, along the drift axis represented by the dotted line.

The projection of the motion in the yOz plane (the plane perpendicular to $\boldsymbol{B}$) is thus a cycloidal trajectory, as is shown in Fig. 2.5: the non-periodic terms $(q_\alpha/m_\alpha\omega_c)E_i t$ $[i = y, z]$ push the particle in a direction perpendicular to $\boldsymbol{E}_\perp$ and $\boldsymbol{B}$ along a virtual straight line, whose parametric equation is given by:

$$y_d = -\frac{q_\alpha}{m_\alpha\omega_c}E_z t\ , \tag{2.88}$$

and:

$$z_d = \frac{q_\alpha}{m_\alpha\omega_c}E_y t\ . \tag{2.89}$$

These relations can be combined to give:

$$z_d = -\frac{E_y}{E_z}y_d\ . \tag{2.90}$$

The average velocity of this shifting motion, called the *electric field drift velocity*, taken from (2.88) and (2.89), is:

$$w_{de} = \sqrt{\left(\frac{q_\alpha E_z}{m_\alpha\omega_c}\right)^2 + \left(\frac{q_\alpha E_y}{m_\alpha\omega_c}\right)^2} = \frac{E_\perp}{B}\ . \tag{2.91}$$

This velocity is independent of the mass of the particle, and of its charge. Further, because the motion is directed perpendicular[62] to $\boldsymbol{E}$ (to both $\boldsymbol{E}_\perp$ and $\boldsymbol{E}_\parallel$ components, see Fig. 2.5), the particle in its drift motion does no work in the field $\boldsymbol{E}$: the drift velocity thus remains constant.
A uniformly accelerated motion in the direction perpendicular to the $y\mathrm{O}z$, plane, following the E_x component of the electric field, must be added to the motion in the $y\mathrm{O}z$ plane.

4. Comparative study of the cycloidal motion of electrons and ions.
We will ignore the motion due to $\boldsymbol{E}_\parallel$. Recall the convention: the motion of positive ions is in the direction of the electric field. At $t = 0$, the electron and the ion are at the origin of the frame, with zero velocity. Immediately afterwards, the ion starts to move in the direction of $\boldsymbol{E}_\perp$ but its trajectory is instantly curved, by the magnetic component of the Lorentz force, following $\boldsymbol{w}_{de}$ (Fig. 2.5). The electron is initially accelerated in the opposite direction, but the Lorentz force leads it to follow the same drift direction as the ion because of the opposite sign of its charge ($\boldsymbol{F}_{Lm} = -e\boldsymbol{w}_e \wedge \boldsymbol{B}$): the two trajectories (if we ignore the influence of $\boldsymbol{E}_\parallel$) are confined in the plane $(\boldsymbol{w}_{de}, \boldsymbol{E}_\perp)$, as is shown in Fig. 2.6.
In (2.86) where $y = -(q_\alpha E_y/\omega_{c\alpha}^2 m_\alpha)\cos\omega_{c\alpha}t + \cdots$, the amplitude of the periodic motion of the particle is proportional to m_α ($\omega_{c\alpha}^2 m_\alpha \propto m_\alpha^{-1}$): the electrons describe much smaller arcs than those of the ions but their number per second is much larger (Fig. 2.6) since the ratio of the masses $m_i/m_e \gg 1$ leads to $\omega_{ce}/\omega_{ci} \gg 1$.

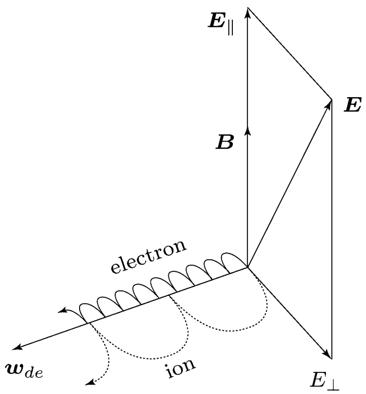

Fig. 2.6 Schematic representation of the motion of electrons and ions in the electric field drift, showing that the arcs described by the electrons have much smaller amplitudes but are more numerous.

[62] To see that $\boldsymbol{w}_{de}$ is perpendicular to $\boldsymbol{E}$, note that the slope of the trajectory describing the particle motion $z = f(y)$ is given by $\Delta x/\Delta y = -E_y/E_z$ (2.90) while the orientation of $\boldsymbol{E}_\perp$ in the same frame (y, z) is expressed by E_z/E_y: these slopes are therefore orthogonal. To distinguish it from the present drift velocity, the drift in a field $\boldsymbol{E}$ including collisions (Sect. 3.8.2) will be called the *collisional drift velocity*.

Remarks:

1. $E_\perp/B$ has the units of velocity (the proof is left to the reader)
2. The maximum amplitude ρ_α of the cycloid of a particle of type α with respect to the drift axis is proportional to $E_\perp/B^2$ (Fig. 2.5). The calculation of this expression is also left to the reader.

The preceding discussion can be treated in a more complete manner by considering more generally that $\boldsymbol{w}_0 \neq 0$: then the influence of the cyclotron gyration is superimposed on the drift velocity in the total motion of the particle. Nonetheless to simplify the calculation, we will assume $\boldsymbol{E} \perp \boldsymbol{B}$.

Perpendicular $\boldsymbol{E}$ and $\boldsymbol{B}$ fields with $\boldsymbol{w_0} \neq 0$: combined drift and cyclotron motion

The $\boldsymbol{B}$ field is still along Ox but this time $\boldsymbol{E}$ is entirely along Oz. This leads to the following equations for the trajectory of the charged particle:

$$x = w_{\|0} t \,, \tag{2.92}$$

$$y = \frac{w_{z0}}{\omega_c} \cos \omega_c t + \left(\frac{w_{y0}}{\omega_c} + \frac{q_\alpha E}{m_\alpha \omega_c^2} \right) \sin \omega_c t - \frac{q_\alpha E}{m_\alpha \omega_c} t - \frac{w_{z0}}{\omega_c} \,, \tag{2.93}$$

$$z = \frac{w_{z0}}{\omega_c} \sin \omega_c t - \left(\frac{w_{y0}}{\omega_c} + \frac{q_\alpha E}{m_\alpha \omega_c^2} \right) \cos \omega_c t + \left(\frac{w_{y0}}{\omega_c} + \frac{q_\alpha E}{m_\alpha \omega_c^2} \right) . \tag{2.94}$$

To illustrate the various forms of the trajectories, one needs to consider the ratio w_{y0}/w_{de}, where $w_{de} = E_\perp/B$ (we will assume $w_{y0} = w_{z0}$) and distinguish three particular cases.
To do this, consider the term:

$$\frac{w_{y0}}{\omega_c} + \frac{q_\alpha E}{m_\alpha \omega_c^2}$$

appearing in the expressions (2.93) and (2.94) for y and z. Taking into account the convention on the sign of $\omega_{c\alpha}$ (2.50), this term can be transformed in terms of the ratio w_{y0}/w_{de} such that:

$$\frac{1}{\omega_c} \left[w_{y0} - \frac{q_\alpha E}{m_\alpha} \frac{m_\alpha}{q_\alpha B} \right] = \frac{1}{\omega_c} \left[w_{y0} - \frac{E}{B} \right] = \frac{1}{\omega_c} \left[w_{y0} - w_{de} \right] . \tag{2.95}$$

If $w_{de} \gg w_{y0}$, then $w_{y0} \simeq 0$ and $w_{z0} \simeq 0$ (no cyclotron motion because $w_{\perp 0} \simeq 0$) and Eq. (2.93) for y reduces to:

$$y = -\frac{q_\alpha E}{m_\alpha \omega_c^2} \left(\omega_c t - \sin \omega_c t \right) \,, \tag{2.96}$$

which obviously leads to (2.86) in the case where $E_y = 0$.
For the same approximation ($w_{y0} \simeq 0$ and $w_{z0} \simeq 0$), Eq. (2.94) for z becomes:

$$z = \frac{q_\alpha E}{m_\alpha \omega_c^2}(1 - \cos \omega_c t) , \tag{2.97}$$

the expression obtained when $E_y = 0$ in (2.87).
We will now consider the three following typical cases:

- $w_{y0}/w_{de} = 100$ (Fig. 2.7a)
- $w_{y0}/w_{de} = 2$ (Fig. 2.7b)
- $w_{y0}/w_{de} \leq 1$ (Fig. 2.7c)

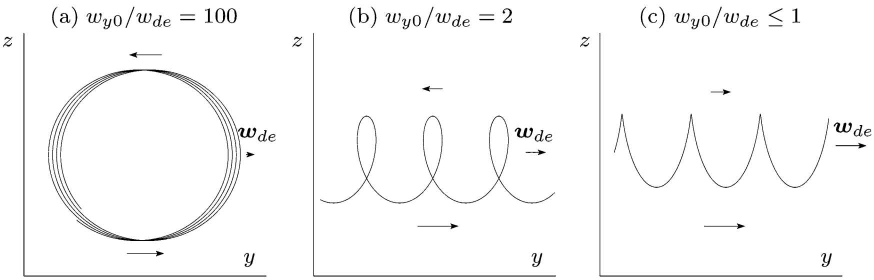

Fig. 2.7 Trajectory of a positive ion in uniform static $\boldsymbol{E}$ and $\boldsymbol{B}$ fields, with the respective components E_z and B_x, for different values of the ratio w_{y0}/w_{de} where $w_{z0} = w_{y0}$ (the $\boldsymbol{B}$ field is directed towards the reader).

Figure 2.7a shows that the cyclotron motion is hardly affected by a weak $\boldsymbol{E}$ field, the guiding centre being slightly displaced in the direction of the electric field drift. Figure 2.7b describes what happens to the cyclotron motion when it is strongly modified by the drag along y due to the electric field drift. Finally, Fig. 2.7c shows that all traces of cyclotron motion disappear when $w_{de} \geq w_{y0}$.

To obtain a simple analytic form for the resulting trajectories, suppose $w_{z0} = 0$ (in Fig. 2.7, note that $w_{z0} = w_{y0} \neq 0$). The resultant trajectory for $w_{y0}/w_{de} = 2$ is that of a quasi trochoid[63], for which the mathematical expression is:

$$y = a\tau - b \sin \tau \qquad z = b(\cos \tau - 1) \tag{2.98}$$

with, following (2.93) and (2.94) and assuming $w_{z0} = 0$:

$$a = \frac{E}{B\omega_c} , \qquad b = -\frac{1}{\omega_c}\left[w_{y0} - \frac{E}{B}\right] \qquad \text{and} \qquad \tau = \omega_c t .$$

[63] A true trochoid requires $y = a\tau - b \sin \tau$ and $z = a - b \cos \tau$.

In the case $w_{y0}/w_{de} < 1$ ($a \simeq b$), the trajectory is that of a cycloid (with a sign inversion):

$$y = a(\tau - \sin\tau) \quad z = a(\cos\tau - 1) \quad \text{with } a = \frac{E}{B\omega_c} . \tag{2.99}$$

Note that setting $w_{z0} = 0$ while $w_{y0}/w_{de} = 1$ (Eqs. (2.93) and (2.94)) suppresses all periodic motion in the y and z ($b = 0$) directions: all that remains is a rectilinear trajectory along y due to the electric field drift.

Remark: In the case $w_{de} \ll w_{\perp 0} = \omega_c r_B$ (weak $E_\perp$ field), as shown in Fig. 2.7a, the trajectories are quasi cyclotronic, with a weak drift velocity of their guiding centres in the direction perpendicular to $\boldsymbol{B}$ and $\boldsymbol{E}_\perp$. The guiding centre of the cyclotron trajectory of a positive ion moves slowly in the direction of the drift, because the cyclotron curvature is smaller when the ion moves in the direction of $\boldsymbol{E}_\perp$ ($w_\perp$ increases, as does r_B) than when it moves in the opposite direction to $\boldsymbol{E}_\perp$. This deformation of the cyclotron motion leads to a shift of the guiding centre and, accordingly, to the particle drift.

Parallel $\boldsymbol{E}$ and $\boldsymbol{B}$ fields: no drift motion

Assume the Ox axis is in the direction of the fields: It is then useful to distinguish two cases:

- The initial velocity is zero.
  From (2.85) to (2.87), we find:

$$x(t) = \frac{q_\alpha}{m_\alpha}\frac{E_x t^2}{2} , \tag{2.100}$$

$$y(t) = 0 , \tag{2.101}$$

$$z(t) = 0 . \tag{2.102}$$

  The motion is only along Ox and uniformly accelerated: since the $\boldsymbol{B}$ field is in the direction of motion, it plays no role on the trajectory of the particle ($\boldsymbol{F}_{Lm} \equiv q_\alpha \boldsymbol{w} \wedge \boldsymbol{B} = 0$ since $\boldsymbol{w} \parallel \boldsymbol{B}$).
- The initial velocity normal to $\boldsymbol{B}$ is non zero ($w_{y0} \neq 0$, $w_{z0} \neq 0$).
  Under these conditions, we can resume the development from (2.71)–(2.73). We then obtain a helical trajectory, as in the previous case of a magnetic field only, but the pitch of the helix increases (or decreases) because the E_x field gives rise to a velocity component w_x:

$$p_h = w_\parallel \mathcal{T}_c = \frac{2\pi}{|\omega_c|} w_\parallel = \frac{2\pi}{q_\alpha B} m_\alpha w_\parallel = \frac{2\pi m_\alpha}{q_\alpha B}\left(\frac{q_\alpha}{m_\alpha}E_x t\right) = \frac{2\pi}{B}E_x t . \tag{2.103}$$

The general solution

By combining the results of the preceding cases, it is possible to obtain the general characteristics of the motion of a charged particle in uniform, static fields, $\boldsymbol{E}$ and $\boldsymbol{B}$. The charged particle describes a trajectory which, in the most general form, consists of:

1. A cyclotron gyration in the plane perpendicular to $\boldsymbol{B}$, provided that $\boldsymbol{w}_{\perp 0} \neq \boldsymbol{0}$. If in addition $\boldsymbol{w}_{\parallel 0} \neq 0$, the particle motion develops in three dimensions, leading to a helical motion, with constant pitch if $\boldsymbol{E} = \boldsymbol{0}$ or increasing (decreasing) pitch if the $\boldsymbol{E}$ field has a component parallel to the $\boldsymbol{B}$ field.
2. A net motion perpendicular to both $\boldsymbol{E}$ and $\boldsymbol{B}$, referred to as the electric field drift trajectory, which is independent of both m_α and q_α, and has a constant velocity $w_{de} = E_\perp / B$.

Examination of the general equation of motion (2.5) will enable us to recover these results. For that purpose, we regroup the terms homogenous in $\boldsymbol{w}$ on the LHS:

$$\dot{\boldsymbol{w}} - \frac{q_\alpha}{m_\alpha} \boldsymbol{w} \wedge \boldsymbol{B} = \frac{q_\alpha}{m_\alpha} \boldsymbol{E} \,, \tag{2.104}$$

The solution of this differential equation consists of the general solution $\boldsymbol{w}_1$ of the homogeneous equation without the RHS (helical motion with constant pitch) to which is added a particular solution $\boldsymbol{w}_2$ that includes the RHS. We want to determine $\boldsymbol{w}$ such that:

$$\boldsymbol{w} = \boldsymbol{w}_1 + \boldsymbol{w}_2 \,. \tag{2.105}$$

- General solution without the RHS ($\boldsymbol{E} = 0$)
  The value of $\boldsymbol{w}_1$ has already been obtained (2.68) in the form:

$$\boldsymbol{w}_1 = \boldsymbol{w}_{\parallel 0} + \boldsymbol{\omega}_c \wedge \boldsymbol{r}_B \,, \tag{2.106}$$

  describing a helical motion, where $\boldsymbol{w}_{\parallel 0}$ is the initial velocity parallel to $\boldsymbol{B}$. Therefore, we only need to calculate $\boldsymbol{w}_2$.
- Particular solution including the RHS: the expression for $\boldsymbol{w}_2$
  We can construct this solution in a completely arbitrary way, provided that the result obtained is a true solution. To guide us in this process, we know that this particular solution must reproduce the drift motion. Because of this, we express $\boldsymbol{w}_2$ in a trihedral coordinate system, whose Cartesian axes are defined (Fig. 2.8) such that:

$$\hat{\mathrm{e}}_z \parallel \boldsymbol{B} \,, \quad \hat{\mathrm{e}}_y \parallel \boldsymbol{E}_\perp \,, \quad \hat{\mathrm{e}}_x \parallel (\boldsymbol{E}_\perp \wedge \boldsymbol{B}) \,.$$

  This method was proposed by J.L. Delcroix.

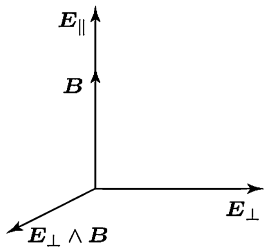

Fig. 2.8 Trihedral coordinate system used to calculate the particular solution (after J.L. Delcroix).

We are thus looking for a solution of the form:

$$\boldsymbol{w}_2 = a\boldsymbol{E}_{\parallel} + b\boldsymbol{E}_{\perp} + c(\boldsymbol{E}_{\perp} \wedge \boldsymbol{B}) \,, \tag{2.107}$$

$$\dot{\boldsymbol{w}}_2 = \dot{a}\boldsymbol{E}_{\parallel} + \dot{b}\boldsymbol{E}_{\perp} + \dot{c}(\boldsymbol{E}_{\perp} \wedge \boldsymbol{B}) \,, \tag{2.108}$$

which we can substitute in the equation of motion (2.5) including the RHS:

$$\begin{aligned} \dot{a}\boldsymbol{E}_{\parallel} + \dot{b}\boldsymbol{E}_{\perp} + \dot{c}(\boldsymbol{E}_{\perp} \wedge \boldsymbol{B}) &= \frac{q_\alpha}{m_\alpha}\left[a\boldsymbol{E}_{\parallel} + b\boldsymbol{E}_{\perp} + c(\boldsymbol{E}_{\perp} \wedge \boldsymbol{B})\right] \wedge \boldsymbol{B} \\ &= \frac{q_\alpha}{m_\alpha}(\boldsymbol{E}_{\parallel} + \boldsymbol{E}_{\perp}) \,. \end{aligned} \tag{2.109}$$

Noting that[64] $(\boldsymbol{E}_{\perp} \wedge \boldsymbol{B}) \wedge \boldsymbol{B} = -\boldsymbol{E}_{\perp} B^2$ and regrouping the terms along the different axes:

$$\left(\dot{a} - \frac{q_\alpha}{m_\alpha}\right)\boldsymbol{E}_{\parallel} + \left(\dot{b} + \frac{q_\alpha c B^2}{m_\alpha} - \frac{q_\alpha}{m_\alpha}\right)\boldsymbol{E}_{\perp} + \left(\dot{c} - \frac{b q_\alpha}{m_\alpha}\right)\boldsymbol{E}_{\perp} \wedge \boldsymbol{B} = 0 \,, \tag{2.110}$$

we obtain:

$$\dot{a} = \frac{q_\alpha}{m_\alpha} \,, \quad \dot{b} = \frac{q_\alpha}{m_\alpha} - \frac{q_\alpha c B^2}{m_\alpha} \,, \quad \dot{c} = \frac{q_\alpha}{m_\alpha} b \,, \tag{2.111}$$

for which a particular solution is obviously $\dot{a} = q_\alpha/m_\alpha$ and $\dot{b} = \dot{c} = 0$ such that:

$$a = \frac{q_\alpha t}{m_\alpha} \,, \quad b = 0 \,, \quad c = \frac{1}{B^2} \,. \tag{2.112}$$

This shows that we have actually chosen as particular solution that for which the initial velocity of the particle in the plane ($\boldsymbol{B}$, $\boldsymbol{E}_{\perp}$) is zero. We then have:

$$\boldsymbol{w}_2 = \frac{q_\alpha t}{m_\alpha}\boldsymbol{E}_{\parallel} + \frac{\boldsymbol{E}_{\perp} \wedge \boldsymbol{B}}{B^2} \,, \tag{2.113}$$

where the first term on the RHS is a uniformly accelerated motion along $\boldsymbol{B}$, the second term represents the electric drift in the direction perpendicular to both $\boldsymbol{E}_{\perp}$ and $\boldsymbol{B}$, for which the modulus of the velocity is $E_{\perp}/B$.

---

[64] Double vectorial product rule: $\boldsymbol{A} \wedge (\boldsymbol{B} \wedge \boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{C} \cdot \boldsymbol{A}) - \boldsymbol{C}(\boldsymbol{A} \cdot \boldsymbol{B})$.

- Solution of the general equation of motion
  By adding $\boldsymbol{w}_1$ (2.106) (noting that $\boldsymbol{\omega}_c \wedge \boldsymbol{r}_B = -(q_\alpha/m_\alpha)\boldsymbol{B} \wedge \boldsymbol{r}_B$ and $\boldsymbol{w}_2$ (2.113), we obtain the full general solution:

$$\boldsymbol{w} = \underbrace{\boldsymbol{w}_{\parallel 0} + \frac{q_\alpha}{m_\alpha}\boldsymbol{r}_B \wedge \boldsymbol{B}}_{\text{Helical motion}} + \underbrace{\frac{q_\alpha t}{m_\alpha}\boldsymbol{E}_\parallel}_{\substack{\uparrow \\ \text{Uniformly} \\ \text{accelerated motion} \\ \text{along } \boldsymbol{E}_\parallel}} + \underbrace{\frac{\boldsymbol{E}_\perp \wedge \boldsymbol{B}}{B^2}}_{\text{Electric drift}} \ . \tag{2.114}$$

Electrical conductivity in the presence of a magnetic field: the need for a tensor representation (a digression from individual trajectories)

In Sect. 2.2.1, we calculated the electrical conductivity of charged particles in a periodic electric field ($\boldsymbol{B} = 0$). We now want to obtain an expression for the conductivity when the particles are subjected to uniform, static magnetic and electric fields.

In order to calculate the current created by the charged particles in the $\boldsymbol{E}$ an $\boldsymbol{B}$ fields, we will move from the trajectory of one particle to an ensemble of individual particle trajectories per unit volume. For this ensemble of particles, we will again make the assumption that their initial velocities are isotropic, such that on average, at $t = 0$, there is no directed motion: $\langle \boldsymbol{w}_{\perp 0} \rangle = 0$, $\langle \boldsymbol{w}_{\parallel 0} \rangle = 0$. In (2.114), it follows that $\boldsymbol{w}_{\parallel 0} = 0$ and $\boldsymbol{r}_B \wedge \boldsymbol{B} = 0$, ($\boldsymbol{r}_B \propto \boldsymbol{w}_{\perp 0}$)[65]. The current density $\boldsymbol{J}_\alpha$ of charged particles of type α then reduces to:

$$\boldsymbol{J}_\alpha \equiv n_\alpha q_\alpha \boldsymbol{w}_\alpha = \frac{n_\alpha q_\alpha^2 t}{m_\alpha}\boldsymbol{E}_\parallel + \frac{n_\alpha q_\alpha}{B^2}(\boldsymbol{E}_\perp \wedge \boldsymbol{B}) \ . \tag{2.115}$$

In the following discussion, until equation (2.121), we shall omit the index α in $\boldsymbol{J}$ and σ.

Conductivity is now a tensor quantity: we will show that, if it is considered a priori as a scalar, it cannot satisfy (2.115). In fact, in the case where $\boldsymbol{J} = \sigma \boldsymbol{E}$, we would have the following components:

$$\boldsymbol{J} = \sigma E_x \hat{\mathbf{e}}_x + \sigma E_y \hat{\mathbf{e}}_y + \sigma E_z \hat{\mathbf{e}}_z \ , \tag{2.116}$$

but in developing (2.115), and since $\boldsymbol{E}_\perp = E_x \hat{\mathbf{e}}_x + E_y \hat{\mathbf{e}}_y$ ($\boldsymbol{B}$ is taken to be along z)[66], we obtain:

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[65] The value of r_B, initially fixed by $\boldsymbol{w}_{\perp 0}$ in the case of the solution to (2.104) without the RHS ($\boldsymbol{E} = 0$), is not affected by the inclusion of the particular solution ($\boldsymbol{E} \neq 0$) because $\boldsymbol{w}_{2\perp} = 0$ ($b = 0$ in (2.112)).

[66] We have not decomposed equation (2.115) following the trihedral coordinate system of Fig. 2.8 because this, being vectorial, can be developed in any chosen coordinate system.

$$\boldsymbol{J} = \frac{n_\alpha q_\alpha}{B^2}(B)\, E_y \hat{\mathbf{e}}_x - \frac{n_\alpha q_\alpha}{B^2}(B)\, E_x \hat{\mathbf{e}}_y + \frac{n_\alpha q_\alpha^2}{m_\alpha} t\, E_z \hat{\mathbf{e}}_z \ , \tag{2.117}$$

because:

$$\boldsymbol{E}_\perp \wedge \boldsymbol{B} = \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ E_x & E_y & 0 \\ 0 & 0 & B \end{vmatrix} \ . \tag{2.118}$$

Note that in (2.117) there is no E_x component along $\hat{\mathbf{e}}_x$ and no E_y component along $\hat{\mathbf{e}}_y$, as is required by (2.116). In fact, in (2.117), for example J_x has the form:

$$J_x = \left(\frac{n_\alpha q_\alpha}{B} \right) E_y \ , \tag{2.119}$$

from which we can conclude that σ cannot be a scalar in the presence of $\boldsymbol{B}$.

We will now seek to write the components of a tensor $\underline{\boldsymbol{\sigma}}$ explicitly, supposing it to be of order 2 (see Appendix VII for a brief introduction to tensors and Appendix VIII for tensor operations), defined by the relation:

$$\boldsymbol{J} = \underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E} \ , \tag{2.120}$$

which can be written explicitly as:

$$J^i = \sigma^{ij} E_j \ , \tag{2.121}$$

where σ^{ij} is a tensor element with two (order 2) superscript (contravariant) indices. Note that the vector $\boldsymbol{J}$ is also contravariant but that $\boldsymbol{E}$ is (by nature) covariant: by convention, there is a summation over the same index when it appears in both the covariant and contravariant positions, and this index is said to be mute. In the following, however, we will not distinguish between the variance of the quantities. Expanding (2.121), we find:

$$\begin{aligned} \boldsymbol{J} = {} & (\sigma_{xx} E_x + \sigma_{xy} E_y + \sigma_{xz} E_z) \hat{\mathbf{e}}_x + (\sigma_{yx} E_x + \sigma_{yy} E_y + \sigma_{yz} E_z) \hat{\mathbf{e}}_y \\ & + (\sigma_{zx} E_x + \sigma_{zy} E_y + \sigma_{zz} E_z) \hat{\mathbf{e}}_z \ . \end{aligned} \tag{2.122}$$

By identification of (2.122) with (2.117),

$$\sigma_{xy} = \frac{n_\alpha q_\alpha}{B} \ , \qquad \sigma_{yx} = -\frac{n_\alpha q_\alpha}{B} \ , \qquad \sigma_{zz} = \frac{n_\alpha q_\alpha^2 t}{m_\alpha} \ , \tag{2.123}$$

such that the tensor can be represented by the matrix:

$$\underline{\boldsymbol{\sigma}} = n_\alpha q_\alpha \begin{pmatrix} 0 & 1/B & 0 \\ -1/B & 0 & 0 \\ 0 & 0 & q_\alpha t / m_\alpha \end{pmatrix} \ . \tag{2.124}$$

In the present case, and assuming a macroscopically neutral plasma ($n_e = n_i$), the total electric current due to the positive ions and the electrons (subscripts i and e respectively) is such that only its component along the direction of the $\boldsymbol{B}$ field is non zero, because along x and y, $\sigma^i_{xy} + \sigma^e_{xy} = (en_i/B) - (en_e/B) = 0$, etc. In fact, the electric field drift motion cannot give rise to a net current because the drift of the ions and electrons takes place in the same direction, so that the net transport of charge is zero[67].

Remarks:

1. In (2.121), the element σ_{ij} of the tensor $\underline{\boldsymbol{\sigma}}$ expresses the fact that the component E_j of the electric field (a force) in a given direction induces a current J^i (an action) in another direction.
2. The reader can calculate the corresponding relative permittivity tensor corresponding to $\underline{\boldsymbol{\sigma}}$ and introduce therein the electron plasma frequency, by generalising (2.40):
$$\underline{\boldsymbol{\epsilon}}_p = \underline{\boldsymbol{I}} + \frac{\underline{\boldsymbol{\sigma}}}{\mathrm{i}\omega\epsilon_0} \,, \tag{2.125}$$
where $\underline{\boldsymbol{I}}$ is the unit tensor (represented by the unit matrix).

UNIFORM STATIC MAGNETIC FIELD AND UNIFORM PERIODIC ELECTRIC FIELD

The problem to be resolved is not very different from that of Eq. (2.104), which led to the general solution of the preceding case ($\boldsymbol{E}$ constant) because now:
$$\dot{\boldsymbol{w}} - \frac{q_\alpha}{m_\alpha}(\boldsymbol{w} \wedge \boldsymbol{B}) = \frac{q_\alpha}{m_\alpha}\boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t} \,. \tag{2.126}$$

We are left to find a particular solution including the RHS[68], still with the trihedral coordinate system of Fig. 2.8, but this time setting:

$$\boldsymbol{w}_2 = a\boldsymbol{E}_{0\parallel}\mathrm{e}^{\mathrm{i}\omega t} + b\boldsymbol{E}_{0\perp}\mathrm{e}^{\mathrm{i}\omega t} + c(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\mathrm{e}^{\mathrm{i}\omega t} \,. \tag{2.127}$$

Substituting this expression into (2.126), we obtain:

$$\begin{aligned}
&\left[\dot{a}\boldsymbol{E}_{0\parallel} + \dot{b}\boldsymbol{E}_{0\perp} + \dot{c}(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right]\mathrm{e}^{\mathrm{i}\omega t} \\
&+\mathrm{i}\omega\left[a\boldsymbol{E}_{0\parallel} + b\boldsymbol{E}_{0\perp} + c(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right]\mathrm{e}^{\mathrm{i}\omega t} \\
&-\frac{q_\alpha}{m_\alpha}\left[\left(a\boldsymbol{E}_{0\parallel} + b\boldsymbol{E}_{0\perp} + c(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right) \wedge \boldsymbol{B}\right]\mathrm{e}^{\mathrm{i}\omega t} = \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{E}_{0\parallel} + \boldsymbol{E}_{0\perp}\right]\mathrm{e}^{\mathrm{i}\omega t} \,.
\end{aligned} \tag{2.128}$$

[67] It constitutes a neutral beam of charged particles!

[68] Remember that this solution $\boldsymbol{w}_2$ is related to the drift motion in $E_\perp$ and $\boldsymbol{B}$.

Noting that $\boldsymbol{E}_{0\parallel} \wedge \boldsymbol{B} = 0$, we obtain, along the different base vectors of the trihedral coordinate system, by identification:

$$\boldsymbol{E}_{0\parallel}\left(\dot{a} + \mathrm{i}\omega a - \frac{q_\alpha}{m_\alpha}\right) = 0 \rightarrow \dot{a} + \mathrm{i}\omega a = \frac{q_\alpha}{m_\alpha}\ , \tag{2.129}$$

$$\boldsymbol{E}_{0\perp}\left(\dot{b} + \mathrm{i}\omega b + \frac{q_\alpha c B^2}{m_\alpha} - \frac{q_\alpha}{m_\alpha}\right) = 0 \rightarrow \dot{b} + \mathrm{i}\omega b = -\frac{q_\alpha B^2}{m_\alpha} c + \frac{q_\alpha}{m_\alpha}\ , \tag{2.130}$$

$$\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}\left(\dot{c} + \mathrm{i}\omega c - \frac{q_\alpha b}{m_\alpha}\right) = 0 \rightarrow \dot{c} + \mathrm{i}\omega c = \frac{q_\alpha b}{m_\alpha}\ . \tag{2.131}$$

To find the solution, we must distinguish two situations:

1. Off-resonance case ($\omega \neq \omega_c$)

   - Solution of (2.129)–(2.131)
     A simple particular solution is then $\dot{a} = \dot{b} = \dot{c} = 0$; the value of the coefficients in this case are:

$$a = \frac{q_\alpha}{\mathrm{i}\omega m_\alpha}\ ,\ b = \frac{q_\alpha}{\mathrm{i}\omega m_\alpha}(1 - B^2 c) \text{ and } c = \frac{q_\alpha b}{\mathrm{i}\omega m_\alpha}\ , \tag{2.132}$$

such that:

$$b = \frac{q_\alpha}{\mathrm{i}\omega m_\alpha}\left(1 - \frac{B^2 q_\alpha b}{\mathrm{i}\omega m_\alpha}\right)\ , \quad \text{i.e.} \quad b\left(1 - \frac{q_\alpha^2 B^2}{m_\alpha^2 \omega^2}\right) = \frac{q_\alpha}{\mathrm{i}\omega m_\alpha}\ , \tag{2.133}$$

where again:

$$b = -\frac{iq_\alpha}{\omega m_\alpha} \frac{1}{\left(1 - \dfrac{\omega_c^2}{\omega^2}\right)}\ . \tag{2.134}$$

Note that the coefficient b is finite on condition that $\omega \neq \omega_c$. Finally:

$$a = -\frac{iq_\alpha}{\omega m_\alpha}\ ,\ b = \frac{iq_\alpha}{m_\alpha}\frac{\omega}{(\omega_c^2 - \omega^2)} \text{ and } c = \frac{q_\alpha^2}{m_\alpha^2}\frac{1}{(\omega_c^2 - \omega^2)}\ , \tag{2.135}$$

such that the general motion, off cyclotron resonance, can be written:

$$\begin{aligned}\boldsymbol{w} = \underset{\substack{\uparrow \\ \text{Helical motion} \\ +\text{ all initial} \\ \text{conditions}}}{\boldsymbol{w}_1} + \Bigg(-\frac{iq_\alpha}{\omega m_\alpha}\ \underset{\substack{\uparrow \\ (-i)}}{\boldsymbol{E}_{0\parallel}} + \frac{\mathrm{i}\omega q_\alpha}{m_\alpha(\omega_c^2 - \omega^2)}\ \underset{\substack{\uparrow \\ (+i)}}{\boldsymbol{E}_{0\perp}} \\ + \frac{q_\alpha^2}{m_\alpha^2(\omega_c^2 - \omega^2)}(\overbrace{\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}}^{\substack{(+1) \\ \downarrow}})\Bigg)\,\mathrm{e}^{\mathrm{i}\omega t}\ .\end{aligned} \tag{2.136}$$

Because this describes a periodic motion with the same frequency along the 3 axes and because of the particular phase relations between the three components of $\boldsymbol{w}_2$, namely (for a positive ion) $-\pi/2$ for $\boldsymbol{E}_{0\parallel}$ and $\pi/2$ for $\boldsymbol{E}_{0\perp}$ with respect to the axis ($\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}$) in the case where $\omega_c > \omega$, the trajectory obtained from (2.136) is closed on itself, corresponding to a helical motion, depending on the initial conditions superimposed on a three dimensional elliptical motion (which is difficult to represent graphically!).

In the particular case where $\omega = 0$ (constant field $\boldsymbol{E}$), we have seen that the velocity $\boldsymbol{w}_2$ describes the motion (axial and lateral) of the guiding centre[69]. In the presence of a harmonically varying $\boldsymbol{E}$ field, the drift motion does not occur: the term containing $\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}$ in (2.136) is not constant and when integrated, cannot yield a linear dependence on t, as is the case in (2.86) and (2.87) where $\boldsymbol{E}$ is constant. This drift is in fact "annihilated", because the $\boldsymbol{E}_{0\perp}$ component and, as a result the drift velocity, oscillate periodically. On the other hand, if ω tends to zero, the term $\boldsymbol{E}_{0\perp}$ in (2.136) disappears and the term in $\boldsymbol{E}_{0\parallel}$ reduces to $(q_\alpha/m_\alpha)E_{0\parallel}t$ because $\sin\omega t \to \omega t$, in complete agreement with the expression (2.113) for $\boldsymbol{w}_2$ obtained for constant $\boldsymbol{E}$.

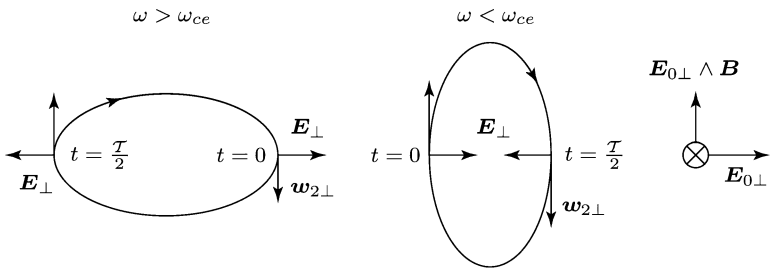


Fig. 2.9 Orientation of $\boldsymbol{w}_{2\perp}$ with respect to the reference frame ($\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}$, $\boldsymbol{E}_{0\perp}$, $\boldsymbol{B}$) for the case of a non-resonant electron cyclotron frequency. See Appendix IX for details.

- Representation of the $\boldsymbol{w}_{2\perp}$ component of the particular solution of (2.136)

  Returning to the coordinate frame in Fig. 2.8, we find, in the plane perpendicular to $\boldsymbol{B}$, an ellipse whose major axis varies according to $\boldsymbol{E}_{0\perp}$ or $\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}$, depending on whether $\omega > \omega_c$ or $\omega < \omega_c$ (Fig. 2.9). To show this, we rewrite the two corresponding components of $\boldsymbol{w}_2$ in (2.136) in the form:

[69] In fact, for constant $\boldsymbol{E}$, $\boldsymbol{w}_2$ (2.114) includes the drift motion (perpendicular to $E_\perp$ and $\boldsymbol{B}$) and the uniformly accelerated motion along $\boldsymbol{B}$, which together describe the cyclotron motion around the guiding centre.

$$\frac{q_\alpha}{m_\alpha(\omega_c^2-\omega^2)}\left\{\mathrm{i}\omega \boldsymbol{E}_{0\perp}-\omega_c\frac{(\boldsymbol{E}_{0\perp}\wedge \boldsymbol{B})}{B}\right\}\mathrm{e}^{\mathrm{i}\omega t}\,, \tag{2.137}$$

noting that the term $\boldsymbol{E}_{0\perp}\wedge \boldsymbol{B}/B$ has the same modulus as $\boldsymbol{E}_{0\perp}$. We can then conclude that for $\omega > \omega_c$, the velocity $\boldsymbol{w}_{2\perp}$ is mainly[70] in phase quadrature (in advance for electrons because $q_\alpha = -e$) with the field $\boldsymbol{E}_\perp$ while for $\omega < \omega_c$, $\boldsymbol{w}_{2\perp}$ is principally in phase: this leads to the representation in Fig. 2.9.

2. Resonant case ($\omega = \omega_c$)
The particular solution can no longer have $\dot{b} = \dot{c} = 0$ because, following (2.135), the coefficients b and c would then tend to infinity. We can, however, retain the solution that corresponds to $\dot{a} = 0$, from (2.132):

$$a = \frac{q_\alpha}{\mathrm{i}\omega m_\alpha}\,. \tag{2.138}$$

To find the value of the coefficient c, we substitute the value of b given by (2.130) in (2.131) and obtain:

$$\dot{c}+\mathrm{i}\omega c = \frac{q_\alpha}{m_\alpha}\left[-\frac{q_\alpha B^2 c}{m_\alpha}+\frac{q_\alpha}{m_\alpha}-\dot{b}\right]\frac{1}{\mathrm{i}\omega} \tag{2.139}$$

and, to eliminate $\dot{b}$, we differentiate (2.131), and rearrange the result to write $\dot{b}$ in the form:

$$\dot{b} = (\ddot{c}+\mathrm{i}\omega\dot{c})\frac{m_\alpha}{q_\alpha}\,, \tag{2.140}$$

which, substituted into (2.139), gives:

$$\mathrm{i}\omega(\dot{c}+\mathrm{i}\omega c) = \frac{q_\alpha}{m_\alpha}\left[-\frac{q_\alpha B^2 c}{m_\alpha}+\frac{q_\alpha}{m_\alpha}-(\ddot{c}+\mathrm{i}\omega\dot{c})\frac{m_\alpha}{q_\alpha}\right]. \tag{2.141}$$

By regrouping the terms in (2.141), we obtain:

$$\ddot{c}+2\mathrm{i}\omega\dot{c} = \frac{q_\alpha^2}{m_\alpha^2}-\omega_c^2 c+\omega^2 c\,, \tag{2.142}$$

such that for resonance ($\omega = \omega_c$):

$$\ddot{c}+2\mathrm{i}\omega\dot{c} = \frac{q_\alpha^2}{m_\alpha^2}\,. \tag{2.143}$$

A valid particular solution for (2.143) is $\ddot{c} = 0$, which leads to $\dot{c} = q_\alpha^2/2\mathrm{i}\omega m_\alpha^2$, from which finally:

[70] The adverb mainly is used to emphasise that the weakest amplitude in (2.137) is not completely negligible, depending on the ratio ω/ω_c.

$$c = \frac{q_\alpha^2 t}{2\mathrm{i}\omega m_\alpha^2} \; . \tag{2.144}$$

The expression (2.144) for c substituted into (2.131) gives for b:

$$b = \frac{m_\alpha}{q_\alpha}\left[\frac{q_\alpha^2}{2\mathrm{i}\omega m_\alpha^2} + \frac{q_\alpha^2 t}{2m_\alpha^2}\right] = \frac{q_\alpha}{2m_\alpha\omega}[\omega t - \mathrm{i}] \; . \tag{2.145}$$

Ultimately, the particular solution can be written:

$$\boldsymbol{w}_2 = \left[-\frac{\mathrm{i}q_\alpha}{m_\alpha\omega}\boldsymbol{E}_{0\parallel} + \frac{q_\alpha}{2m_\alpha\omega}(\omega t - \mathrm{i})\boldsymbol{E}_{0\perp} - \frac{\mathrm{i}q_\alpha^2 t}{2\omega m_\alpha^2}(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right] \mathrm{e}^{\mathrm{i}\omega t} \; . \tag{2.146}$$

Discussion of the solution

- the motion parallel to $\boldsymbol{B}$ is the same as that for non-resonance (and it is obviously independent of $\boldsymbol{B}$).
- the motion in the plane perpendicular to $\boldsymbol{B}$ is completely different. The terms involving $\boldsymbol{E}_{0\perp}$ and $(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})$ increase indefinitely with time, and this motion tends towards an infinite amplitude: this is the phenomenon of gyro-magnetic resonance or *cyclotron resonance.*

The motion in the plane perpendicular to $\boldsymbol{B}$ can, in fact, be decomposed into 2 parts:

- a motion along $\boldsymbol{E}_{0\perp}$, purely oscillatory, with limited amplitude;
- a motion along $\boldsymbol{E}_{0\perp}$ and a motion along $\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B}$, $\pi/2$ out of phase with respect to each other and with increasing amplitude: the result is a spiral of increasing radius r_B, as can readily be verified, but with constant rotation frequency (because $\omega_{c\alpha} = -q_\alpha B/m_\alpha$ is independent of the particle velocities).

Remarks:

1. If the $\boldsymbol{E}_\perp$ component of the electric field rotates in the opposite direction to the particle cyclotron motion, and at the same frequency, i.e. $\omega = -\omega_c$, there can be no resonance (see exercise 2.7).
2. It is obvious that the amplitude of the cyclotron motion cannot increase indefinitely because:
   - collisions can interrupt the electron (ion) motion, limiting the gain in energy,
   - in any case, the increase of the electron (ion) gyro-radius is limited by the dimensions of the vessel.

2.2.3 *Magnetic field either (slightly) non uniform or (slightly) varying in time*

The treatment of the equations of motion until now has been purely analytical, with no approximation. To deal with cases where particles are subjected to magnetic fields which are no longer uniform or no longer static, we must limit ourselves to $\boldsymbol{B}$ fields which are only slightly spatially non uniform, or slowly varying in time. This restriction allows us to consider a helical trajectory about an initial line of force, which imperceptibly modifies the orbit during a cyclotron rotation: in other words, a number of complete gyrations are required before the axial velocity of the guiding centre or its initial position in the direction perpendicular to $\boldsymbol{B}$ is significantly modified[71]. This slow variation of the guiding centre motion allows us to introduce the *guiding centre approximation*, also called the *adiabatic approximation* (in the sense that the particle energy varies very slowly), this concept being developed using a perturbation method.

Characteristics of the guiding centre approximation

- To zeroth order in this approximation, the trajectory in the plane perpendicular to $\boldsymbol{B}$ is circular. At a given point on the line of the field $\boldsymbol{B}$ defining the guiding centre axis, the field $\boldsymbol{B}$ is assumed to be uniform both in the plane containing the cyclotron trajectory and axially: this is the *local uniformity approximation*. At another point on this field line, the field $\boldsymbol{B}$ can be different, but it is once again assumed to be uniform transversely and axially. In the absence of an applied electric field, the motion in the direction of $\boldsymbol{B}$ is uniform. The complete trajectory is helical.
- To first order, the "inhomogeneties" (spatial or temporal) introduce variations in the guiding centre motion in both the direction of $\boldsymbol{B}$ (we are looking in particular for the axial velocity) and that perpendicular to $\boldsymbol{B}$ (of particular interest is the lateral position). These inhomogeneities occur locally, transversally as well as axially, as perturbations in the $\boldsymbol{B}$ field, assumed to be uniform to zeroth order.

The orbital magnetic moment associated with the cyclotron motion as a constant of motion defining the guiding centre approximation

The local uniformity approximation method that we have just introduced can be justified physically, and developed using a simple mathematical method,

[71] Recall that the *guiding centre axis* is defined instantaneously by the line of force of the field $\boldsymbol{B}$ around which the cyclotron motion occurs.

making use of the *orbital magnetic moment*, an invariant associated with the cyclotron component of the helical motion of the charged particles.

Definition: The magnetic moment μ of a current loop of intensity I bounding a surface S is equal to SI. In the context of our approximation, to order zero, we have $S = \pi r_B^2$ and $I = q_\alpha N_{\mathcal{T}_c}$, where $N_{\mathcal{T}_c}$ is the number of turns per second which are effected by the charged particle on its cyclotron orbit. Since $N_{\mathcal{T}_c} \equiv f_c = \omega_c/2\pi$, the modulus of μ is given by:

$$|\mu| = \pi r_B^2 \frac{q_\alpha|\omega_c|}{2\pi} \tag{2.147}$$

and:

$$|\mu| = \pi\left(\frac{w_\perp^2}{\omega_c^2}\right)\frac{q_\alpha|\omega_c|}{2\pi} = \frac{w_\perp^2 q_\alpha}{2|\omega_c|} = \frac{1}{2}\frac{m_\alpha w_\perp^2}{B} = \frac{\mathcal{E}_{\mathrm{kin}\perp}}{B}\ , \tag{2.148}$$

where $\mathcal{E}_{\mathrm{kin}\perp}$ is the kinetic energy of the particle in the plane perpendicular to $\boldsymbol{B}$. Since the magnetic field created by the cyclotron motion of the particle tends to oppose the applied field $\boldsymbol{B}$ (see p. 117, and the remark on diamagnetism), $\boldsymbol{\mu}$ is a vector anti-parallel to $\boldsymbol{B}$.

The magnetic moment is a constant of motion (to order zero)

Consider the case where the variation in $\boldsymbol{B}$ is simply a function of time[72]. From Maxwell's equations, this leads to the appearance of an electric field:

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}\ , \tag{2.149}$$

which can accelerate (decelerate) the particles (without modifying the total kinetic energy). Thus, in the direction perpendicular to $\boldsymbol{B}$, we can write (2.10) such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_\alpha w_\perp^2\right) \equiv q_\alpha \boldsymbol{E}\cdot\boldsymbol{w}_\perp\ , \tag{2.150}$$

where $\boldsymbol{E}$ is the field induced by the variation of $\boldsymbol{B}$ with time ($\partial\boldsymbol{B}/\partial t$). In this case, the variation in kinetic energy over a period $2\pi/\omega_c$ is given by:

$$\delta\left(\frac{1}{2}m_\alpha w_\perp^2\right) = \int_0^{2\pi/\omega_c} q_\alpha \boldsymbol{E}\cdot\frac{\mathrm{d}\boldsymbol{\ell}}{\mathrm{d}t}\,\mathrm{d}t\ , \tag{2.151}$$

[72] We could equally define the adiabaticity of $\boldsymbol{\mu}$ considering a spatial inhomogeneity: this is a question of reference frame. If $\boldsymbol{B}$ is inhomogeneous in the laboratory frame, in the frame of the particle, $\boldsymbol{B}$ varies with time.

where $\mathrm{d}\boldsymbol{\ell}/\mathrm{d}t$ is the instantaneous curvilinear velocity vector, tangent to the trajectory at each point. If we now suppose that the velocity parallel to $\boldsymbol{B}$ is not very large and that the guiding centre is only slightly displaced perpendicular to $\boldsymbol{B}$, notably because the field $\boldsymbol{B}$ does not greatly vary (the basic assumption for this calculation method), we can replace the integral over the helical trajectory by a line integral along the circular orbit (not perturbed by the inhomogeneity). Then, calling on Stokes theorem, which states that "the line integral of a vector along a closed contour is equal to the rotational flux of this vector traversing any surface bounded by this contour", we obtain:

$$\delta\left(\frac{1}{2}m_\alpha w_\perp^2\right) = \oint q_\alpha \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{\ell} = q_\alpha \iint_S (\boldsymbol{\nabla}\wedge\boldsymbol{E})\cdot\mathrm{d}\boldsymbol{S} \tag{2.152}$$

and:

$$\delta\left(\frac{1}{2}m_\alpha w_\perp^2\right) = -q_\alpha \iint_S \frac{\partial \boldsymbol{B}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} = \pm q_\alpha \frac{\partial B}{\partial t}\pi r_B^2\ , \tag{2.153}$$

since $\partial\boldsymbol{B}/\partial t$ is a flux perpendicular to the plane of the cyclotron motion (adiabatic approximation) and therefore to the surface element $\mathrm{d}\boldsymbol{S}$. The sign of the cosine of the angle between the direction of the normal to the elementary surface and the vector $\partial\boldsymbol{B}/\partial t$ determines the sign of the integrand.

The variation of the kinetic energy **per unit time** then takes the form ($\mathcal{T}_c$ being the period of gyration):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_\alpha w_\perp^2\right) = \pm q_\alpha \frac{\partial B}{\partial t}\frac{\pi r_B^2}{\mathcal{T}_c} \equiv \frac{\partial B}{\partial t}\frac{\pi r_B^2 q_\alpha |\omega_c|}{2\pi} \tag{2.154}$$

and from (2.49), by definition, we find simply that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_\alpha w_\perp^2\right) = \mu\frac{\partial B}{\partial t}\ . \tag{2.155}$$

Also, following (2.148), it is equally possible to write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m_\alpha w_\perp^2\right) = \frac{\mathrm{d}}{\mathrm{d}t}(\mu B) \equiv \frac{\partial \mu}{\partial t}B + \mu\frac{\partial B}{\partial t}\ , \tag{2.156}$$

such that, by comparing (2.155) and (2.156), it is obvious that $\partial\mu/\partial t = 0$, which shows that the moment μ is a constant in time.

This constant of motion is called the *first adiabatic invariant*. Remember that the magnetic moment is strictly constant only if $\boldsymbol{B}$ is completely uniform and $\boldsymbol{w}_{0\parallel} = 0$; it is constant, to a first approximation, if the change in $\boldsymbol{B}$ is slow, that is to say adiabatic.

Remark: In so far as one can consider the moment μ to be constant, the corresponding ratio $\mathcal{E}_{\text{kin}\perp}/B$ also remains constant and therefore whenever $\boldsymbol{B}$ varies, $\mathcal{E}_{\text{kin}\perp}$ should also vary in the same way and proportionally. Since the total kinetic energy is conserved (in the absence of an applied field $\boldsymbol{E}$), the values of $w_{\|}$ and $w_{\perp}$ will be modified in such a way that $w_{\perp}$ decreases and $w_{\|}$ increases and vice versa.

Static magnetic field, but non uniform in the direction parallel to $\boldsymbol{B}$ ($\boldsymbol{E} = \boldsymbol{0}$)

We will continue to suppose that there is no applied field $\boldsymbol{E}$ [73]. A priori, we are led to represent the magnetic field as being purely axial:

$$\boldsymbol{B} = B(z)\hat{\mathrm{e}}_z \ , \tag{2.157}$$

which will be proved to be incorrect: the gradient in $\boldsymbol{B}$ along z necessarily requires the existence of a component B_r. To see this, we assume a field $\boldsymbol{B}$ which is axially symmetric, as is shown, as an example, in Fig. 2.10.

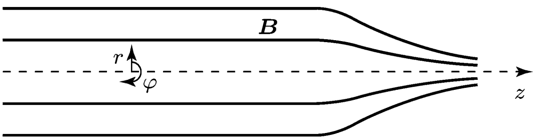


Fig. 2.10 Approximate representation of the lines of force in the case where the field $\boldsymbol{B}$ is axially symmetric and axially non uniform. The contraction of the lines of force indicates an increase in the intensity of B.

We need simply to consider the Maxwell equation:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{2.158}$$

(which signifies that the magnetic field lines should close) and to expand it in cylindrical coordinates as suggested by the symmetry of the problem. The units of local length are $e_1 = 1$, $e_2 = 1$ et $e_3 = r$, for the coordinates z, r, φ respectively[74]. We then obtain:

[73] Since $\boldsymbol{B}$ is constant in the laboratory frame, $\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\partial \boldsymbol{B}/\partial t$ is zero and there is no electric field, which is not the case in the frame of the particle!

[74] Quite generally, the divergence of a vector can be expressed as (see Appendix XX):

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = \frac{1}{e_1 e_2 e_3} \left[\partial_1(e_2 e_3 B_1) + \partial_2(e_1 e_3 B_2) + \partial_3(e_1 e_2 B_3)\right] \ ,$$

where $\boldsymbol{\nabla} \cdot \boldsymbol{B}$ is in fact a pseudo-scalar (see Appendix VII).

$$\nabla \cdot \boldsymbol{B} = \frac{\partial}{\partial z} B_z + \frac{1}{r}\frac{\partial}{\partial r}(r B_r) + \frac{1}{r}\frac{\partial}{\partial \varphi} B_\varphi = 0. \tag{2.159}$$

By construction, Fig. 2.10 shows an axial symmetry of the $\boldsymbol{B}$ field, that is to say $\partial B_\varphi / \partial \varphi = 0$, such that':

$$\frac{1}{r}\frac{\partial}{\partial r}(r B_r) = -\frac{\partial}{\partial z} B_z \,, \tag{2.160}$$

which implies that the inhomogeneity of the field $\boldsymbol{B}$ in its own direction cannot exist without the presence of a transverse component, which is B_r in the present case.

1. The expression for $\boldsymbol{B}$ in the neighbourhood of its axis of symmetry, for a weakly non-uniform field
   Assume that we know a priori the expression for $B_z(z)$ and its gradient $(\partial B_z/\partial z)_{r=0}$ at $r = 0$. In addition, we can use Fig. 2.10 to see that B_z passes radially through a maximum on the axis of symmetry and that at $r = 0$, $\partial B_z/\partial r = 0$. Based on this, we assume that in the region close to the axis, $(\partial B/\partial z)_{r \simeq 0} \simeq$ constant, such that the B_z component is independent of r to second order. Under these conditions, by integration of (2.160) over r in the neighbourhood of the axis:

$$r B_r \approx -\int_0^r r' \left(\frac{\partial B_z}{\partial z}\right)_{r'=0} \mathrm{d}r' = -\frac{1}{2} r^2 \left(\frac{\partial B_z}{\partial z}\right)_{r=0} . \tag{2.161}$$

   The complete and correct expression for the field $\boldsymbol{B}$ when it is non uniform in its own direction, and with the assumption of axial symmetry, is not (2.157), but rather:

$$\boldsymbol{B} = \hat{\mathbf{e}}_z B_z(z) - \hat{\mathbf{e}}_r \frac{r}{2} \left(\frac{\partial B_z}{\partial z}\right)_{r=0} . \tag{2.162}$$

   Note that the correction introduced by the B_r component becomes more important when the axial gradient is large, and as we move away from the axis. Under the basic assumptions of our calculation, this correction is of first order, and is in fact linear in r in the vicinity of the axis.
   Because the B_φ component is zero, and therefore $\boldsymbol{B} = \hat{\mathbf{e}}_r B_r + \hat{\mathbf{e}}_z B_z$, we can express $\boldsymbol{B}$ in Cartesian coordinates in the following way:

$$\boldsymbol{B} = -\frac{1}{2} x \left(\frac{\partial B_z}{\partial z}\right)_{x=y=0} \hat{\mathbf{e}}_x \; - \frac{1}{2} y \left(\frac{\partial B_z}{\partial z}\right)_{0,0} \hat{\mathbf{e}}_y + B_z \hat{\mathbf{e}}_z \, . \tag{2.163}$$

2. The trajectory of a charged particle in the calculated field $\boldsymbol{B}$
   We must solve:

$$m_\alpha \dot{\boldsymbol{w}} = q_\alpha (\boldsymbol{w} \wedge \boldsymbol{B}) \, . \tag{2.164}$$

From our assumptions, the component of velocity perpendicular to $\boldsymbol{B}$ can be obtained, to first approximation, by supposing that the cyclotron motion takes place in a locally uniform field. All that remains is to calculate $\boldsymbol{w}_\parallel$.

3. The equation of motion in the direction of B_z
Since the field $\boldsymbol{B}$ is not completely uniform along z, the velocity of the guiding centre in the same direction does not remain constant.
To calculate this, set $\boldsymbol{w} = w_x\hat{\mathbf{e}}_x + w_y\hat{\mathbf{e}}_y + w_z\hat{\mathbf{e}}_z$, and consider (2.164):

$$m_\alpha \dot{\boldsymbol{w}}_\parallel = \hat{\mathbf{e}}_z q_\alpha [B_y w_x - B_x w_y] \, . \tag{2.165}$$

The variation of the guiding centre axial velocity described by (2.165) stems from the first order of our calculation method. It is therefore correct to use the zero order velocities in the plane perpendicular to the z axis to develop (2.165):

$$m_\alpha \dot{\boldsymbol{w}}_\parallel \approx \hat{\mathbf{e}}_z q_\alpha \left[-\frac{1}{2} y \left(\frac{\partial B_z}{\partial z} \right)_{0,0} w_x + \frac{1}{2} x \left(\frac{\partial B_z}{\partial z} \right)_{0,0} w_y \right] , \tag{2.166}$$

where the term $(\partial B_z/\partial z)_{0,0}$ is, by assumption, of first order while x, y, w_x and w_y are of order zero; the term on the RHS of (2.166) is thus of first order.

4. Solution of the equation of motion
The expressions for the position and velocity in the plane perpendicular to $\boldsymbol{B}$ are, from the assumptions of the approximation method, those already obtained in a uniform field $\boldsymbol{B}$ (Sect. 2.2.2, $\boldsymbol{E} = 0$). They can be written more succinctly:

$$w_x = A \sin(\omega_c t - \varphi) \, , \qquad x = -\frac{A}{\omega_c} \cos(\omega_c t - \varphi) \, , \tag{2.167}$$

$$w_y = A \cos(\omega_c t - \varphi) \, , \qquad y = \frac{A}{\omega_c} \sin(\omega_c t - \varphi) \, . \tag{2.168}$$

Setting $w_x(0) = 0$ and $w_y(0) = w_{y0}$, which leads to $\varphi = 0$ and $A = w_{y0}$, respectively, we obtain:

$$w_x = w_{y0} \sin \omega_c t \, , \qquad x = -\frac{w_{y0}}{\omega_c} \cos \omega_c t \, , \tag{2.169}$$

$$w_y = w_{y0} \cos \omega_c t \, , \qquad y = \frac{w_{y0}}{\omega_c} \sin \omega_c t \, . \tag{2.170}$$

This solution is such that, with $\omega_c > 0$ and $\boldsymbol{B}$ entering the page, the electrons are seen to rotate in the anti-clockwise direction; to check it, consider the values of x and y at $t = 0$ and $t = \pi/2\omega_c$. There is thus a change in convention, and to re-establish the motion in the true direction, we need to set $\omega_{ce} = -eB/m$ instead of $\omega_{ce} = eB/m$.

In order to come back to our initial conventions (Sect. 2.2.2, $\boldsymbol{E}=0$), we must take $w_x = A\cos(\omega_c t-\varphi)$ and $w_y = A\sin(\omega_c t-\varphi)$ with $w_y(0)=0$ and $w_x(0)=w_{x0}$ at $t=0$. This yields:

$$w_x = w_{x0}\cos\omega_c t\ , \qquad x = \frac{w_{x0}}{\omega_c}\sin\omega_c t\ , \tag{2.171}$$

$$w_y = w_{x0}\sin\omega_c t\ , \qquad y = -\frac{w_{x0}}{\omega_c}\cos\omega_c t\ . \tag{2.172}$$

We can easily verify that (2.169) and (2.170) lead to $x^2+y^2=(w_{y0}/\omega_c)^2 = r_B^2$. Thus, by substituting (2.169) and (2.170) into (2.166):

$$m_\alpha \dot{\boldsymbol{w}}_\parallel = \hat{\mathbf{e}}_z\frac{q_\alpha}{2}\left(\frac{\partial B_z}{\partial z}\right)_{0,0}\left[-\frac{w_{y0}^2}{\omega_c}\sin^2\omega_c t-\frac{w_{y0}^2}{\omega_c}\cos^2\omega_c t\right]\ , \tag{2.173}$$

$$\begin{aligned} m_\alpha \dot{w}_\parallel &= -\frac{q_\alpha}{2}\left(\frac{\partial B_z}{\partial z}\right)_{0,0}\left(\frac{w_{y0}^2}{\omega_c}\right) \\ &= -\frac{q_\alpha}{2}\left(\frac{\partial B_z}{\partial z}\right)_{0,0}\left(\frac{r_B^2\omega_c^2 m_\alpha}{q_\alpha B_\parallel}\right)\ , \end{aligned} \tag{2.174}$$

where, to allow for the sign of ω_c, we have chosen, exceptionally, $\omega_c = (q_\alpha/m_\alpha)B_\parallel$[75]. Simplifying:

$$\dot{w}_\parallel = -\frac{1}{2}\frac{r_B^2\omega_c^2}{B_\parallel}\left(\frac{\partial B_z}{\partial z}\right)_{0,0} \tag{2.175}$$

from which, finally, after integration:

$$\boldsymbol{w}_\parallel(t) = \boldsymbol{w}_\parallel(0) - \frac{\hat{\mathbf{e}}_z}{2}r_B^2\omega_c^2\frac{1}{B_\parallel}\left(\frac{\partial B_z}{\partial z}\right)_{0,0} t\ . \tag{2.176}$$

This is the velocity, entirely parallel to $\boldsymbol{B}$, of the guiding centre in the case where the gradient in $\boldsymbol{B}$ is principally in the direction of the field.
From (2.174), we can also derive an expression that will be useful later:

$$F_z = m_\alpha\dot{w}_\parallel = -\frac{1}{2}m_\alpha w_{\perp 0}^2\frac{1}{B_\parallel}\left(\frac{\partial B_z}{\partial z}\right)_{0,0} \equiv -\mu\left(\frac{\partial B_z}{\partial z}\right)_{0,0}\ . \tag{2.177}$$

Appendix X suggests another demonstration of expression (2.177). In addition, Appendix XI uses (2.177) to show, with a different method than that developed from (2.149) to (2.156), that μ is a constant of motion in the guiding centre approximation.

[75] $B_\parallel$ represents the value of $B_z(z)$ along $z=0$ (region of uniform $\boldsymbol{B}$).

5. Analysis of the motion $\boldsymbol{w}_{\parallel}$: retardation or acceleration of charged particles along an axial gradient in $\boldsymbol{B}$
Following (2.176), the gradient $\partial B_z/\partial z$ subjects the charged particles to:
   - either a retardation if $\partial B_z/\partial z > 0$ because in this case $w_{\parallel}(t)$ slows down as a function of time, and finally changes the sign of the RHS of (2.176) with respect to the LHS. If B_0 is the value in the uniform $\boldsymbol{B}$ region and $B_{\max}$ the maximum value of B (Fig. 2.11), the region $B_0 < B < B_{\max}$ where the particles are subject to reflection is called a *magnetic mirror*.
   - or an acceleration if $\partial B_z/\partial z < 0$, as is the case after reflection by a mirror, for example.

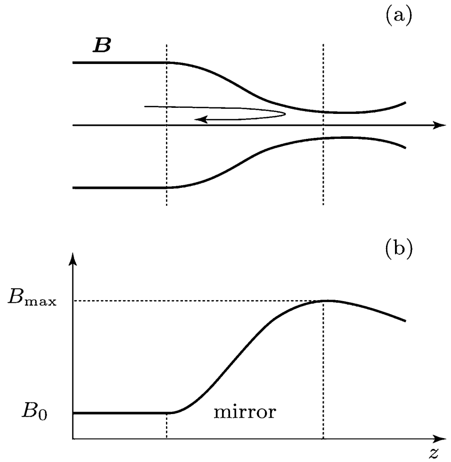


Fig. 2.11 **a** Magnetic field for the confinement of charged particles showing the mirror zone where they are reflected. **b** The value of B increases as the lines of force (figure a) get closer together.

The type of action exercised by $\partial B_z/\partial z$ on the velocity depends neither on the charge of the particle or its mass, because from (2.176):

$$\boldsymbol{w}_{\parallel} = \boldsymbol{w}_{\parallel 0} - \frac{\hat{\mathbf{e}}_z}{2} \frac{w_{\perp}^2}{B_{\parallel}} \left(\frac{\partial B_z}{\partial z} \right)_{0,0} t \,, \tag{2.178}$$

and there is thus the possibility of confining all the charged particles. The efficiency of the confinement depends, finally, on the ratio $w_{\parallel}(0)/w_{\perp}(0)$: if it is too large, the mirror cannot play its role, as we will show below.

Remark: The role of the magnetic mirror (Fig. 2.11) can also be understood from the fact that, in the absence of an applied $\boldsymbol{E}$ field and within our guiding centre approximation, the total kinetic energy of the particle is conserved:

$$W_{\perp} + W_{\parallel} = \text{constant} \tag{2.179}$$

and only the ratio $W_\parallel/W_\perp$ can vary, thus:

$$\mathrm{d}W_\parallel = -\mathrm{d}W_\perp \ . \tag{2.180}$$

In addition, from (2.177), we can write the infinitesimal element of work effected by the particle on the field $\boldsymbol{B}$ in terms of the kinetic energy parallel to $\boldsymbol{B}$[76]:

$$F\mathrm{d}z \equiv \mathrm{d}W_\parallel = -\mu\mathrm{d}B_\parallel \ . \tag{2.181}$$

Inserting (2.180) in (2.181) and because $\mu = W_\perp/B$ (2.148), we have:

$$\mathrm{d}W_\perp = \mu\mathrm{d}B_\parallel = \frac{W_\perp}{B_\parallel}\mathrm{d}B_\parallel \tag{2.182}$$

or also:

$$\frac{\mathrm{d}W_\perp}{\mathrm{d}B_\parallel} = \frac{W_\perp}{B_\parallel} \equiv \mu \ . \tag{2.183}$$

This result signifies that if $B_\parallel$ increases, $W_\perp$ must increase, such that the ratio $W_\perp/B_\parallel$ remains constant. When the particle enters into the mirror zone, its energy $W_\parallel$ will decrease, if need be to zero, after which it will increase again after being "reflected". Since $W_\perp$ increases in the mirror neck (Fig. 2.11a), and because $r_B = W_\perp/B$, the question is whether the value of r_B could become so large that the particle reaches the wall. In fact, the value of r_B in the mirror zone decreases because the value of $\boldsymbol{B}$ increases more rapidly[77] than $W_\perp$.

6. The loss cone in the magnetic mirror of a linear machine
Consider the typical configuration of a linear magnetic confinement machine, with a mirror at each extremity such as that shown in Fig. 2.12. We are looking for the conditions such that the incident particles "cross the mirror", i.e. are lost.
Consider a particle traversing the uniform zone with a velocity $\boldsymbol{w}_0$ (making an angle α_0 with $\boldsymbol{B}$), as is shown in Fig. 2.13a. Let us now separate the velocity of this particle into parallel and perpendicular components with respect to the field $\boldsymbol{B}$. Thus in the region of uniform field (Fig. 2.13b), $\boldsymbol{w}_0 = \boldsymbol{w}_{0\parallel} + \boldsymbol{w}_{0\perp}$ (the subscript 0 indicates that the particle is in the homogeneous field region of the machine) where:

[76] In (2.177), we found $F = -\mu\partial B_z/\partial z$, from which $F\mathrm{d}z \simeq -\mu\mathrm{d}B_\parallel$.

[77] To verify this assertion, it is sufficient to differentiate $r_B^2 = w_\perp^2/\omega_c^2$. Taking (2.182) into account, we find

$$\mathrm{d}r_B = -\frac{m_\alpha W_\perp}{r_B q_\alpha^2 B_\parallel^2}\left(\frac{\mathrm{d}B_\parallel}{B_\parallel}\right) .$$

In consequence, if the gradient of $B_\parallel$ is positive (mirror zone), the Larmor radius effectively decreases when $B_\parallel$ increases ($\mathrm{d}r_B$ is negative).

$$w_{0\|} = w_0 \cos\alpha_0 \;, \tag{2.184}$$

$$w_{0\perp} = w_0 \sin\alpha_0 \;, \tag{2.185}$$

with $w_0 = \sqrt{w_{0\|}^2 + w_{0\perp}^2}$.

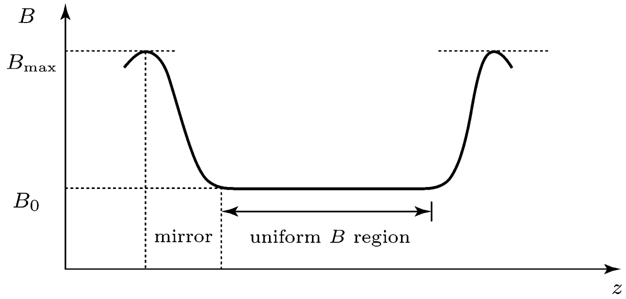


Fig. 2.12 Typical configuration of the confining magnetic field of a linear discharge in which each extremity is closed by a magnetic mirror (a configuration referred to as "minimum B").

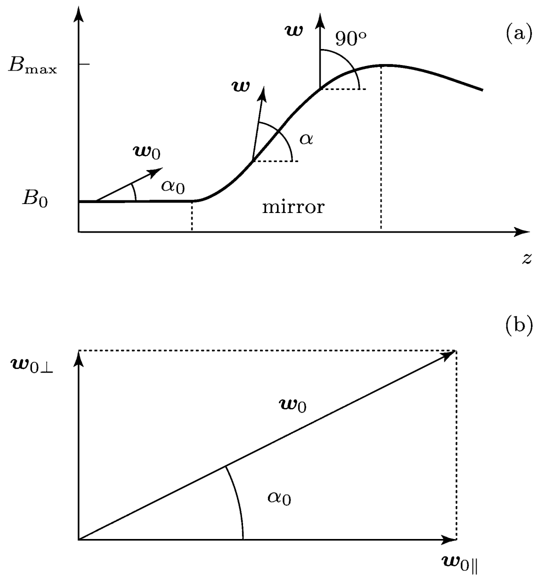


Fig. 2.13 a Orientation of the velocity vector with respect to the z axis in the zone of uniform B (α_0) and the mirror zone (α). **b** Decomposition of the velocity $\boldsymbol{w}_0$ along the z axis ($\boldsymbol{w}_{0\|}$) and perpendicular to it ($\boldsymbol{w}_{0\perp}$).

In the absence of an applied $\boldsymbol{E}$ field and with the assumption that $\boldsymbol{B}$ varies slowly along z, we know that $m_\alpha w_0^2/2$ = constant (where only the ratio $w_\perp/w_\|$ can vary) and that the magnetic moment μ is constant to first order.

We can thus establish a relation between the velocity in the uniform field region and that in the mirror, noting that from (2.168):

$$\mu = \frac{\frac{1}{2} m_\alpha w_0^2 \sin^2 \alpha_0}{B_0} = \frac{\frac{1}{2} m_\alpha w_0^2 \sin^2 \alpha}{B} , \tag{2.186}$$

where $w_\perp = w_0 \sin \alpha$ in the mirror region, such that:

$$\sin \alpha = \sin \alpha_0 \sqrt{\frac{B}{B_0}} \, . \tag{2.187}$$

There is a reflection of the particle in the case when $\alpha > \pi/2$. Equation (2.187) shows that if α_0 is sufficiently small (corresponding to a large enough "parallel" component of velocity of the particle in the homogeneous field region), the value of B/B_0 cannot be large enough to reach at least $\alpha = \pi/2$ ($\sin \alpha = 1$); it is certainly true for $\alpha_0 = 0$! When this is the case, the particle will cross the mirror and be neutralised on the end walls, and it will be "lost" for the plasma. We will denote α_{0m} as the minimum angle of α_0 for which there is still a reflection of particles at the maximum of the field B_{max}. If we define the *mirror ratio* by:

$$\mathcal{R} \equiv B_{\text{max}}/B_0 \, , \tag{2.188}$$

the value α_{0m} is obtained for $\sin \alpha = 1$ in (2.187):

$$1 = \sin \alpha_{0m} \sqrt{\frac{B_{\text{max}}}{B_0}} \tag{2.189}$$

and finally:

$$\sin \alpha_{0m} = \frac{1}{\sqrt{\mathcal{R}}} \, . \tag{2.190}$$

The angle α_{0m} defines a cone, in the interior of which the particles leave the plasma at the end of the machine. Note that the efficiency of a magnetic mirror to reflect charged particles is independent of the modulus of the velocity of the particles (w_0) as well as their charge and mass.

7. The percentage of incident particles reflected by a magnetic mirror
We will consider the preceding magnetic field configuration (Fig. 2.13a) and suppose that the angular distribution of the particle velocities is isotropic in the uniform region: in other words, the density $n(\alpha_0)$ of particles with an angle α_0 is the same for each value of α_0. We wish to calculate $C_r = \Gamma_r/\Gamma_{\text{inc}}$, the fraction of incident flux Γ_{inc} reflected by the mirror, knowing that there is a reflection if $\alpha_0 > \alpha_{0m}$.
To do this, we must calculate the number of particles per second that are directed towards the mirror, Γ_{inc}, and then subtract the number of them for which $\alpha_0 < \alpha_{0m}$ (and which are not reflected), which will lead us to Γ_r. It is sufficient to establish such a balance for each value of α_0 on an elementary solid angle $\mathrm{d}\Omega$, independently of the value of the azimuthal

angle φ owing to axial symmetry. We therefore consider the solid angle $\mathrm{d}\Omega(\alpha_0, \varphi)$ in which the particles enter (Fig. 2.14). To this angle $\mathrm{d}\Omega(\alpha_0, \varphi)$, there corresponds an elementary surface $\mathrm{d}\sigma(\alpha_0)$ directed along α_0, whose projection perpendicular to the mirror axis, $\mathrm{d}\sigma(\alpha_0)\cos\alpha_0$[78], constitutes the effective surface traversed by the incident flux in the direction of the mirror.

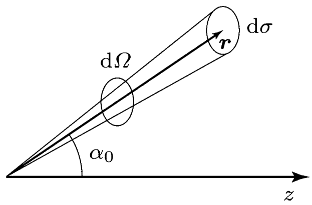


Fig. 2.14 Elementary surface $\mathrm{d}\sigma(\alpha_0)$ collecting particles of velocity $\boldsymbol{w}_0$ directed along α_0 and entering the solid angle $\mathrm{d}\Omega$.

We then have:

$$\Gamma_{\mathrm{inc}}(\alpha_0) = n w_0 \,\mathrm{d}\sigma(\alpha_0) \cos\alpha_0 \;, \tag{2.191}$$

where, as we have seen, n does not depend on α_0.
By definition $\mathrm{d}\sigma = r^2 \mathrm{d}\Omega$ where $\mathrm{d}\Omega$ can be expressed in spherical coordinates $(r, c\alpha_0, \varphi)$ as:

$$\mathrm{d}\Omega = \sin\alpha_0 \,\mathrm{d}\alpha_0 \,\mathrm{d}\varphi \;. \tag{2.192}$$

The axial symmetry implies that the integration over φ yields 2π. We can then write:

$$C_r \equiv \frac{\Gamma_r}{\Gamma_{\mathrm{inc}}} = \frac{n w_0 r^2 (2\pi) \int_{\alpha_{0m}}^{\pi/2} \cos\alpha_0 \sin\alpha_0 \mathrm{d}\alpha_0}{n w_0 r^2 (2\pi) \int_0^{\pi/2} \cos\alpha_0 \sin\alpha_0 \mathrm{d}\alpha_0} \;. \tag{2.193}$$

The result is independent of the magnitude of the velocity, thus it is valid for all particle energy distributions.
Simplifying, and after a trigonometric transformation:

$$C_r \equiv \frac{\int_{\alpha_{0m}}^{\pi/2} \sin 2\alpha_0 \mathrm{d}\alpha_0}{\int_0^{\pi/2} \sin 2\alpha_0 \mathrm{d}\alpha_0} = \frac{-\cos 2\alpha_0 \Big|_{\alpha_{0m}}^{\pi/2}}{-\cos 2\alpha_0 \Big|_0^{\pi/2}} \;, \tag{2.194}$$

which gives:

$$\begin{aligned} C_r &= \frac{1 + \cos 2\alpha_{0m}}{2} = \frac{\left[1 + (1 - 2\sin^2\alpha_{0m})\right]}{2} = 1 - \sin^2\alpha_{0m} \\ &= 1 - \frac{B_0}{B_{\max}} \;, \end{aligned} \tag{2.195}$$

[78] Recall that a flux is by definition always evaluated normal to the surface that it traverses.

from which, finally:

$$C_r = 1 - \frac{1}{\mathcal{R}} \; . \tag{2.196}$$

Remarks:

1. The fraction of reflected particles becomes larger as $\mathcal{R}$ increases, that is to say as $B_{\max}$ becomes more important relative to B_0.
2. Satellite measurements have provided evidence for the existence of belts (layers) of high energy charged particles surrounding the earth. These particles, essentially electrons and protons from the solar wind, are trapped in the earth magnetic field and reflected at the poles: the lines of force of the $\boldsymbol{B}$ field become tighter at the poles, forming a mirror.
3. The particles confined in a system with a mirror at each extremity will oscillate between the two mirrors (see exercises 2.15 and 2.16).

Constant magnetic field, but non uniform in the direction perpendicular to $\boldsymbol{B}$

The following section is divided into two parts: 1) the field lines are assumed rectilinear; 2) the curvature of the field lines is taken into account.

1. Field lines assumed rectilinear
   We consider $\boldsymbol{B}$ entirely directed along the z axis and uniform along this axis. The gradient which affects it is, by hypothesis, perpendicular to it and uniquely directed along the y axis: $\boldsymbol{\nabla} B = (\partial B/\partial y)\hat{\mathbf{e}}_y$ and thus $\partial B/\partial x = 0$. In this case, we will assume that B increases slowly with y such that $\boldsymbol{B}$ can be expressed by:

$$\boldsymbol{B}(y) = \hat{\mathbf{e}}_z B_0(1 + \beta y) \; , \quad 0 < \beta \ll 1 \; . \tag{2.197}$$

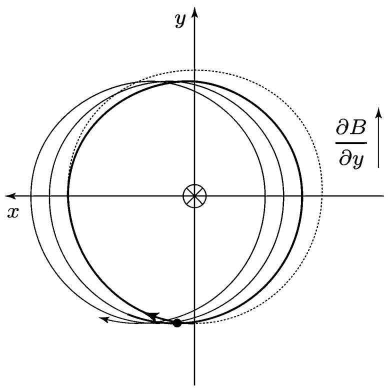


Fig. 2.15 (Trochoidal) trajectory of an electron in the plane perpendicular to the field $B\hat{\mathbf{e}}_z$, which is non uniform in the direction Oy (2.197). There is a magnetic field drift along x.

If the field were uniform ($\beta = 0$), we would have a cyclotron gyration of constant radius in the plane xOy (the dotted trajectory in Fig. 2.15). Due to the inhomogemeity of the field in this plane ($\beta \neq 0$), the trajectory is no longer an exact circle, and it does not close on itself, as is shown in Fig. 2.15[79]: this is due to the fact that the Larmor radius decreases, and with it, the radius of curvature of the trajectory, whenever the particle is moving towards increasing values of y (in the example considered), with the result that the guiding centre shifts. The guiding centre drifts, on average, along increasing x if the particle rotates in the clockwise direction as shown in Fig. 2.15; this average motion (over many periods) is called the *magnetic field drift*. It occurs in the direction perpendicular to $\boldsymbol{B}$ and to $\boldsymbol{\nabla}|\boldsymbol{B}|$, hence its alternative designation as the $\boldsymbol{\nabla}|\boldsymbol{B}|$ *drift*. We will now calculate the velocity $\boldsymbol{w}_{dm}$ of this magnetic field drift.

- The instantaneous velocity of the guiding centre
  To find $\mathrm{d}\boldsymbol{R}_g/\mathrm{d}t$, where $\boldsymbol{R}_g$ is the instantaneous position of the guiding centre (Fig. 2.16), we will call on our adiabatic approximation: the motion of the particle is determined to zeroth order by the cyclotron gyration in the field $\boldsymbol{B}$, when the effects of its non-uniformity are ignored: this motion is perturbed, to first order, by the magnetic field drift.

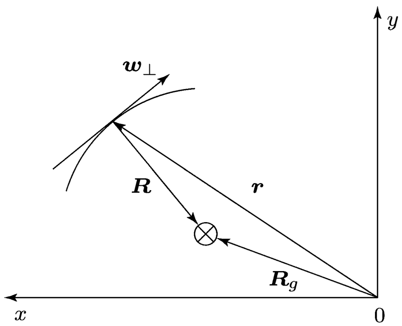


Fig. 2.16 The vector $\boldsymbol{R}$ describes the position of the guiding centre in the frame of the particle (in this case an electron), which itself is at position $\boldsymbol{r}$ in the laboratory frame. Note that $\boldsymbol{R}$ is perpendicular to the cyclotron trajectory at the point considered and that $\boldsymbol{R}_g = \boldsymbol{r} + \boldsymbol{R}$.

Zeroth order motion: calculation of $\boldsymbol{R}$

The radius of the gyration vector $\boldsymbol{R}$ gives the position of the guiding centre with respect to the particle, as illustrated in Fig. 2.16, and we will show that:

$$\boldsymbol{R} = \frac{m_\alpha}{q_\alpha B^2}(\boldsymbol{w} \wedge \boldsymbol{B}) \ . \tag{2.198}$$

To demonstrate this expression, we need only recall that, in general, for a particle situated at $\boldsymbol{r}'$ with respect to the axis about which it is

[79] According to our adiabatic approximation, many complete gyrations are required for this phenomenon to manifest itself.

rotating with a frequency ω, the tangential velocity obeys $\boldsymbol{w} = \boldsymbol{\omega} \wedge \boldsymbol{r}'$. In the present case, this translates into:

$$\boldsymbol{w} = +\frac{q_\alpha \boldsymbol{B}}{m_\alpha} \wedge \boldsymbol{R} \,. \tag{2.199}$$

Multiplying this expression vectorially on the right by $\boldsymbol{B}$, yields:

$$\boldsymbol{w} \wedge \boldsymbol{B} = \frac{q_\alpha}{m_\alpha}(\boldsymbol{B} \wedge \boldsymbol{R}) \wedge \boldsymbol{B} \,. \tag{2.200}$$

Recalling that the double vectorial product obeys the following rule:

$$\boldsymbol{P} \wedge (\boldsymbol{Q} \wedge \boldsymbol{T}) = \boldsymbol{Q}(\boldsymbol{T} \cdot \boldsymbol{P}) - \boldsymbol{T}(\boldsymbol{P} \cdot \boldsymbol{Q}) \,, \tag{2.201}$$

hence:

$$(\boldsymbol{Q} \wedge \boldsymbol{T}) \wedge \boldsymbol{P} = \boldsymbol{T}(\boldsymbol{P} \cdot \boldsymbol{Q}) - \boldsymbol{Q}(\boldsymbol{T} \cdot \boldsymbol{P}) \,, \tag{2.202}$$

we find that:

$$\boldsymbol{w} \wedge \boldsymbol{B} = \frac{q_\alpha}{m_\alpha}\left[\boldsymbol{R}(\boldsymbol{B} \cdot \boldsymbol{B}) - \boldsymbol{B}(\boldsymbol{R} \cdot \boldsymbol{B})\right] \,, \tag{2.203}$$

where the term $\boldsymbol{R} \cdot \boldsymbol{B}$ is zero, because to zeroth order the vector radius of gyration $\boldsymbol{R}$ is necessarily perpendicular to the guiding axis. Then (2.203) leads to (2.198)[80]:

$$\boldsymbol{R} = \frac{m_\alpha}{q_\alpha B^2}(\boldsymbol{w} \wedge \boldsymbol{B}) \,. \tag{2.198}$$

First order motion: calculation of $\boldsymbol{R}_g$

We have assumed till now that the lines of force are rectilinear. In order to avoid repeating the calculation when tackling point 2) where the lines are curvilinear, we set $\boldsymbol{B} = B\hat{\mathbf{e}}_B$ rather than $\boldsymbol{B} = B\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_B$ is the unit vector tangent to the field line, which takes into account the possible curvature of these lines.
Following Fig. 2.16:

$$\boldsymbol{R}_g = \boldsymbol{r} + \boldsymbol{R} \,, \tag{2.204}$$

where $\boldsymbol{R}$ describes the guiding centre motion in the frame of the particle, which is itself at position $\boldsymbol{r}$ in the laboratory frame. We can then rewrite $\boldsymbol{R}$ (2.198) in the form:

$$\boldsymbol{R} = \frac{m_\alpha}{q_\alpha B}(\boldsymbol{w} \wedge \hat{\mathbf{e}}_B) \,. \tag{2.205}$$

[80] In fact, it is sufficient to note that $|\boldsymbol{R}| = m_\alpha w_\perp / q_\alpha B$ ($|R| = r_B$) and that $\boldsymbol{R}$ is perpendicular to $\boldsymbol{w}$ and $\boldsymbol{B}$.

The derivative of (2.204), taking (2.205) into account, gives[81]:

$$\frac{\mathrm{d}\boldsymbol{R}_g}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}t} + \frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{w} - \frac{m_\alpha}{q_\alpha B^2}\frac{\mathrm{d}B}{\mathrm{d}t}(\boldsymbol{w} \wedge \hat{\mathbf{e}}_B) + \frac{m_\alpha}{q_\alpha B}\left(\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \wedge \hat{\mathbf{e}}_B\right) + \frac{m_\alpha}{q_\alpha B}\left(\boldsymbol{w} \wedge \frac{\mathrm{d}\hat{\mathbf{e}}_B}{\mathrm{d}t}\right) , \tag{2.206}$$

where $\mathrm{d}\hat{\mathbf{e}}_B/\mathrm{d}t = 0$ when we assume that $\boldsymbol{B}$ is directed parallel to the z axis (case 1). In the context of point 2) which follows, where we make the assumption of a weak field curvature, we will neglect the term comprising $\mathrm{d}\hat{\mathbf{e}}_B/\mathrm{d}t$ [82]. We can therefore take $\boldsymbol{B} = \hat{\mathbf{e}}_z B$ and (2.206) reduces to:

$$\frac{\mathrm{d}\boldsymbol{R}_g}{\mathrm{d}t} = \boldsymbol{w} - \frac{m_\alpha}{q_\alpha B^3}\frac{\mathrm{d}B}{\mathrm{d}t}(\boldsymbol{w} \wedge \boldsymbol{B}) + \frac{m_\alpha}{q_\alpha B^2}\left(\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \wedge \boldsymbol{B}\right) . \tag{2.207}$$

In order to modify the third term on the RHS, we will take the equation of motion $m_\alpha \mathrm{d}\boldsymbol{w}/\mathrm{d}t = q_\alpha(\boldsymbol{w} \wedge \boldsymbol{B})$ and multiply it on the right vectorially by $\boldsymbol{B}$:

$$m_\alpha \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \wedge \boldsymbol{B} = q_\alpha(\boldsymbol{w} \wedge \boldsymbol{B}) \wedge \boldsymbol{B} . \tag{2.208}$$

Owing to the properties of the double vectorial product (2.199):

$$(\boldsymbol{w} \wedge \boldsymbol{B}) \wedge \boldsymbol{B} = \boldsymbol{B}(\boldsymbol{B} \cdot \boldsymbol{w}) - \boldsymbol{w}(\boldsymbol{B} \cdot \boldsymbol{B}) \equiv \boldsymbol{B}(B w_\parallel) - \boldsymbol{w}B^2 , \tag{2.209}$$

we obtain:

$$m_\alpha \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} \wedge \boldsymbol{B} = q_\alpha(\boldsymbol{w}_\parallel - \boldsymbol{w})B^2 . \tag{2.210}$$

This expression can be substituted in the third term on the RHS of (2.207), which after some reorganisation, becomes:

$$\frac{\mathrm{d}\boldsymbol{R}_g}{\mathrm{d}t} = \boldsymbol{w} + \frac{1}{q_\alpha B^2}\left[q_\alpha(-\boldsymbol{w} + \boldsymbol{w}_\parallel)B^2\right] - \frac{m_\alpha}{q_\alpha B^3}\frac{\mathrm{d}B}{\mathrm{d}t}(\boldsymbol{w} \wedge \boldsymbol{B}) . \tag{2.211}$$

After simplification, we find an expression for the (instantaneous) velocity of the guiding centre in the laboratory frame:

$$\frac{\mathrm{d}\boldsymbol{R}_g}{\mathrm{d}t} = \boldsymbol{w}_\parallel - \frac{m_\alpha}{q_\alpha B^3}\frac{\mathrm{d}B}{\mathrm{d}t}(\boldsymbol{w}_\perp \wedge \boldsymbol{B}) , \tag{2.212}$$

[81] If $\boldsymbol{B}$ is spatially non-uniform in the laboratory frame, it varies with time in the frame of the particle, as already mentioned.

[82] If we include the term $\mathrm{d}\hat{\mathbf{e}}_B/\mathrm{d}t$, its contribution will be of second order in an expression which is of first order. In effect, $\mathrm{d}\hat{\mathbf{e}}_B/\mathrm{d}t = (\partial\hat{\mathbf{e}}_B/\partial y)\partial y/\partial t$ is a second order term.

where the first term represents the guiding centre velocity along the lines of force of the field $\boldsymbol{B}$ (zeroth order expression) and the second term is that in the direction perpendicular to $\boldsymbol{w}_\perp$ and $\boldsymbol{B}$ (first order expression), a motion that varies with time as a result of the cyclotron trajectory of the particle.

- The average velocity of the guiding centre in the plane perpendicular to $\boldsymbol{w}_\perp$ and to $\boldsymbol{B}$: the gradient of the magnetic field drift velocity
In order to calculate the temporal average of the second term on the RHS of (2.212), we rearrange it in the following form:

$$\begin{aligned} \frac{-m_\alpha}{q_\alpha B^3}\frac{\mathrm{d}B}{\mathrm{d}t}(\boldsymbol{w}\wedge\boldsymbol{B}) &= \frac{-m_\alpha}{q_\alpha B^3}\frac{\partial B}{\partial y}w_y\,(w_x\hat{\mathbf{e}}_x + w_y\hat{\mathbf{e}}_y + w_z\hat{\mathbf{e}}_z)\wedge B_z\hat{\mathbf{e}}_z \\ &= \frac{-m_\alpha}{q_\alpha B^3}\frac{\partial B}{\partial y}w_y(\hat{\mathbf{e}}_x w_y B_z - \hat{\mathbf{e}}_y w_x B_z)\,, \end{aligned} \tag{2.213}$$

where the RHS is now expressed in the laboratory frame. Since the temporal average of $w_x w_y$ is zero[83] and that:

$$\overline{w_y^2} = \frac{1}{2}w_\perp^2\,, \qquad (w_\perp^2 \equiv \overline{w_x^2} + \overline{w_y^2})\,, \tag{2.214}$$

the velocity associated with the average motion of the magnetic field drift finally reduces to:

$$\boldsymbol{w}_{dm} = \frac{-m_\alpha}{q_\alpha B^2}\frac{\partial B}{\partial y}\frac{w_\perp^2}{2}\hat{\mathbf{e}}_x\,. \tag{2.215}$$

This expression can be transformed, since in a direct trihedral coordinate system (contrary to a indirect one) $-\hat{\mathbf{e}}_x = \hat{\mathbf{e}}_z \wedge \hat{\mathbf{e}}_y$, into:

$$\boldsymbol{w}_{dm} = m_\alpha \frac{w_\perp^2}{2}\frac{1}{q_\alpha B^3}(\boldsymbol{B}\wedge\boldsymbol{\nabla}B) \tag{2.216}$$

or, equivalently:

$$\boldsymbol{w}_{dm} = \frac{\mu}{q_\alpha}\frac{(\boldsymbol{B}\wedge\boldsymbol{\nabla}B)}{B^2}\,, \tag{2.217}$$

which is the *magnetic field drift velocity* of a particle in the presence of a gradient in the field perpendicular to $\boldsymbol{B}$ and assumed to have no curvature[84].
The relation (2.217) could have been obtained directly from the general expression giving the drift velocity of charged particles subjected to a magnetic field in the presence of a given force, as is shown in Ap-

[83] Larmor motion: if w_x is proportional to $\sin\omega_c t$ and w_y is proportional to $\cos\omega_c t$, since these two functions are orthogonal, the time integral of $w_x w_y$ over a period is zero.

[84] In fact, this gradient is related to the lines of force because $\beta \simeq 1/\rho$ (XIII.18).

pendix XII. Appendix XIII allows us, in addition, to write (2.217) in the form:

$$\boldsymbol{w}_{dm} = -\frac{1}{\omega_c B}\frac{w_\perp^2}{2}\left(\frac{\boldsymbol{\rho}}{\rho^2} \wedge \boldsymbol{B}\right) , \tag{2.218}$$

where $\boldsymbol{\rho}$ is the radius of curvature (see Fig. XIII.1). It will be useful to compare this expression with that of the curvature drift velocity, which we will now calculate.

2. Accounting for the curvature of the field lines
The magnetic field drift, for which we have just established the equations of motion, cannot exist alone, because the lines of force of $\boldsymbol{B}$ which we have supposed to be rectilinear by setting:

$$\boldsymbol{B} = \hat{\mathbf{e}}_z B_0(1 + \beta y) , \tag{2.219}$$

where $\beta \ll 1$, are not really so! In fact, although Maxwell's equation for the divergence of $\boldsymbol{B}$:

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} \equiv \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \tag{2.220}$$

can be trivially verified, in contrast to Maxwell's equation $\boldsymbol{\nabla} \wedge \boldsymbol{B} = 0$[85], since the curl operates on (2.219). It requires that the field $\boldsymbol{B}$ be in the form:

$$\boldsymbol{B} = \hat{\mathbf{e}}_y(\beta B_0 z) + \hat{\mathbf{e}}_z[B_0(1 + \beta y)] , \tag{2.221}$$

as is shown in (XIII.7). Note that the component along y is of first order ($\beta \ll 1$). These field lines, which we find in a toroidal configuration, are schematically drawn in Fig. 2.17: the greater the distance from the origin of the frame, the more the contribution from B_y becomes important.

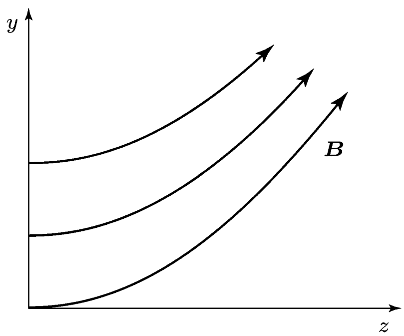


Fig. 2.17 Lines of force $y(z)$ in the presence of a gradient in B in the direction perpendicular to $\boldsymbol{B}$.

[85] In general $\boldsymbol{\nabla} \wedge \boldsymbol{H} = \boldsymbol{J} + \epsilon_0 \partial \boldsymbol{E}/\partial t$: however, in the context of the individual trajectories model, we neglect the current associated with the charged particle motion, $\boldsymbol{J} = 0$, as well as the corresponding displacement current $\epsilon_0 \partial \boldsymbol{E}/\partial t$. This last term is non zero in the case of a variable electric field $\boldsymbol{E}$ applied from outside.

The expression (2.221) represents a field $\boldsymbol{B}$ with lines of force characterised by a local curvature ρ. Recalling that the *radius of curvature* at a given point A of a curve is the distance between that point and the point of intersection of two normal vectors to the curve situated immediately on either side of A (Fig. XIII.1 of Appendix XIII), one can show that ρ is approximately $1/\beta$ (XIII.18).

- Magnetic curvature drift velocity
  This field curvature is associated with a particular drift motion, perpendicular to the lines of force (hence a velocity perpendicular to $\boldsymbol{B}$, as with the other drift velocities already defined). We will determine the average temporal velocity of this drift called the *magnetic curvature drift*, by resorting to the general expression for the drift of a charged particle subject to a given force $\boldsymbol{F}_D$ in a magnetic field $\boldsymbol{B}$ (Appendix XII).
  For this, we need to know the expression for the force exerted on the particle by the curvature of the lines: during its helical motion around the lines of force, the particle experiences a centrifugal force, for which the corresponding inertia term is of the classical form:

$$\boldsymbol{F}_{cd} = -\frac{m_\alpha w_\parallel^2}{\rho}\,\hat{\mathbf{e}}_y \ , \tag{2.222}$$

  where $\boldsymbol{w}_\parallel$ is the velocity parallel to the line of $\boldsymbol{B}$ at a given point and $\hat{\mathbf{e}}_y$ is the base vector linked to the coordinate system of the particle and directed towards the "instantaneous centre of rotation": we then have $\boldsymbol{\rho} = -\rho\,\hat{\mathbf{e}}_y$. Following (XII.2), the drift velocity in the curved magnetic field is then:

$$\boldsymbol{w}_{dc} = \frac{m_\alpha w_\parallel^2}{q_\alpha \rho^2 B^2}\,\boldsymbol{\rho}\wedge\boldsymbol{B} \tag{2.223}$$

  or equivalently:

$$\boldsymbol{w}_{dc} = -\frac{w_\parallel^2}{\omega_c}\,\frac{\boldsymbol{\rho}\wedge\boldsymbol{B}}{\rho^2 B} \ . \tag{2.224}$$

- Total drift velocity due to the presence of a gradient in B in the direction perpendicular to $\boldsymbol{B}$
  From (2.218) and (2.224), we obtain finally:

$$\boldsymbol{w}_{dm} + \boldsymbol{w}_{dc} = -\underset{\substack{\uparrow\\ \text{Charge}\\ \text{sign}}}{\frac{\boldsymbol{\rho}\wedge\boldsymbol{B}}{B\omega_c\rho^2}}\left[\underset{\substack{\uparrow\\ \text{Magnetic}\\ \text{field drift}}}{\frac{1}{2}w_\perp^2} + \underset{\substack{\uparrow\\ \text{Magnetic}\\ \text{curvature drift}}}{w_\parallel^2}\right] \ . \tag{2.225}$$

  Remark: These two contributions to the drift motion are in the same direction, defined by the vector $-\boldsymbol{\rho}\wedge\boldsymbol{B}$, but whose sense depends on the sign of the charged particle. This drift can therefore create a separation of charges in the plasma, generating an electric field[86]. This effect causes a loss of charged particles in tokomaks, because they are directed to the walls, as we see in the following.

- The evolution of the drift motion tied to the magnetic field in a tokomak
Figure 2.18a is a schematic representation of the configuration of the coils producing the toroidal field in a tokomak: this magnetic field, imposed by the machine, is directed along the z axis. Because the coils are closer towards the central axis of the torus, than at the outer radius, the $\boldsymbol{B}$ field is inhomogeneous as a function of x[87] and, due to this, it acquires a curvature. We will examine the different effects to which the particles are subjected in the presence of this toroidal field by referring to Fig. 2.18b.

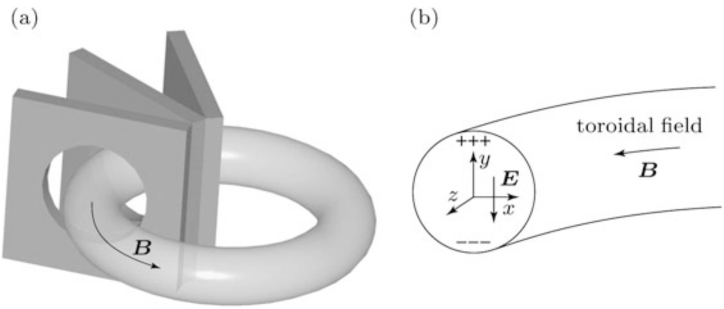


Fig. 2.18 **a** Schematic representation, showing the positioning of some magnetic field coils around a toroidal vessel: because they become closer towards the central axis of the torus, the field $\boldsymbol{B}$ increases along x. **b** Section of the toroidal vessel showing the charge separation created by the particle drifts in the toroidal magnetic field.

- The two magnetic drifts create a separation of charges along y (electrons downwards, ions upwards: the direction of the drift is that of q_α in (2.217)).
- This separation of charges creates a field $\boldsymbol{E}$ (perpendicular to z and x), directed downwards, opposed to the magnetic field drift current, giving rise to a weak current.
- The fields $\boldsymbol{E}$ and $\boldsymbol{B}$ then create an electric field drift, which is oriented according to $\boldsymbol{E} \wedge \boldsymbol{B}$ (crossed-field case, (2.222)). In the electric field drift, positive ions and negative electrons are displaced in the same direction: in the present case, they are directed towards the external wall of the torus (the vector product rule applied to the

[86] Except in structures where the magnetic configuration is closed on itself (a magnetic structure with rotational symmetry, for example).

[87] Note: direction designated by y in the previous discussion.

right trihedral coordinate system yields $-\hat{\mathbf{e}}_y \wedge \hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_x$). These charged particles are then "lost" to the fusion plasma: they recombine at the wall losing their energy, additionally inflicting damage on the wall.

Remark: The plasma particles subjected to a simple toroidal magnetic field do not remain confined, as we have just seen, and a supplementary magnetic field, called a "poloidal" field, is used to reduce the drift effects. This second magnetic field provides a slight azimuthal variation to the toroidal magnetic field line configuration, forming a helix around the minor axis of the torus, in order to prevent particles from traveling to the walls.

Problems

2.1. Consider electrons and ions, of mass m_e and m_i respectively, subject to a constant electric field. Assuming that the average time τ between two collisions is the same for electrons and ions, show that the (average) kinetic energy acquired by an electron in the time τ is m_i/m_e times greater than that acquired by an ion in the same time.

Answer

We know that:

$$F \equiv |q|E = m_e \frac{\mathrm{d}w_e}{\mathrm{d}t} = m_i \frac{\mathrm{d}w_i}{\mathrm{d}t} \ . \tag{2.226}$$

The same force, with opposite sign, acts on the ion and the electron. From this, for a time τ between two collisions:

$$\int_0^\tau F\,\mathrm{d}t = m_e[w_e(\tau) - w_e(0)] \ , \tag{2.227}$$

$$= m_i[w_i(\tau) - w_i(0)] \ , \tag{2.228}$$

and, to simplify the problem, setting $w_e(0) = w_i(0) = 0$, we obtain:

$$w_e = \frac{m_i}{m_e} w_i \ . \tag{2.229}$$

The ratio of the kinetic energy of the electron to that of the ion is then:

$$\frac{\mathcal{E}_{ce}}{\mathcal{E}_{ci}} = \frac{m_e w_e^2/2}{m_i w_i^2/2} = \frac{m_e \left(\frac{m_i}{m_e}\right)^2 w_i^2}{m_i w_i^2} = \frac{m_i}{m_e}\ ! \tag{2.230}$$

2.2. Consider the motion of an electron ($\alpha = e$) or that of an ion ($\alpha = i$) in the plane perpendicular to a magnetic field $\boldsymbol{B} = B\hat{\mathbf{e}}_z$ present in the plasma, as shown in Fig. 2.4.

a) Calculate the direction and amplitude of the magnetic moment μ_α of an electron and an ion in cyclotron rotation about $\boldsymbol{B}$.
b) Calculate the macroscopic magnetisation $\boldsymbol{\mathcal{M}}_\alpha$ (magnetic moment per unit volume, expressed in A/m) induced by the electron ($\alpha = e$) and ion ($\alpha = i$) populations rotating in the field $\boldsymbol{B}$:

$$\mathcal{M}_{z\alpha} = \int \mu_{z\alpha} f_\alpha(w)\, \mathrm{d}\boldsymbol{w}\ . \tag{2.231}$$

Assume that the velocity distributions $f_\alpha(w)$ for the electrons and ions are Maxwellian, with temperatures T_e and T_i, respectively.
c) Calculate the total macroscopic magnetisation $\boldsymbol{\mathcal{M}}$ and discuss the respective contributions of the electron and ion populations to the plasma diamagnetism.
d) Assuming that $n_e = n_i = n$ and $T_e = T_i = T$, deduce the magnetic induction $\boldsymbol{B}$ resulting from both the applied magnetic induction $\boldsymbol{B}_0 = B_0\hat{\mathbf{e}}_z$ and the component $\mu_0\boldsymbol{\mathcal{M}}$ due to the diamagnetism of the plasma. Write the condition for which the intensity of the field $\boldsymbol{B}$ in the plasma becomes equal to half B_0.

Numerical application: $B_0 = 2 \times 10^{-2}$ tesla (200 gauss), $T_e = T_i = 35000\,\mathrm{K}$.

Answer

a) The cyclotron motion of an electron and an ion in the plane perpendicular to the magnetic induction $\boldsymbol{B}$ can be described by equation (2.68):

$$\boldsymbol{w}_\alpha = \boldsymbol{\omega}_{c\alpha} \wedge \boldsymbol{r}_{B\alpha} \tag{2.232}$$

where:

$$\boldsymbol{\omega}_{c\alpha} = -\frac{q_\alpha \boldsymbol{B}}{m_\alpha}\ . \tag{2.233}$$

For the electrons, $\boldsymbol{\omega}_{ce}$ and $\boldsymbol{B}$ are collinear with the same sign, i.e. the same direction, while for the ions, $\boldsymbol{\omega}_{ci}$ and $\boldsymbol{B}$ have opposite sign. On the other hand, the currents $\boldsymbol{i}_e$ and $\boldsymbol{i}_i$ associated with the cyclotron motion of electrons and ions are in the same direction, because of their opposite charge, as shown in the Fig. 2.4. They induce a magnetic field $\boldsymbol{B}'$ in the

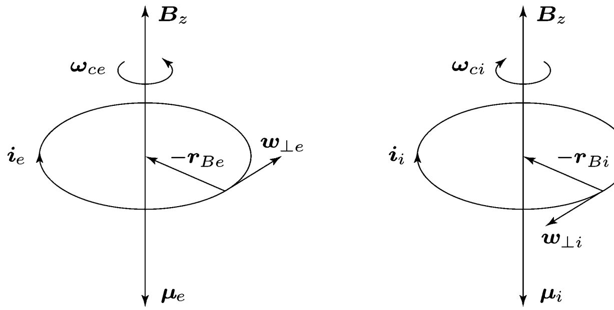


Fig. 2.19 Characteristic parameters of the cyclotron motion of an electron (left) and an ion (right) in a magnetic field.

direction opposite to $\boldsymbol{B}$ (Biot-Savart Law (2.70)); for the same reasons, $\boldsymbol{\mu}_i$ and $\boldsymbol{\mu}_e$ are anti-parallell to $\boldsymbol{B}$, as shown in Fig. 2.19.
The modulus of the magnetic moment μ is defined as the product of a current density i circulating in a closed loop with surface S. In the case of a cyclotron gyration:

$$\mu_{z\alpha} = \pi r_{B\alpha}^2 i_\alpha \; . \tag{2.234}$$

The current induced by the rotational motion is then:

$$i_\alpha = \frac{q_\alpha \omega_{c\alpha}}{2\pi} \; , \tag{2.235}$$

such that (2.148):

$$\mu_{z\alpha} = \frac{m_\alpha w_{\perp\alpha}^2}{2B} \; . \tag{2.236}$$

b) For an ensemble of particles of type α, the average value $\mathcal{M}_z$ of the macroscopic magnetisation (in the hydrodynamic sense) is given by (3.39):

$$\mathcal{M}_{z\alpha} = \int_w \mu_z f_\alpha(w)\, \mathrm{d}\boldsymbol{w} \; , \tag{2.231}$$

i.e. (neglecting the subscript α):

$$\mathcal{M}_z = \frac{1}{B} \int_w \frac{m w_\perp^2}{2} n \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp\left(-\frac{m w^2}{2 k_B T} \right) \mathrm{d}\boldsymbol{w} \; , \tag{2.237}$$

which can be expanded:

$$\begin{aligned}\mathcal{M}_z =& \frac{nm}{2B}\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \left[\int_{-\infty}^{\infty} w_x^2 \exp\left(-\frac{mw_x^2}{2k_BT}\right) \mathrm{d}w_x \right. \\ & \times \int_{-\infty}^{\infty} \exp\left(-\frac{mw_y^2}{2k_BT}\right) \mathrm{d}w_y \int_{-\infty}^{\infty} \exp\left(-\frac{mw_z^2}{2k_BT}\right) \mathrm{d}w_z \\ & + \int_{-\infty}^{\infty} \exp\left(-\frac{mw_x^2}{2k_BT}\right) \mathrm{d}w_x \int_{-\infty}^{\infty} w_y^2 \exp\left(-\frac{mw_y^2}{2k_BT}\right) \mathrm{d}w_y \\ & \left. \times \int_{-\infty}^{\infty} \exp\left(-\frac{mw_z^2}{2k_BT}\right) \mathrm{d}w_z \right] \end{aligned} \tag{2.238}$$

and this reduces to:

$$\mathcal{M}_{z\alpha} = \frac{n_\alpha k_B T_\alpha}{B} \; . \tag{2.239}$$

In vector form, taking account of the sense of the induced magnetisation $\boldsymbol{\mathcal{M}}_\alpha$ with respect to $\boldsymbol{B}$:

$$\boldsymbol{\mathcal{M}}_\alpha = -\frac{n_\alpha k_B T_\alpha}{B^2}\boldsymbol{B} \; . \tag{2.240}$$

c) The total magnetisation is the sum of those induced by the ions and the electrons, i.e.:

$$\boldsymbol{\mathcal{M}} = -\frac{\boldsymbol{B}}{B^2}\left(n_e k_B T_e + n_i k_B T_i\right) \; . \tag{2.241}$$

If $T_i \ll T_e$, $\mathcal{M}$ is induced solely by the electrons.
If $T_i = T_e$, the contribution from the ions is equal to that from the electrons.

d) For $n_e = n_i = n$ and $T_e = T_i = T$, we have from (2.241):

$$\boldsymbol{\mathcal{M}} = -\frac{2nk_BT}{B^2}\boldsymbol{B} = -\frac{2p}{B^2}\boldsymbol{B} \; , \tag{2.242}$$

where p is the pressure exerted by the charged particles, referred to as the (scalar) kinetic pressure (p. 205).
$\boldsymbol{B}$ is the magnetic induction in the plasma and results from the vector addition of the applied field $\boldsymbol{B}_0$ (which exists in the absence of the plasma) and the field created by the motion of charged particles, i.e. $\mu_0\boldsymbol{\mathcal{M}}$ (the magnetisation $\boldsymbol{\mathcal{M}}$ giving the magnetic field, the corresponding magnetic induction is thus obtained by multiplying the field $\boldsymbol{\mathcal{M}}$ by μ_0, the vacuum magnetic permeability). We then have:

$$\boldsymbol{B} = \boldsymbol{B}_0 + \mu_0 \boldsymbol{\mathcal{M}} = \boldsymbol{B}_0 - \frac{p}{B^2/2\mu_0}\boldsymbol{B} \,, \tag{2.243}$$

$$\boldsymbol{B}\left(1 + \frac{p}{B^2/2\mu_0}\right) = \boldsymbol{B}_0 \,. \tag{2.244}$$

The diamagnetism of the plasma causes a reduction of the applied magnetic field, due to the motion of the charged particles in the same field. The diamagnetism can be neglected ($\boldsymbol{B} \simeq \boldsymbol{B}_0$) if:

$$p \ll \frac{B^2}{2\mu_0} \,, \tag{2.245}$$

i.e. the kinetic pressure p remains much smaller than the *magnetic pressure* $B^2/2\mu_0$.
The magnetic induction in the plasma is half the applied magnetic field ($B = B_0/2$) if:

$$p = \frac{B^2}{2\mu_0} \,, \tag{2.246}$$

that is:

$$n = \frac{B^2}{2\mu_0 k_B T} \,. \tag{2.247}$$

Numerical application

From (2.247), we obtain:

$$\begin{aligned} n &= \frac{10^{-4}}{2 \times 4\pi \times 10^{-7} \times 1.38 \times 10^{-23} \times 35000} = 8.24 \times 10^{19}\,\mathrm{m}^{-3} \,, \\ &= 8.24 \times 10^{13}\,\mathrm{cm}^{-3} \,. \end{aligned} \tag{2.248}$$

Remark: Maxwell's Law for the curl of $\boldsymbol{H}$ applied to the field $\boldsymbol{\mathcal{M}}$ leads to $\boldsymbol{J}_{\mathcal{M}} = \boldsymbol{\nabla} \wedge \boldsymbol{\mathcal{M}}$. Since $\boldsymbol{\mathcal{M}}$ is uniform in the plasma:

$$\boldsymbol{\nabla} \wedge \boldsymbol{\mathcal{M}} = 0 \,, \tag{2.249}$$

thus $\boldsymbol{J}_{\mathcal{M}} = 0$: no macroscopic current is induced. On the other hand, in regions with gradients in $\boldsymbol{\mathcal{M}}$ (boundaries of enclosed plasmas), the diamagnetism of the plasma actually induces magnetisation currents ($\boldsymbol{J}_{\mathcal{M}} \neq 0$).

2.3. Consider a particle of charge q subject to uniform, static magnetic and electric fields which are perpendicular to each other. The particle velocity $\boldsymbol{w}$ can be decomposed according to $\boldsymbol{w} = \boldsymbol{w}_D + \boldsymbol{w}'$, where $\boldsymbol{w}_D$ is the electric field drift. From the equation of motion, show analytically that $\boldsymbol{w}'$ represents the motion the particle would have in the magnetic field alone.

Answer

The equation of motion can be written (2.5):

$$m\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t} = q\left[\boldsymbol{E}_\perp + \boldsymbol{w} \wedge \boldsymbol{B}\right] \tag{2.250}$$

in which we substitute $\boldsymbol{w}_D + \boldsymbol{w}'$ for $\boldsymbol{w}$ knowing that:

$$\boldsymbol{w}_D = \frac{\boldsymbol{E}_\perp \wedge \boldsymbol{B}}{B^2}\ , \tag{IX.2}$$

noting that $\mathrm{d}\boldsymbol{w}_D/\mathrm{d}t = 0$ because $\boldsymbol{E}$ and $\boldsymbol{B}$ are constant in time. We find:

$$m\frac{\mathrm{d}\boldsymbol{w}'}{\mathrm{d}t} = q\left[\boldsymbol{E}_\perp + \left(\frac{\boldsymbol{E}_\perp \wedge \boldsymbol{B}}{B^2}\right) \wedge \boldsymbol{B} + \boldsymbol{w}' \wedge \boldsymbol{B}\right]\ . \tag{2.251}$$

From the double vector product:

$$(\boldsymbol{Q} \wedge \boldsymbol{T}) \wedge \boldsymbol{P} = \boldsymbol{T}(\boldsymbol{P} \cdot \boldsymbol{Q}) - \boldsymbol{Q}(\boldsymbol{T} \cdot \boldsymbol{P})\ , \tag{2.252}$$

this becomes:

$$\left(\frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B^2}\right) \wedge \boldsymbol{B} = \frac{\boldsymbol{B}(\boldsymbol{B} \cdot \boldsymbol{E}) - \boldsymbol{E}(B^2)}{B^2} = -\boldsymbol{E}\frac{B^2}{B^2} \tag{2.253}$$

from the assumption that $\boldsymbol{E}$ is perpendicular to $\boldsymbol{B}$.
Finally:

$$\frac{\mathrm{d}\boldsymbol{w}'}{\mathrm{d}t} = q(\boldsymbol{w}' \wedge \boldsymbol{B})\ , \tag{2.254}$$

which is precisely the motion of a particle in a magnetic field alone.

2.4. Consider the motion of a charged particle in a uniform and static magnetic field $\boldsymbol{B}$, and a uniform electric field $\boldsymbol{E}$, directed perpendicular to $\boldsymbol{B}$ and slowly varying in time. The velocity of the particle is denoted by $\boldsymbol{w}$.

a) Show, by expressing the velocity $\boldsymbol{w}$ in terms of the following three velocity components:

$$\boldsymbol{w} = \boldsymbol{w}_D + \boldsymbol{w}' + \boldsymbol{w}_p\ , \tag{2.255}$$

where $\boldsymbol{w}_D$ is the electric field drift velocity and:

$$\boldsymbol{w}_p = \frac{m}{qB^2}\frac{\partial \boldsymbol{E}}{\partial t}\ , \tag{2.256}$$

that $\boldsymbol{w}'$ and $\boldsymbol{w}_p$ then obey the equation of motion:

$$m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p = q(\boldsymbol{w}' \wedge \boldsymbol{B})\ , \tag{2.257}$$

where m is the mass of the particle and q its charge.

b) Consider a periodically varying field $E(t)$, with an angular frequency ω. Show that if the frequency of the field oscillation is small compared to ω_c for the cyclotron gyration, then the $\boldsymbol{w}'$ component describes the cyclotron motion of the particle in the $\boldsymbol{B}$ field alone.
c) Show that there is no net current (ions and electrons) associated with $\boldsymbol{w}_D$ while, on the other hand, $\boldsymbol{w}_p$ leads to a current which is referred to as the polarisation current:

$$\boldsymbol{J}_p = \frac{\rho_m}{B^2}\dot{\boldsymbol{E}} \,, \tag{2.258}$$

where $\rho_m = (m_e + m_i)n$ is the mass density of the electrons and ions (of masses m_e and m_i respectively) and n, the charged particle density. The velocity $\boldsymbol{w}_p$ is called the polarisation drift velocity.
d) By considering the total charge current (the conduction current and the displacement current $\partial\boldsymbol{D}/\partial t$), show that the relative permittivity of the medium with respect to vacuum is given by:

$$\epsilon_p = 1 + \frac{\rho_m}{\epsilon_0 B^2} \,. \tag{2.259}$$

To do this, recall from (2.45) that:

$$\boldsymbol{J}_T = \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_c = \frac{\partial \boldsymbol{D}'}{\partial t} \,, \tag{2.260}$$

where $\boldsymbol{D}' = \epsilon_0\epsilon_p\boldsymbol{E}$ is the displacement current in the dielectric description (see (2.43)).

Answer

a) Equation (2.257) signifies that the presence of the field $\boldsymbol{E}$ does not qualitatively modify the helical motion (described by $\boldsymbol{w}'$: to be shown in b) of the particle.
We know that the equation of motion is linked to the Lorentz force by:

$$m\dot{\boldsymbol{w}} = q[\boldsymbol{E} + \boldsymbol{w} \wedge \boldsymbol{B}] \,, \tag{2.261}$$

independent of the form of $\boldsymbol{E}$ and $\boldsymbol{B}$.
In the present case, we make the assumption that the total velocity can be expressed in terms of the three vectors given by (2.255). Developing (2.261) in terms of these different velocities, we obtain:

$$m\dot{\boldsymbol{w}}_D + m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p = q[\boldsymbol{E} + (\boldsymbol{w}_D \wedge \boldsymbol{B}) + (\boldsymbol{w}' \wedge \boldsymbol{B}) + (\boldsymbol{w}_p \wedge \boldsymbol{B})] \tag{2.262}$$

and, replacing $\boldsymbol{w}_p$ on the RHS by equation (2.256) and $\boldsymbol{w}_D$ by its vector form:

$$\boldsymbol{w}_D = \frac{\boldsymbol{E} \wedge \boldsymbol{B}}{B^2} \,, \tag{2.263}$$

we obtain:

$$m\dot{\boldsymbol{w}}_D + m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p = q\left\{\boldsymbol{E} + \underbrace{\left[\left(\frac{\boldsymbol{E}\wedge\boldsymbol{B}}{B^2}\right)\wedge\boldsymbol{B}\right]}_{\substack{\text{double vector product}\\ -\boldsymbol{E}}} + (\boldsymbol{w}'\wedge\boldsymbol{B}) + \frac{m}{q}\frac{\mathrm{d}}{\mathrm{d}t}\underbrace{\left(\frac{\boldsymbol{E}\wedge\boldsymbol{B}}{B^2}\right)}_{=m\dot{\boldsymbol{w}}_D}\right\} . \tag{2.264}$$

From the double vector product (see problem 2.3), this becomes:

$$m\dot{\boldsymbol{w}}_D + m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p = q\left\{\boldsymbol{E} - \boldsymbol{E} + (\boldsymbol{w}'\wedge\boldsymbol{B})\right\} + m\dot{\boldsymbol{w}}_D \tag{2.265}$$

and finally:

$$m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p = q(\boldsymbol{w}'\wedge\boldsymbol{B}) . \tag{2.257}$$

b) We need to show, starting from (2.256) and (2.257), that $\left|\dot{\boldsymbol{w}}_p/\dot{\boldsymbol{w}}'\right| \ll 1$. Setting $E = E_0\mathrm{e}^{\mathrm{i}\omega t}$, we can write:

$$\left|\frac{m\dot{\boldsymbol{w}}_p}{m\dot{\boldsymbol{w}}' + m\dot{\boldsymbol{w}}_p}\right| = \left|\frac{m^2\omega^2}{qB^2}E_0\right|\left|\frac{1}{qw'B}\right| = \left|\frac{\omega^2}{\omega_c^2}\right|\left|\frac{E_0}{Bw'}\right| = \left|\frac{\omega^2}{\omega_c^2}\right|\left|\frac{w_D}{w'}\right| , \tag{2.266}$$

which indicates that we require not only $\omega/\omega_c \leq 1$ but also, preferably, $w_D \lesssim w'$, which is an acceptable hypothesis.

c) Since the velocity $\boldsymbol{w}_D$ does not depend on the charge of the particles, the corresponding conduction current density is zero, because:

$$\boldsymbol{J}_D = \sum_\alpha n_\alpha q_\alpha \boldsymbol{w}_D = n\boldsymbol{w}_D(e-e) = 0 . \tag{2.267}$$

For the *conduction* current referred to as the *polarisation* current, we have:

$$\boldsymbol{J}_p \equiv \sum_\alpha n_\alpha q_\alpha \boldsymbol{w}_{p\alpha} = \frac{\dot{\boldsymbol{E}}}{B^2}\left[\frac{n_e(-e)m_e}{-e} + \frac{n_i(e)m_i}{e}\right] = \frac{\dot{\boldsymbol{E}}}{B^2}\,n(m_i+m_e) , \tag{2.268}$$

$$= \frac{\dot{\boldsymbol{E}}}{B^2}\,\rho_m . \tag{2.269}$$

d) The conduction current $\boldsymbol{J}_c$ reduces to $\boldsymbol{J}_p$, as we have just shown. In addition, because quite generally:

$$\boldsymbol{J}_T \equiv \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J}_c = \frac{\partial \boldsymbol{D}'}{\partial t} \ , \tag{2.260}$$

in the present case this reduces to:

$$\boldsymbol{J}_T \equiv \epsilon_0 \dot{\boldsymbol{E}} + \boldsymbol{J}_p = \epsilon_p \epsilon_0 \dot{\boldsymbol{E}} \tag{2.270}$$

and from (2.269):

$$\boldsymbol{J}_T \equiv \epsilon_0 \dot{\boldsymbol{E}} + \frac{\dot{\boldsymbol{E}}}{B^2}\rho_m = \epsilon_p \epsilon_0 \dot{\boldsymbol{E}} \ , \tag{2.271}$$

such that we obtain, as required:

$$\epsilon_p = 1 + \frac{\rho_m}{B^2 \epsilon_0} \ . \tag{2.259}$$

2.5. Consider a plasma subject to a high frequency electric field $\boldsymbol{E}_0 \mathrm{e}^{\mathrm{i}\omega t}$, directed arbitrarily with respect to a static magnetic field of intensity B, both fields being spatially uniform. In the framework of the "individual motion of charged particles" description, calculate the conductivity and permittivity tensors for electrons whose motion is associated with the particular solution of the non-resonant equation of motion. Assume $\boldsymbol{B}$ is directed along the Oz axis and express $\boldsymbol{E}_\perp$ in terms of the Cartesian coordinates x and y. The multiplying factor for the tensor $\underline{\boldsymbol{\sigma}}$ should be such that it reduces to a unitary matrix for $B = 0$.

Answer

In order to obtain $\boldsymbol{w}_2$, the particular solution to this problem (see Sect. 2.2.2, p. 123), we used the frame of Fig. 2.10, which led us to the expression:

$$\boldsymbol{w}_2 = \left[a\boldsymbol{E}_{0\|} + b\boldsymbol{E}_{0\perp} + c(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right] \mathrm{e}^{\mathrm{i}\omega t} \ . \tag{2.127}$$

To transpose this result into Cartesian coordinates (x, y, z), as posed by the question, we write:

$$\boldsymbol{w}_2 = \hat{\mathbf{e}}_z(aE_{0\|}) + \hat{\mathbf{e}}_x(bE_{0x}) + \hat{\mathbf{e}}_y(bE_{0y}) + c(\hat{\mathbf{e}}_x E_{0x} + \hat{\mathbf{e}}_y E_{0y}) \wedge \hat{\mathbf{e}}_z B \ , \tag{2.272}$$

where we have set $\boldsymbol{E}_{0\perp} = \hat{\mathbf{e}}_x E_{0x} + \hat{\mathbf{e}}_y E_{0y}$ and have canceled, for simplicity, the dependence on $\mathrm{e}^{\mathrm{i}\omega t}$. After regrouping the terms along the three base vectors, we find:

$$\boldsymbol{w}_2 = \hat{\mathbf{e}}_x[bE_{0x} + cE_{0y}B] + \hat{\mathbf{e}}_y[bE_{0y} - cE_{0x}B] + \hat{\mathbf{e}}_z[aE_z] \ . \tag{2.273}$$

Including the coefficients a, b and c for the non-resonance solution from Sect. 2.2.2 (Eq. (2.135)), we obtain:

$$\boldsymbol{w}_2 = \hat{\mathbf{e}}_x \left[\frac{\mathrm{i}q}{m_\alpha} \frac{\omega}{\omega_c^2 - \omega^2} E_{0x} + \frac{q^2}{m_\alpha^2} \frac{B}{\omega_c^2 - \omega^2} E_{0y} \right] + \hat{\mathbf{e}}_y \left[\frac{\mathrm{i}q}{m_\alpha} \frac{\omega}{\omega_c^2 - \omega^2} E_{0y} - \frac{q^2}{m_\alpha^2} \frac{B}{\omega_c^2 - \omega^2} E_{0x} \right] + \hat{\mathbf{e}}_z \left[-\frac{\mathrm{i}q}{\omega m_\alpha} E_z \right] . \quad (2.274)$$

To develop the electrical conductivity tensor, recall that $J^i = \sigma^{ij} E_j$ (2.121). By definition, the current density $J^i = nqw^i$ and we have, after introducing the factor $-\mathrm{i}nq^2/m_\alpha\omega$:

$$\boldsymbol{J} = -\frac{\mathrm{i}nq^2}{m_\alpha\omega} \left\{ \hat{\mathbf{e}}_x \left[\left(-\frac{\omega^2}{\omega_c^2 - \omega^2} E_{0x} \right) + \frac{\mathrm{i}q}{m_\alpha} \frac{\omega B}{\omega_c^2 - \omega^2} E_{0y} \right] + \hat{\mathbf{e}}_y \left[-\frac{\omega^2}{\omega_c^2 - \omega^2} E_{0y} - \frac{\mathrm{i}qB\omega}{m_\alpha(\omega_c^2 - \omega^2)} E_{0x} \right] + \hat{\mathbf{e}}_z E_z \right\} . \quad (2.275)$$

There are two components of $\underline{\boldsymbol{\sigma}}$ along $\hat{\mathbf{e}}_x$, and recalling that $qB/m_\alpha = -\omega_c$, we find:

$$\sigma_{xx} = \frac{\omega^2}{\omega^2 - \omega_c^2} , \qquad \sigma_{xy} = \frac{\mathrm{i}\omega_c\omega}{\omega^2 - \omega_c^2} , \quad (2.276)$$

then, along $\hat{\mathbf{e}}_y$:

$$\sigma_{yy} = \frac{\omega^2}{\omega^2 - \omega_c^2} , \qquad \sigma_{yx} = -\frac{\mathrm{i}\omega_c\omega}{\omega^2 - \omega_c^2} , \quad (2.277)$$

and, finally, along $\hat{\mathbf{e}}_z$:

$$\sigma_{zz} = 1 . \quad (2.278)$$

The tensor $\underline{\boldsymbol{\sigma}}$, represented as a 3×3 matrix, has the value:

$$\underline{\boldsymbol{\sigma}} = -\frac{\mathrm{i}nq^2}{m_\alpha\omega} \begin{pmatrix} \dfrac{\omega^2}{\omega^2 - \omega_c^2} & \dfrac{\mathrm{i}\omega_c\omega}{\omega^2 - \omega_c^2} & 0 \\ -\dfrac{\mathrm{i}\omega_c\omega}{\omega^2 - \omega_c^2} & \dfrac{\omega^2}{\omega^2 - \omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (2.279)$$

and we can verify that if $B = 0$ ($\omega_c = 0$), the matrix becomes unitary: the plasma is no longer anisotropic.

For the relative permittivity tensor $\underline{\boldsymbol{\epsilon}}_p$, we have for $E_0\mathrm{e}^{\mathrm{i}\omega t}$:

$$\underline{\boldsymbol{\epsilon}}_p = \underline{\boldsymbol{I}} + \frac{\underline{\boldsymbol{\sigma}}}{\mathrm{i}\omega\epsilon_0} , \quad (2.125)$$

such that:

$$\underline{\boldsymbol{\epsilon}}_p = \begin{pmatrix} 1 - \dfrac{nq^2}{m\epsilon_0\omega^2}\dfrac{\omega^2}{\omega^2-\omega_c^2} & -\dfrac{nq^2}{m\epsilon_0\omega^2}\dfrac{\mathrm{i}\omega_c\omega}{\omega^2-\omega_c^2} & 0 \\ \dfrac{nq^2}{m\epsilon_0\omega^2}\dfrac{\mathrm{i}\omega_c\omega}{\omega^2-\omega_c^2} & 1 - \dfrac{nq^2}{m\epsilon_0\omega^2}\dfrac{\omega^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 - \dfrac{nq^2}{m\epsilon_0\omega^2} \end{pmatrix} , \quad (2.280)$$

from which, finally:

$$\underline{\boldsymbol{\epsilon}}_p = \begin{pmatrix} 1 - \dfrac{\omega_{pe}^2}{\omega^2-\omega_c^2} & -\dfrac{\mathrm{i}\omega_{pe}^2}{\omega^2-\omega_c^2}\dfrac{\omega_c}{\omega} & 0 \\ \dfrac{\mathrm{i}\omega_{pe}^2}{\omega^2-\omega_c^2}\dfrac{\omega_c}{\omega} & 1 - \dfrac{\omega_{pe}^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 - \dfrac{\omega_{pe}^2}{\omega^2} \end{pmatrix} . \quad (2.281)$$

2.6. Consider a uniform, alternating electric field of the form $\boldsymbol{E}_0\mathrm{e}^{-\mathrm{i}\omega t}$, together with a uniform, static magnetic field $\boldsymbol{B}$ along the z axis (entering into the page). We wish to study the phenomenon of cyclotron resonance using the coordinate frame rotating in the plane perpendicular to $\boldsymbol{B}$. Expressing the field $\boldsymbol{E}$ in the laboratory Cartesian frame in the form:

$$\boldsymbol{E} = \hat{\mathbf{e}}_x E_x + \hat{\mathbf{e}}_y E_y + \hat{\mathbf{e}}_z E_z \ , \quad (2.282)$$

the same field in the rotating coordinates frame is written:

$$\begin{aligned} \boldsymbol{E} =& \hat{\mathbf{e}}_+ E_+ + \hat{\mathbf{e}}_- E_- + \hat{\mathbf{e}}_z E_z \\ =& \frac{(\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y)}{\sqrt{2}} E_+ + \frac{(\hat{\mathbf{e}}_x - \mathrm{i}\hat{\mathbf{e}}_y)}{\sqrt{2}} E_- + \hat{\mathbf{e}}_z E_z \ . \end{aligned} \quad (2.283)$$

a) Express the components E_+ and E_- in terms of the components E_x and E_y. Determine which of the two unit vectors, $\hat{\mathbf{e}}_+$ or $\hat{\mathbf{e}}_-$, rotate in the same direction as the electrons during their cyclotron motion around $\boldsymbol{B}$.
b) The conductivity tensor, expressed in the laboratory frame, for $\omega \neq \omega_c$, and $E_0\mathrm{e}^{-\mathrm{i}\omega t}$, has the following elements (exercise 2.5):

$$\underline{\boldsymbol{\sigma}} = \sigma_0 \begin{pmatrix} \dfrac{\omega^2}{\omega^2-\omega_c^2} & \dfrac{-\mathrm{i}\omega\omega_c}{\omega^2-\omega_c^2} & 0 \\ \dfrac{\mathrm{i}\omega\omega_c}{\omega^2-\omega_c^2} & \dfrac{\omega^2}{\omega^2-\omega_c^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} . \quad (2.284)$$

Show, by calculating the corresponding terms σ_+ and σ_-, that this tensor is diagonalised in the rotating frame.

c) Show that electron cyclotron resonance leads to an increase in velocity of the particles as a function of time, according to the relation:

$$\boldsymbol{w}_2 = \frac{q}{m}\boldsymbol{E}_+ t \; . \tag{2.285}$$

In other words, the electrons in their own frame "see" a continuous electric field ($\omega = 0$) which accelerates them continuously between two collisions. For this, develop the relation (2.146) in the laboratory frame.

Answer

a) We can develop (2.283) by regrouping the terms along the unit vectors in the laboratory frame:

$$\boldsymbol{E} = \frac{1}{\sqrt{2}}\hat{\mathbf{e}}_x[E_+ + E_-] + \frac{\mathrm{i}}{\sqrt{2}}\hat{\mathbf{e}}_y[E_+ - E_-] + \hat{\mathbf{e}}_z E_z \; , \tag{2.286}$$

which must be equal to the same vector $\boldsymbol{E}$ expressed in the laboratory frame:

$$\boldsymbol{E} = \hat{\mathbf{e}}_x E_x + \hat{\mathbf{e}}_y E_y + \hat{\mathbf{e}}_z E_z \; , \tag{2.287}$$

from which:

$$E_x = \frac{1}{\sqrt{2}}[E_+ + E_-] \; , \tag{2.288}$$

$$E_y = \frac{\mathrm{i}}{\sqrt{2}}[E_+ - E_-] \; . \tag{2.289}$$

By combining (2.288) and (2.289), we obtain the components of the field $\boldsymbol{E}$ in the rotating frame:

$$E_+ = \frac{E_x - \mathrm{i}E_y}{\sqrt{2}} \tag{2.290}$$

and, similarly:

$$E_- = \frac{E_x + \mathrm{i}E_y}{\sqrt{2}} \; . \tag{2.291}$$

The components E_+ and E_-, in terms of the components E_x and E_y, thus correspond to the concept of a rotating field: the superposition of two oscillating fields with the same frequency, perpendicular to each other and in phase quadrature.

Since the rotating field $\boldsymbol{E}_+$ can be written:

$$\boldsymbol{E}_+ = \frac{\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}} \, E_+ \mathrm{e}^{-\mathrm{i}\omega t} \; , \tag{2.292}$$

taking the real part, we obtain:

$$\boldsymbol{E}_+ = \frac{E_+}{\sqrt{2}} \left(\hat{\mathbf{e}}_x \cos\omega t + \hat{\mathbf{e}}_y \sin\omega t\right) \ , \tag{2.293}$$

from which it is easy to verify, in the $x\mathrm{O}y$ plane, that the vector $\hat{\mathbf{e}}_+$ (and thus the field $\boldsymbol{E}_+$) rotates in the clockwise direction, thus, according to our convention (field $\boldsymbol{B}$ entering the page and $\omega_c > 0$), in the same direction as the electrons, as shown in the figure. Note that the intensity in expression (2.293) originating from (2.290), is constant in the rotating frame.
Make sure that the orientation of the axes x and y is such that the field along z enters into the page (right-handed frame).

b) Equation (2.284) gives us the components of the tensor in the laboratory frame. Note that $\sigma_{xx} = \sigma_{yy}$ and $\sigma_{xy} = -\sigma_{yx}$. Thus, by expanding the current density $\boldsymbol{J} = \underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E}$ in the laboratory frame, we can write:

$$\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E} = \sigma_{xx}\hat{\mathbf{e}}_x E_x + \sigma_{xy}\hat{\mathbf{e}}_x E_y - \sigma_{xy}\hat{\mathbf{e}}_y E_x + \sigma_{xx}\hat{\mathbf{e}}_y E_y + \sigma_{zz}\hat{\mathbf{e}}_z E_z \ . \tag{2.294}$$

Replacing the components of the fields in the laboratory frame with those of the rotating frame ((2.288) and (2.289)), we have:

$$\begin{aligned}\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E} = \quad & \sigma_{xx}\hat{\mathbf{e}}_x \frac{1}{\sqrt{2}}[E_+ + E_-] + \sigma_{xy}\hat{\mathbf{e}}_x \frac{\mathrm{i}}{\sqrt{2}}[E_+ - E_-] \\ & - \sigma_{xy}\hat{\mathbf{e}}_y \frac{1}{\sqrt{2}}[E_+ + E_-] + \sigma_{xx}\hat{\mathbf{e}}_y \frac{\mathrm{i}}{\sqrt{2}}[E_+ - E_-] + \sigma_{zz}\hat{\mathbf{e}}_z E_z \ .\end{aligned} \tag{2.295}$$

Regrouping the terms in E_+ and those in E_-:

$$\frac{E_+}{\sqrt{2}} \left[\hat{\mathbf{e}}_x\, \sigma_{xx} + \mathrm{i}\hat{\mathbf{e}}_x\, \sigma_{xy} - \hat{\mathbf{e}}_y\, \sigma_{xy} + \mathrm{i}\hat{\mathbf{e}}_y\, \sigma_{xx}\right] \ , \tag{2.296}$$

$$\frac{E_-}{\sqrt{2}} \left[\hat{\mathbf{e}}_x\, \sigma_{xx} - \mathrm{i}\hat{\mathbf{e}}_x\, \sigma_{xy} - \hat{\mathbf{e}}_y\, \sigma_{xy} - \mathrm{i}\hat{\mathbf{e}}_y\, \sigma_{xx}\right] \ , \tag{2.297}$$

and introducing the base vectors $\hat{\mathbf{e}}_+$ and $\hat{\mathbf{e}}_-$, we find:

$$\begin{aligned} & E_+ \left[\frac{\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}}\, \sigma_{xx} + \mathrm{i}\frac{\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}}\, \sigma_{xy}\right] + \\ & E_- \left[\frac{\hat{\mathbf{e}}_x - \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}}\, \sigma_{xx} - \mathrm{i}\frac{\hat{\mathbf{e}}_x - \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}}\, \sigma_{xy}\right] \ . \end{aligned} \tag{2.298}$$

Finally, we obtain:

$$\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E} = E_+\hat{\mathbf{e}}_+\sigma_{xx} + \mathrm{i}E_+\hat{\mathbf{e}}_+\sigma_{xy} + E_-\hat{\mathbf{e}}_-\sigma_{xx} - \mathrm{i}E_-\hat{\mathbf{e}}_-\sigma_{xy} + \sigma_{zz}\hat{\mathbf{e}}_z E_z \ . \tag{2.299}$$

We can now introduce the elements of the tensor $\underline{\boldsymbol{\sigma}}$ in the rotating frame:

$$\underline{\boldsymbol{\sigma}} \cdot \boldsymbol{E} = \left[E_+\hat{\mathbf{e}}_+\underbrace{(\sigma_{xx} + \mathrm{i}\sigma_{xy})}_{\sigma_+} + E_-\hat{\mathbf{e}}_-\underbrace{(\sigma_{xx} - \mathrm{i}\sigma_{xy})}_{\sigma_-}\right] + \sigma_{zz}\hat{\mathbf{e}}_z E_z \ , \quad (2.300)$$

which is an expression showing that, in the rotating frame, the tensor has been diagonalised (there are no mixed components $E_+\hat{\mathbf{e}}_-$ or $E_-\hat{\mathbf{e}}_+$). The matrix can now be represented by:

$$\underline{\boldsymbol{\sigma}} = \sigma_0 \begin{pmatrix} \dfrac{\omega}{\omega - \omega_c} & 0 & 0 \\ 0 & \dfrac{\omega}{\omega + \omega_c} & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad (2.301)$$

where the base vectors are $\hat{\mathbf{e}}_+$, $\hat{\mathbf{e}}_-$ and $\hat{\mathbf{e}}_z$, respectively.
We can thus verify that the electron resonance is indeed directed along $\hat{\mathbf{e}}_+$ ($\omega = \omega_c$).

c) Consider (2.146):

$$\boldsymbol{w}_2 = \left[-\frac{\mathrm{i}q_\alpha}{m_\alpha\omega}\boldsymbol{E}_{0\parallel} + \frac{q_\alpha}{2m_\alpha\omega}(\omega t - \mathrm{i})\boldsymbol{E}_{0\perp} - \frac{\mathrm{i}q_\alpha^2 t}{2\omega m_\alpha^2}(\boldsymbol{E}_{0\perp} \wedge \boldsymbol{B})\right] \mathrm{e}^{\mathrm{i}\omega t} \ , \quad (2.302)$$

where we have replaced the dependence $\mathrm{e}^{\mathrm{i}\omega t}$ of the field $\boldsymbol{E}$ by $\mathrm{e}^{-\mathrm{i}\omega t}$, as indicated in the introduction.
We will replace i by $-\mathrm{i}$ in (2.302). Since we are only interested in the resonance phenomenon, we will ignore the term in the $\hat{\mathbf{e}}_z$ direction and the second term in $\boldsymbol{E}_{0\perp}$, which is time independent (which rapidly becomes negligible). Writing this in Cartesian coordinates:

$$\boldsymbol{E}_{0\perp} = \hat{\mathbf{e}}_x E_{0x} + \hat{\mathbf{e}}_y E_{0y} \ , \quad (2.303)$$

this becomes:

$$\boldsymbol{w}_{20\perp} = \frac{qt}{2m}\left(\hat{\mathbf{e}}_x E_{0x} + \hat{\mathbf{e}}_y E_{0y}\right) + \frac{\mathrm{i}q^2 Bt}{2\omega m^2}\left(\hat{\mathbf{e}}_x E_{0y} - \hat{\mathbf{e}}_y E_{0x}\right) . \quad (2.304)$$

Knowing that at resonance, $-qB/m = \omega_c$ and $\omega = \omega_c$, from (2.304):

$$\boldsymbol{w}_{20\perp} = \frac{qt}{2m}\left[(\hat{\mathbf{e}}_x E_{0x} + \hat{\mathbf{e}}_y E_{0y}) - \mathrm{i}(\hat{\mathbf{e}}_x E_{0y} - \hat{\mathbf{e}}_y E_{0x})\right] , \quad (2.305)$$

$$\boldsymbol{w}_{20\perp} = \frac{qt}{2m}\left[\hat{\mathbf{e}}_x(E_{0x} - \mathrm{i}E_{0y}) + \hat{\mathbf{e}}_y(E_{0y} + \mathrm{i}E_{0x})\right] , \quad (2.306)$$

$$\boldsymbol{w}_{20\perp} = \frac{qt}{m}\left[\left(\frac{\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y}{2}\right)(E_{0x} - \mathrm{i}E_{0y})\right] , \quad (2.307)$$

such that, from (2.283) and (2.290), and multiplying by $\mathrm{e}^{-\mathrm{i}\omega t}$:

$$\boldsymbol{w}_{2\perp} = \frac{qt}{m}E_+\left(\frac{\hat{\mathbf{e}}_x + \mathrm{i}\hat{\mathbf{e}}_y}{\sqrt{2}}\right)\mathrm{e}^{-\mathrm{i}\omega t} = \frac{qt}{m}\hat{\mathbf{e}}_+ E_+ \mathrm{e}^{-\mathrm{i}\omega t} = \frac{qt}{m}\boldsymbol{E}_+ \ . \quad (2.308)$$

2.7. Consider a homogeneous, static magnetic field $\boldsymbol{B} = \hat{\mathbf{e}}_z B_0$ and a homogeneous alternating electric field $\boldsymbol{E} = \hat{\mathbf{e}}_x E_0 \cos\omega t$ ($\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ are the unit vectors along the Cartesian axes x, y and z).

a) Show that at cyclotron resonance, the contribution of this effect to the velocity of particles of mass m is given by:

$$\boldsymbol{w} = \frac{q}{2m} E_0 t \left[\cos(\omega t)\hat{\mathbf{e}}_x + \sin(\omega t)\hat{\mathbf{e}}_y\right] + \frac{q}{2m\omega} E_0 \sin(\omega t)\hat{\mathbf{e}}_x \ . \tag{2.309}$$

b) Write the form of this motion at resonance explicitly. What does it represent?

c) In an alternating electric field $\boldsymbol{E}$ directed instead along $\hat{\mathbf{e}}_y$:

$$\boldsymbol{E} = E_0 \sin(\omega t)\hat{\mathbf{e}}_y \ , \tag{2.310}$$

show that the contribution to the particle velocity at cyclotron resonance can be written in the form:

$$\boldsymbol{w}' = \frac{q}{2m} E_0 \left(t - \frac{\pi}{2\omega}\right) \left[\sin(\omega t)\hat{\mathbf{e}}_y + \cos(\omega t)\hat{\mathbf{e}}_x\right] - \frac{q}{2m\omega} E_0 \cos(\omega t)\hat{\mathbf{e}}_y \ . \tag{2.311}$$

d) A rotating electric field is applied in the $x\mathrm{O}y$ plane with an amplitude such that $E_x = E_y = E_0$. Following the chosen direction of rotation, we have:

$$\boldsymbol{E}_+ = E_0 \left[\cos(\omega t)\hat{\mathbf{e}}_x + \sin(\omega t)\hat{\mathbf{e}}_y\right] \tag{2.312}$$

or:

$$\boldsymbol{E}_- = E_0 \left[\cos(\omega t)\hat{\mathbf{e}}_x - \sin(\omega t)\hat{\mathbf{e}}_y\right] \ . \tag{2.313}$$

Based on the expressions for $\boldsymbol{w}$ and $\boldsymbol{w}'$, calculate the resultant velocity for a particle at cyclotron resonance $\omega_c > 0$ in a field rotating to the right (E_+) and then a field rotating to the left (E_-). What can you conclude?

Answer

a) In the presence of a magnetic field:

$$\boldsymbol{B} = B_0 \hat{\mathbf{e}}_z \tag{2.314}$$

and in a periodic electric field perpendicular to it:

$$\boldsymbol{E} = E_0 \mathrm{e}^{\mathrm{i}\omega t} \hat{\mathbf{e}}_y \ , \tag{2.315}$$

the particular solution (2.146) of the equation of motion at $\omega = \omega_c$ represents the effect of the cyclotron resonance on the particle velocity. We will ignore the contribution to the velocity in the direction parallel to the field $\boldsymbol{B}$, since it is not affected by the field $\boldsymbol{E}$, because it is perpendicular to $\boldsymbol{B}$.

In the $x\mathrm{O}y$ plane (2.142) we have:

$$\boldsymbol{w}_2 = \left[\frac{q}{2m\omega}(\omega t - \mathrm{i})E_0\hat{\mathbf{e}}_x - \frac{\mathrm{i}q^2 t}{2\omega m^2}(E_0\hat{\mathbf{e}}_x \wedge B_0\hat{\mathbf{e}}_z)\right]\mathrm{e}^{\mathrm{i}\omega t} . \tag{2.316}$$

We can then take the real part of this expression (knowing that $\mathrm{e}^{\mathrm{i}\omega t} = \cos(\omega t) + \mathrm{i}\sin(\omega t)$), from which:

$$\begin{aligned}\boldsymbol{w}_2 = \frac{q}{2m\omega}\omega t E_0\cos(\omega t)\hat{\mathbf{e}}_x &+ \frac{q}{2m\omega}E_0\sin(\omega t)\hat{\mathbf{e}}_x \\ &+ \frac{q^2 t}{2\omega m^2}E_0\sin(\omega t)(\hat{\mathbf{e}}_x \wedge \hat{\mathbf{e}}_z)B_0 .\end{aligned} \tag{2.317}$$

Since $\hat{\mathbf{e}}_x \wedge \hat{\mathbf{e}}_z = -\hat{\mathbf{e}}_y$ (right-handed frame) and $\omega = \omega_c \equiv -qB_0/m$, this becomes:

$$\begin{aligned}\boldsymbol{w}_2 = \frac{q}{2m}E_0 t &\left[\cos(\omega t)\hat{\mathbf{e}}_x - \frac{qm}{(-1)qB_0 m}\sin(\omega t)B_0\hat{\mathbf{e}}_y\right] \\ &+ \frac{q}{2m\omega}E_0\sin(\omega t)\hat{\mathbf{e}}_x .\end{aligned} \tag{2.318}$$

By setting $\boldsymbol{w}_2 = \boldsymbol{w}$, we obtain the relation from statement a):

$$\boldsymbol{w} = \frac{q}{2m}E_0 t[\cos(\omega t)\hat{\mathbf{e}}_x + \sin(\omega t)\hat{\mathbf{e}}_y] + \frac{q}{2m\omega}E_0\sin(\omega t)\hat{\mathbf{e}}_x . \tag{2.309}$$

b) The third term on the RHS of (2.309) describes a periodic motion in the direction of the field $\boldsymbol{E}$ (this is normal: it would also exist in the absence of $\boldsymbol{B}$) while the first and the second conjugate terms represent a periodic rotational motion, of frequency ω_c, whose amplitude continuously increases with time: in other words, the particle describes a spiral. The modulus of the corresponding velocity of this motion is $w_0 = qE_0 t/2m$, because the contribution to the periodic motion along $\hat{\mathbf{e}}_x$ rapidly becomes negligible. This increase in amplitude comes precisely from the resonance between ω and ω_c, called cyclotron resonance.

c) We use equation (2.317), in which the direction of the field $\boldsymbol{E}$ is along $\hat{\mathbf{e}}_y$ rather than $\hat{\mathbf{e}}_x$. Since $\hat{\mathbf{e}}_y \wedge \hat{\mathbf{e}}_z = \hat{\mathbf{e}}_x$, we have:

$$\begin{aligned}\boldsymbol{w}' &= \frac{qt}{2m}E_0\cos(\omega t)\hat{\mathbf{e}}_y + \frac{q}{2m\omega}E_0\sin(\omega t)\hat{\mathbf{e}}_y + \frac{q^2 t}{2\omega m^2}E_0 B_0\sin(\omega t)\hat{\mathbf{e}}_x \\ &= \frac{q}{2m}E_0 t\left[\cos(\omega t)\hat{\mathbf{e}}_y - \sin(\omega t)\hat{\mathbf{e}}_x\right] + \frac{q}{2\omega m}E_0\sin(\omega t)\hat{\mathbf{e}}_y .\end{aligned} \tag{2.319}$$

We now change the time origin by replacing t by $t - \pi/2\omega$: as $\cos(\omega t - \pi/2) = \sin(\omega t)$ and $\sin(\omega t - \pi/2) = -\cos(\omega t)$, we obtain from (2.319):

$$\boldsymbol{w}' = \frac{q}{2m}E_0\left(t - \frac{\pi}{2\omega}\right)[\sin(\omega t)\hat{\mathbf{e}}_y + \cos(\omega t)\hat{\mathbf{e}}_x] - \frac{q}{2m\omega}E_0\cos(\omega t)\hat{\mathbf{e}}_y \ . \quad (2.311)$$

d) To get the resulting particle velocity in the field E_+ (defined by (2.312)), we simply add the velocities $\boldsymbol{w} + \boldsymbol{w}'$ from equations (2.309) and (2.311):

$$\boldsymbol{w}_{\mathrm{TOT+}} = \frac{q}{2m}E_0 t[2\cos(\omega t)\hat{\mathbf{e}}_x + 2\sin(\omega t)\hat{\mathbf{e}}_y] - \frac{qE_0\pi}{4m\omega}[\sin(\omega t)\hat{\mathbf{e}}_y + \cos(\omega t)\hat{\mathbf{e}}_x] + \frac{qE_0}{2m\omega}[\sin(\omega t)\hat{\mathbf{e}}_x - \cos(\omega t)\hat{\mathbf{e}}_y] \ . \quad (2.320)$$

To obtain the resulting particle velocity in the field E_-, we add the velocity $\boldsymbol{w}$ with $\boldsymbol{w}''$, where $\boldsymbol{w}'' = -\boldsymbol{w}$, since then $\boldsymbol{E} = -E_0\sin(\omega t)\hat{\mathbf{e}}_y$. This yields:

$$\begin{aligned}\boldsymbol{w}_{\mathrm{TOT-}} &\equiv \boldsymbol{w} - \boldsymbol{w}' \\ &= \frac{q}{2m}E_0 t\,[\,\overbrace{\cos(\omega t)\hat{\mathbf{e}}_x + \sin(\omega t)\hat{\mathbf{e}}_y - \sin(\omega t)\hat{\mathbf{e}}_y - \cos(\omega t)\hat{\mathbf{e}}_x}^{=0}\,] \\ &\quad + \frac{q}{4m}\frac{\pi}{\omega}E_0[\sin(\omega t)\hat{\mathbf{e}}_y + \cos(\omega t)\hat{\mathbf{e}}_x] \\ &\quad + \frac{q}{2m\omega}E_0[\sin(\omega t)\hat{\mathbf{e}}_x + \cos(\omega t)\hat{\mathbf{e}}_y] \ . \end{aligned} \quad (2.321)$$

We can conclude that if the field turns in the positive direction of ω_c (i.e. if the particle velocity is not in the direction of the rotating field), the velocity of the particle does not increase linearly in time and, therefore, there is no cyclotron resonance.

2.8. Consider a magnetic field of the form:

$$\boldsymbol{B} = B_0(1 - \epsilon\cos kx)\hat{\mathbf{e}}_x \ , \quad (2.322)$$

where ϵ is a parameter smaller than unity and k is a constant. This field is used to axially confine charged particles at each end of a linear machine whose centre is at $x = 0$.

a) Find the expression for $\boldsymbol{w}_\parallel$ as a function of $\boldsymbol{w}_\parallel(0)$, $\boldsymbol{w}_\perp(0)$, ϵ and k.
b) Show that particles will be effectively trapped if:

$$w_\parallel^2(0) \le w_\perp^2(0)\frac{2\epsilon}{1-\epsilon} \ . \quad (2.323)$$

c) Assuming an isotropic velocity distribution at $x = 0$, calculate the number of trapped particles as a fraction of the total number of particles.

Answer

a) This is a case in which the magnetic field has a (weak) non uniformity in its own direction, which corresponds to the situation treated in Sect. 2.2.3 (page 138); note that we are using here the x axis as the direction of $\boldsymbol{B}$ rather than z. We know that the expression:

$$\boldsymbol{B} = B_0(1 - \epsilon \cos kx)\hat{\mathbf{e}}_x \tag{2.324}$$

is not complete, because the component of $\boldsymbol{B}$ (of order 1), required to satisfy Maxwell's equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, is missing. Nevertheless, this correction does not appear in the calculation of the $\boldsymbol{w}_\parallel$ component. Finally, note that the minimum value of $|\boldsymbol{B}|$, $|\boldsymbol{B}| = B_0(1 - \epsilon)$ is found at $x = 0$ and thus corresponds to the region situated between the mirrors: correctly speaking, there is no region of uniform field in this machine, only two mirrors at each end with a minimum in the magnetic field in between.
By transposing equation (2.176) along $\hat{\mathbf{e}}_x$ and setting $B_\parallel = B(x = 0)$, we have:

$$\boldsymbol{w}_\parallel(t) = \boldsymbol{w}_\parallel(0) - \frac{\hat{\mathbf{e}}_x}{2} r_B^2 \omega_c^2 \frac{1}{B(x=0)} \left(\frac{\partial B_x}{\partial x}\right)_{y=z=0} t\,, \tag{2.325}$$

where ω_c corresponds to the value at $B(x = 0)$. Since $\partial B_x/\partial x = B_0 \epsilon k \sin kx$ and $r_B^2 \omega_c^2 = w_\perp^2(0)$, we find:

$$\boldsymbol{w}_\parallel(t) = \boldsymbol{w}_\parallel(0) - \frac{\hat{\mathbf{e}}_x}{2} w_\perp^2(0) \frac{\epsilon k \sin kx}{1 - \epsilon} t \tag{2.326}$$

(we could neglect the parameter ϵ compared to 1 in the denominator).

b) We have shown in Sect. 2.2.3 (p. 138) that the particles coming from the central region of the machine ($x = 0$) are reflected by the magnetic mirror if the angle α_0 of their vector velocity with respect to the axis (of components $w_{0\parallel} = w_0 \cos\alpha_0$ and $w_{0\perp} = w_0 \sin\alpha_0$) in the section of uniform field (the region of the weakest field between the two mirrors) has a value greater than α_{0m}, defined by (2.189):

$$\sin\alpha_{0m} = \sqrt{\frac{B(x=0)}{B_{\max}}}\,. \tag{2.327}$$

$B_{\max}$ is reached for $\cos kx = -1$, from which $B_{\max} = 1 + \epsilon$ and $B(x = 0) = 1 - \epsilon$, such that (2.327) gives:

$$\sin^2\alpha_{0m} = \frac{1 - \epsilon}{1 + \epsilon}\,. \tag{2.328}$$

Noting that the ratio of the velocities $w_\perp(x = 0)/w(x = 0)$ corresponds to $\sin\alpha_{0m}$, we have:

$$\frac{w_\perp^2(0)}{w^2(0)} = \frac{1-\epsilon}{1+\epsilon} , \tag{2.329}$$

such that:

$$w_\perp^2(0)(1+\epsilon) = (w_\perp^2(0) + w_\parallel^2(0))(1-\epsilon) , \tag{2.330}$$

$$w_\perp^2(0)[(1+\epsilon) + (\epsilon - 1)] = w_\parallel^2(0)(1-\epsilon) , \tag{2.331}$$

and finally:

$$w_\parallel^2(0) = w_\perp^2(0)\frac{2\epsilon}{1-\epsilon} . \tag{2.332}$$

The condition for reflection is defined by the inequality:

$$w_\parallel^2(0) \le w_\perp^2(0)\frac{2\epsilon}{1-\epsilon} . \tag{2.333}$$

c) The reflection coefficient C_r of particles in a magnetic mirror (2.196) leads us to:

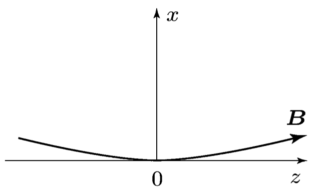
$$C_r \equiv \frac{\Gamma_r}{\Gamma_{\mathrm{inc}}} = 1 - \frac{1}{\mathcal{R}} = 1 - \frac{1}{B_{\max}/B(x=0)} = 1 - \frac{1-\epsilon}{1+\epsilon} = \frac{2\epsilon}{1+\epsilon} . \tag{2.334}$$

2.9. Consider a magnetic field directed **principally** along z but subject to a slight curvature, represented by the term $\partial B_x/\partial z$ (we suppose that the curvature is in two dimensions only, in the plane xz). The origin of the Cartesian frame is chosen such that the field $\boldsymbol{B}$ is directed along the z axis while, on each side of the origin, there is a contribution from the x and z components of the field, as shown in the figure. The radius of curvature ρ of this field line is assumed to be much greater than the particle Larmor radius (of charge q and mass m).

a) Show that, in the immediate region of the origin, $\boldsymbol{B}$ is described by:

$$\boldsymbol{B} = B_0\left(\hat{\mathbf{e}}_x\frac{z}{\rho} + \hat{\mathbf{e}}_z\right) , \tag{2.335}$$

where B_0 is the intensity of the field at $z = 0$ and $\hat{\mathbf{e}}_x$, $\hat{\mathbf{e}}_z$ are the unit base vectors in the Cartesian frame (x, y, z); $\rho^{-1} = \mathrm{d}^2x/\mathrm{d}z^2$ in the case that $\mathrm{d}x/\mathrm{d}z$ is not too large.



b) Assume $\boldsymbol{w}$ is the particle velocity. Using the field given by (2.335), express the components of $\dot{\boldsymbol{w}}$ to first order, in Cartesian coordinates.

c) Determine, to order zero, the three components of velocity and position, using the following initial conditions:

$$x^0 = y^0 = z^0 = 0 , \tag{2.336}$$

$$\boldsymbol{w}^0 = w_{\perp 0}\hat{\mathbf{e}}_x + w_{z0}\hat{\mathbf{e}}_z , \tag{2.337}$$

(the zero superscript in A^0 signifies that the quantity A is expressed to order zero).

d) Show that the calculation to first order leads to:

$$w_x = \left(\frac{w_{z0}^2}{\rho}\right) t + w_{\perp 0} \cos \omega_c t \ , \tag{2.338}$$

$$w_y = -\left(\frac{w_{z0}^2}{\rho \omega_c}\right) + w_{\perp 0} \sin \omega_c t \ , \tag{2.339}$$

where the constants of integration are fixed such that if we set $\rho^{-1} = 0$, we will recover the zeroth order solution. In this first order calculation, we have replaced the values of w_z and z, which appear in the expressions obtained for w_x and w_y in b), by their values approaching $z = 0$, i.e. to order zero, namely w_{z0} and $w_{z0}t$ respectively.

e) Finally, show that the position of the guiding centre satisfies the expression:

$$x = \left(\frac{1}{2\rho}\right) z^2 \ . \tag{2.340}$$

Answer

a) The magnetic field is directed, at the origin of the system, along the z axis and is affected by a (symmetric) curvature of the lines of force in the $x\mathrm{O}z$ plane. Equation (2.335) in the terms of the problem suggests that the B_x component is a correction term for B_z in the neighbourhood of $z = 0$. A similar question was treated in Sect. 2.2.3, p. 147 and p. 152 (2.221) with the inhomogeneity along $\hat{\mathbf{e}}_y$ instead of $\hat{\mathbf{e}}_x$. Achieving this transposition yields:

$$\boldsymbol{B} = \hat{\mathbf{e}}_x(\beta B_0 z) + \hat{\mathbf{e}}_z[B_0(1 + \beta x)] \ . \tag{2.341}$$

The principal component B_z is subject to a first order correction, which we can neglect in comparison to B_0, of order zero; the correction along the x axis introduces a (small) term β, where $\beta \simeq \rho^{-1}$ (XIII.18): equation (2.335) of the question is therefore demonstrated.

We can also treat the problem from the beginning, without recourse to the results of Sect. 2.2.3 as we have just done. Since B_x is a first order correction, a development in series, limited to first order, is justified:

$$B_x = \underbrace{B_x(z=0)}_{=0} + \frac{\partial B_x}{\partial x} x + \frac{\partial B_x}{\partial y} y + \frac{\partial B_x}{\partial z} z \simeq \frac{\partial B_x}{\partial z} z \ , \tag{2.342}$$

the B_x component depending only on z (see figure). Since $\boldsymbol{B} = \hat{\mathbf{e}}_x B_x + \hat{\mathbf{e}}_z B_z$ and $B_z = B_0$, we obtain:

$$\boldsymbol{B} = \hat{\mathbf{e}}_x \frac{\partial B_x}{\partial z} z + \hat{\mathbf{e}}_z B_0 \ . \tag{2.343}$$

To calculate $\partial B_x/\partial z$, we consider a magnetic field line whose slope $\mathrm{d}x/\mathrm{d}z$ is, as a general rule, equal to the ratio B_x/B_z of the local magnetic field components:

$$\frac{\mathrm{d}x}{\mathrm{d}z} = \frac{B_x}{B_z} \ . \tag{2.344}$$

From (2.344), we have:

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} = \frac{\partial B_x}{\partial z}\frac{1}{B_z} - \frac{\partial B_z}{\partial z}\frac{B_x}{B_z^2} \ , \tag{2.345}$$

where the second term on the RHS is negligible compared to the first term because it is of order 2: from (2.341) and (2.342), $\partial B_x/\partial z$ and B_x are both non zero only to order 1.
By assumption, $\mathrm{d}x/\mathrm{d}z$ is not too large in the neighbourhood of the origin, then (XIII.2):

$$\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} \simeq \frac{1}{\rho} \ . \tag{2.346}$$

Finally, from (2.345) and (2.346):

$$\frac{\partial B_x}{\partial z} = \frac{B_0}{\rho} \ , \tag{2.347}$$

from which, following (2.343), we have thus demonstrated the validity of (2.335).

b) We simply return to the general equations shown in Sect. 2.1, which we can write, taking $-qB_0/m = \omega_c$ and setting $E = 0$ (Eqs. (2.6)–(2.8)) in the form:

$$\dot{w}_x = \frac{q}{m}[B_z w_y - B_y w_z] = \frac{q}{m}[B_0 w_y] = -\omega_c w_y \ , \tag{2.348}$$

$$\begin{aligned}\dot{w}_y = \frac{q}{m}[B_x w_z - B_z w_x] &= \frac{q}{m}\left[\frac{B_0 z}{\rho} w_z - B_0 w_x\right] \\ &= -\omega_c \left[\underline{\frac{z}{\rho} w_z} - w_x\right] \ ,\end{aligned} \tag{2.349}$$

$$\dot{w}_z = \frac{q}{m}[B_y w_x - B_x w_y] = -\frac{q}{m}\left[\frac{B_0 z}{\rho} w_y\right] = \omega_c \left[\underline{\frac{z}{\rho} w_y}\right] \ , \tag{2.350}$$

where the underlined terms are of first order, because they tend to zero if the radius of curvature tends to infinity (rectilinear lines of force).

c) To obtain the expression for the components of $\dot{\boldsymbol{w}}$ to order zero, we only need to set $\rho \to \infty$ in (2.348)–(2.350). We then get:

$$\dot{w}_x = -\omega_c w_y \;, \tag{2.351}$$

$$\dot{w}_y = \omega_c w_x \;, \tag{2.352}$$

$$\dot{w}_z = 0 \;. \tag{2.353}$$

We can start by calculating the expressions for w_x and x. To do this, we differentiate (2.351):

$$\dot{w}_x = -\omega_c \dot{w}_y \tag{2.354}$$

and substituting $\dot{w}_y$ from (2.352) gives:

$$\dot{w}_x = -\omega_c(\omega_c w_x) \;, \tag{2.355}$$

from which:

$$\dot{w}_x + \omega_c^2 w_x = 0 \;, \tag{2.356}$$

for which the solution is clearly:

$$w_x = A_1 \cos \omega_c t + A_2 \sin \omega_c t \;, \tag{2.357}$$

where A_1 and A_2 are constants, depending on the initial conditions (equations (2.336) and (2.337)). This leads to:

$$w_x = w_{\perp 0} \cos \omega_c t \tag{2.358}$$

and:

$$x = \frac{w_{\perp 0}}{\omega_c} \sin \omega_c t \;. \tag{2.359}$$

To calculate the expressions for w_y and y, we will take (2.352) and substitute therein the value of w_x from (2.358). After integration, and application of initial conditions, we obtain:

$$w_y = w_{\perp 0} \sin \omega_c t \tag{2.360}$$

and:

$$y = \frac{w_{\perp 0}}{\omega_c}(1 - \cos \omega_c t) \;. \tag{2.361}$$

As for w_z and z, we deduce from (2.353):

$$w_z = w_{z0} \quad \text{and} \quad z = w_{z0} t \;. \tag{2.362}$$

d) The system (2.348)–(2.350) can be resolved quite easily provided that, in the terms in $1/\rho$ (terms of first order), the expressions for velocity and position along $\hat{\mathbf{e}}_z$ are those of (2.362), which are of order zero. Note that, as a result, these terms remain of order one. Then from (2.349), we write down:

$$\dot{w}_y = -\omega_c \left[\frac{w_{z0}^2 t}{\rho} - w_x \right] \;. \tag{2.363}$$

To obtain w_x, we start by differentiating (2.348):

$$\dot{w}_x = -\omega_c \dot{w}_y \ , \tag{2.364}$$

which we can render homogeneous in w_x by substituting the expression for $\dot{w}_y$ from (2.363):

$$\dot{w}_x = \omega_c^2 \left[\frac{w_{z0}^2 t}{\rho} - w_x \right] \ , \tag{2.365}$$

from which:

$$\dot{w}_x + \omega_c^2 w_x = \frac{\omega_c^2 w_{z0}^2 t}{\rho} \ . \tag{2.366}$$

The solution is the general solution with no RHS, namely (2.358), to which must be added a particular solution (of first order): $w_x^{(1)} = w_{z0}^2 t/\rho$ (following (2.366)), from which, in total:

$$w_x = w_{\perp 0} \cos \omega_c t + \frac{w_{z0}^2 t}{\rho} \ . \tag{2.367}$$

For w_y, differentiate (2.363) and we have:

$$\dot{w}_y = -\omega_c \left[\frac{w_{z0}^2}{\rho} - \dot{w}_x \right] \ , \tag{2.368}$$

an expression in which we can replace $\dot{w}_x$ by its expression in (2.348), such that:

$$\dot{w}_y = -\omega_c \left[\frac{w_{z0}^2}{\rho} + \omega_c w_y \right] \ , \tag{2.369}$$

from which:

$$\dot{w}_y + \omega_c^2 w_y = -\frac{\omega_c}{\rho} w_{z0}^2 \ , \tag{2.370}$$

whose solution, following the method of (2.367), is:

$$w_y = w_{\perp 0} \sin \omega_c t - \frac{1}{\rho \omega_c} w_{z0}^2 \ . \tag{2.371}$$

e) To calculate the guiding centre velocity, we simply ignore the sinusoidal terms in (2.367) and (2.371), such that:

$$w_x = \frac{w_{z0}^2 t}{\rho} \ , \tag{2.372}$$

$$w_y = -\frac{w_{z0}^2}{\rho \omega_c} \ , \tag{2.373}$$

and, clearly ($\boldsymbol{E} = 0$):

$$w_z = w_{z0} \ . \tag{2.374}$$

The position of the guiding centre is then described by:

$$x = \frac{w_{z0}^2 t^2}{2\rho} \,, \tag{2.375}$$

$$y = -\frac{w_{z0}^2 t}{\rho\omega_c} \,, \tag{2.376}$$

$$z = w_{z0} t \,. \tag{2.377}$$

Combining (2.375) and (2.377) for x and z, we obtain:

$$x = \frac{z^2}{2\rho} \,. \tag{2.378}$$

This guiding centre motion is due to the magnetic curvature drift (Sect. 2.2.3, p. 147). To obtain the contribution from the magnetic drift, as was shown in that section, we need to include the correction terms (to first order) in B_z, which we have neglected in our calculation so far (see (2.363)).

Remark: We can also obtain (2.378) by recalling that the motion of the guiding centre due to the curvature drift is effected along the lines of force $\boldsymbol{B}$. It is sufficient to use the parameterisation of the lines of force calculated in Appendix XIII. From equation (XIII.7), we have $\mathrm{d}y/\mathrm{d}z = z/\rho$ which, adapted to the direction of curvature in the present exercise, becomes $dx/dz = z/\rho$, from which $x = z^2/2\rho$, since the constant of integration is zero, because at $z = 0$, $B_x = 0$.

2.10. Consider an axial, linear magnetic confinement machine whose magnetic field in the homogeneous region has a magnitude B_0. This machine is fitted with a magnetic mirror situated at $z \geq 0$ such that the value of the magnetic field is:

$$\boldsymbol{B} = \hat{\mathbf{e}}_z B_0 \left[1 + \left(\frac{z}{a} \right)^2 \right] \,, \tag{2.379}$$

where a is a constant. Find an expression giving the position z, in the mirror zone, at which a particle is reflected if its vector velocity makes an angle α_0 with the direction of the magnetic field in the homogeneous region.

Answer

The position at which a particle is reflected in the mirror zone is independent of the magnitude of the velocity, but depends on the angle α_0 (in the homogeneous field region) of its vector velocity with respect to the axis of the machine. More exactly, we have seen that:

$$\sin \alpha = \sin \alpha_0 \sqrt{\frac{B(z)}{B_0}} \,, \tag{2.187}$$

where the angle α is the angle of its vector velocity with respect to the axis of the machine in the mirror region at the position z, where the magnetic field is $B(z)$ (see Fig. 2.15). The value of z at which the particle is reflected is simply obtained by setting $\sin \alpha = 1$ in the preceding equation. Since:

$$B(z) = B_0 \left[1 + \left(\frac{z}{a} \right)^2 \right] \,, \tag{2.379}$$

we then obtain:

$$1 = \sin \alpha_0 \sqrt{\frac{B_0 \left[1 + \left(\frac{z}{a} \right)^2 \right]}{B_0}} \tag{2.380}$$

from which:

$$\left(\frac{1}{\sin \alpha_0} \right)^2 = 1 + \left(\frac{z}{a} \right)^2 \,, \tag{2.381}$$

i.e.:

$$z = a \left[\left(\frac{1}{\sin \alpha_0} \right)^2 - 1 \right]^{\frac{1}{2}} = \pm a \cot \alpha_0 \,. \tag{2.382}$$

Remark: Although the field $\boldsymbol{B}$ has components B_x and B_y, which are necessary to satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$, the important component for the mirror effect is B_z (remember that in the adiabatic approximation, only F_z enters into the conservation of μ_z).

2.11. Consider the motion of an electron in a uniform magnetic field $\boldsymbol{B}$ directed along the z axis and symmetric about this axis. This magnetic field varies slowly as a function of time, such that $B_z = B_0(1 - \alpha t)$, where the intensity B_0 is constant and α is a very small parameter.

a) Verify that the field $\boldsymbol{B}$ satisfies Maxwell's equations
b) Show that, in a Cartesian frame, the equation of motion of the electron can be written in the form:

$$\dot{w}_x = -\omega_{ce} \left[w_y (1 - \alpha t) - \frac{\alpha y}{2} \right] \,, \tag{2.383}$$

$$\dot{w}_y = \omega_{ce} \left[w_x (1 - \alpha t) - \frac{\alpha x}{2} \right] \,, \tag{2.384}$$

$$\dot{w}_z = 0 \,, \tag{2.385}$$

where $\boldsymbol{w} = (w_x, w_y, w_z)$ is the velocity of the electron, and ω_{ce} its cyclotron frequency in the field B_0.
In this calculation, why is it important that α be small?

Answer

a) Conformity of the field $\boldsymbol{B}$ with Maxwell's equations.
Due to the fact that the magnetic field varies with time, it creates an associated electric field $\boldsymbol{E}$, described by Maxwell's equation:

$$\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} , \tag{2.386}$$

and this field $\boldsymbol{E}$ is perpendicular to $\boldsymbol{B}$. Since $\boldsymbol{B} = \hat{\mathbf{e}}_z B_0(1-\alpha t)$, expanding equation (2.386) gives:

$$\hat{\mathbf{e}}_x \left(-\frac{\partial E_y}{\partial z}\right) + \hat{\mathbf{e}}_y \left(\frac{\partial E_x}{\partial z}\right) + \hat{\mathbf{e}}_z \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = \alpha B_0 \hat{\mathbf{e}}_z . \tag{2.387}$$

This requires that $\partial E_y/\partial z = \partial E_x/\partial z = 0$ and $\partial E_y/\partial x - \partial E_x/\partial y = \alpha B_0$, i.e. that E_x and E_y are independent of z. Under these conditions, due to the axial symmetry with respect to z, we can infer that the rotational term along the z axis can be written:

$$E_x = -\frac{y}{2}(\alpha B_0) , \tag{2.388}$$

$$E_y = \frac{x}{2}(\alpha B_0) . \tag{2.389}$$

In fact, the vector $\boldsymbol{E}$ describes a circle in the $x\mathrm{O}y$ plane.
For the equation $\boldsymbol{\nabla} \cdot \boldsymbol{D} = \rho$, where $\rho = 0$ in the framework of individual trajectories, following (2.388) and (2.389), we have $\partial E_x/\partial x + \partial E_y/\partial y = 0$, and this equation is verified.
The verification of $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ is trivial, since $\boldsymbol{B}$ is independent of position.
Finally, for $\boldsymbol{\nabla} \wedge \boldsymbol{H} = \boldsymbol{J} + \partial \boldsymbol{D}/\partial t$, where $\boldsymbol{J} = 0$ (individual trajectory), $\boldsymbol{\nabla} \wedge \boldsymbol{H} = 0$ because $\boldsymbol{H}$ is uniform, and the term $\partial \boldsymbol{D}/\partial t = \epsilon_0 \partial \boldsymbol{E}/\partial t = 0$.

b) Equation of motion. Following (2.6)–(2.8), we have:

$$m_e \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = q_e \left[E_x + B_z \frac{\mathrm{d}y}{\mathrm{d}t}\right] = -q_e \frac{\alpha B_0}{2} y + q_e B_0 (1-\alpha t)\dot{y} , \tag{2.390}$$

$$m_e \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = q_e \left[E_y - B_z \frac{\mathrm{d}x}{\mathrm{d}t}\right] = q_e \frac{\alpha B_0}{2} x - q_e B_0 (1-\alpha t)\dot{x} , \tag{2.391}$$

$$m_e \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = 0 . \tag{2.392}$$

Since the motion is uniform along z, setting $\omega_{ce} = -q_e B/m$ and $w_x \equiv \mathrm{d}x/\mathrm{d}t$, $w_y \equiv \mathrm{d}y/\mathrm{d}t$, we have then verified (2.383) and (2.384):

$$\dot{w}_x = -\omega_{ce}\left[(1-\alpha t)w_y - \frac{\alpha y}{2}\right] , \tag{2.393}$$

$$\dot{w}_y = \omega_{ce}\left[(1-\alpha t)w_x - \frac{\alpha x}{2}\right] , \tag{2.394}$$

$$\dot{w}_z = 0 . \tag{2.395}$$

The parameter α must be chosen to be small in order to keep a positive intensity of $\boldsymbol{B}$ with time ($\alpha t < 1$).

2.12. Consider the motion of an electron near the origin of a given frame. The electron is subject to a magnetic field which is constant in time, but slightly inhomogeneous along the field lines. Its equation is given by:

$$\boldsymbol{B} = \hat{\mathbf{e}}_z B_0(1+\alpha z) , \tag{2.396}$$

where α is sufficiently small, such that $\alpha z \ll 1$. Assume that the field is axially symmetric about the z axis. The initial conditions are as follows: $x(0) = -x_0$, $y(0) = 0$, $z(0) = 0$, $w_x(0) = 0$, $w_y(0) = w_{\perp 0}$, $w_z(0) = w_{z0}$.

a) Show that the x, y and z components of the equation of motion due to the Lorentz force are described by the following equations:

$$\ddot{x} = -\omega_{ce}\left[\dot{y} + \alpha\left(z\dot{y} + \frac{1}{2}y\dot{z}\right)\right] , \tag{2.397}$$

$$\ddot{y} = \omega_{ce}\left[\dot{x} + \alpha\left(z\dot{x} + \frac{1}{2}x\dot{z}\right)\right] , \tag{2.398}$$

$$\ddot{z} = -\omega_{ce}\left(\frac{\alpha}{2}\right)[x\dot{y} - y\dot{x}] . \tag{2.399}$$

b) Supposing that the initial velocity is given by:

$$\boldsymbol{w}_0 = w_{\perp 0}\hat{\mathbf{e}}_y + w_{z0}\hat{\mathbf{e}}_z , \tag{2.400}$$

show that:

$$\dot{z} \equiv w_z = -\frac{1}{2}\alpha w_{\perp 0}^2 t + w_{z0} . \tag{2.401}$$

Answer

a) The field we are considering has a slight inhomogeneity in its own direction. Under these conditions, and due to the axial symmetry of $\boldsymbol{B}$, we know (Sect. 2.2.3, p. 138) there is, in fact, a component B_r (of order 1 with respect to the component B_z, of order zero) which has been ignored in (2.396).
Including an expression for this in Cartesian coordinates (2.163), the complete expression for the field $\boldsymbol{B}$ is now:

$$\boldsymbol{B} = -\frac{1}{2}x\left(\frac{\partial B_z}{\partial z}\right)_{x=y=0}\hat{\mathbf{e}}_x - \frac{1}{2}y\left(\frac{\partial B_z}{\partial z}\right)_{0,0}\hat{\mathbf{e}}_y + B_0(1+\alpha z)\hat{\mathbf{e}}_z\ , \quad (2.402)$$

where $\partial B_z/\partial z$ is calculated from (2.396), and finally:

$$\boldsymbol{B} = -\frac{1}{2}xB_0\,\alpha\hat{\mathbf{e}}_x - \frac{1}{2}yB_0\,\alpha\hat{\mathbf{e}}_y + B_0(1+\alpha z)\hat{\mathbf{e}}_z\ . \quad (2.403)$$

Developing the equation of motion along the three axes of the Cartesian frame, and setting $E = 0$, we have:

along x:
$$\dot{x} = \frac{q_e}{m_e}(B_z\dot{y} - B_y\dot{z}) = \frac{q_e}{m_e}\left[B_0(1+\alpha z)\dot{y} + \frac{B_0}{2}\alpha y\dot{z}\right]$$
$$= -\omega_{ce}\left[\dot{y} + \alpha\left(z\dot{y} + \frac{y\dot{z}}{2}\right)\right]\ , \quad (2.404)$$

along y:
$$\dot{y} = \omega_{ce}\left[\dot{x} + \alpha\left(z\dot{x} + \frac{x\dot{z}}{2}\right)\right]\ , \quad (2.405)$$

and along z:
$$\dot{z} = -\omega_{ce}\frac{\alpha}{2}(x\dot{y} - y\dot{x})\ . \quad (2.406)$$

b) To calculate the velocity along the guiding axis, we can observe, following (2.406), that it is of order 1, due to the presence of the small parameter α (assuming $\alpha z \ll 1$), which agrees with the assumption of the guiding centre approximation. We therefore simply need to replace the positions and velocities of the cyclotron motion appearing in (2.406) by their expansion, limited to order zero.
The equations describing the zeroth order motion in the perpendicular plane are obtained by setting $\alpha = 0$ in (2.404) and (2.405):

$$\dot{x} = -\omega_{ce}\dot{y}\ , \quad (2.407)$$

$$\dot{y} = \omega_{ce}\dot{x}\ . \quad (2.408)$$

Now we must resolve the system of 2 equation in 2 unknowns.
To calculate the motion along y, we perform a first integration of (2.407) over time:

$$\dot{x} = -\omega_{ce}y + C_1\ , \quad (2.409)$$

where the constant $C = 0$, because $\dot{x}(0) = 0$ and $y(0) = 0$, such that:

$$\dot{x} = -\omega_{ce}y\ . \quad (2.410)$$

Substituting this equation in (2.408), we obtain:

$$\dot{y} + \omega_{ce}^2 y = 0\ , \quad (2.411)$$

for which the solution is:

$$y = A_1 \cos \omega_{ce} t + A_2 \sin \omega_{ce} t \ . \tag{2.412}$$

To determine the constants A_1 and A_2, we note that:

$$y(0) \equiv 0 = A_1 \ , \tag{2.413}$$

such that:

$$w_y = A_2 \omega_{ce} \cos \omega_{ce} t \ , \tag{2.414}$$

with the initial condition $w_y(0) = w_{\perp 0}$.
Finally:

$$w_y = w_{\perp 0} \cos \omega_{ce} t \tag{2.415}$$

and:

$$y = \frac{w_{\perp 0}}{\omega_{ce}} \sin \omega_{ce} t \ . \tag{2.416}$$

To calculate the motion along the x axis, we continue in an analogous fashion. Integrating (2.408), we obtain:

$$\dot{y} = \omega_{ce} x + C_2 \ , \tag{2.417}$$

where, because of the initial conditions, the constant of integration $C_2 = w_{\perp 0} - \omega_{ce} x_0$.
Substituting (2.417) in (2.407):

$$\dot{x} + \omega_{ce}^2 x = -\omega_{ce} w_{\perp 0} + \omega_{ce}^2 x_0 \ , \tag{2.418}$$

whose complete solution has the form:

$$x = A_1 \cos \omega_{ce} t + A_2 \sin \omega_{ce} t - \frac{w_{\perp 0}}{\omega_{ce}} - x_0 \ . \tag{2.419}$$

Using the initial conditions $x(0) = -x_0$ and $w_x(0) = 0$, we find successively $A_1 = w_{\perp 0}/\omega_{ce}$, from which:

$$x = \frac{w_{\perp 0}}{\omega_{ce}} \cos \omega_{ce} t \tag{2.420}$$

and $A_2 = 0$, from which:

$$w_x = -w_{\perp 0} \sin \omega_{ce} t \ . \tag{2.421}$$

To calculate the motion along z, we substitute the values for the motion along x and y in (2.406):

$$\dot{z} = -\frac{\omega_{ce}\alpha}{2} \left[\frac{w_{\perp 0}^2}{\omega_{ce}} \cos^2 \omega_{ce} t + \frac{w_{\perp 0}^2}{\omega_{ce}} \sin^2 \omega_{ce} t \right] = -\frac{\alpha}{2} w_{\perp 0}^2 \ , \tag{2.422}$$

which, on integration gives:

$$\dot{z} = -\frac{\alpha}{2} w_{\perp 0}^2 t + C_3 \,, \tag{2.423}$$

where $\dot{z}(0) \equiv w_{z0} = C_3$, which indeed leads us to (2.401):

$$\dot{z} = -\frac{\alpha}{2} w_{\perp 0}^2 t + w_{z0} \,.$$

Remark: The choice of $x(0) = -x_0$ rather than $x(0) = 0$ as initial condition enables us to obtain simpler expressions for the components of velocity and position!

2.13. In the context of the individual trajectories model, consider an applied magnetic field:

$$\boldsymbol{B} = B_0 h(t) \hat{\mathbf{e}}_z \,, \tag{2.424}$$

where B_0 is a constant and $h(t)$ is a slowly varying function of time.

a) Verify that Maxwell's equations are satisfied if the field $\boldsymbol{E}$ induced by $\mathrm{d}\boldsymbol{B}/\mathrm{d}t$ is expressed by:

$$\boldsymbol{E} = \frac{1}{2} B_0 \dot{h} (y \hat{\mathbf{e}}_x - x \hat{\mathbf{e}}_y) \,, \tag{2.425}$$

where $\dot{h} = \mathrm{d}h(t)/\mathrm{d}t$. Specify the restrictions, if necessary.

b) Using the values of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$, show that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} m w_\perp^2 \right) = -\frac{1}{2} m \omega_c \dot{h} (y w_x - x w_y) \,, \tag{2.426}$$

where $\boldsymbol{w}_\perp = w_x \hat{\mathbf{e}}_x + w_y \hat{\mathbf{e}}_y$, and ω_c is the cyclotron frequency of the charged particle.

c) Find the solutions for x, y, w_x, w_y, to zeroth order (initial conditions $x = y = z = 0$; $w_x(0) = w_{x0}$, $w_y(0) = w_{y0}$, $w_z(0) = w_{z0}$).

d) Show that the quantity $m w_\perp^2 / 2B$ remains constant to order zero in h.

Answer

a) We want to check whether the field $\boldsymbol{E}$ induced by the time variation of $\boldsymbol{B}$ and the field $\boldsymbol{B}$ itself obey Maxwell's four equations:

1. $$\boldsymbol{\nabla} \wedge \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \,. \tag{2.427}$$

The calculation of the LHS gives:

$$\nabla \wedge \boldsymbol{E} \equiv \nabla \wedge \left[\frac{1}{2} B_0 \dot{h}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\right] = \frac{1}{2} B_0 \dot{h} \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} ,$$

$$\equiv \frac{1}{2} B_0 \dot{h} \hat{\mathbf{e}}_z(-1-1) = -B_0 \dot{h} \hat{\mathbf{e}}_z \ , \tag{2.428}$$

but:

$$- B_0 \dot{h} \hat{\mathbf{e}}_z \equiv -\frac{\partial \boldsymbol{B}}{\partial t} \ , \tag{2.429}$$

which corresponds to the RHS of (2.427): this equation is therefore satisfied.

2. $$\nabla \cdot \epsilon_0 \boldsymbol{E} = 0 \ . \tag{2.430}$$

From Poisson's equation $\nabla \cdot \boldsymbol{D} \equiv \nabla \cdot \epsilon_0 \boldsymbol{E} = \rho$, where $\rho = 0$ in the individual trajectories model; in effect, the assumption in this description is that there are no charges to induce the field $\boldsymbol{E}$. Equation (2.430) is effectively verified since:

$$\begin{aligned} \nabla \cdot \epsilon_0 \boldsymbol{E} &\equiv \nabla \cdot \left[\epsilon_0 B_0 \dot{h}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\right] \\ &= \epsilon_0 B_0 \dot{h} \left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x) + \frac{\partial}{\partial z}(0)\right] \equiv 0 \ . \end{aligned} \tag{2.431}$$

3. $$\nabla \wedge \boldsymbol{B} = \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \boldsymbol{J} \ . \tag{2.432}$$

where $\boldsymbol{J}$, the conduction current, is zero in the framework of individual trajectories. Expanding the LHS of this equation gives:

$$\begin{aligned} \nabla \wedge \boldsymbol{B} &\equiv \begin{vmatrix} \hat{\mathbf{e}}_x & \hat{\mathbf{e}}_y & \hat{\mathbf{e}}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0 & 0 & B_0 h(t) \end{vmatrix} \\ &\equiv \frac{\partial}{\partial y}(B_0 h(t))\hat{\mathbf{e}}_x - \frac{\partial}{\partial x}(B_0 h(t))\hat{\mathbf{e}}_y \equiv 0 \ , \end{aligned} \tag{2.433}$$

since $B_0 h(t)$ is independent of position.
It remains to show that the RHS, $\partial \boldsymbol{E}/\partial t$, is also zero. From (2.425), we have:

$$\begin{aligned}\frac{\partial \boldsymbol{E}}{\partial t} &= \frac{\partial}{\partial t}\left[\frac{1}{2}B_0\dot{h}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\right] \\ &= \frac{B_0}{2}\left[\dot{h}(w_y\hat{\mathbf{e}}_x - w_x\hat{\mathbf{e}}_y) + \ddot{h}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\right] .\end{aligned} \tag{2.434}$$

This expression is zero to zeroth order ($\dot{h} \simeq \ddot{h} \simeq 0$), but not to first order.

Remark: We cannot set a priori, $\partial \boldsymbol{E}/\partial t = 0$ because this requires us, in the present case, to neglect the field $\boldsymbol{E}$ induced by $\partial \boldsymbol{B}/\partial t$. On the other hand, the field $\boldsymbol{E}$ induced by the particle motion ($\boldsymbol{J} = \sigma \boldsymbol{E}$) is effectively zero in the framework of individual particles since $J = 0$.

4. $$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 . \tag{2.435}$$

This equation is trivially verified because B_0 and $h(t)$ are independent of position. In effect:

$$\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(0) + \frac{\partial}{\partial z}(B_0 h(t)) = 0 . \tag{2.436}$$

The four Maxwell equations are satisfied, but only to order zero ($\dot{h} = \ddot{h} = 0$) for $\boldsymbol{\nabla} \wedge \boldsymbol{B} = \partial \boldsymbol{D}/\partial t$.

b) We know that only the electric field affects the work done (Sect. 2.1); in the present case, the actual electric field is that induced by the variation of $\boldsymbol{B}$. The component of the kinetic energy perpendicular to the field $\boldsymbol{B}$ is given by:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mw_\perp^2\right) = q\boldsymbol{E}\cdot\boldsymbol{w}_\perp , \tag{2.437}$$

but this is also the total work effected because $\boldsymbol{E}$ is entirely in the plane perpendicular to $\boldsymbol{B}$.
We then have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mw_\perp^2\right) &= q\frac{1}{2}B_0\dot{h}(y\hat{\mathbf{e}}_x - x\hat{\mathbf{e}}_y)\cdot(w_x\hat{\mathbf{e}}_x + w_y\hat{\mathbf{e}}_y) , \\ &= \left(\frac{qB_0}{m}\right)\frac{m\dot{h}}{2}(yw_x - xw_y)\end{aligned} \tag{2.438}$$

and, noting $\omega_c = -qB_0/m$, we then retrieve:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}mw_\perp^2\right) = -\frac{1}{2}m\omega_c\dot{h}(yw_x - xw_y) . \tag{2.426}$$

c) The equations of motion in Cartesian coordinates (Sect. 2.1) are:

$$\dot{w}_x = \frac{q}{m}\left[\frac{1}{2}B_0\dot{h}y + B_0h(t)w_y\right] , \quad (2.439)$$

$$\dot{w}_y = \frac{q}{m}\left[-\frac{1}{2}B_0\dot{h}x - B_0h(t)w_x\right] , \quad (2.440)$$

$$\dot{w}_z = \frac{q}{m}[0] . \quad (2.441)$$

To integrate these equations, we will use the guiding centre approximation, and consider that there are two timescales, that of the cyclotron motion, and that of the much slower motion due to $h(t)$: thus $h(t)$ will be a constant to zeroth order and we will then put $\dot{h} = 0$ in Eqs. (2.439) and (2.440). Setting $-qB_0h/m = \Omega$, we can, by identification, use the solutions for a constant field B, taken from Sect. 2.2.2, p. 113.

However, in this case, $\boldsymbol{B}$ is along $\hat{\mathbf{e}}_x$ while here it needs to be along $\hat{\mathbf{e}}_z$: exchanging $x \leftrightarrow z$ and recalling that, to maintain a right-handed frame, we must have $\hat{\mathbf{e}}_x \wedge \hat{\mathbf{e}}_y = \hat{\mathbf{e}}_z$, it is necessary to replace z by $-x$ in such a permutation.

Finally:

$$x = -\frac{w_{x0}}{\Omega}\sin\Omega t + \frac{w_{y0}}{\Omega}\cos\Omega t - \frac{w_{y0}}{\Omega} , \quad (2.442)$$

$$y = \frac{w_{x0}}{\Omega}\cos\Omega t + \frac{w_{y0}}{\Omega}\sin\Omega t - \frac{w_{x0}}{\Omega} , \quad (2.443)$$

$$z = w_{z0}t , \quad (2.444)$$

$$w_x = -w_{x0}\cos\Omega t - w_{y0}\sin\Omega t , \quad (2.445)$$

$$w_y = -w_{x0}\sin\Omega t + w_{y0}\cos\Omega t , \quad (2.446)$$

$$w_z = w_{z0} . \quad (2.447)$$

We can easily verify that the initial conditions are respected: we recover $x = y = z = 0$ at $t = 0$ and $w_x(0) = w_{x0}$, $w_y(0) = w_{y0}$, $w_z(0) = w_{z0}$.

Another solution

The initial conditions are the same except for $w_x(0) = 0$. The expressions are then much simpler:

$$x = \frac{w_{y0}}{\Omega}\cos\Omega t - \frac{w_{y0}}{\Omega} , \quad (2.448)$$

$$y = \frac{w_{y0}}{\Omega}\sin\Omega t , \quad (2.449)$$

$$z = w_{z0}t , \quad (2.450)$$

$$w_x = -w_{y0} \sin \Omega t \; , \tag{2.451}$$

$$w_y = w_{y0} \cos \Omega t \; , \tag{2.452}$$

$$w_z = w_{z0} \; . \tag{2.453}$$

d) We need to verify that the first adiabatic invariant is, in effect, constant to zeroth order, in the present configuration.
We will consider the temporal variation of the magnetic moment:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu \equiv \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mw_\perp^2}{2B}\right) = \frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{mw_\perp^2}{2}\right) - \frac{1}{B^2}\left(\frac{\mathrm{d}B}{\mathrm{d}t}\right)\left(\frac{mw_\perp^2}{2}\right) . \tag{2.454}$$

The derivative appearing in the first term on the RHS is given by (2.426), while the second term is calculated by making use of the zeroth order velocities ((2.445) and (2.446)), conforming to the concept of adiabatic invariance:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu = \frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}t}\left[-\frac{1}{2}m\omega_c\dot{h}(yw_x - xw_y)\right] - \frac{1}{B_0^2h^2}B_0\dot{h}\left[\frac{1}{2}m(w_{x0}^2 + w_{y0}^2)\right] . \tag{2.455}$$

It is clear that the RHS, due to the presence of $\dot{h}$, is of first order, therefore zero to zeroth order, which leads to the fact that μ is indeed a constant to order zero.

2.14. Consider a static magnetic field (toroidal field):

$$\boldsymbol{B} = \hat{\mathbf{e}}_x B_0 \alpha z + \hat{\mathbf{e}}_z B_0(1 + \alpha x) \; , \tag{2.456}$$

where α is a constant much less than unity.

a) Express the equation of motion of an individual particle in Cartesian coordinates. Underline the terms which are linked to the curvature of the lines of force of the magnetic field $\boldsymbol{B}$.
b) Find the solutions of the motion to zeroth order, knowing that the initial conditions are:

$$\begin{aligned} &w_x(0) = 0 && x(0) = r_B \\ &w_y(0) = w_{y0} && y(0) = 0 \\ &w_z(0) = w_{z0} && z(0) = 0 \end{aligned} \tag{2.457}$$

with $r_B = w_{y0}/\omega_c$, where ω_c is the cyclotron frequency and r_B, the cyclotron gyro-radius.
c) Then show that to order one, the following equation is obtained for w_x.

$$\ddot{w}_x + \omega_c^2 w_x \simeq \alpha\omega_c^2 \left(\frac{3}{2} r_B w_{y0} \sin 2\omega_c t + w_{z0}^2 t \right) . \tag{2.458}$$

d) Find the solution for w_x from (2.458).

Answer

a) This problem corresponds to the case studied in Sect. 2.2.3, p. 152. The component of $\boldsymbol{B}$ in the direction of $\hat{\mathbf{e}}_x$ is that responsible for the curvature of the field lines of $\boldsymbol{B}$. The field already satisfies Maxwell's equations, since $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ (Sect. 2.2.3, p. 138) and clearly from (2.457), $\boldsymbol{\nabla} \wedge \boldsymbol{B} = 0$.
The equation of motion in Cartesian coordinates, with $E = 0$, $B_y = 0$ and $\omega_c = -qB/m$, is obtained from (2.6)–(2.8):

$$\dot{w}_x = \frac{q}{m}\left[B_z w_y\right] = \frac{q}{m}\left[B_0(1+\alpha x) w_y\right] = -\omega_c w_y (1+\alpha x) , \tag{2.459}$$

$$\begin{aligned}\dot{w}_y &= \frac{q}{m}\left[B_x w_z - B_z w_x\right] = \frac{q}{m}\left[B_0 \alpha z w_z - B_0(1+\alpha x) w_x\right] \\ &= \omega_c \left[w_x + \alpha(x w_x - \underline{z w_z})\right] ,\end{aligned} \tag{2.460}$$

$$\dot{w}_z = \frac{q}{m}\left[-B_x w_y\right] = -\frac{q}{m}\left[B_0 \alpha z w_y\right] = \underline{\omega_c \alpha z w_y} , \tag{2.461}$$

where the first order quantities underlined are related to the curvature of the field (contribution from the B_x component).

b) In order to reduce (2.459)–(2.461) to order zero, we only need to set $\alpha = 0$:

$$\dot{w}_x = -\omega_c w_y , \tag{2.462}$$

$$\dot{w}_y = \omega_c w_x , \tag{2.463}$$

$$\dot{w}_z = 0 . \tag{2.464}$$

To calculate the w_x component, we start by differentiating equation (7), to introduce $\dot{w}_y$:

$$\dot{w}_x = -\omega_c \dot{w}_y , \tag{2.465}$$

then use (2.463) to obtain:

$$\dot{w}_x + \omega_c^2 w_x = 0 , \tag{2.466}$$

which has the (harmonic oscillator) solution:

$$w_x = A_1 \sin \omega_c t + A_2 \cos \omega_c t , \tag{2.467}$$

where the constants A_1 and A_2 must be determined from the initial conditions. Since $w_x(0) = 0$, $A_2 = 0$. For A_1, we have, by integrating (2.467):

$$x = -\frac{A_1}{\omega_c}\cos\omega_c t \tag{2.468}$$

and since $x(0) = r_B$, $A_1 = -r_B\omega_c$:

$$w_x = -r_B\omega_c \sin\omega_c t\ , \tag{2.469}$$

$$x = r_B \cos\omega_c t\ . \tag{2.470}$$

w_y, it is found by the same method:

$$w_y = A_1 \sin\omega_c t + A_2 \cos\omega_c t\ , \tag{2.471}$$

which leads to:

$$w_y = w_{y0}\cos\omega_c t \tag{2.472}$$

and:

$$y = r_B \sin\omega_c t\ . \tag{2.473}$$

Finally, for the w_z component, since $\dot{w}_z = 0$ (2.461), we obtain:

$$w_z = w_{z0} \tag{2.474}$$

and:

$$z = w_{z0}t\ . \tag{2.475}$$

c) By substituting the values of the zero order components of $\boldsymbol{w}$ and $\boldsymbol{r}$ in the terms involving α ((2.459)–(2.461)), we find:

$$\dot{w}_x = -\omega_c\left[w_y + \alpha w_{y0} r_B \cos^2\omega_c t\right]\ , \tag{2.476}$$

$$\dot{w}_y = \omega_c\left[w_x + \alpha(-r_B^2\omega_c \sin\omega_c t\cos\omega_c t - w_{z0}^2 t)\right]\ , \tag{2.477}$$

$$\dot{w}_z = \alpha\omega_c w_{y0} w_{z0} t\cos\omega_c t\ . \tag{2.478}$$

To obtain an homogeneous equation for w_x, we proceed in the same fashion as b), differentiating (2.476) with respect to time t, then replacing $\dot{w}_y$ by its value taken from (2.477):

$$\begin{aligned}\ddot{w}_x &= -\omega_c\left[\dot{w}_y - 2\alpha w_{y0} r_B\omega_c\cos\omega_c t\sin\omega_c t\right]\ ,\\ \ddot{w}_x &= -\omega_c\left\{\omega_c\left[w_x + \alpha\left(-\frac{r_B w_{y0}}{2}\sin 2\omega_c t - w_{z0}^2 t\right)\right]\right\}\\ &\quad +\omega_c^2\alpha w_{y0} r_B \sin 2\omega_c t\ ,\end{aligned}$$

i.e.:

$$\ddot{w}_x + \omega_c^2 w_x = \alpha\omega_c^2\left[\frac{3}{2}r_B w_{y0}\sin 2\omega_c t + w_{z0}^2 t\right]\ . \tag{2.479}$$

d) The solution of the differential equation (2.479) is the sum of the general solution without the RHS:

$$w_x = A_1 \cos\omega_c t + A_2 \sin\omega_c t \tag{2.480}$$

and a particular solution with the RHS (not obvious!):

$$w_x = -\frac{1}{2}\alpha r_B w_{y0} \sin 2\omega_c t + \alpha w_{z0}^2 t \ . \tag{2.481}$$

We can verify that (2.481) is indeed a particular solution of (2.479). From (2.479) and (2.480),

$$\begin{aligned}\dot{w}_x + \omega_c^2 w_x &\equiv 2\alpha\omega_c^2 r_B w_{y0} \sin 2\omega_c t - \frac{\omega_c^2}{2}\alpha r_B w_{y0} \sin 2\omega_c t + \omega_c^2 \alpha w_{z0}^2 t \\ &\equiv \frac{3}{2}\alpha\omega_c^2 r_B w_{y0} \sin 2\omega_c t + \omega_c^2 \alpha w_{z0}^2 t \ ,\end{aligned}$$

which corresponds exactly to the RHS of (2.479).
We can fix the constants A_1 and A_2 in (2.480) by the values they have at $t = 0$. Since $w_x(0) = 0$, then $A_1 = 0$. To obtain A_2, we integrate the complete solution to obtain x:

$$x = -\frac{A_2}{\omega_c}\cos\omega_c t + \frac{1}{2}\frac{\alpha r_B w_{y0}}{2\omega_c}\cos 2\omega_c t + \frac{\alpha w_{z0}^2 t^2}{2} \ , \tag{2.482}$$

from which:

$$x(0) = -\frac{A_2}{\omega_c} + \frac{1}{4}\frac{\alpha r_B w_{y0}}{\omega_c} = r_B \tag{2.483}$$

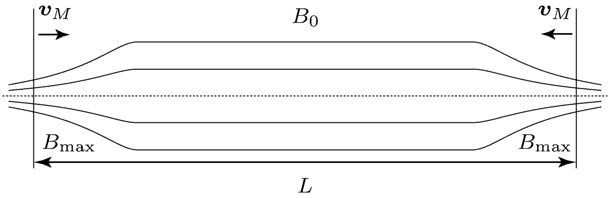
and, thus, since $r_B = \dfrac{w_{0\perp}}{\omega_c} = \dfrac{w_{0y}}{\omega_c}$:

$$A_2 = \left(\frac{1}{4}\frac{\alpha r_B w_{y0}}{\omega_c} - r_B\right)\omega_c = \frac{1}{4}\alpha r_B w_{y0} - w_{y0} = w_{y0}\left(\frac{\alpha r_B}{4} - 1\right) \ , \tag{2.484}$$

i.e.:

$$w_x = w_{y0}\left(\frac{\alpha r_B}{4} - 1\right)\sin\omega_c t - \frac{1}{2}\alpha r_B w_{y0} \sin 2\omega_c t + \alpha w_{z0}^2 t \ . \tag{2.485}$$

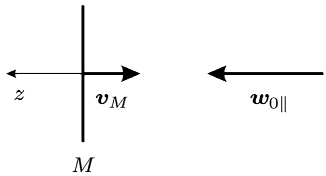
2.15. Consider a linear magnetic confinement machine, limited at each end by magnetic mirrors. We can arrange for the two mirrors to be displaced with respect to each other, each being given a velocity v_M in the laboratory frame. We will consider a particle of charge q and mass m which, in the homogeneous magnetic field region of the machine, is characterised initially by a velocity $\boldsymbol{w}_0$, such that $w_{0\perp} = w_{0\parallel}$, and by its kinetic energy $\mathcal{E}_i$.



a) Show that at each reflection by the mirrors, the magnitude of the parallel velocity of the particle is increased by $2v_M$.
b) Explain why the particle will eventually leave the mirror.
c) Calculate the energy $\mathcal{E}_p$ of the particle when it leaves the mirror: express $\mathcal{E}_p$ as a function of the initial energy $\mathcal{E}_i$ and the mirror ratio $\mathcal{R} = B_{\max}/B_0$.
d) Find the expression giving the number of reflections n that the particle experiences before leaving the system, as a function of v_M, $\mathcal{E}_i$, $\mathcal{R}$, and m.

Answer

a) In the laboratory frame, the particle with velocity $\boldsymbol{w}_{0\parallel}$ is directed towards the mirror M, which is itself in motion, with a velocity $\boldsymbol{v}_M$, in the direction of the incident particle, as illustrated in the figure. The velocity of the particle along the z axis, in this frame, is:



$$\boldsymbol{w}_z = \boldsymbol{w}_{0\parallel} \ , \tag{2.486}$$

taking the positive sign as the direction to the mirror.
An observer tied to the mirror sees the particle coming towards him with an increased velocity:

$$\boldsymbol{w}_{zM} = \boldsymbol{w}_{0\parallel} - \boldsymbol{v}_M \ , \tag{2.487}$$

since $\boldsymbol{v}_M$ is negative.
In the same frame, after reflection, the particle returns with the opposite velocity:

$$\boldsymbol{w}'_{zM} = -(\boldsymbol{w}_{0\parallel} - \boldsymbol{v}_M) \ . \tag{2.488}$$

Returning to the laboratory frame, we must, this time, add $\boldsymbol{v}_M$ to the particle velocity (inverse operation as for (2.487)):

$$\boldsymbol{w}'_z = -(\boldsymbol{w}_{0\parallel} - \boldsymbol{v}_M) + \boldsymbol{v}_M = -(\boldsymbol{w}_{0\parallel} - 2\boldsymbol{v}_M) \ . \tag{2.489}$$

The speed of the particle after reflection is thus augmented by $|2\boldsymbol{v}_M|$ (compare (2.486) and (2.489)).

Another solution for a)

We can consider the reflection of the particle on the mirror as a head-on elastic collision between a light particle of mass m and a moving wall, i.e. a heavy particle with a quasi-infinite mass M ($m \ll M$).
The relative velocity $\boldsymbol{w}_{mM}$ of the two particles, independent of the frame, is expressed by:

$$\boldsymbol{w}_{mM} = \boldsymbol{w}_{\parallel 0} - \boldsymbol{v}_M \ . \tag{1.70}$$

After reflection, considering an elastic collision, the relative velocity is opposite and with the same modulus (after (1.79)):

$$\boldsymbol{w}'_{mM} = -\boldsymbol{w}_{mM} = -(\boldsymbol{w}_{\parallel 0} - \boldsymbol{v}_M) \ . \tag{2.490}$$

In the laboratory frame, the velocity $\boldsymbol{w}'_{\parallel 0}$ of the light particle after reflection becomes (1.74):

$$\boldsymbol{w}'_{\parallel 0} = \boldsymbol{w}'_{CM} + \left(\frac{M}{m+M}\right)\boldsymbol{w}'_{mM} \ , \tag{2.491}$$

where $\boldsymbol{w}'_{CM}$ is the velocity of the center of mass after reflection. Then, from (1.69) and (1.70):

$$\boldsymbol{w}'_{CM} = \boldsymbol{w}_{CM} = \frac{m\boldsymbol{w}_{\parallel 0} + M\boldsymbol{v}_M}{m+M} \ . \tag{2.492}$$

From (2.491) and (2.492), we obtain:

$$\boldsymbol{w}'_{\parallel 0} = \frac{m\boldsymbol{w}_{\parallel 0} + M\boldsymbol{v}_M + M\boldsymbol{v}_M - M\boldsymbol{w}_{\parallel 0}}{m+M} \ , \tag{2.493}$$

which, assuming $M \gg m$, gives:

$$\boldsymbol{w}'_{\parallel 0} = -\boldsymbol{w}_{\parallel 0} + 2\boldsymbol{v}_M \ . \tag{2.494}$$

b) We have just shown in a) that $\boldsymbol{w}_{\parallel}$, the component of the particle velocity parallel to the field $\boldsymbol{B}$, is increased at each reflection, while the component $\boldsymbol{w}_{\perp}$ remains unchanged, with $\boldsymbol{w}_{\perp}^{(n)} = \boldsymbol{w}_{\perp}^{(0)}$ after a number of reflections n. As a result, assuming that in the uniform region before the first reflection we have:

$$w_{0\perp}^{(0)} = w_0 \sin\alpha_0 \ , \tag{2.495}$$

then after the first reflection, this becomes:

$$w_{0\perp}^{(1)} \equiv w_0^{(1)} \sin\alpha_0^{(1)} = w_0^{(0)} \sin\alpha_0^{(0)} \ . \tag{2.496}$$

Since w_0, the modulus of $\boldsymbol{w}$, increases, the angle $\alpha_0^{(1)}$ must decrease, for $w_{0\perp}$ to remain constant: after n reflections, the angle $\alpha_0^{(n)}$ will be smaller than α_{0m} and, following (2.189), the particle will find itself in the loss cone.

c) By assumption, initially:

$$\mathcal{E}_i = \frac{1}{2} m (w_{0\perp}^2 + w_{0\|}^2) = m w_{0\|}^2 \tag{2.497}$$

and, as discussed in b), the particle will leave the machine when α_0 is sufficiently small, i.e. when:

$$\frac{w_{0\perp}}{w_0^{(n)}} \leq \sin \alpha_{0m} = \frac{1}{\sqrt{\mathcal{R}}} \; . \tag{2.498}$$

Setting the equality in (2.498), where w_0 is increased at each reflection, such that after n reflections:

$$w_0^{(n)} = \left[w_{0\perp}^2 + (w_{0\|} + 2n\, v_M)^2 \right]^{\frac{1}{2}} \; . \tag{2.499}$$

Owing to our initial assumption $w_{0\|} = w_{0\perp}$, from (2.498) we have:

$$w_0^{(n)} = w_{0\perp} \sqrt{\mathcal{R}} = w_{0\|} \sqrt{\mathcal{R}} \; , \tag{2.500}$$

such that, by multiplying the square of equation (2.499) by $m/2$ and substituting (2.500), we have:

$$\mathcal{E}_p \equiv \left[\frac{m w_{0\perp}^2}{2} + \frac{m w_{0\|}^2}{2} + 2mn\, w_{0\|} v_M + 2mn^2 v_M^2 \right] = \frac{1}{2} m w_{0\|}^2 \mathcal{R} \; , \tag{2.501}$$

thus:

$$\mathcal{E}_p = \mathcal{E}_i \frac{\mathcal{R}}{2} \; .$$

d) We can calculate n from its quadratic expression in (2.501):

$$n^2 + \frac{w_{0\|}}{v_M} n - \mathcal{E}_i \left(\frac{\mathcal{R}}{2} - 1 \right) \frac{1}{2 m v_M^2} = 0 \; , \tag{2.502}$$

from which we obtain:

$$n = -\frac{w_{0\|}}{2 v_M} + \frac{1}{2} \sqrt{\left(\frac{w_{0\|}}{v_M} \right)^2 + 2\mathcal{E}_i \left(\frac{\mathcal{R}}{2} - 1 \right) \frac{1}{m v_M^2}} \; , \tag{2.503}$$

because the solution with a negative sign in front of the square root would give a negative value for n. Continuing the development further, we have:

$$n = -\frac{w_{0\|}}{2v_M} + \frac{1}{2}\sqrt{\left(\frac{w_{0\|}}{v_M}\right)^2 + 2\left(\frac{w_{0\|}}{v_M}\right)^2\left(\frac{\mathcal{R}}{2} - 1\right)}\,, \tag{2.504}$$

$$n = -\frac{w_{0\|}}{2v_M} + \frac{w_{0\|}}{2v_M}\sqrt{1 + \mathcal{R} - 2}\,, \tag{2.505}$$

$$n = \frac{w_{0\|}}{2v_M}\left[(\mathcal{R} - 1)^{\frac{1}{2}} - 1\right] = \left(\frac{\mathcal{E}_i}{m}\right)^{\frac{1}{2}} \frac{1}{2v_M}\left[(\mathcal{R} - 1)^{\frac{1}{2}} - 1\right]\,, \tag{2.506}$$

of which we will take the nearest upper integer value!

Another solution for d)

We have:

$$w_{f\|} = w_{0\|} + 2nv_M\,, \tag{2.507}$$

where $w_{f\|}$ is the parallel velocity of the particle on leaving the machine, defined by:

$$\sin\alpha_{0m} \geq \frac{w_{0\perp}}{(w_{f\|}^2 + w_{0\perp}^2)^{\frac{1}{2}}}\,. \tag{2.508}$$

Considering equality (2.508), we have:

$$\sin\alpha_{0m}^2 = \frac{w_{0\perp}^2}{w_{f\|}^2 + w_{0\perp}^2} = \frac{1}{\mathcal{R}}\,, \tag{2.509}$$

which becomes:

$$w_{0\perp}^2(\mathcal{R} - 1) = w_{f\|}^2\,. \tag{2.510}$$

From (2.507), we can then write (with, by assumption, $w_{0\|} = w_{0\perp}$ in the present case):

$$n = \frac{w_{f\|} - w_{0\perp}}{2v_M} = \frac{w_{0\|}[(\mathcal{R} - 1)^{\frac{1}{2}} - 1]}{2v_M}$$

and, using (2.497):

$$n = \left(\frac{\mathcal{E}_i}{m}\right)^{\frac{1}{2}} \frac{1}{2v_M}\left[(\mathcal{R} - 1)^{\frac{1}{2}} - 1\right]\,. \tag{2.506}$$

2.16. A magnetic mirror is situated at each end of a machine, and its axis is along the z axis. The magnetic configuration of these mirrors is symmetric with respect to the plane $z = 0$. The planes of the mirrors are situated at z_M and $-z_M$.
The magnetic field has been constructed such that, along the z axis, we have:

$$B(z) = B_0\left[1 + \left(\frac{z}{a}\right)^2\right]\,, \tag{2.511}$$

where a is a constant.

a) Show that the period of oscillation of a particle of mass m between the two mirrors is given by:

$$\mathcal{T}_p = 2\pi a \left[\frac{m}{2\mu B_0}\right]^{\frac{1}{2}} , \tag{2.512}$$

where μ is the orbital magnetic moment. Assume the adiabatic approximation is valid along the whole trajectory.

b) Calculate the particle velocity along z (the particle is assumed to remain within the machine) and show that:

$$w_z = \left(\frac{2B_0\mu}{m}\right)^{\frac{1}{2}} \frac{z_M}{a} \cos\left(\frac{2\pi t}{\mathcal{T}_p}\right) \tag{2.513}$$

and finally:

$$w_z = \left(\frac{2\mu}{m}\right)^{\frac{1}{2}} [B(z_M) - B(z)]^{\frac{1}{2}} . \tag{2.514}$$

Answer

a) To the extent that the magnetic field is only slightly non uniform (adiabatic condition) in its own direction, we can write (2.177):

$$m\dot{\boldsymbol{w}} = \boldsymbol{\mu} \cdot \boldsymbol{\nabla B} \tag{2.515}$$

because $\boldsymbol{\mu} = -\mu\hat{\mathbf{e}}_z$ (p. 136). Along z, in the present case:

$$m\dot{w}_z = -\mu\frac{\partial B_z}{\partial z} = -\frac{2\mu B_0 z}{a^2} , \tag{2.516}$$

where equivalently:

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{2\mu B_0}{ma^2} z , \tag{2.517}$$

which has a permissible solution, for $z = 0$ at $t = 0$:

$$z = z_M \sin\left(\frac{2\mu B_0}{ma^2}\right)^{\frac{1}{2}} t , \tag{2.518}$$

from which the period:

$$\mathcal{T}_p = 2\pi a\sqrt{\frac{m}{2\mu B_0}} . \tag{2.519}$$

b) At $z = \pm z_M$, due to the assumption that the particle does not leave the mirror, we must have $w_{\|}(z_M) = 0$, that is:

$$\frac{1}{2}mw_{\|}^2(z_M) = 0 . \tag{2.520}$$

The conservation of kinetic energy gives:

$$\frac{1}{2}mw_{\parallel}^2(z) + \frac{1}{2}mw_{\perp}^2(z) = \frac{1}{2}mw_{\perp}^2(z_M) \ . \tag{2.521}$$

Knowing that μ is a constant of the motion, we have:

$$\mu = \frac{1}{2}\frac{mw_{\perp}^2(z)}{B(z)} = \text{constant} \ , \tag{2.522}$$

such that, from (2.521) and (2.522):

$$\frac{1}{2}mw_{\parallel}^2(z) = \mu[B(z_M) - B(z)] \ , \tag{2.523}$$

from which:

$$w_{\parallel}(z) = \sqrt{\frac{2\mu}{m}[B(z_M) - B(z)]} \ . \tag{2.524}$$

If we now replace $B(z_M)$ and $B(z)$ by their respective values:

$$w_{\parallel}(z) = \sqrt{\frac{2\mu B_0}{m}} \left[1 + \left(\frac{z_M}{a}\right)^2 - 1 - \left(\frac{z}{a}\right)^2\right]^{\frac{1}{2}} \ , \tag{2.525}$$

$$= \sqrt{\frac{2\mu B_0}{m}} \frac{z_M}{a} \left[1 - \left(\frac{z}{z_M}\right)^2\right]^{\frac{1}{2}} \ , \tag{2.526}$$

then, from (2.518) with (2.519):

$$\begin{aligned} w_{\parallel}(z) &= \sqrt{\frac{2\mu B_0}{m}} \frac{z_M}{a} \left[1 - \sin^2\left(\frac{2\pi t}{\mathcal{T}_p}\right)\right]^{\frac{1}{2}} \\ &= \sqrt{\frac{2\mu B_0}{m}} \frac{z_M}{a} \cos\left(\frac{2\pi t}{\mathcal{T}_p}\right) \ . \end{aligned} \tag{2.527}$$

2.17. The Hamilton-Jacobi formalism of classical mechanics allows us to introduce the concept of an adiabatic invariant I as the average of an action integral (LHS of (2.528) below). In the case where the particle is subject to a periodic motion, this integral takes a constant value:

$$\frac{1}{2\pi}\oint p\mathrm{d}q = I \ , \tag{2.528}$$

where q is a generalised canonical coordinate[88] (for example a position variable) and p is the conjugate canonical moment (for example, the momentum

[88] Do not confuse the variables p and q of the Hamilton-Jacobi formalism with pressure and charge.

corresponding to the position q). Given the kinetic energy $\mathcal{E}_c$ of the system, the value of p is obtained from the equation:

$$p = \frac{\partial \mathcal{E}_c}{\partial \dot{q}} \, . \tag{2.529}$$

Consider a linear discharge, confined by a static magnetic field $\boldsymbol{B}$, directed axially, and terminated at its two ends by magnetic mirrors.

a) Show that the action integral taken over the cyclotron motion of a charged particle leads, within a constant coefficient, to a constant value of the orbital magnetic motion μ of the particle. Specify the assumptions used for your calculation.
b) Show that the charged particle oscillates between the mirrors with a period $\mathcal{T}$, given by:

$$\mathcal{T} = \oint \frac{\mathrm{d}z}{\left[\left(\frac{2}{m}\right)(\mathcal{E}_c - \mu B)\right]^{\frac{1}{2}}} \, . \tag{2.530}$$

Specify your assumptions.
c) Calculate the second invariant I_2, linked to the axial motion, to show that:

$$I_2 = \frac{1}{2\pi} \oint \left[2m(\mathcal{E}_c - V)\right]^{\frac{1}{2}} \mathrm{d}z \, , \tag{2.531}$$

where we have introduced the pseudo potential $V \equiv \mu B$, to give the integral a more general aspect.
d) Consider the case where the discharge is confined by a magnetic field:

$$\boldsymbol{B} = B_0 \left[1 + \left(\frac{z}{a}\right)^2\right] \hat{\mathbf{e}}_z \, , \tag{2.532}$$

where a is a constant such that $\partial B/\partial z$ varies very slowly along z: calculate the period of oscillation of a particle in such a linear machine.

Answer

a) Since this concerns cyclotron motion, we will describe the system in cylindrical coordinates (r, φ, z). We then have $x = r_B \cos\varphi$, $y = r_B \sin\varphi$, where r_B is the Larmor radius, which we assume to be constant because B varies slowly along the axis.
Since:

$$\mathcal{E}_c = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \, , \tag{2.533}$$

this can be written in cylindrical coordinates as:

$$\mathcal{E}_c = \frac{1}{2} m(r_B^2 \dot{\varphi}^2 + \dot{z}^2) \, . \tag{2.534}$$

Because $\dot{x}^2 + \dot{y}^2 = (-r_B \sin\varphi\, \dot{\varphi})^2 + (r_B \cos\varphi\, \dot{\varphi})^2$ and setting $q = \varphi$, we obtain for p_φ:

$$p_\varphi = \frac{\partial \mathcal{E}_c}{\partial \dot{\varphi}} = m r_B^2 \dot{\varphi} \ , \tag{2.535}$$

where $\dot{\varphi} \equiv \mathrm{d}\varphi/\mathrm{d}t = \omega_c$, and then:

$$I_1 \equiv \frac{1}{2\pi} \oint p_\varphi \mathrm{d}\varphi = \frac{1}{2\pi} \int\limits_{\substack{\text{cyclotron}\\ \text{period}}} m r_B^2 \dot{\varphi}\, \mathrm{d}\varphi = \frac{1}{2\pi} m r_B^2 \omega_c \int \mathrm{d}\varphi = \frac{m w_\perp^2}{\omega_c} \ , \tag{2.536}$$

recalling that $\omega_c = w_\perp / r_B$. Knowing that $\mu \equiv m w_\perp^2 / 2B$ and $\omega_c = |q|B/m$, we obtain:

$$I_1 = \left(\frac{2m}{|q|} \right) \mu \ . \tag{2.537}$$

b) The period of oscillation $\mathcal{T}$ along the axis of the discharge is found by integrating the closed (cyclic) motion along z:

$$\mathcal{T} = \oint \frac{\mathrm{d}z}{w_\parallel} = \int \frac{\mathrm{d}z}{\dot{z}} \ . \tag{2.538}$$

In addition, we have from (2.534):

$$\mathcal{E}_c = \frac{1}{2} m r_B^2 \omega_c^2 + \frac{1}{2} m \dot{z}^2 = \frac{1}{2} m w_\perp^2 + \frac{1}{2} m \dot{z}^2 = \mu B + \frac{1}{2} m \dot{z}^2 \ , \tag{2.539}$$

which allows us to calculate the axial velocity $\dot{z}$:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = \dot{z} = \left[\frac{2}{m} (\mathcal{E}_c - \mu B) \right]^{\frac{1}{2}} \ , \tag{2.540}$$

thus, over a cycle back and forth, the period of oscillation of the particle is:

$$\mathcal{T} \equiv \int_0^{\mathcal{T}} \mathrm{d}t = \oint \frac{\mathrm{d}z}{\left[\frac{2}{m} (\mathcal{E}_c - \mu B) \right]^{\frac{1}{2}}} \ . \tag{2.541}$$

c) Taking advantage of the definition (2.528), we can introduce the adiabatic invariant linked to the axial motion, by setting $q = z$ and $p = p_z$:

$$I_2 = \frac{1}{2\pi} \oint p_z \, \mathrm{d}z \ . \tag{2.542}$$

Since $p_z \equiv m\dot{z}$, we obtain:

$$I_2 = \frac{1}{2\pi} \oint m\dot{z} \, \mathrm{d}z \ , \tag{2.543}$$

which can be evaluated from (2.540), giving:

$$I_2 = \frac{1}{2\pi} \oint m \left[\frac{2}{m} (\mathcal{E}_c - \mu B) \right]^{\frac{1}{2}} \mathrm{d}z \; ,$$
$$= \frac{1}{2\pi} \oint [2m(\mathcal{E}_c - \mu B)]^{\frac{1}{2}} \mathrm{d}z \; . \tag{2.544}$$

d) We know from (2.177) that:

$$F_z = -\mu \frac{\partial B_z}{\partial z} \; , \tag{2.545}$$

such that, in the present case:

$$F_z = -\mu \frac{B_0 2z}{a^2} \tag{2.546}$$

and, since $F_z = m\dot{z}$, this leads to:

$$\dot{z} + \frac{\mu}{m} \frac{B_0 2}{a^2} z = 0 \; , \tag{2.547}$$

which is the equation of periodic motion. Its oscillation frequency ω is given by:

$$\omega = \left(\frac{2B_0\mu}{m} \right)^{\frac{1}{2}} \frac{1}{a} \tag{2.548}$$

and since $\omega = 2\pi/\mathcal{T}$:

$$\mathcal{T} = 2\pi a \left(\frac{m}{2B_0\mu} \right)^{\frac{1}{2}} \; . \tag{2.549}$$

See also problem 2.16.

2.18. Calculate the current density of ions and electrons in the ionosphere, due to the gravitational gradient in the terrestrial magnetic field at an altitude of 300 km. Assume that the magnetic induction vector $\boldsymbol{B}$ is perpendicular to the terrestrial gravitational field.
The general relation for the gravitational force exerted on a mass m due to a mass M, when the masses are separated by a distance r, is:

$$\boldsymbol{F}_g = -\frac{\boldsymbol{G}Mm}{r^2} \; , \tag{2.550}$$

where $\boldsymbol{G}$ is the universal gravitational constant. By definition, at the surface of the earth, of mass M, a mass m is subject to a force:

$$\boldsymbol{F}_{gt} = -\frac{\boldsymbol{G}Mm}{R^2} = -\boldsymbol{g}_0 m \; , \tag{2.551}$$

where R is the radius of the earth (the mass M is localised at the centre of the earth!).
Numerically, consider the O^+ ions to have a density, $1.8 \times 10^{12}\,\mathrm{m}^{-3}$, equal to that of the electrons; $|\boldsymbol{B}| = 10^{-4}$ tesla (1 gauss). The mass m_e of the electrons is 9.11×10^{-28} g and that of O^+, is $m_i = m_e \times 1836 \times 16$. The earth radius is $4 \times 10^7/2\pi$ metres and g_0, the gravitational field at the surface of the earth, is $9.8\,\mathrm{m\,s^{-2}}$.

Answer

a) The expression for the drift velocity in the gravitational field
It has been shown (Appendix XII) that the drift velocity of a charged particle in a magnetic field $\boldsymbol{B}$ due to an arbitrary force $\boldsymbol{F}_D$ is given by:

$$\boldsymbol{w}_D = \frac{\boldsymbol{F}_D \wedge \boldsymbol{B}}{qB^2} , \tag{2.552}$$

where $\boldsymbol{F}_D$ is the gravitational force, in the present case.
For the electrons $\boldsymbol{F}_{De} = -\boldsymbol{G}Mm_e/r^2$ and for the ions $\boldsymbol{F}_{Di} = -\boldsymbol{G}Mm_i/r^2$. As can be seen from (2.552), the two types of particles, because of the opposite sign of their charge, drift in opposite directions: there will be a net current.

b) Calculation of the gravitational force at an altitude of 300 km with respect to the surface of the earth
At a given altitude z with respect to the surface of the earth such that $z \ll R$ (which is the case here: $(300/40000)2\pi \simeq 0.05$), we can develop (2.550) to first order with respect to the surface of the earth ($z = 0$):

$$\boldsymbol{F}_g = -\frac{\boldsymbol{G}Mm}{(R+z)^2} = -\frac{\boldsymbol{G}Mm}{R^2\left(1+\dfrac{z}{R}\right)^2} \simeq -\frac{\boldsymbol{G}Mm}{R^2}\left(1-\frac{2z}{R}\right) . \tag{2.553}$$

Since, at the surface of the earth:

$$\boldsymbol{F}_{gt} = -\frac{\boldsymbol{G}Mm}{R^2} = -m\boldsymbol{g}_0 , \tag{2.551}$$

then from (2.551) and (2.553):

$$F_g = -mg_0\left(1-\frac{2z}{R}\right) \tag{2.554}$$

and finally, because $\boldsymbol{B} \perp \boldsymbol{F}_g$, the expression of drift velocity ((2.551) and (2.552)) is:

$$w_D = \frac{mg_0\left(1-\dfrac{2z}{R}\right)}{qB} . \tag{2.555}$$

c) The current density of the gravitational drift is due to the contribution of the ions and the electrons which, moving in opposite directions (contrary to the electric field drift in electric and magnetic fields), creates a net current:

$$\boldsymbol{J} = ne\boldsymbol{v}_{Di} - ne\boldsymbol{v}_{De} \ , \tag{2.556}$$

where $\boldsymbol{v}_{Di}$ and $\boldsymbol{v}_{De}$ are the drift velocities of the ions and electrons respectively. This leads to:

$$J = ne\left(1 - \frac{2z}{R}\right)\left[\frac{m_i g_0}{eB} - \frac{m_e g_0}{-eB}\right] \ , \tag{2.557}$$

$$J = \frac{n g_0}{B}\left(1 - \frac{2z}{R}\right)(m_i + m_e) \tag{2.558}$$

and, because $m_i \gg m_e$:

$$J \simeq \frac{n g_0}{B}\left(1 - \frac{2z}{R}\right) m_i \ . \tag{2.559}$$

This drift is perpendicular to the direction of the earth radius and to B.

d) Numerical application:

$$J \simeq \frac{1.8 \times 10^{12} \times 9.8\left(1 - \dfrac{2 \times 300 \times 10^3 \times 2\pi}{4 \times 10^7}\right)}{10^{-4}} 1837 \times 16 \times 9.11 \times 10^{-31}$$

$$= 4.3 \times 10^{-9}\,\mathrm{A\,m^{-2}} \simeq 4\,\mathrm{nA\,m^{-2}}\ !$$

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