Chapter 2
The Spectrum of a Closed Operator

The main themes of this chapter are the most important concepts concerning general closed operators, spectrum and resolvent. Section 2.2 is devoted to basic properties of these notions for arbitrary closed operators. In Sect. 2.3 we treat again differentiation operators as illustrating examples. First however, in Sect. 2.1 we introduce regular points and defect numbers and derive some technical results that are useful for the study of spectra (Sect. 2.2) and for self-adjointness criteria (Sect. 13.2).

2.1 Regular Points and Defect Numbers of Operators

Let T be a linear operator on a Hilbert space $\mathcal{H}$.

Definition 2.1 A complex number λ is called a *regular point* for T if there exists a number $c_\lambda > 0$ such that

$$\left\|(T-\lambda I)x\right\| \geq c_\lambda \|x\| \quad \text{for all } x \in \mathcal{D}(T). \tag{2.1}$$

The set of regular points of T is the *regularity domain* of T and denoted by $\pi(T)$.

Remark There is no unique symbol for the regularity domain of an operator in the literature. It is denoted by $\hat{\rho}(T)$ in [BS], by $\varPi(T)$ in [EE], and by $\varGamma(T)$ in [We]. Many books have no special symbol for this set.

Recall that the dimension of a Hilbert space $\mathcal{H}$, denoted by $\dim \mathcal{H}$, is defined by the cardinality of an orthonormal basis of $\mathcal{H}$.

Definition 2.2 For $\lambda \in \pi(T)$, we call the linear subspace $\mathcal{R}(T-\lambda I)^\perp$ of $\mathcal{H}$ the *deficiency subspace* of T at λ and its dimension $d_\lambda(T) := \dim \mathcal{R}(T-\lambda I)^\perp$ the *defect number* of T at λ.

Deficiency spaces and defect numbers will play a crucial role in the theory of self-adjoint extensions of symmetric operators developed in Chap. 13.

A number of properties of these notions are collected in the next proposition.

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DOI 10.1007/978-94-007-4753-1_2,

Proposition 2.1 *Let T be a linear operator on $\mathcal{H}$, and $\lambda \in \mathbb{C}$.*

(i) *$\lambda \in \pi(T)$ if and only if $T - \lambda I$ has a bounded inverse $(T - \lambda I)^{-1}$ defined on $\mathcal{R}(T - \lambda I)$. In this case inequality (2.1) holds with $c_\lambda = \|(T - \lambda I)^{-1}\|^{-1}$.*

(ii) *$\pi(T)$ is an open subset of $\mathbb{C}$. More precisely, if $\lambda_0 \in \pi(T)$, $\lambda \in \mathbb{C}$, and $|\lambda - \lambda_0| < c_{\lambda_0}$, where c_{λ_0} is a constant satisfying (2.1) for λ_0, then $\lambda \in \pi(T)$.*

(iii) *If T is closable, then $\pi(\overline{T}) = \pi(T)$, $d_\lambda(\overline{T}) = d_\lambda(T)$, and $\mathcal{R}(\overline{T} - \lambda I)$ is the closure of $\mathcal{R}(T - \lambda I)$ in $\mathcal{H}$ for each $\lambda \in \pi(T)$.*

(iv) *If T is closed and $\lambda \in \pi(T)$, then $\mathcal{R}(T - \lambda I)$ is a closed linear subspace of $\mathcal{H}$.*

Proof (i): First suppose that $\lambda \in \pi(T)$. Then $\mathcal{N}(T - \lambda I) = \{0\}$ by (2.1), so the inverse $(T - \lambda I)^{-1}$ exists. Let $y \in \mathcal{D}((T - \lambda I)^{-1}) = \mathcal{R}(T - \lambda I)$. Then we have $y = (T - \lambda I)x$ for some $x \in \mathcal{D}(T)$, and hence,

$$\left\|(T - \lambda I)^{-1} y\right\| = \|x\| \le c_\lambda^{-1} \left\|(T - \lambda I)x\right\| = c_\lambda^{-1} \|y\|$$

by (2.1). That is, $(T - \lambda I)^{-1}$ is bounded, and $\|(T - \lambda I)^{-1}\| \le c_\lambda^{-1}$.

Assume now that $(T - \lambda I)^{-1}$ has a bounded inverse. Then, with x and y as above,

$$\|x\| = \left\|(T - \lambda I)^{-1} y\right\| \le \left\|(T - \lambda I)^{-1}\right\| \|y\| = \left\|(T - \lambda I)^{-1}\right\| \left\|(T - \lambda I)x\right\|.$$

Hence, (2.1) holds with $c_\lambda = \|(T - \lambda I)^{-1}\|^{-1}$.

(ii): Let $\lambda_0 \in \pi(T)$ and $\lambda \in \mathbb{C}$. Suppose that $|\lambda - \lambda_0| < c_{\lambda_0}$, where c_{λ_0} is a constant such that (2.1) holds. Then for $x \in \mathcal{D}(T)$, we have

$$\begin{aligned}\left\|(T - \lambda I)x\right\| &= \left\|(T - \lambda_0 I)x - (\lambda - \lambda_0)x\right\| \ge \left\|(T - \lambda_0 I)x\right\| - |\lambda - \lambda_0| \|x\| \\ &\ge \bigl(c_{\lambda_0} - |\lambda - \lambda_0|\bigr) \|x\|.\end{aligned}$$

Thus, $\lambda \in \pi(T)$, since $|\lambda - \lambda_0| < c_{\lambda_0}$. This shows that the set $\pi(T)$ is open.

(iii): Let y be in the closure of $\mathcal{R}(T - \lambda I)$. Then there is a sequence $(x_n)_{n \in \mathbb{N}}$ of vectors $x_n \in \mathcal{D}(T)$ such that $y_n := (T - \lambda I)x_n \to y$ in $\mathcal{H}$. By (2.1) we have

$$\|x_n - x_k\| \le c_\lambda^{-1} \left\|(T - \lambda I)(x_n - x_k)\right\| = c_\lambda^{-1} \|y_n - y_k\|.$$

Hence, (x_n) is a Cauchy sequence in $\mathcal{H}$, because (y_n) is a Cauchy sequence. Let $x := \lim_n x_n$. Then $\lim_n T x_n = \lim_n (y_n + \lambda x_n) = y + \lambda x$. Since T is closable, $x \in \mathcal{D}(\overline{T})$ and $\overline{T}x = y + \lambda x$, so that $y = (\overline{T} - \lambda I)x \in \mathcal{R}(\overline{T} - \lambda I)$. This proves that $\overline{\mathcal{R}(T - \lambda I)} \subseteq \mathcal{R}(\overline{T} - \lambda I)$. The converse inclusion follows immediately from the definition of the closure $\overline{T}$. Thus, $\overline{\mathcal{R}(T - \lambda I)} = \mathcal{R}(\overline{T} - \lambda I)$.

Clearly, $\pi(\overline{T}) = \pi(T)$ by (2.1). Since $\mathcal{R}(\overline{T} - \lambda I)$ is the closure of $\mathcal{R}(T - \lambda I)$, both have the same orthogonal complements, so $d_\lambda(\overline{T}) = d_\lambda(T)$ for $\lambda \in \pi(T)$.

(iv) follows at once from (iii). □

Combining Proposition 2.1(iii) and formula (1.7), we obtain

Corollary 2.2 *If T is a closable densely defined linear operator, and $\lambda \in \pi(T)$, then $\mathcal{H} = \mathcal{R}(\overline{T} - \lambda I) \oplus \mathcal{N}(T^* - \overline{\lambda} I)$.*

The following technical lemma is needed in the proof of the next proposition.

Lemma 2.3 *If $\mathcal{F}$ and $\mathcal{G}$ are closed linear subspaces of a Hilbert space $\mathcal{H}$ such that* $\dim \mathcal{F} < \dim \mathcal{G}$, *then there exists a nonzero vector $y \in \mathcal{G} \cap \mathcal{F}^\perp$.*

Proof In this proof we denote by $|M|$ the cardinality of a set M. First, we suppose that $k = \dim \mathcal{F}$ is finite. We take a $(k+1)$-dimensional subspace $\mathcal{G}_0$ of $\mathcal{G}$ and define the mapping $\Phi : \mathcal{G}_0 \to \mathcal{F}$ by $\Phi(x) = Px$, where P is the projection of $\mathcal{H}$ onto $\mathcal{F}$. If Φ would be injective, then $k+1 = \dim \mathcal{G}_0 = \dim \Phi(\mathcal{G}_0) \le \dim \mathcal{F} = k$, which is a contradiction. Hence, there is a nonzero vector $y \in \mathcal{N}(\Phi)$. Clearly, $y \in \mathcal{G} \cap \mathcal{F}^\perp$.

Now suppose that $\dim \mathcal{F}$ is infinite. Let $\{f_k : k \in K\}$ and $\{g_l : l \in L\}$ be orthonormal bases of $\mathcal{F}$ and $\mathcal{G}$, respectively. Set $L_k := \{l \in L : \langle f_k, g_l \rangle \ne 0\}$ for $k \in K$ and $L' = \bigcup_{k\in K} L_k$. Since each set L_k is at most countable and $\dim \mathcal{F} = |K|$ is infinite, we have $|L'| \le |K||\mathbb{N}| = |K|$. Since $|K| = \dim \mathcal{F} < \dim \mathcal{G} = |L|$ by assumption, we deduce that $L' \ne L$. Each vector g_l with $l \in L \setminus L'$ is orthogonal to all f_k, $k \in K$, and hence, it belongs to $\mathcal{G} \cap \mathcal{F}^\perp$. □

The next proposition is a classical result of *M.A. Krasnosel'skii* and *M.G. Krein*.

Proposition 2.4 *Suppose that T is a closable linear operator on $\mathcal{H}$. Then the defect number $d_\lambda(T)$ is constant on each connected component of the open set $\pi(T)$.*

Proof By Proposition 2.1(iii), we can assume without loss of generality that T is closed. Then $\mathcal{R}(T - \mu I)$ is closed for all $\mu \in \pi(T)$ by Proposition 2.1(iv). Therefore, setting $\mathcal{K}_\mu := \mathcal{R}(T - \mu I)^\perp$, we have

$$(\mathcal{K}_\mu)^\perp = \mathcal{R}(T - \mu I) \quad \text{for } \mu \in \pi(T). \tag{2.2}$$

Suppose that $\lambda_0 \in \pi(T)$ and $\lambda \in \mathbb{C}$ are such that $|\lambda - \lambda_0| < c_{\lambda_0}$. Then $\lambda \in \pi(T)$ by Proposition 2.1(ii). The crucial step is to prove that $d_\lambda(T) = d_{\lambda_0}(T)$.

Assume to the contrary that $d_\lambda(T) \ne d_{\lambda_0}(T)$. First suppose that $d_\lambda(T) < d_{\lambda_0}(T)$. By Lemma 2.3 there exists a nonzero vector $y \in \mathcal{K}_{\lambda_0}$ such that $y \in (\mathcal{K}_\lambda)^\perp$. Then $y \in \mathcal{R}(T - \lambda I)$ by (2.2), say $y = (T - \lambda I)x$ for some nonzero $x \in \mathcal{D}(T)$. Since $y = (T - \lambda I)x \in \mathcal{K}_{\lambda_0}$, we have

$$\big\langle (T - \lambda I)x, (T - \lambda_0 I)x \big\rangle = 0. \tag{2.3}$$

Equation (2.3) is symmetric in λ and λ_0, so it holds also when $d_{\lambda_0}(T) < d_\lambda(T)$. Using (2.3), we derive

$$\begin{aligned} \big\|(T - \lambda_0 I)x\big\|^2 &= \big\langle (T - \lambda I)x + (\lambda - \lambda_0)x, (T - \lambda_0 I)x \big\rangle \\ &\le |\lambda - \lambda_0| \|x\| \big\|(T - \lambda_0 I)x\big\|. \end{aligned}$$

Thus, $\|(T - \lambda_0 I)x\| \le |\lambda - \lambda_0| \|x\|$. Since $x \ne 0$ and $|\lambda - \lambda_0| < c_{\lambda_0}$, we obtain

$$|\lambda - \lambda_0| \|x\| < c_{\lambda_0} \|x\| \le \big\|(T - \lambda_0 I)x\big\| \le |\lambda - \lambda_0| \|x\|$$

by (2.1), which is a contradiction. Thus, we have proved that $d_\lambda(T) = d_{\lambda_0}(T)$.

The proof will be now completed by using a well-known argument from elementary topology. Let α and β be points of the same connected component $\mathcal{U}$ of the open set $\pi(T)$ in the complex plane. Then there exists a polygonal path $\mathcal{P}$ contained in $\mathcal{U}$ from α to β. For $\lambda \in \mathcal{P}$, let $\mathcal{U}_\lambda = \{\lambda' \in \mathbb{C} : |\lambda' - \lambda| < c_\lambda\}$. Then $\{\mathcal{U}_\lambda : \lambda \in \mathcal{P}\}$ is an open cover of the compact set $\mathcal{P}$, so there exists a finite subcover $\{\mathcal{U}_{\lambda_1}, \dots, \mathcal{U}_{\lambda_s}\}$ of $\mathcal{P}$. Since $d_\lambda(T)$ is constant on each open set $\mathcal{U}_{\lambda_k}$ as shown in the preceding paragraph, we conclude that $d_\alpha(T) = d_\beta(T)$. □

The *numerical range* of a linear operator T in $\mathcal{H}$ is defined by

$$\Theta(T) = \bigl\{\langle Tx, x\rangle : x \in \mathcal{D}(T),\ \|x\| = 1\bigr\}.$$

A classical result of F. Hausdorff (see, e.g., [K2, V, Theorem 3.1]) says that $\Theta(T)$ is a convex set. In general, the set $\Theta(T)$ is neither closed nor open for a bounded or closed operator. However, we have the following simple but useful fact.

Lemma 2.5 *Let T be a linear operator on $\mathcal{H}$. If $\lambda \in \mathbb{C}$ is not in the closure of $\Theta(T)$, then $\lambda \in \pi(T)$.*

Proof Set $\gamma_\lambda := \operatorname{dist}(\lambda, \Theta(T)) > 0$. For $x \in \mathcal{D}(T)$, $\|x\| = 1$, we have

$$\bigl\|(T - \lambda I)x\bigr\| \ge \bigl|\bigl\langle (T - \lambda I)x, x\bigr\rangle\bigr| = \bigl|\langle Tx, x\rangle - \lambda\bigr| \ge \gamma_\lambda,$$

so that $\|(T - \lambda I)y\| \ge \gamma_\lambda \|y\|$ for arbitrary $y \in \mathcal{D}(T)$. Hence, $\lambda \in \pi(T)$. □

2.2 Spectrum and Resolvent of a Closed Operator

In this section we assume that T is a *closed* linear operator on a Hilbert space $\mathcal{H}$.

Definition 2.3 A complex number λ belongs to the *resolvent set* $\rho(T)$ of T if the operator $T - \lambda I$ has a bounded everywhere on $\mathcal{H}$ defined inverse $(T - \lambda I)^{-1}$, called the *resolvent* of T at λ and denoted by $R_\lambda(T)$.

The set $\sigma(T) := \mathbb{C} \setminus \rho(T)$ is called the *spectrum* of the operator T.

Remarks 1. Formally, the preceding definition could be also used to define the spectrum for a not necessarily closed operator T. But if $\lambda \in \rho(T)$, then the bounded everywhere defined operator $(T - \lambda I)^{-1}$ is closed, so is its inverse $T - \lambda I$ by Theorem 1.8(vi) and hence T. Therefore, if T is *not closed*, we would always have that $\rho(T) = \emptyset$ and $\sigma(T) = \mathbb{C}$ according to Definition 2.3, so the notion of spectrum becomes trivial. For this reason, we assumed above that the operator T is closed.

2. The reader should notice that in the literature the resolvent $R_\lambda(T)$ is often defined by $(\lambda I - T)^{-1}$ rather than $(T - \lambda I)^{-1}$ as we do.

By Definition 2.3, a complex number λ is in $\rho(T)$ if and only if there is an operator $B \in \mathbf{B}(\mathcal{H})$ such that

$$B(T - \lambda I) \subseteq I \quad \text{and} \quad (T - \lambda I)B = I.$$

The operator B is then uniquely determined and equal to the resolvent $R_\lambda(T)$.

Proposition 2.6

(i) $\rho(T) = \{\lambda \in \pi(T) : d_\lambda(T) = 0\}$.
(ii) $\rho(T)$ *is an open subset, and* $\sigma(T)$ *is a closed subset of* $\mathbb{C}$.

Proof (i) follows at once from Proposition 2.1, (i) and (iv). Since $\pi(T)$ is open and $d_\lambda(T)$ is locally constant on $\pi(T)$ by Proposition 2.4, the assertion of (i) implies that $\rho(T)$ is open. Hence, $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is closed. □

The requirement that the inverse $(T - \lambda I)^{-1}$ is *bounded* can be omitted in Definition 2.3. This is the first assertion of the next proposition.

Proposition 2.7 *Let T be a closed operator on $\mathcal{H}$.*

(i) $\rho(T)$ *is the set of all numbers* $\lambda \in \mathbb{C}$ *such that* $T - \lambda I$ *is a bijective mapping of* $\mathcal{D}(T)$ *on* $\mathcal{H}$ (*or equivalently*, $\mathcal{N}(T - \lambda I) = \{0\}$ *and* $\mathcal{R}(T - \lambda I) = \mathcal{H}$).
(ii) *Suppose that* $\mathcal{D}(T)$ *is dense in* $\mathcal{H}$ *and let* $\lambda \in \mathbb{C}$. *Then* $\lambda \in \sigma(T)$ *if and only if* $\overline{\lambda} \in \sigma(T^*)$. *Moreover,* $R_\lambda(T)^* = R_{\overline{\lambda}}(T^*)$ *for* $\lambda \in \rho(T)$.

Proof (i): Clearly, $T - \lambda I$ is bijective if and only if the inverse $(T - \lambda I)^{-1}$ exists and is everywhere defined on $\mathcal{H}$. It remains to prove that $(T - \lambda I)^{-1}$ is bounded if $T - \lambda I$ is bijective. Since T is closed, $T - \lambda I$ is closed, and so is its inverse $(T - \lambda I)^{-1}$ by Theorem 1.8(vi). That is, $(T - \lambda I)^{-1}$ is a closed linear operator defined on the whole Hilbert space $\mathcal{H}$. Hence, $(T - \lambda I)^{-1}$ is bounded by the closed graph theorem.

(ii): It suffices to prove the corresponding assertion for the resolvent sets.

Let $\lambda \in \rho(T)$. Then, by Theorem 1.8(iv), $(T - \lambda I)^* = T^* - \overline{\lambda} I$ is invertible, and $(T^* - \overline{\lambda} I)^{-1} = ((T - \lambda I)^{-1})^*$. Since $(T - \lambda I)^{-1} \in \mathbf{B}(\mathcal{H})$ by $\lambda \in \rho(T)$, we have $((T - \lambda I)^{-1})^* \in \mathbf{B}(\mathcal{H})$, and hence $(T^* - \overline{\lambda} I)^{-1} \in \mathbf{B}(\mathcal{H})$, that is, $\overline{\lambda} \in \rho(T^*)$.

Replacing T by T^* and λ by $\overline{\lambda}$ and using the fact that $T = T^{**}$, it follows that $\overline{\lambda} \in \rho(T^*)$ implies $\lambda \in \rho(T)$. Thus, $\lambda \in \rho(T)$ if and only if $\overline{\lambda} \in \rho(T^*)$. □

Proposition 2.8 *Let T be a closed operator on $\mathcal{H}$. Let $\mathcal{U}$ be a connected open subset of $\mathbb{C} \backslash \overline{\Theta(T)}$. If there exists a number $\lambda_0 \in \mathcal{U}$ which is contained in $\rho(T)$, then $\mathcal{U} \subseteq \rho(T)$. Moreover,* $\|(T - \lambda I)^{-1}\| \leq (\operatorname{dist}(\lambda, \Theta(T)))^{-1}$ *for* $\lambda \in \mathcal{U}$.

Proof By Lemma 2.5 we have $\mathcal{U} \subseteq \pi(T)$. Therefore, since T is closed, it follows from Proposition 2.1(iv) that $\mathcal{R}(T - \lambda I)$ is closed in $\mathcal{H}$ for all $\lambda \in \mathcal{U}$. By Proposition 2.4, the defect number $d_\lambda(T)$ is constant on the connected open set $\mathcal{U}$. But $d_{\lambda_0}(T) = 0$ for $\lambda_0 \in \mathcal{U}$, since $\lambda_0 \in \rho(T)$. Hence, $d_\lambda(T) = 0$ on the whole set $\mathcal{U}$. Consequently, $\mathcal{U} \subseteq \rho(T)$ by Proposition 2.6(i).

From the inequality $\|(T - \lambda I)y\| \geq \gamma_\lambda \|y\|$ for $y \in \mathcal{D}(T)$ shown in the proof of Lemma 2.5 we get $\|(T - \lambda I)^{-1}\| \leq \gamma_\lambda^{-1}$ for $\lambda \in \mathcal{U}$, where $\gamma_\lambda = \operatorname{dist}(\lambda, \Theta(T))$. □

Next we define an important subset of the spectrum.

Definition 2.4 $\sigma_p(T) := \{\lambda \in \mathbb{C} : \mathcal{N}(T - \lambda I) \neq \{0\}\}$ is the *point spectrum* of T. We call $\lambda \in \sigma_p(T)$ an *eigenvalue* of T, the dimension of $\mathcal{N}(T - \lambda I)$ its *multiplicity*, and any nonzero element of $\mathcal{N}(T - \lambda I)$ an *eigenvector* of T at λ.

Let λ be a point of the spectrum $\sigma(T)$. Then, by Proposition 2.7(i), the operator $(T - \lambda I) : \mathcal{D}(T) \to \mathcal{H}$ is *not bijective*. This means that $T - \lambda I$ is not injective *or* $T - \lambda I$ is not surjective. Clearly, the point spectrum $\sigma_p(T)$ is precisely the set of all $\lambda \in \sigma(T)$ for which $T - \lambda I$ is not injective. Let us look now at the numbers where the surjectivity of the operator $T - \lambda I$ fails.

The set of all $\lambda \in \mathbb{C}$ for which $T - \lambda I$ has a bounded inverse which is not defined on the whole Hilbert space $\mathcal{H}$ is called the *residual spectrum* of T and denoted by $\sigma_r(T)$. Note that $\sigma_r(T) = \{\lambda \in \pi(T) : d_\lambda(T) \neq 0\}$. By Proposition 2.4 this description implies that $\sigma_r(T)$ is an open set. It follows from Proposition 3.10 below that for self-adjoint operators T, the residual spectrum $\sigma_r(T)$ is empty.

Further, the set of $\lambda \in \mathbb{C}$ for which the range of $T - \lambda I$ is not closed, that is, $\mathcal{R}(T - \lambda I) \neq \overline{\mathcal{R}(T - \lambda I)}$, is called the *continuous spectrum* $\sigma_c(T)$ of T. Then $\sigma(T) = \sigma_p(T) \cup \sigma_r(T) \cup \sigma_c(T)$, but the sets $\sigma_c(T)$ and $\sigma_p(T)$ are in general *not disjoint*, see Exercise 5.

Remark The reader should be cautioned that some authors (for instance, [RN, BEH]) define $\sigma_c(T)$ as the complement of $\sigma_p(T) \cup \sigma_r(T)$ in $\sigma(T)$; then $\sigma(T)$ becomes the *disjoint* union of the three parts.

Example 2.1 (*Example* 1.3 *continued*) Let φ be a continuous function on an interval $\mathcal{J}$. Recall that the operator M_φ was defined by $M_\varphi f = \varphi \cdot f$ for f in the domain $\mathcal{D}(M_\varphi) = \{f \in L^2(\mathcal{J}) : \varphi \cdot f \in L^2(\mathcal{J})\}$.

Statement *$\sigma(M_\varphi)$ is the closure of the set $\varphi(\mathcal{J})$.*

Proof Let $\lambda \in \varphi(\mathcal{J})$, say $\lambda = \varphi(t_0)$ for $t_0 \in \mathcal{J}$. Given $\varepsilon > 0$, by the continuity of φ there exists an interval $K \subseteq \mathcal{J}$ of positive length such that $|\varphi(t) - \varphi(t_0)| \leq \varepsilon$ for all $t \in K$. Then $\|(M_\varphi - \lambda I)\chi_K\| \leq \varepsilon \|\chi_K\|$. If λ would be in $\rho(M_\varphi)$, then

$$\|\chi_K\| = \big\|R_\lambda(M_\varphi)(M_\varphi - \lambda I)\chi_K\big\| \leq \big\|R_\lambda(M_\varphi)\big\|\varepsilon\|\chi_K\|,$$

which is impossible if $\varepsilon\|R_\lambda(M_\varphi)\| < 1$. Thus, $\lambda \in \sigma(M_\varphi)$ and $\varphi(\mathcal{J}) \subseteq \sigma(M_\varphi)$. Hence, $\overline{\varphi(\mathcal{J})} \subseteq \sigma(M_\varphi)$.

Suppose that $\lambda \notin \overline{\varphi(\mathcal{J})}$. Then there is a $c > 0$ such that $|\lambda - \varphi(t)| \geq c$ for all $t \in \mathcal{J}$. Hence, $\psi(t) := (\varphi(t) - \lambda)^{-1}$ is a bounded function on $\mathcal{J}$, so M_ψ is bounded, $\mathcal{D}(M_\psi) = L^2(\mathcal{J})$, and $M_\psi = (M_\varphi - \lambda I)^{-1}$. Therefore, $\lambda \in \rho(M_\varphi)$. □ ∘

Now we turn to the resolvents. Suppose that T and S are closed operators on $\mathcal{H}$ such that $\mathcal{D}(S) \subseteq \mathcal{D}(T)$. Then the following *resolvent identities* hold:

$$R_\lambda(T) - R_\lambda(S) = R_\lambda(T)(S-T)R_\lambda(S) \quad \text{for } \lambda \in \rho(S) \cap \rho(T), \tag{2.4}$$

$$R_\lambda(T) - R_{\lambda_0}(T) = (\lambda - \lambda_0)R_\lambda(T)R_{\lambda_0}(T) \quad \text{for } \lambda, \lambda_0 \in \rho(T). \tag{2.5}$$

Indeed, if $\lambda \in \rho(S) \cap \rho(T)$ and $x \in \mathcal{H}$, we have $R_\lambda(S)x \in \mathcal{D}(S) \subseteq \mathcal{D}(T)$ and

$$\begin{aligned} R_\lambda(T)(S-T)R_\lambda(S)x &= R_\lambda(T)\big((S-\lambda I) - (T-\lambda I)\big)R_\lambda(S)x \\ &= R_\lambda(T)x - R_\lambda(S)x, \end{aligned}$$

which proves (2.4). The second formula (2.5) follows at once from the first (2.4) by setting $S = T + (\lambda - \lambda_0)I$ and using the relation $R_\lambda(S) = R_{\lambda_0}(T)$.

Both identities (2.4) and (2.5) are very useful for the study of operator equations. In particular, (2.5) implies that $R_\lambda(T)$ and $R_{\lambda_0}(T)$ commute.

The next proposition shows that the resolvent $R_\lambda(T)$ is an analytic function on the resolvent set $\rho(T)$ with values in the Banach space $(\mathbf{B}(\mathcal{H}), \|\cdot\|)$.

Proposition 2.9 *Suppose that $\lambda_0 \in \rho(T)$, $\lambda \in \mathbb{C}$, and $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$. Then we have $\lambda \in \rho(T)$ and*

$$R_\lambda(T) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^{n+1}, \tag{2.6}$$

where the series converges in the operator norm. In particular,

$$\lim_{\lambda \to \lambda_0} \big\| R_\lambda(T) - R_{\lambda_0}(T) \big\| = 0 \quad \textit{for } \lambda_0 \in \rho(T). \tag{2.7}$$

Proof As stated in Proposition 2.1(i), (2.1) holds with $c_{\lambda_0} = \|R_{\lambda_0}(T)\|^{-1}$, so that $|\lambda - \lambda_0| < c_{\lambda_0}$ by our assumption. Therefore, $\lambda \in \pi(T)$ and $d_\lambda(T) = d_{\lambda_0}(T) = 0$, and hence, $\lambda \in \rho(T)$ by Propositions 2.4 and 2.6.

Since $\|(\lambda - \lambda_0)R_{\lambda_0}(T)\| < 1$ by assumption, the operator $I - (\lambda - \lambda_0)R_{\lambda_0}(T)$ has a bounded inverse on $\mathcal{H}$ which is given by the Neumann series

$$\big(I - (\lambda - \lambda_0)R_{\lambda_0}(T)\big)^{-1} = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_{\lambda_0}(T)^n. \tag{2.8}$$

On the other hand, we have $R_\lambda(T)(I - (\lambda - \lambda_0)R_{\lambda_0}(T)) = R_{\lambda_0}(T)$ by (2.5), and hence, $R_\lambda(T) = R_{\lambda_0}(T)(I - (\lambda - \lambda_0)R_{\lambda_0}(T))^{-1}$. Multiplying (2.8) by $R_{\lambda_0}(T)$ from the left and using the latter identity, we obtain $R_\lambda(T)$. This proves (2.6).

Since analytic operator-valued functions are continuous, (2.6) implies (2.7). □

From formula (2.6) it follows in particular that for arbitrary vectors $x, y \in \mathcal{H}$, the complex function $\lambda \to \langle R_\lambda(T)x, y\rangle$ is analytic on the resolvent set $\rho(T)$.

For $T \in \mathbf{B}(\mathcal{H})$, it is well known (see, e.g., [RS1, Theorem VI.6]) that the spectrum $\sigma(T)$ is not empty and contained in a circle centered at the origin with radius

$$r(T) := \lim_{n \to \infty} \big\| T^n \big\|^{1/n}.$$

This number $r(T)$ is called the *spectral radius* of the operator T. Clearly, we have $r(T) \le \|T\|$. If $T \in \mathbf{B}(\mathcal{H})$ is self-adjoint, then $r(T) = \|T\|$.

By Proposition 2.6 the spectrum of a closed operator is a closed subset of $\mathbb{C}$. Let us emphasize that *any closed subset* (!) of the complex plane arises in this manner. Example 2.2 shows that each *nonempty* closed subset is spectrum of some closed operator. A closed operator with *empty* spectrum is given in Example 2.4 below.

Example 2.2 Suppose that M is a nonempty closed subset of $\mathbb{C}$. Since $\mathbb{C}$ is separable, so is M, that is, there exists a countable subset $\{r_n : n \in \mathbb{N}\}$ of M which is dense in M. Define the operator T on $l^2(\mathbb{N})$ by $\mathcal{D}(T) = \{(x_n) \in l^2(\mathbb{N}) : (r_n x_n) \in l^2(\mathbb{N})\}$ and $T(x_n) = (r_n x_n)$ for $(x_n) \in \mathcal{D}(T)$. It is easily seen that $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $T^*(x_n) = (\overline{r_n} x_n)$ for $(x_n) \in \mathcal{D}(T^*)$. Hence, $T = T^{**}$, so T is closed. Each number r_n is an eigenvalue of T, and we have $\sigma(T) = \overline{\{r_n : n \in \mathbb{N}\}} = M$. ∘

The next propositions relate the spectrum of the resolvent to the spectrum of the operator. Closed operators with compact resolvents will play an important role in several later chapters of this book.

Proposition 2.10 *Let* λ_0 *be a fixed number of* $\rho(T)$*, and let* $\lambda \in \mathbb{C}$, $\lambda \ne \lambda_0$.

(i) $\lambda \in \rho(T)$ *if and only if* $(\lambda - \lambda_0)^{-1} \in \rho(R_{\lambda_0}(T))$.
(ii) λ *is an eigenvalue of* T *if and only if* $(\lambda - \lambda_0)^{-1}$ *is an eigenvalue of* $R_{\lambda_0}(T)$. *In this case both eigenvalues have the same multiplicities.*

Proof Both assertions are easy consequences of the following identity:

$$T - \lambda I = \big(R_{\lambda_0}(T) - (\lambda - \lambda_0)^{-1} I\big)(T - \lambda_0 I)(\lambda_0 - \lambda). \tag{2.9}$$

(i): Since $(T - \lambda_0 I)(\lambda_0 - \lambda)$ is a bijection from $\mathcal{D}(T)$ to $\mathcal{H}$, it follows from (2.9) that $T - \lambda I$ is a bijection from $\mathcal{D}(T)$ to $\mathcal{H}$ if and only if $R_{\lambda_0}(T) - (\lambda - \lambda_0)^{-1} I$ is a bijection of $\mathcal{H}$. By Proposition 2.7(i) this gives the assertion.

(ii): From (2.9) we conclude that $(T - \lambda_0 I)(\lambda_0 - \lambda)$ is a bijection of $\mathcal{N}(T - \lambda I)$ on $\mathcal{N}(R_{\lambda_0}(T) - (\lambda - \lambda_0)^{-1} I)$. □

We shall say that a closed operator T has a *purely discrete spectrum* if $\sigma(T)$ consists only of eigenvalues of finite multiplicities which have no finite accumulation point.

Proposition 2.11 *Suppose that there exists a* $\lambda_0 \in \rho(T)$ *such that* $R_{\lambda_0}(T)$ *is compact. Then* $R_\lambda(T)$ *is compact for all* $\lambda \in \rho(T)$*, and* T *has a purely discrete spectrum.*

Proof The compactness of $R_\lambda(T)$ follows at once from the resolvent identity (2.5). By Theorem A.3 all nonzero numbers in the spectrum of the compact operator $R_{\lambda_0}(T)$ are eigenvalues of finite multiplicities which have no nonzero accumulation point. By Proposition 2.10 this implies that the operator T has a purely discrete spectrum. □

2.3 Examples: Differentiation Operators II

In this section we determine spectra and resolvents of the differentiation operators $-\mathrm{i}\frac{d}{dx}$ on intervals from Sect. 1.3.1.

Example 2.3 (*Example* 1.4 *continued*: *bounded interval* (a,b)) Recall that $\mathcal{D}(T^*) = H^1(a,b)$ and $T^* f = -\mathrm{i}f'$ for $f \in \mathcal{D}(T^*)$. For each $\lambda \in \mathbb{C}$, $f_\lambda(x) := e^{\mathrm{i}\lambda x}$ is in $\mathcal{D}(T^*)$, and $T^* f_\lambda = \lambda f_\lambda$, so $\lambda \in \sigma_{\mathrm{p}}(T^*)$. Thus, $\sigma(T^*) = \mathbb{C}$. Since $T = (T^*)^*$, Proposition 2.7(ii) implies that $\sigma(T) = \mathbb{C}$. ∘

Example 2.4 (*Example* 1.5 *continued*)

Statement $\sigma(S_z) = \{\lambda \in \mathbb{C} : e^{\mathrm{i}\lambda(a-b)}z = 1\}$ *for* $z \in \mathbb{C}$ *and* $\sigma(S_\infty) = \emptyset$.

Proof Let $\lambda \in \mathbb{C}$ and $g \in L^2(a,b)$. In order to "guess" the formula for the resolvent of S_z, we try to find an element $f \in \mathcal{D}(S_z)$ such that $(S_z - \lambda I)f \equiv -\mathrm{i}f' - \lambda f = g$. The general solution of the differential equation $-\mathrm{i}f' - \lambda f = g$ is

$$f(x) = \mathrm{i}e^{\mathrm{i}\lambda x}\left(\int_a^x e^{-\mathrm{i}\lambda t} g(t)\,dt + c_{\lambda,g}\right), \quad \text{where } c_{\lambda,g} \in \mathbb{C}. \tag{2.10}$$

Clearly, $f \in H^1(a,b)$, since $g \in L^2(a,b)$ and hence $e^{-\mathrm{i}\lambda t}g(t) \in L^1(a,b)$. Hence, f is in $\mathcal{D}(S_z)$ if and only if f satisfies the boundary condition $f(b) = zf(a)$ for $z \in \mathbb{C}$ resp. $f(a) = 0$ for $z = \infty$.

First suppose that $z \in \mathbb{C}$ and $e^{\mathrm{i}\lambda(a-b)}z \neq 1$. Then $f \in \mathcal{D}(S_z)$ if and only if

$$c_{\lambda,g} = \left(e^{\mathrm{i}\lambda(a-b)}z - 1\right)^{-1} \int_a^b e^{-\mathrm{i}\lambda t} g(t)\,dt. \tag{2.11}$$

We therefore define

$$\bigl(R_\lambda(S_z)g\bigr)(x) = \mathrm{i}e^{\mathrm{i}\lambda x}\left(\int_a^x e^{-\mathrm{i}\lambda t} g(t)\,dt + \left(e^{\mathrm{i}\lambda(a-b)}z - 1\right)^{-1} \int_a^b e^{-\mathrm{i}\lambda t} g(t)\,dt\right).$$

Next suppose that $z = \infty$. Then $f \in \mathcal{D}(S_\infty)$ if and only if $c_{\lambda,g} = 0$, so we define

$$\bigl(R_\lambda(S_\infty)g\bigr)(x) = \mathrm{i}e^{\mathrm{i}\lambda x}\int_a^x e^{-\mathrm{i}\lambda t} g(t)\,dt.$$

We prove that $R_\lambda(S_z)$, $z \in \mathbb{C} \cup \{\infty\}$, is the resolvent of S_z. Let $g \in L^2(a,b)$ and set $f := R_\lambda(S_z)g$. By the preceding considerations, we have $f \in \mathcal{D}(S_z)$ and $(S_z - \lambda I)f = (S_z - \lambda I)R_\lambda(S_z)g = g$. Hence, $S_z - \lambda I$ is surjective. From (2.10) and (2.11) we conclude that $g = 0$ implies that $f = 0$, so $S_z - \lambda I$ is injective. Therefore, by Proposition 2.7(i), $\lambda \in \rho(S_z)$ and $(S_z - \lambda I)^{-1} = R_\lambda(S_z)$. Thus, we have shown that $\{\lambda : e^{\mathrm{i}\lambda(a-b)}z \neq 1\} \subseteq \rho(S_z)$ for $z \in \mathbb{C}$ and $\rho(S_\infty) = \mathbb{C}$.

Suppose that $z \in \mathbb{C}$ and $e^{\mathrm{i}\lambda(a-b)}z = 1$. Then $f_\lambda(x) := e^{\mathrm{i}\lambda x}$ belongs to $\mathcal{D}(S_z)$, and $S_z f_\lambda = \lambda f_\lambda$. Hence, $\lambda \in \sigma(S_z)$. This completes the proof of the statement. □

Let us consider the special case where $|z|=1$, say $z=e^{\mathrm{i}\mu(b-a)}$ with $\mu\in\mathbb{R}$. Then the operator S_z is self-adjoint (by Example 1.5) and the above statement yields

$$\sigma(S_z)=\bigl\{\mu+(b-a)^{-1}2\pi k : k\in\mathbb{Z}\bigr\}. \qquad \circ$$

Example 2.5 (*Example* 1.6 *continued*: *half-axis*) Recall that $\mathcal{D}(T)=H_0^1(0,+\infty)$. We prove that $\sigma(T)=\{\lambda\in\mathbb{C}:\operatorname{Im}\lambda\le 0\}$.

Assume that $\operatorname{Im}\lambda<0$. Then $f_\lambda(x):=e^{\mathrm{i}\overline{\lambda}x}\in\mathcal{D}(T^*)$ and $T^*f_\lambda=\overline{\lambda}f_\lambda$, so that $\overline{\lambda}\in\sigma_{\mathrm{p}}(T^*)$ and $\lambda\in\sigma(T)$ by Proposition 2.7(ii). Hence, $\{\lambda:\operatorname{Im}\lambda\le 0\}\subseteq\sigma(T)$.

Suppose now that $\operatorname{Im}\lambda>0$ and define

$$\bigl(R_\lambda(T)g\bigr)(x)=\mathrm{i}\int_0^x e^{\mathrm{i}\lambda(x-t)}g(t)\,dt,\qquad g\in L^2(0,+\infty).$$

That is, $R_\lambda(T)$ is the convolution operator with the function $h(t):=\mathrm{i}e^{\mathrm{i}\lambda t}$ on the half-axis $[0,+\infty)$. Since $\operatorname{Im}\lambda>0$ and hence $h\in L^1(0,+\infty)$, $R_\lambda(T)$ is a bounded operator on $L^2(0,+\infty)$. Indeed, using the Cauchy–Schwarz inequality, we derive

$$\begin{aligned}
\bigl\|\bigl(R_\lambda(T)g\bigr)\bigr\|^2&=\int_0^\infty\left|\int_0^x h(x-t)g(t)\,dt\right|^2dx\\
&\le\int_0^\infty\left(\int_0^x\bigl|h(x-t)\bigr|\,dt\right)\left(\int_0^x\bigl|h(x-t)\bigr|\bigl|g(t)\bigr|^2dt\right)dx\\
&\le\|h\|_{L^1(0,+\infty)}\int_0^\infty\int_0^x\bigl|h(x-t)\bigr|\bigl|g(t)\bigr|^2dt\,dx\\
&\le\|h\|_{L^1(0,+\infty)}\int_0^\infty\int_0^\infty\bigl|h\bigl(x'\bigr)\bigr|\bigl|g(t)\bigr|^2dt\,dx'=\|h\|^2_{L^1(0,+\infty)}\|g\|^2.
\end{aligned}$$

Set $f:=R_\lambda(T)g$. Clearly, $f\in AC[a,b]$ for all intervals $[a,b]\subseteq(0,+\infty)$. Since $f\in L^2(0,+\infty)$, $f'=\mathrm{i}(\lambda f+g)\in L^2(0,+\infty)$ and $f(0)=0$, we have $f\in H_0^1(0,+\infty)=\mathcal{D}(T)$ and $(T-\lambda I)f=(T-\lambda I)R_\lambda(T)g=g$. This shows that $T-\lambda I$ is surjective. Since $\operatorname{Im}\lambda>0$, $\mathcal{N}(T-\lambda I)=\{0\}$. Thus, $T-\lambda I$ is bijective, and hence $\lambda\in\rho(T)$ by Proposition 2.7(i). From the equality $(T-\lambda I)R_\lambda(T)g=g$ for $g\in L^2(0,+\infty)$ it follows that $R_\lambda(T)=(T-\lambda I)^{-1}$ is the resolvent of T.

By the preceding we have proved that $\sigma(T)=\{\lambda:\operatorname{Im}\lambda\le 0\}$. $\circ$

Example 2.6 (*Example* 1.7 *continued*: *real line*) Then the operator $T=-\mathrm{i}\frac{d}{dx}$ on $H^1(\mathbb{R})$ is self-adjoint. We show that $\sigma(T)=\mathbb{R}$.

Suppose that $\lambda\in\mathbb{R}$. Let us choose a function $\omega\in C_0^\infty(\mathbb{R})$, $\omega\neq 0$, and put $h_\epsilon(x):=\epsilon^{1/2}e^{\mathrm{i}\lambda x}\omega(\epsilon x)$ for $\epsilon>0$. Since $\|h_\epsilon\|=\|\omega\|$ and $\|(T-\lambda I)h_\epsilon\|=\epsilon\|\omega'\|$, it follows that λ is not in $\pi(T)$ and so not in $\rho(T)$. Hence, $\lambda\in\sigma(T)$. Since T is self-adjoint, $\sigma(T)\subseteq\mathbb{R}$ by Corollary 3.14 below. Thus, $\sigma(T)=\mathbb{R}$.

The resolvents of T for $\lambda\in\mathbb{C}\backslash\mathbb{R}$ are given by the formulas

$$\bigl(R_\lambda(T)g\bigr)(x)=\mathrm{i}\int_{-\infty}^x e^{\mathrm{i}\lambda(x-t)}g(t)\,dt,\quad \operatorname{Im}\lambda>0,\tag{2.12}$$

$$\bigl(R_\lambda(T)g\bigr)(x)=-\mathrm{i}\int_x^{+\infty} e^{\mathrm{i}\lambda(x-t)}g(t)\,dt,\quad \operatorname{Im}\lambda<0.\tag{2.13}$$

$\circ$

2.4 Exercises

1. Find a bounded operator T such that $\Theta(T)$ is not the convex hull of $\sigma(T)$.
   Hint: Look for some lower triangular 2×2-matrix.
2. Let $\alpha = (\alpha_n)_{n\in\mathbb{N}}$ be a complex sequence. Define an operator T_α on $l^2(\mathbb{N})$ with domain $\mathcal{D}(T) = \{(\varphi_n) \in l^2(\mathbb{N}) : (\alpha_n\varphi_n) \in l^2(\mathbb{N})\}$ by $T_\alpha(\varphi_n) = (\alpha_n\varphi_n)$.
   a. Determine the spectrum $\sigma(T_\alpha)$ and the point spectrum $\sigma_{\mathrm{p}}(T_\alpha)$.
   b. When has T_α a discrete spectrum?
3. Let M_φ be the multiplication operator from Example 2.1. Find necessary and/or sufficient conditions for a number belonging to the point spectrum $\sigma_{\mathrm{p}}(M_\varphi)$.
4. Let T_1 and T_2 be closed operators on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.
   a. Show that $T_1 \oplus T_2$ is a closed operator on $\mathcal{H}_1 \oplus \mathcal{H}_2$.
   b. Show that $\sigma(T_1 \oplus T_2) = \sigma(T_1) \cup \sigma(T_2)$.
5. Find a bounded operator T and a $\lambda \in \sigma_{\mathrm{p}}(T)$ such that $\mathcal{R}(T - \lambda I) \neq \overline{\mathcal{R}(T - \lambda I)}$.
   Hint: Look for some operator $T = T_1 \oplus T_2$.
6. Let $T = -\mathrm{i}\frac{d}{dx}$ on $\mathcal{D}(T) = \{f \in H^1(0,1) : f(0) = 0\}$ in $\mathcal{H} = L^2(0,1)$.
   a. Show that T is a closed operator.
   b. Determine the adjoint operator T^*.
   c. Show that $\rho(T) = \mathbb{C}$ and determine the operator $R_\lambda(T)$ for $\lambda \in \mathbb{C}$.
7. Prove the two resolvent formulas (2.12) and (2.13) in Example 2.6. Show that none of these operators is compact.
8. Let q be a real-valued continuous function on $[a,b]$, $a, b \in \mathbb{R}$, $a < b$. For $z \in \mathbb{T}$, define an operator T_z on $L^2(a,b)$ by $(T_z f)(x) = -\mathrm{i}f'(x) + q(x)f(x)$ with domain $\mathcal{D}(T_z) = \{f \in H^1(a,b) : f(b) = zf(a)\}$.
   a. Show that T_z is a self-adjoint operator on $L^2(a,b)$.
   b. Determine the spectrum and the resolvent $R_\lambda(T_z)$ for $\lambda \in \rho(T_z)$.
   Hint: Find a unitary operator U on $L^2(a,b)$ such that $T_z = UTU^*$, where T is the operator from Example 2.3.
9. Find a densely defined closed operator T such that each complex number is an eigenvalue of T^*, but T has no eigenvalue.
10. Let T be a closed operator on $\mathcal{H}$. Use formula (2.6) to prove that

$$\frac{dR_\lambda(T)}{d\lambda} := \lim_{h\to 0} \frac{R_{\lambda+h}(T) - R_\lambda(T)}{h} = R_\lambda(T)^2, \quad \lambda \in \rho(T),$$

   in the operator norm on $\mathcal{H}$.
11. Prove that $\sigma(TS)\cup\{0\} = \sigma(ST)\cup\{0\}$ for $T \in \mathbf{B}(\mathcal{H}_1, \mathcal{H}_2)$ and $S \in \mathbf{B}(\mathcal{H}_2, \mathcal{H}_1)$.
   Hint: Verify that $(ST - \lambda I)^{-1} = \lambda^{-1}[S(TS - \lambda I)^{-1}T - I]$ for $\lambda \neq 0$.

*12. (*Volterra integral operator*)
   Let K be a bounded measurable function on $\{(x,y) \in \mathbb{R}^2 : 0 \le y \le x \le 1\}$. Prove that the spectrum of the Volterra operator V_K is equal to $\{0\}$, where

$$(V_K f)(x) = \int_0^x K(x,t)f(t)\,dt, \quad f \in L^2(0,1).$$

Hints: Show that $(V_K)^n$ is an integral operator with kernel K_n satisfying

$$\left|K_n(x, y)\right| \le M^n |x - y|^{n-1}/(n-1)!, \quad \text{where } M := \|K\|_{L^\infty(0,1)}.$$

Then deduce that $\|(V_K)^n\| \le M^n/(n-1)!$ and hence $\lim_n \|(V_K)^n\|^{1/n} = 0$.

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