Chapter 2

Multidimensional Grüss-Čebyšev and-Trapezoid-type inequalities

2.1 Introduction

During the last two decades many researchers have given considerable attention to the famous inequalities (1), (3) and (4) associated to the names of Čebyšev, Grüss and Trapezoid. In view of the usefulness of these inequalities and their applications, many authors have investigated a large number of new multidimensional, Grüss, Čebyšev and Trapezoid type inequalities. Some of these results provide simple and elegant extensions of the inequalities (1), (3) and (4) and have a wider scope of applicability. These results did not just add new objects of study, but also brought new insights and techniques to handle such inequalities. This chapter deals with a number of new multidimensional inequalities discovered by various investigators, which claim their origin to the well-known inequalities in (1), (3) and (4). Some applications are given to illustrate the usefulness of certain inequalities.

2.2 Some Grüss-type inequalities in inner product spaces

In this section we offer some fundamental Grüss-type inequalities established by Dragomir [32,43,53] and Dragomir, Pečarić and Tepeš [56] in inner product spaces.
We start with the following Grüss-type inequality investigated in [32].

Theorem 2.2.1. Let $(H,(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$) and $e\in H$, $\|e\|=1$. If $\phi,\ \gamma,\ \Phi,\ \Gamma$ are real or complex numbers and $x,\ y$ are vectors in H such that the condition

$$\mathrm{Re}(\Phi e-x,x-\phi e)\geqslant 0,\quad \mathrm{Re}(\Gamma e-y,y-\gamma e)\geqslant 0, \tag{2.2.1}$$

holds, then we have the inequality

$$|(x,y)-(x,e)(e,y)|\leqslant \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|. \tag{2.2.2}$$

The constant $\frac{1}{4}$ is the best possible.

Proof. It is obvious that (see [156])

$$(x,y)-(x,e)(e,y)=(x-(x,e)e,y-(y,e)e). \tag{2.2.3}$$

Using Schwarz's inequality in inner product spaces, we have

$$\begin{aligned} |(x,y)-(x,e)(e,y)|^2 &= |(x-(x,e)e,y-(y,e)e)|^2 \\ &\leqslant \|x-(x,e)e\|^2\|y-(y,e)e\|^2 \\ &= \left(\|x\|^2-|(x,e)|^2\right)\left(\|y\|^2-|(y,e)|^2\right). \end{aligned} \tag{2.2.4}$$

On the other hand, a simple computation shows that

$$(\Phi-(x,e))\left(\overline{(x,e)}-\overline{\phi}\right)-(\Phi e-x,x-\phi e)=\|x\|^2-|(x,e)|^2, \tag{2.2.5}$$

and

$$(\Gamma-(y,e))\left(\overline{(y,e)}-\overline{\gamma}\right)-(\Gamma e-y,y-\gamma e)=\|y\|^2-|(y,e)|^2. \tag{2.2.6}$$

Taking the real part in both the above equalities, we can write

$$\operatorname{Re}\left[(\Phi-(x,e))\left(\overline{(x,e)}-\overline{\phi}\right)\right]-\operatorname{Re}(\Phi e-x,x-\phi e)=\|x\|^2-|(x,e)|^2, \tag{2.2.7}$$

and

$$\operatorname{Re}\left[(\Gamma-(y,e))\left(\overline{(y,e)}-\overline{\gamma}\right)\right]-\operatorname{Re}(\Gamma e-y,y-\gamma e)=\|y\|^2-|(y,e)|^2. \tag{2.2.8}$$

From the condition (2.2.1), we deduce

$$\|x\|^2-|(x,e)|^2\leqslant \operatorname{Re}\left[(\Phi-(x,e))\left(\overline{(x,e)}-\overline{\phi}\right)\right], \tag{2.2.9}$$

and

$$\|y\|^2-|(y,e)|^2\leqslant \operatorname{Re}\left[(\Gamma-(y,e))\left(\overline{(y,e)}-\overline{\gamma}\right)\right]. \tag{2.2.10}$$

Using the elementary inequality $4\operatorname{Re}\left(a\overline{b}\right)\leqslant|a+b|^2$ holding for real or complex numbers a,b, for $a:=\Phi-(x,e)$ and $b:=(x,e)-\phi$, we get

$$\operatorname{Re}\left[(\Phi-(x,e))\left(\overline{(x,e)}-\overline{\phi}\right)\right]\leqslant\frac{1}{4}|\Phi-\phi|^2, \tag{2.2.11}$$

and, similarly

$$\operatorname{Re}\left[(\Gamma-(y,e))\left(\overline{(y,e)}-\overline{\gamma}\right)\right]\leqslant\frac{1}{4}|\Gamma-\gamma|^2. \tag{2.2.12}$$

Consequently, using (2.2.3)–(2.2.12), we have successively

$$|(x,y)-(x,e)(e,y)|^2\leqslant\left(\|x\|^2-|(x,e)|^2\right)\left(\|y\|^2-|(y,e)|^2\right)$$

$$\leqslant\operatorname{Re}\left[(\Phi-(x,e))\left(\overline{(x,e)}-\overline{\phi}\right)\right]\operatorname{Re}\left[(\Gamma-(y,e))\left(\overline{(y,e)}-\overline{\gamma}\right)\right]$$

$$\leqslant \frac{1}{16}|\Phi-\phi|^2|\Gamma-\gamma|^2,$$

from which we get the desired inequality (2.2.2).

To prove that the constant $\frac{1}{4}$ is sharp, we can restrict ourselves to the real case. Let $e,m \in H$ with $\|e\| = \|m\| = 1$, $e \perp m$ and assume that ϕ,γ,Φ,Γ are real numbers. Define the vectors

$$x := \frac{\Phi+\phi}{2}e + \frac{\Phi-\phi}{2}m, y := \frac{\gamma+\Gamma}{2}e + \frac{\Gamma-\gamma}{2}m.$$

Then

$$(\Phi e - x, x - \phi e) = \left(\frac{\Phi-\phi}{2}\right)^2 (e-m, e+m) = 0,$$

and similarly $(\Gamma e - y, y - \gamma e) = 0$, i.e., the condition (2.2.1) holds. Now, observe that

$$(x,y) = \left(\frac{\Phi+\phi}{2}\right)\left(\frac{\gamma+\Gamma}{2}\right) + \left(\frac{\Phi-\phi}{2}\right)\left(\frac{\Gamma-\gamma}{2}\right),$$

and

$$(x,e)(e,y) = \left(\frac{\Phi+\phi}{2}\right)\left(\frac{\gamma+\Gamma}{2}\right).$$

Consequently,

$$|(x,y)-(x,e)(e,y)| = \frac{1}{4}|\Phi-\phi||\Gamma-\gamma|,$$

which shows that the constant $\frac{1}{4}$ is sharp.

In [53], the author gave an alternative proof of (2.2.2) by using the following lemmas.

Lemma 2.2.1. Let a,x,A be vectors in the inner product space $(H,(\cdot,\cdot))$ over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R},\mathbb{C}$) with $a \neq A$. Then

$$\mathrm{Re}(A-x, x-a) \geqslant 0,$$

if and only if

$$\left\| x - \frac{a+A}{2} \right\| \leqslant \frac{1}{2}\|A-a\|.$$

Proof. Define

$$I_1 := \mathrm{Re}(A-x,x-a), \quad I_2 := \frac{1}{4}\|A-a\|^2 - \left\| x - \frac{a+A}{2}\right\|^2.$$

A simple calculation shows that

$$I_1 = I_2 = \mathrm{Re}[(x,a)+(A,x)] - \mathrm{Re}(A,a) - \|x\|^2,$$

and thus obviously, $I_1 \geqslant 0$ if and only if $I_2 \geqslant 0$, showing the required equivalence.

The following corollary is obvious.

Corollary 2.2.1. Let $x,\ e \in H$ with $\|e\| = 1$ and δ, $Delta \in \mathbb{K}$ with $\delta \neq \Delta$. Then

$$\operatorname{Re}(\Delta e - x, x - \delta e) \geqslant 0,$$

if and only if

$$\left\| x - \frac{\delta + \Delta}{2} e \right\| \leqslant \frac{1}{2} \|\Delta - \delta\| .$$

Lemma 2.2.2. Let $x, e \in H$ with $\|e\| = 1$. Then one has the following representation

$$0 \leqslant \|x\|^2 - |(x,e)|^2 = \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2, \tag{2.2.13}$$

where $\mathbb{K}$ is as in Lemma 2.2.1.

Proof. For any $\lambda \in \mathbb{K}$ observe that

$$(x - \lambda e, x - (x,e)) = \|x\|^2 - |(x,e)|^2 - \lambda \left[(e,x) - (e,x)\|e\|^2 \right] = \|x\|^2 - |(x,e)|^2.$$

Using Schwarz inequality, we have

$$\left[\|x\|^2 - |(x,e)|^2 \right]^2 = |(x - \lambda e, x - (x,e)e)|^2$$

$$\leqslant \|x - \lambda e\|^2 \|x - (x,e)e\|^2 = \|x - \lambda e\|^2 \left[\|x\|^2 - |(x,e)|^2 \right],$$

giving the bound

$$\|x\|^2 - |(x,e)|^2 \leqslant \|x - \lambda e\|^2. \tag{2.2.14}$$

Taking the infimum in (2.2.14) over $\lambda \in \mathbb{K}$, we deduce

$$\|x\|^2 - |(x,e)|^2 \leqslant \inf_{\lambda \in \mathbb{K}} \|x - \lambda e\|^2.$$

Since, for $\lambda_0 = (x,e)$, we get $\|x - \lambda_0 e\|^2 = \|x\|^2 - |(x,e)|^2$, then the representation (2.2.13) is proved.

The following result is proved in [53].

Theorem 2.2.2. Let $(H, (\cdot,\cdot))$ be an inner product space over K ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$) and $e \in H$, $\|e\| = 1$. If ϕ, γ, Φ, Γ are real or complex numbers and x, y are vectors in H such that the conditions (2.2.1) hold, or, equivalently, the following assumptions

$$\left\| x - \frac{\phi + \Phi}{2} e \right\| \leqslant \frac{1}{2} |\Phi - \phi|, \quad \left\| y - \frac{\gamma + \Gamma}{2} e \right\| \leqslant \frac{1}{2} |\Gamma - \gamma|, \tag{2.2.15}$$

are valid, then one has the inequality

$$|(x,y) - (x,e)(e,y)| \leqslant \frac{1}{4} |\Phi - \phi| |\Gamma - \gamma|. \tag{2.2.16}$$

The constant $\frac{1}{4}$ is the best possible.

Proof. As in the proof of Theorem 2.2.1, we have (2.2.3) and (2.2.4). Using Lemma 2.2.2 and conditions (2.2.15), we obviously have

$$\left[\|x\|^2 - |(x,e)|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|x-\lambda e\| \leqslant \left\| x - \frac{\phi+\Phi}{2}e \right\| \leqslant \frac{1}{2}|\Phi-\phi|, \tag{2.2.17}$$

and

$$\left[\|y\|^2 - |(y,e)|^2\right]^{\frac{1}{2}} = \inf_{\lambda\in\mathbb{K}} \|y-\lambda e\| \leqslant \left\| y - \frac{\gamma+\Gamma}{2}e \right\| \leqslant \frac{1}{2}|\Gamma-\gamma|. \tag{2.2.18}$$

Using (2.2.17), (2.2.18) in (2.2.4), the desired inequality in (2.2.16) follows. The fact that $\frac{1}{4}$ is the best possible constant, has been shown in the proof of Theorem 2.2.1 and hence we omit the details.

The refinement of the inequality (2.2.2) proved in [53] is embodied in the following theorem.

Theorem 2.2.3. Let $(H,(\cdot,\cdot))$ be an inner product space over K ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$) and $e\in H$, $\|e\|=1$. If $\phi,\ \gamma,\ \Phi,\ \Gamma$ are real or complex numbers and $x,\ y$ are vectors in H such that the condition (2.2.1) or equivalently, (2.2.15) hold, then we have the inequality

$$\begin{aligned} &|(x,y)-(x,e)(e,y)| \\ &\leqslant \frac{1}{4}|\Phi-\phi||\Gamma-\gamma| - \left[\mathrm{Re}(\Phi e - x, x-\phi e)\right]^{\frac{1}{2}}\left[\mathrm{Re}(\Gamma e - y, y-\gamma e)\right]^{\frac{1}{2}}. \end{aligned} \tag{2.2.19}$$

Proof. Following the proof of Theorem 2.2.1, we have (2.2.3), (2.2.7), (2.2.8), (2.2.11), (2.2.12) and consequently, we observe that

$$\begin{aligned} |(x,y)-(x,e)(e,y)|^2 &\leqslant \left[\frac{1}{4}|\Phi-\phi|^2 - \left(\left[\mathrm{Re}(\Phi e - x, x-\phi e)\right]^{\frac{1}{2}}\right)^2\right] \\ &\times \left[\frac{1}{4}|\Gamma-\gamma|^2 - \left(\left[\mathrm{Re}(\Gamma e - y, y-\gamma e)\right]^{\frac{1}{2}}\right)^2\right]. \end{aligned} \tag{2.2.20}$$

By a suitable application of the elementary inequality

$$\left(m^2-n^2\right)\left(p^2-q^2\right) \leqslant (mp-nq)^2, \tag{2.2.21}$$

for $m,n,p,q\in\mathbb{R}$, to the right hand side of (2.2.20), we have

$$\begin{aligned} &|(x,y)-(x,e)(e,y)|^2 \\ &\leqslant \left[\frac{1}{4}|\Phi-\phi||\Gamma-\gamma| - \left(\left[\mathrm{Re}(\Phi e - x, x-\phi e)\right]^{\frac{1}{2}}\left[\mathrm{Re}(\Gamma e - y, y-\gamma e)\right]^{\frac{1}{2}}\right)^2\right]^2, \end{aligned}$$

from which the desired inequality in (2.2.19) follows.

The following Theorem given in [56] deals with the inequalities of the pre-Grüss-type in inner product spaces.

Theorem 2.2.4. Let $(H,(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$) and $e\in H$, $\|e\|=1$. If ϕ, Φ are real or complex numbers and x,y are vectors in H such that either the condition

$$\operatorname{Re}(\Phi e-x,x-\phi e)\geqslant 0,$$

or equivalently,

$$\left\|x-\frac{\phi+\Phi}{2}e\right\|\leqslant\frac{1}{2}|\Phi-\phi|, \tag{2.2.22}$$

holds true, then we have the inequalities

$$|(x,y)-(x,e)(e,y)|\leqslant\frac{1}{2}|\Phi-\phi|\left(\|y\|^2-|(y,e)|^2\right)^{\frac{1}{2}}, \tag{2.2.23}$$

and

$$|(x,y)-(x,e)(e,y)|\leqslant\frac{1}{2}|\Phi-\phi|\|y\|-\left(\operatorname{Re}(\Phi e-x,x-\phi e)\right)^{\frac{1}{2}}|(y,e)|. \tag{2.2.24}$$

Proof. As in the proof of Theorem 2.2.1, we have (2.2.3) and (2.2.4). Now, the inequality (2.2.23) is a simple consequence of (2.2.2) for $x=y$ or of Lemma 2.2.2 and (2.2.22). Furthermore, from the proof of Theorem 2.2.1, we have (2.2.7) and (2.2.11). Using (2.2.7) and (2.2.11), we have

$$\|x\|^2-|(x,e)|^2\leqslant\left(\frac{1}{2}|\Phi-\phi|\right)^2-\left(\left(\operatorname{Re}(\Phi e-x,x-\phi e)\right)^{\frac{1}{2}}\right)^2. \tag{2.2.25}$$

From (2.2.4) and (2.2.25), we get

$$\begin{aligned}&|(x-(x,e)e,y-(y,e)e)|^2\\ &\leqslant\left(\left(\frac{1}{2}|\Phi-\phi|\right)^2-\left(\left(\operatorname{Re}(\Phi e-x,x-\phi e)\right)^{\frac{1}{2}}\right)^2\right)\left(\|y\|^2-|(y,e)|^2\right).\end{aligned} \tag{2.2.26}$$

Now, by a suitable application of the elementary inequality (2.2.21) to the right hand side of (2.2.26) and rewriting, we get the desired inequality in (2.2.24). The proof is complete.

Before closing this section, we present a Grüss-type inequality for sequences of vectors in inner product spaces given in [43].

The following lemma given in [43] is of interest in itself.

Lemma 2.2.3. Let $(H,(\cdot,\cdot))$ be an inner product space over the real or complex number field $\mathbb{K}$, $x_i\in H$ and $p_i\geqslant 0$ ($i=1,\ldots,n$) such that $\sum_{i=1}^n p_i=1$ ($n\geqslant 2$). If $x,\ X\in H$ are such that

$$\operatorname{Re}(X-x_i,x_i-x)\geqslant 0, \tag{2.2.27}$$

for all $i\in\{1,\ldots,n\}$, then we have the inequality

$$0\leqslant\sum_{i=1}^n p_i\|x_i\|^2-\left\|\sum_{i=1}^n p_ix_i\right\|^2\leqslant\frac{1}{4}\|X-x\|^2. \tag{2.2.28}$$

The constant $\frac{1}{4}$ is sharp.

Proof. Define

$$I_1 := \left(X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right),$$

and

$$I_2 := \sum_{i=1}^{n} p_i (X - x_i, x_i - x).$$

Then

$$I_1 = \sum_{i=1}^{n} p_i (X, x_i) - (X, x) - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 + \sum_{i=1}^{n} p_i (x_i, x),$$

and

$$I_2 = \sum_{i=1}^{n} p_i (X, x_i) - (X, x) - \sum_{i=1}^{n} p_i \|x_i\|^2 + \sum_{i=1}^{n} p_i (x_i, x).$$

Consequently,

$$I_1 - I_2 = \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2. \tag{2.2.29}$$

Taking the real value in (2.2.29), we can state

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 = \operatorname{Re} \left(X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right) - \sum_{i=1}^{n} p_i \operatorname{Re}(X - x_i, x_i - x), \tag{2.2.30}$$

which is an identity of interest in itself.

Using the assumption (2.2.27), we can conclude by (2.2.30), that

$$\sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leqslant \operatorname{Re} \left(X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right). \tag{2.2.31}$$

It is known that if $y,\ z \in H$, then

$$4 \operatorname{Re}(z, y) \leqslant \|z + y\|^2, \tag{2.2.32}$$

with equality if and only if $z = y$. Now, by (2.2.32), we can state that

$$\operatorname{Re} \left(X - \sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i x_i - x \right) \leqslant \frac{1}{4} \left\| X - \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} p_i x_i - x \right\|^2 = \frac{1}{4} \|X - x\|^2.$$

Using (2.2.31), we can easily deduce (2.2.28).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that the inequality (2.2.28) holds with a constant $c > 0$, i.e.,

$$0 \leqslant \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{n} p_i x_i \right\|^2 \leqslant c\|X - x\|^2, \tag{2.2.33}$$

for all p_i, x_i and x, X as in the hypotheses. Assume that $n = 2$, $p_1 = p_2 = \frac{1}{2}$, $x_1 = x$ and $x_2 = X$ with x, $X \in H$ and $x \neq X$. Then, obviously,

$$(X - x_1, x_1 - x) = (X - x_2, x_2 - x) = 0,$$

which shows that the condition (2.2.27) holds. If we replace n, p_1, p_2, x_1, x_2 in (2.2.33) as above, we obtain

$$\sum_{i=1}^{2} p_i \|x_i\|^2 - \left\| \sum_{i=1}^{2} p_i x_i \right\|^2 = \frac{1}{2}\left(\|x\|^2 + \|X\|^2 - \left\| \frac{x+X}{2} \right\|^2 \right) = \frac{1}{4}\|X - x\|^2 \leqslant c\|X - x\|^2,$$

from where we deduce $c \geqslant \frac{1}{4}$ which proves the sharpness of the constant factor $\frac{1}{4}$.
The following Grüss-type inequality holds (see [43]).

Theorem 2.2.5. Let $(H, (\cdot, \cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$) and x_i, $y_i \in H$, $p_i \geqslant 0$, $(i = 1, \ldots, n)$ $(n \geqslant 2)$ with $\sum_{i=1}^{n} p_i = 1$. If x, X, y, $Y \in H$ are such that

$$\operatorname{Re}(X - x_i, x_i - x) \geqslant 0, \quad \operatorname{Re}(Y - y_i, y_i - y) \geqslant 0,$$

for all $i \in \{1, \ldots, n\}$, then we have the inequality

$$\left| \sum_{i=1}^{n} p_i (x_i, y_i) - \left(\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right) \right| \leqslant \frac{1}{4}\|X - x\|\|Y - y\|. \tag{2.2.34}$$

The constant $\frac{1}{4}$ is sharp.

Proof. A simple calculation shows that

$$\sum_{i=1}^{n} p_i (x_i, y_i) - \left(\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right) = \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j (x_i - x_j, y_i - y_j). \tag{2.2.35}$$

Taking modulus in both parts of (2.2.35), and using the generalized triangle inequality, we obtain

$$\left| \sum_{i=1}^{n} p_i (x_i, y_i) - \left(\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i y_i \right) \right| \leqslant \frac{1}{2} \sum_{i,j=1}^{n} p_i p_j \left| (x_i - x_j, y_i - y_j) \right|. \tag{2.2.36}$$

By using Schwarz's inequality in inner product spaces we have

$$\left| (x_i - x_j, y_i - y_j) \right| \leqslant \|x_i - x_j\| \|y_i - y_j\|, \tag{2.2.37}$$

for $i,\ j\in\{1,\ldots,n\}$, and therefore

$$\left|\sum_{i=1}^{n}p_i(x_i,y_i)-\left(\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right)\right|\leqslant\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|x_i-x_j\|\|y_i-y_j\|. \tag{2.2.38}$$

Using the Cauchy-Schwarz inequality for double sums, we can state that

$$\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|x_i-x_j\|\|y_i-y_j\|$$

$$\leqslant\left(\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|x_i-x_j\|^2\right)^{\frac{1}{2}}\left(\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|y_i-y_j\|^2\right)^{\frac{1}{2}}, \tag{2.2.39}$$

and a simple calculation shows that

$$\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|x_i-x_j\|^2=\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2,$$

and

$$\frac{1}{2}\sum_{i,j=1}^{n}p_ip_j\|y_i-y_j\|^2=\sum_{i=1}^{n}p_i\,\|y_i\|^2-\left\|\sum_{i=1}^{n}p_iy_i\right\|^2.$$

We obtain

$$\left|\sum_{i=1}^{n}p_i(x_i,y_i)-\left(\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right)\right|$$

$$\leqslant\left(\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right)^{\frac{1}{2}}\left(\sum_{i=1}^{n}p_i\,\|y_i\|^2-\left\|\sum_{i=1}^{n}p_iy_i\right\|^2\right)^{\frac{1}{2}}. \tag{2.2.40}$$

Using Lemma 2.2.3, we know that

$$\left(\sum_{i=1}^{n}p_i\|x_i\|^2-\left\|\sum_{i=1}^{n}p_ix_i\right\|^2\right)^{\frac{1}{2}}\leqslant\frac{1}{2}\|X-x\|, \tag{2.2.41}$$

and

$$\left(\sum_{i=1}^{n}p_i\,\|y_i\|^2-\left\|\sum_{i=1}^{n}p_iy_i\right\|^2\right)^{\frac{1}{2}}\leqslant\frac{1}{2}\|Y-y\|. \tag{2.2.42}$$

Using (2.2.41) and (2.2.42) in (2.2.40), we get the desired inequality in (2.2.34).

To prove the sharpness of the constant $\frac{1}{4}$, let us assume that (2.2.34) holds with a constant $c>0$, i.e.,

$$\left|\sum_{i=1}^{n}p_i(x_i,y_i)-\left(\sum_{i=1}^{n}p_ix_i,\sum_{i=1}^{n}p_iy_i\right)\right|\leqslant c\|X-x\|\|Y-y\|, \tag{2.2.43}$$

under the above assumptions $p_i,\ x_i,\ y_i,\ x,\ X,\ y,\ Y$ and $n \geqslant 2$. If we choose $n = 2$, $x_1 = x$, $x_2 = X$, $y_1 = y$, $y_2 = Y$ $(x \neq X,\ y \neq Y)$ and $p_1 = p_2 = \frac{1}{2}$, then

$$\sum_{i=1}^{2} p_i(x_i, y_i) - \left(\sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i \right) = \frac{1}{2} \sum_{i,j=1}^{2} p_i p_j (x_i - x_j, y_i - y_j)$$

$$= \sum_{1 \leqslant i < j \leqslant 2} p_i p_j (x_i - x_j, y_i - y_j) = \frac{1}{4}(x - X, y - Y),$$

and then

$$\left| \sum_{i=1}^{2} p_i(x_i, y_i) - \left(\sum_{i=1}^{2} p_i x_i, \sum_{i=1}^{2} p_i y_i \right) \right| = \frac{1}{4} |(x - X, y - Y)|.$$

Choose $X - x = z$, $Y - y = z$, $z \neq 0$. Then using (2.2.43), we derive

$$\frac{1}{4}\|z\|^2 \leqslant c\|z\|^2, z \neq 0,$$

which implies that $c \geqslant \frac{1}{4}$, and the Theorem is proved.

2.3 Grüss-and Čebyšev-type inequalities in two and three variables

This section deals with some Grüss and Čebyšev-type inequalities established by Pachpatte in [89,91,122,129], involving functions of two and three independent variables.

Let $\Delta = [a,b] \times [c,d]$, $a,\ b,\ c,\ d \in \mathbb{R}$. The partial derivatives of a function $h(x,y)$ defined on Δ are denoted by $D_1 h(x,y) = \frac{\partial}{\partial x} h(x,y)$, $D_2 h(x,y) = \frac{\partial}{\partial y} h(x,y)$, $D_2 D_1 h(x,y) = \frac{\partial^2}{\partial y \partial x} h(x,y)$. We denote by $C(\Delta)$ the class of continuous functions $h : \Delta \to \mathbb{R}$ for which $D_1 h(x,y)$, $D_2 h(x,y)$, $D_2 D_1 h(x,y)$ exist and are continuous on Δ and belong to $L_\infty(\Delta)$. For any function $h(x,y) \in L_\infty(\Delta)$, we define $\|h\|_\infty = \sup_{(x,y)\in\Delta} |h(x,y)| < \infty$. For convenience, we introduce the following notation to simplify the details of presentation:

$$k = (b-a)(d-c),$$

$$H_1(x) = \left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2} \right)^2 \right],$$

$$H_2(y) = \left[\frac{1}{4}(d-c)^2 + \left(y - \frac{c+d}{2} \right)^2 \right],$$

$$F(x,y) = \left[(d-c) \int_a^b f(t,y)dt + (b-a) \int_c^d f(x,s)ds \right],$$

$$G(x,y) = \left[(d-c) \int_a^b g(t,y)dt + (b-a) \int_c^d g(x,s)ds \right],$$

$$A_0(x,y) = g(x,y)\int_a^b\int_c^d f(t,s)dsdt + f(x,y)\int_a^b\int_c^d g(t,s)dsdt,$$

$$A_1(x,y) = g(x,y)\int_a^b\int_c^d p(x,t)D_1f(t,s)dsdt + f(x,y)\int_a^b\int_c^d p(x,t)D_1g(t,s)dsdt,$$

$$A_2(x,y) = g(x,y)\int_a^b\int_c^d q(y,s)D_2f(t,s)dsdt + f(x,y)\int_a^b\int_c^d q(y,s)D_2g(t,s)dsdt,$$

$$A_3(x,y) = g(x,y)\int_a^b\int_c^d p(x,t)q(y,s)D_2D_1f(t,s)dsdt$$

$$+f(x,y)\int_a^b\int_c^d p(x,t)q(y,s)D_2D_1g(t,s)dsdt,$$

$$M_1(x,y) = |g(x,y)|\|D_1f\|_\infty + |f(x,y)|\|D_1g\|_\infty,$$

$$M_2(x,y) = |g(x,y)|\|D_2f\|_\infty + |f(x,y)|\|D_2g\|_\infty,$$

$$M_3(x,y) = |g(x,y)|\|D_2D_1f\|_\infty + |f(x,y)|\|D_2D_1g\|_\infty,$$

$$A(x,y) = \|D_1f\|_\infty(d-c)H_1(x) + \|D_2f\|_\infty(b-a)H_2(y) + \|D_2D_1f\|_\infty H_1(x)H_2(y),$$

$$B(x,y) = \|D_1g\|_\infty(d-c)H_1(x) + \|D_2g\|_\infty(b-a)H_2(y) + \|D_2D_1g\|_\infty H_1(x)H_2(y),$$

for some suitable functions $f,\ g$ defined on Δ, and $p:[a,b]^2\to\mathbb{R}$, $q:[c,d]^2\to\mathbb{R}$ are given by

$$p(x,t) = \begin{cases} t-a, & t\in[a,x] \\ t-b, & t\in(x,b] \end{cases}$$

$$q(y,s) = \begin{cases} s-c, & s\in[c,y] \\ s-d, & s\in(y,d] \end{cases}$$

and set

$$L[h(x,y)] = \int_a^b\int_c^d p(x,t)D_1h(t,s)dsdt$$

$$+\int_a^b\int_c^d q(y,s)D_2h(t,s)dsdt + \int_a^b\int_c^d p(x,t)q(y,s)D_2D_1h(t,s)dsdt,$$

$$M[h(x,y)] = \int_a^b\int_c^d p(x,t)q(y,s)D_2D_1h(t,s)dsdt,$$

for some suitable function h defined on Δ.

We begin with proving some auxiliary results.

Lemma 2.3.1 (see [37]). Let $h : \Delta \to \mathbb{R}$ be such that the partial derivatives $D_1 h(x,y)$, $D_2 h(x,y)$, $D_2 D_1 h(x,y)$ exist and are continuous on Δ. Then for all $(x,y) \in \Delta$, we have the representation

$$kh(x,y) - \int_a^b \int_c^d h(t,s)dsdt = L[h(x,y)]. \tag{2.3.1}$$

Proof. We use the following identity, which can be easily proved by integration by parts,

$$g(u) = \frac{1}{\beta - \alpha} \int_\alpha^\beta g(z)dz + \frac{1}{\beta - \alpha} \int_\alpha^\beta e(u,z)g'(z)dz, \tag{2.3.2}$$

where $e : [\alpha, \beta]^2 \to \mathbb{R}$ is given by

$$e(u,z) = \begin{cases} z - \alpha, & z \in [\alpha, u] \\ z - \beta, & z \in (u, \beta] \end{cases}$$

and g is absolutely continuous on $[\alpha, \beta]$. Now, write the identity (2.3.2) for the partial map $h(\cdot, y)$, $y \in [c,d]$, to obtain

$$h(x,y) = \frac{1}{b-a} \int_a^b h(t,y)dt + \frac{1}{b-a} \int_a^b p(x,t)D_1 h(t,y)dt, \tag{2.3.3}$$

for all $(x,y) \in \Delta$. Also, if we write (2.3.2) for the map $h(t, \cdot)$, we get

$$h(t,y) = \frac{1}{d-c} \int_c^d h(t,s)ds + \frac{1}{d-c} \int_c^d q(y,s)D_2 h(t,s)ds, \tag{2.3.4}$$

for all $(t,y) \in \Delta$. The same formula (2.3.2) applied for the partial derivative $D_1 h(\cdot, y)$ will produce

$$D_1 h(t,y) = \frac{1}{d-c} \int_c^d D_1 h(t,s)ds + \frac{1}{d-c} \int_c^d q(y,s)D_2 D_1 h(t,s)ds, \tag{2.3.5}$$

for all $(t,y) \in \Delta$. Substituting (2.3.4) and (2.3.5) in (2.3.3), and using the Fubini's theorem, we have

$$\begin{aligned} h(x,y) &= \frac{1}{b-a} \int_a^b \left[\frac{1}{d-c} \int_c^d h(t,s)ds + \frac{1}{d-c} \int_c^d q(y,s)D_2 h(t,s)ds \right] dt \\ &+ \frac{1}{b-a} \int_a^b p(x,t) \left[\frac{1}{d-c} \int_c^d D_1 h(t,s)ds + \frac{1}{d-c} \int_c^d q(y,s)D_2 D_1 h(t,s)ds \right] dt \\ &= \frac{1}{(b-a)(d-c)} \left[\int_a^b \int_c^d h(t,s)dsdt + \int_a^b \int_c^d q(y,s)D_2 h(t,s)dsdt \right. \\ &\left. + \int_a^b \int_c^d p(x,t)D_1 h(t,s)dsdt + \int_a^b \int_c^d p(x,t)q(y,s)D_2 D_1 h(t,s)dsdt \right]. \end{aligned} \tag{2.3.6}$$

Rewriting (2.3.6), we get the required identity in (2.3.1).

Lemma 2.3.2 (see [8]). Let $h:\Delta\to\mathbb{R}$ be a continuous mapping on Δ and $D_2D_1h(x,y)$ exists on $(a,b)\times(c,d)$. Then, we have the identity

$$kh(x,y)-H(x,y)=M\left[h(x,y)\right], \tag{2.3.7}$$

where

$$H(x,y)=(d-c)\int_a^b h(t,y)dt+(b-a)\int_c^d h(x,s)ds-\int_a^b\int_c^d h(t,s)dsdt.$$

Proof. Integrating by parts twice, we can state:

$$\int_a^x\int_c^y (t-a)(s-c)D_2D_1h(t,s)dsdt=(x-a)(y-c)h(x,y)$$

$$-(y-c)\int_a^x h(t,y)dt-(x-a)\int_c^y h(x,s)ds+\int_a^x\int_c^y h(t,s)dsdt, \tag{2.3.8}$$

$$\int_a^x\int_y^d (t-a)(s-d)D_2D_1h(t,s)dsdt=(x-a)(d-y)h(x,y)$$

$$-(d-y)\int_a^x h(t,y)dt-(x-a)\int_y^d h(x,s)ds+\int_a^x\int_y^d h(t,s)dsdt, \tag{2.3.9}$$

$$\int_x^b\int_y^d (t-b)(s-d)D_2D_1h(t,s)dsdt=(b-x)(d-y)h(x,y)$$

$$-(d-y)\int_x^b h(t,y)dt-(b-x)\int_y^d h(x,s)ds+\int_x^b\int_y^d h(t,s)dsdt, \tag{2.3.10}$$

$$\int_x^b\int_c^y (t-b)(s-c)D_2D_1h(t,s)dsdt=(b-x)(y-c)h(x,y)$$

$$-(y-c)\int_x^b h(t,y)dt-(b-x)\int_c^y h(x,s)ds+\int_x^b\int_c^y h(t,s)dsdt. \tag{2.3.11}$$

Adding (2.3.8)–(2.3.11) and rewriting, we easily deduce (2.3.7).

In the following theorems, we present the inequalities investigated in [89,122].

Theorem 2.3.1. Let $f,\ g\in C(\Delta)$. Then

$$|E(f,g)|\leqslant\frac{1}{2k^2}\int_a^b\int_c^d\left[M_1(x,y)(d-c)H_1(x)+M_2(x,y)(b-a)H_2(y)\right.$$

$$\left.+M_3(x,y)H_1(x)H_2(y)\right]dydx, \tag{2.3.12}$$

and

$$|E(f,g)|\leqslant\frac{1}{k^3}\int_a^b\int_c^d A(x,y)B(x,y)dydx, \tag{2.3.13}$$

where

$$E(f,g)=\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx$$

$$-\left(\frac{1}{k}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{k}\int_a^b\int_c^d g(x,y)dydx\right). \tag{2.3.14}$$

Proof. From the hypotheses, we have the following identities (see, Lemma 2.3.1):

$$kf(x,y) = \int_a^b \int_c^d f(t,s)dsdt + \int_a^b \int_c^d p(x,t)D_1 f(t,s)dsdt$$

$$+ \int_a^b \int_c^d q(y,s)D_2 f(t,s)dsdt + \int_a^b \int_c^d p(x,t)q(y,s)D_2 D_1 f(t,s)dsdt, \tag{2.3.15}$$

and

$$kg(x,y) = \int_a^b \int_c^d g(t,s)dsdt + \int_a^b \int_c^d p(x,t)D_1 g(t,s)dsdt$$

$$+ \int_a^b \int_c^d q(y,s)D_2 g(t,s)dsdt + \int_a^b \int_c^d p(x,t)q(y,s)D_2 D_1 g(t,s)dsdt, \tag{2.3.16}$$

for $(x,y) \in \Delta$. Multiplying (2.3.15) by $g(x,y)$, (2.3.16) by $f(x,y)$, and adding the resulting identities, we get

$$2kf(x,y)g(x,y) = A_0(x,y) + A_1(x,y) + A_2(x,y) + A_3(x,y). \tag{2.3.17}$$

Integrating (2.3.17) over Δ and rewriting, we get

$$E(f,g) = \frac{1}{2k^2} \int_a^b \int_c^d [A_1(x,y) + A_2(x,y) + A_3(x,y)]\, dydx. \tag{2.3.18}$$

It is easy to observe that

$$|A_1(x,y)| \leqslant M_1(x,y)(d-c)H_1(x), \tag{2.3.19}$$

$$|A_2(x,y)| \leqslant M_2(x,y)(b-a)H_2(y), \tag{2.3.20}$$

$$|A_3(x,y)| \leqslant M_3(x,y)H_1(x)H_2(y), \tag{2.3.21}$$

for $(x,y) \in \Delta$. From (2.3.18)–(2.3.21), we get

$$|E(f,g)| \leqslant \frac{1}{2k^2} \int_a^b \int_c^d [|A_1(x,y)| + |A_2(x,y)| + |A_3(x,y)|]\, dydx$$

$$\leqslant \frac{1}{2k^2} \int_a^b \int_c^d [M_1(x,y)(d-c)H_1(x) \; + M_2(x,y)(b-a)H_2(y) + M_3(x,y)H_1(x)H_2(y)]\, dydx.$$

This is the required inequality in (2.3.12).

The identities (2.3.15) and (2.3.16) can be rewritten as

$$kf(x,y) - \int_a^b \int_c^d f(t,s)dsdt = L[f(x,y)], \tag{2.3.22}$$

and

$$kg(x,y) - \int_a^b \int_c^d g(t,s)dsdt = L[g(x,y)], \tag{2.3.23}$$

for $(x,y) \in \Delta$. Multiplying the left hand sides and right hand sides of (2.3.22) and (2.3.23), we have

$$k^2 f(x,y)g(x,y) - kf(x,y)\int_a^b\int_c^d g(t,s)dsdt - kg(x,y)\int_a^b\int_c^d f(t,s)dsdt$$

$$+\left(\int_a^b\int_c^d f(t,s)dsdt\right)\left(\int_a^b\int_c^d g(t,s)dsdt\right) = L[f(x,y)]L[g(x,y)]. \tag{2.3.24}$$

Integrating (2.3.24) over Δ and rewriting, we get

$$E(f,g) = \frac{1}{k^3}\int_a^b\int_c^d L[f(x,y)]L[g(x,y)]dydx. \tag{2.3.25}$$

From (2.3.25) and using the properties of modulus, we get

$$|E(f,g)| \leqslant \frac{1}{k^3}\int_a^b\int_c^d |L[f(x,y)]||L[g(x,y)]|dydx. \tag{2.3.26}$$

It is easy to observe that

$$|L[f(x,y)]| \leqslant \int_a^b\int_c^d |p(x,t)||D_1 f(t,s)|dsdt$$

$$+\int_a^b\int_c^d |q(y,s)||D_2 f(t,s)|dsdt + \int_a^b\int_c^d |p(x,t)||q(y,s)||D_2D_1 f(t,s)|dsdt$$

$$\leqslant \|D_1 f\|_\infty \int_a^b\int_c^d |p(x,t)|dsdt$$

$$+\|D_2 f\|_\infty \int_a^b\int_c^d |q(y,s)|dsdt + \|D_2D_1 f\|_\infty \int_a^b\int_c^d |p(x,t)||q(y,s)|dsdt$$

$$= \|D_1 f\|_\infty (d-c)\int_a^b |p(x,t)|dt$$

$$+\|D_2 f\|_\infty (b-a)\int_c^d |q(y,s)|ds + \|D_2D_1 f\|_\infty \int_a^b\int_c^d |p(x,t)||q(y,s)|dsdt$$

$$= \|D_1 f\|_\infty (d-c)H_1(x) + \|D_2 f\|_\infty (b-a)H_2(y) + \|D_2D_1 f\|_\infty H_1(x)H_2(y)$$

$$= A(x,y). \tag{2.3.27}$$

Similarly, we have

$$|L[g(x,y)]| \leqslant B(x,y). \tag{2.3.28}$$

Using (2.3.27) and (2.3.28) in (2.3.26), we get the desired inequality in (2.3.13). The proof is complete.

Remark 2.3.1. From (2.3.17), (2.3.19)–(2.3.21), it is easy to obtain the inequality

$$|2kf(x,y)g(x,y)-A_0(x,y)| \leqslant M_1(x,y)(d-c)H_1(x)$$

$$+M_2(x,y)(b-a)H_2(y)+M_3(x,y)H_1(x)H_2(y), \tag{2.3.29}$$

for $(x,y) \in \Delta$ and from (2.3.24), (2.3.27), (2.3.28), it is easy to see that the following inequality

$$\left| f(x,y)g(x,y) - \frac{1}{k}\left[f(x,y)\int_a^b\int_c^d g(t,s)dsdt + g(x,y)\int_a^b\int_c^d f(t,s)dsdt \right.\right.$$

$$\left.\left. -\frac{1}{k}\left(\int_a^b\int_c^d f(t,s)dsdt\right)\left(\int_a^b\int_c^d g(t,s)dsdt\right)\right]\right|$$

$$\leqslant \frac{1}{k^2}A(x,y)B(x,y), \tag{2.3.30}$$

holds for $(x,y) \in \Delta$.

Theorem 2.3.2. Let $f,\ g \in C(\Delta)$. Then

$$\left| \frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx + \left(\frac{1}{k}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{k}\int_a^b\int_c^d g(x,y)dydx\right)\right.$$

$$\left. -\frac{1}{2k^2}\int_a^b\int_c^d [g(x,y)F(x,y)+f(x,y)G(x,y)]\,dydx \right|$$

$$\leqslant \frac{1}{2k^2}\int_a^b\int_c^d M_3(x,y)H_1(x)H_2(y)dydx, \tag{2.3.31}$$

and

$$\left| \frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx \right.$$

$$\left. -\frac{1}{k^2}\int_a^b\int_c^d \left[f(x,y)\overline{G}(x,y)+g(x,y)\overline{F}(x,y)-\frac{1}{k}\overline{F}(x,y)\overline{G}(x,y)\right]dydx \right|$$

$$\leqslant \frac{1}{k^3}\|D_2D_1f\|_\infty\|D_2D_1g\|_\infty\int_a^b\int_c^d [H_1(x)H_2(y)]^2\,dydx, \tag{2.3.32}$$

where

$$\overline{F}(x,y) = F(x,y) - \int_a^b\int_c^d f(t,s)dsdt, \overline{G}(x,y) = G(x,y) - \int_a^b\int_c^d g(t,s)dsdt.$$

Proof. From the hypotheses, we have the following identities (see, Lemma 2.3.2):

$$kf(x,y) = F(x,y) - \int_a^b \int_c^d f(t,s)dsdt + \int_a^b \int_c^d p(x,t)q(y,s)D_2D_1f(t,s)dsdt, \quad (2.3.33)$$

and

$$kg(x,y) = G(x,y) - \int_a^b \int_c^d g(t,s)dsdt + \int_a^b \int_c^d p(x,t)q(y,s)D_2D_1g(t,s)dsdt, \quad (2.3.34)$$

for $(x,y) \in \Delta$. Multiplying (2.3.33) by $g(x,y)$, (2.3.34) by $f(x,y)$, and adding the resulting identities, we get

$$2kf(x,y)g(x,y) = g(x,y)F(x,y) + f(x,y)G(x,y) - A_0(x,y) + A_3(x,y). \quad (2.3.35)$$

Integrating (2.3.35) over Δ and rewriting, we have

$$\int_a^b \int_c^d f(x,y)g(x,y)dydx = \frac{1}{2k} \int_a^b \int_c^d [g(x,y)F(x,y) + f(x,y)G(x,y)]\, dydx$$

$$-\frac{1}{k}\left(\int_a^b \int_c^d f(x,y)dydx\right)\left(\int_a^b \int_c^d g(x,y)dydx\right) + \frac{1}{2k}\int_a^b \int_c^d A_3(x,y)dydx. \quad (2.3.36)$$

We note that, here (2.3.21) holds for $(x,y) \in \Delta$. From (2.3.36) and (2.3.21), we observe that

$$\left|\frac{1}{k}\int_a^b \int_c^d f(x,y)g(x,y)dydx + \left(\frac{1}{k}\int_a^b \int_c^d f(x,y)dydx\right)\left(\frac{1}{k}\int_a^b \int_c^d g(x,y)dydx\right)\right.$$

$$\left. -\frac{1}{2k^2}\int_a^b \int_c^d [g(x,y)F(x,y) + f(x,y)G(x,y)]\, dydx\right|$$

$$\leqslant \frac{1}{2k^2}\int_a^b \int_c^d |A_3(x,y)|\, dydx$$

$$\leqslant \frac{1}{2k^2}\int_a^b \int_c^d M_3(x,y)H_1(x)H_2(y)dydx.$$

This is the required inequality in (2.3.31).

The identities (2.3.33) and (2.3.34) can be rewritten as

$$kf(x,y) - \overline{F}(x,y) = M[f(x,y)], \quad (2.3.37)$$

and

$$kg(x,y) - \overline{G}(x,y) = M[g(x,y)], \quad (2.3.38)$$

for $(x,y) \in \Delta$. Multiplying the left hand sides and right hand sides of (2.3.37) and (2.3.38), we have

$$k^2 f(x,y)g(x,y) - kf(x,y)\overline{G}(x,y) - kg(x,y)\overline{F}(x,y)$$

$$+\overline{F}(x,y)\overline{G}(x,y) = M[f(x,y)]M[g(x,y)]. \quad (2.3.39)$$

Rewriting (2.3.39) and integrating over Δ and using the properties of modulus, we have

$$\left|\frac{1}{k}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right.$$

$$\left.-\frac{1}{k^2}\int_a^b\int_c^d\left[f(x,y)\overline{G}(x,y)+g(x,y)\overline{F}(x,y)-\frac{1}{k}\overline{F}(x,y)\overline{G}(x,y)\right]dydx\right|$$

$$\leqslant \frac{1}{k^3}\int_a^b\int_c^d |M[f(x,y)]||M[g(x,y)]|dydx. \tag{2.3.40}$$

It is easy to observe that

$$|M[f(x,y)]| \leqslant \|D_2D_1f\|_\infty \int_a^b\int_c^d |p(x,t)||q(y,s)|dsdt = \|D_2D_1f\|_\infty H_1(x)H_2(y). \tag{2.3.41}$$

Similarly, we get

$$|M[g(x,y)]| \leqslant \|D_2D_1g\|_\infty H_1(x)H_2(y). \tag{2.3.42}$$

Using (2.3.41) and (2.3.42) in (2.3.40), we get the desired inequality in (2.3.32). The proof is complete.

Remark 2.3.2. From (2.3.35) and (2.3.21), it is easy to observe that the following inequality holds,

$$|2kf(x,y)g(x,y)+A_0(x,y)-[g(x,y)F(x,y)+f(x,y)G(x,y)]|$$

$$\leqslant M_3(x,y)H_1(x)H_2(y), \tag{2.3.43}$$

for $(x,y)\in\Delta$ and from (2.3.39), (2.3.41), (2.3.42), one can very easily obtain the following inequality

$$\left|f(x,y)g(x,y)-\frac{1}{k}\left[f(x,y)\overline{G}(x,y)+g(x,y)\overline{F}(x,y)-\frac{1}{k}\overline{F}(x,y)\overline{G}(x,y)\right]\right|$$

$$\leqslant \frac{1}{k^2}\|D_2D_1f\|_\infty\|D_2D_1g\|_\infty\left[H_1(x)H_2(y)\right]^2, \tag{2.3.44}$$

for $(x,y)\in\Delta$.

In our further discussion, the following notation will also be used to simplify the details of presentation.

Let $\Omega=[a,k]\times[b,m]\times[c,n], a,b,c,k,m,n\in\mathbb{R}$. The partial derivative $\frac{\partial^3}{\partial z\partial y\partial x}e(x,y,z)$ of a function e defined on Ω is denoted by $D_3D_2D_1e(x,y,z)$ and the function e is said to be bounded if $\|e\|_\infty=\sup_{(x,y,z)\in\Omega}|e(x,y,z)|<\infty$. For some suitable functions $h:\Delta\to\mathbb{R}$, $e:\Omega\to\mathbb{R}$, we set

$$A(D_2D_1h(x,y))=A[a,c;x,y;b,d;D_2D_1h(s,t)]$$

$$= \int_a^x \int_c^y D_2 D_1 h(t,s)dsdt - \int_a^x \int_y^d D_2 D_1 h(t,s)dsdt$$

$$- \int_x^b \int_c^y D_2 D_1 h(t,s)dsdt + \int_x^b \int_y^d D_2 D_1 h(t,s)dsdt,$$

$$E(h(x,y)) = E[a,c;x,y;b,d;h]$$

$$= \frac{1}{2}[h(x,c)+h(x,d)+h(a,y)+h(b,y)] - \frac{1}{4}\left[h(a,c)+h(a,d)+h(b,c)+h(b,d)\right],$$

$$B(D_3D_2D_1e(r,s,t)) = B[a,b,c;r,s,t;k,m,n;D_3D_2D_1e(u,v,w)]$$

$$= \int_a^r \int_b^s \int_c^t D_3D_2D_1e(u,v,w)dwdvdu - \int_a^r \int_b^s \int_t^n D_3D_2D_1e(u,v,w)dwdvdu$$

$$- \int_a^r \int_s^m \int_c^t D_3D_2D_1e(u,v,w)dwdvdu - \int_r^k \int_b^s \int_c^t D_3D_2D_1e(u,v,w)dwdvdu$$

$$+ \int_a^r \int_s^m \int_t^n D_3D_2D_1e(u,v,w)dwdvdu + \int_r^k \int_s^m \int_c^t D_3D_2D_1e(u,v,w)dwdvdu$$

$$+ \int_r^k \int_b^s \int_t^n D_3D_2D_1e(u,v,w)dwdvdu - \int_r^k \int_s^m \int_t^n D_3D_2D_1e(u,v,w)dwdvdu,$$

$$L(e(r,s,t)) = L[a,b,c;r,s,t;k,m,n;e]$$

$$= \frac{1}{8}[e(a,b,c)+e(k,m,n)]$$

$$-\frac{1}{4}[e(r,b,c)+e(r,m,n)+e(r,m,c)+e(r,b,n)]$$

$$-\frac{1}{4}[e(a,s,c)+e(k,s,n)+e(a,s,n)+e(k,s,c)]$$

$$-\frac{1}{4}[e(a,b,t)+e(k,m,t)+e(k,b,t)+e(a,m,t)]$$

$$+\frac{1}{2}[e(a,s,t)+e(k,s,t)] + \frac{1}{2}[e(r,b,t)+e(r,m,t)]$$

$$+\frac{1}{2}[e(r,s,c)+e(r,s,n)].$$

The Grüss- and Čebyšev-type inequalities established in [91,129] are given in the following theorems.

Theorem 2.3.3. Let $f,\ g : \Delta \to \mathbb{R}$ be continuous functions on Δ and $D_2D_1f(x,y)$, $D_2D_1g(x,y)$ exist, continuous and bounded on Δ. Then

$$\left|\int_a^b\int_c^d \left[f(x,y)g(x,y)-\frac{1}{2}\left[E(f(x,y))g(x,y)+E(g(x,y))f(x,y)\right]\right]dydx\right|$$

$$\leqslant \frac{1}{8}(b-a)(d-c)\int_a^b\int_c^d \left[|g(x,y)|\|D_2D_1f\|_\infty + |f(x,y)|\|D_2D_1g\|_\infty\right]dydx, \tag{2.3.45}$$

and

$$\Bigg|\int_a^b\int_c^d [f(x,y)g(x,y)-[E(f(x,y))g(x,y)+E(g(x,y))f(x,y)$$

$$-E(f(x,y))E(g(x,y))]]\,dydx\Bigg|$$

$$\leqslant \frac{1}{16}\{(b-a)(d-c)\}^2\|D_2D_1f\|_\infty\|D_2D_1g\|_\infty. \tag{2.3.46}$$

Proof. From the hypotheses, it is easy to observe that the following identities hold for $(x,y)\in\Delta$ (see [86,91]).

$$f(x,y) = -f(a,c)+f(x,c)+f(a,y)+\int_a^x\int_c^y D_2D_1f(t,s)dsdt,$$

$$f(x,y) = -f(a,d)+f(x,d)+f(a,y)-\int_a^x\int_y^d D_2D_1f(t,s)dsdt,$$

$$f(x,y) = -f(b,c)+f(x,c)+f(b,y)-\int_x^b\int_c^y D_2D_1f(t,s)dsdt,$$

$$f(x,y) = -f(b,d)+f(x,d)+f(b,y)+\int_x^b\int_y^d D_2D_1f(t,s)dsdt.$$

Adding the above identities and rewriting, we have

$$f(x,y)-E(f(x,y)) = \frac{1}{4}A(D_2D_1f(x,y)), \tag{2.3.47}$$

for $(x,y)\in\Delta$. Similarly, we have

$$g(x,y)-E(g(x,y)) = \frac{1}{4}A\left(D_2D_1g(x,y)\right), \tag{2.3.48}$$

for $(x,y)\in\Delta$. Multiplying (2.3.47) by $g(x,y)$ and (2.3.48) by $f(x,y)$ and adding the resulting identities, rewriting and then integrating over Δ, we have

$$\int_a^b\int_c^d \left[f(x,y)g(x,y)-\frac{1}{2}\left[E(f(x,y))g(x,y)+E(g(x,y))f(x,y)\right]\right]dydx$$

$$= \frac{1}{8}\int_a^b\int_c^d \left[A(D_2D_1f(x,y))g(x,y)+A\left(D_2D_1g(x,y)\right)f(x,y)\right]dydx. \tag{2.3.49}$$

From the properties of modulus and integrals, it is easy to see that

$$|A(D_2D_1f(x,y))| \leqslant \int_a^b\int_c^d |D_2D_1f(t,s)|dsdt \leqslant \|D_2D_1f\|_\infty (b-a)(d-c), \tag{2.3.50}$$

$$|A(D_2D_1g(x,y))| \leqslant \int_a^b\int_c^d |D_2D_1g(t,s)|dsdt \leqslant \|D_2D_1g\|_\infty (b-a)(d-c). \tag{2.3.51}$$

From (2.3.49)–(2.3.51), we observe that

$$\left|\int_a^b\int_c^d \left[f(x,y)g(x,y) - \frac{1}{2}\left[E(f(x,y))g(x,y) + E(g(x,y))f(x,y)\right]\right]dydx\right|$$
$$\leqslant \frac{1}{8}\int_a^b\int_c^d \left[|g(x,y)|\,|A(D_2D_1f(x,y))| \ + |f(x,y)|\,|A(D_2D_1g(x,y))|\right]dydx$$
$$\leqslant \frac{1}{8}\int_a^b\int_c^d \left[|g(x,y)|\int_a^b\int_c^d |D_2D_1f(t,s)|dsdt \ + |f(x,y)|\int_a^b\int_c^d |D_2D_1g(t,s)|\,dsdt\right]dydx$$
$$\leqslant \frac{1}{8}(b-a)(d-c)\int_a^b\int_c^d \left[|g(x,y)|\|D_2D_1f\|_\infty \ + |f(x,y)|\|D_2D_1g\|_\infty\right]dydx,$$

which is the required inequality in (2.3.45).

Multiplying the left hand sides and right hand sides of (2.3.47) and (2.3.48), we get

$$f(x,y)g(x,y) - [f(x,y)E(g(x,y)) + g(x,y)E(f(x,y)) - E(f(x,y))E(g(x,y))]$$
$$= \frac{1}{16}A(D_2D_1f(x,y))A(D_2D_1g(x,y)). \tag{2.3.52}$$

Integrating (2.3.52) over Δ and using the properties of modulus, we have

$$\left|\int_a^b\int_c^d [f(x,y)g(x,y) - [f(x,y)E(g(x,y)) + g(x,y)E(f(x,y)) - E(f(x,y))E(g(x,y))]]dydx\right|$$
$$\leqslant \frac{1}{16}\int_a^b\int_c^d |A(D_2D_1f(x,y))||A(D_2D_1g(x,y))|dydx. \tag{2.3.53}$$

Now, using (2.3.50) and (2.3.51) in (2.3.53), we get (2.3.46). The proof is complete.

Theorem 2.3.4. Let $f,\ g:\Omega\to\mathbb{R}$ be continuous functions on Ω and $D_3D_2D_1f(r,s,t)$, $D_3D_2D_1g(r,s,t)$, exist, continuous and bounded on Ω. Then

$$\left|\int_a^k\int_b^m\int_c^n \left[f(r,s,t)g(r,s,t) - \frac{1}{2}[L(f(r,s,t))g(r,s,t) + L(g(r,s,t))f(r,s,t)]\right]dtdsdr\right|$$
$$\leqslant \frac{1}{16}(k-a)(m-b)(n-c)\int_a^k\int_b^m\int_c^n [|g(r,s,t)|\|D_3D_2D_1f\|_\infty$$
$$+|f(r,s,t)|\|D_3D_2D_1g\|_\infty]\,dtdsdr, \tag{2.3.54}$$

and

$$\left|\int_a^k\int_b^m\int_c^n [f(r,s,t)g(r,s,t) - [L(f(r,s,t))g(r,s,t)\right.$$
$$\left.+L(g(r,s,t))f(r,s,t) - L(f(r,s,t))L(g(r,s,t))]]\,dtdsdr\right|$$
$$\leqslant \frac{1}{64}\left\{(k-a)(m-b)(n-c)\right\}^2\|D_3D_2D_1f\|_\infty\|D_3D_2D_1g\|_\infty. \tag{2.3.55}$$

Proof. From the hypotheses, it is easy to observe that the following identities hold (see [83,90]):

$$f(r,s,t) = f(a,b,c) + f(a,s,t) + f(r,s,c) + f(r,b,t)$$

$$-f(a,b,t) - f(a,s,c) - f(r,b,c) + \int_a^r \int_b^s \int_c^t D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(r,s,n) + f(a,s,t) + f(r,b,t) + f(a,b,n)$$

$$-f(a,b,t) - f(a,s,n) - f(r,b,n) - \int_a^r \int_b^s \int_t^n D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(r,m,t) + f(r,s,c) + f(a,m,c) + f(a,s,t)$$

$$-f(r,m,c) - f(a,m,t) - f(a,s,c) - \int_a^r \int_s^m \int_c^t D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(k,s,t) + f(k,b,c) + f(r,s,c) + f(r,b,t)$$

$$-f(k,s,c) - f(k,b,t) - f(r,b,c) - \int_r^k \int_b^s \int_c^t D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(r,m,t) + f(r,s,n) + f(a,m,n) + f(a,s,t)$$

$$-f(r,m,n) - f(a,m,t) - f(a,s,n) + \int_a^r \int_s^m \int_t^n D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(r,m,t) + f(r,s,c) + f(k,s,t) + f(k,m,c)$$

$$-f(k,m,t) - f(k,s,c) - f(r,m,c) + \int_r^k \int_s^m \int_c^t D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(k,s,t) + f(k,b,n) + f(r,s,n) + f(r,b,t)$$

$$-f(k,s,n) - f(k,b,t) - f(r,b,n) + \int_r^k \int_b^s \int_t^n D_3D_2D_1 f(u,v,w)dwdvdu,$$

$$f(r,s,t) = f(k,m,n) + f(k,s,t) + f(r,m,t) + f(r,s,n)$$

$$-f(k,m,t) - f(k,s,n) - f(r,m,n) - \int_r^k \int_s^m \int_t^n D_3D_2D_1 f(u,v,w)dwdvdu.$$

Adding the above identities and rewriting, we have

$$f(r,s,t) - L(f(r,s,t)) = \frac{1}{8} B(D_3D_2D_1 f(r,s,t)), \tag{2.3.56}$$

for $(r,s,t) \in \Omega$. Similarly, we have

$$g(r,s,t) - L(g(r,s,t)) = \frac{1}{8} B\left(D_3D_2D_1 g(r,s,t)\right), \tag{2.3.57}$$

for $(r,s,t) \in \Omega$. Multiplying (2.3.56) by $g(r,s,t)$ and (2.3.57) by $f(r,s,t)$ and adding the resulting identities, then integrating over Ω and rewriting, we have

$$\int_a^k \int_b^m \int_c^n \left[f(r,s,t)g(r,s,t) - \frac{1}{2}\left[f(r,s,t)L(g(r,s,t)) \ + g(r,s,t)L(f(r,s,t)) \right] \right] dtdsdr$$

$$= \frac{1}{16} \int_a^k \int_b^m \int_c^n \left[g(r,s,t)B(D_3D_2D_1f(r,s,t)) \ + f(r,s,t)B(D_3D_2D_1g(r,s,t)) \right] dtdsdr. \tag{2.3.58}$$

From the properties of modulus and integrals, we observe that

$$|B(D_3D_2D_1f(r,s,t))| \leqslant \int_a^k \int_b^m \int_c^n |D_3D_2D_1f(r,s,t)| dwdvdu$$

$$\leqslant \|D_3D_2D_1f\|_\infty (k-a)(m-b)(n-c). \tag{2.3.59}$$

Similarly, we get

$$|B(D_3D_2D_1g(r,s,t))| \leqslant \|D_3D_2D_1g\|_\infty (k-a)(m-b)(n-c). \tag{2.3.60}$$

Now, from (2.3.58)-(2.3.60) and following the same arguments as in the proof of inequality (2.3.45) with suitable changes, we get the required inequality in (2.5.54).
Multiplying the left hand sides and right hand sides of (2.3.56) and (2.3.57) and integrating over Ω, we get

$$\int_a^k \int_b^m \int_c^n [f(r,s,t)g(r,s,t) - [f(r,s,t)L(g(r,s,t))$$

$$+ g(r,s,t)L(f(r,s,t)) - L(f(r,s,t))L(g(r,s,t))]] \, dtdsdr$$

$$= \frac{1}{64} \int_a^k \int_b^m \int_c^n B(D_3D_2D_1f(r,s,t)) \, B(D_3D_2D_1g(r,s,t)) \, dtdsdr. \tag{2.3.61}$$

From (2.3.61), using the properties of modulus and (2.3.59) and (2.3.60), we get the desired inequality in (2.3.55). The proof is complete.

2.4 Trapezoid-type inequalities in two variables

In this section we present some Trapezoid-type inequalities involving functions of two independent variables, recently established by Dragomir, Barnett and Pearce [39], Barnett and Dragomir [6] and Pachpatte [86]. In our subsequent discussion, we make use of some of the notations and definitions given in Section 2.3 without further mention.
We start with the Trapezoid-type inequality established in [86].

Theorem 2.4.1. Let $f : \Delta \to \mathbb{R}$ be a continuous function on Δ, $D_2D_1f(x,y)$ exists and continuous on Δ. Then

$$\left|\int_a^b\int_c^d f(t,s)dsdt - \frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt \;+(b-a)\int_c^d [f(a,s)+f(b,s)]ds\right]\right.$$
$$\left. +\frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right|$$
$$\leqslant \frac{1}{4}(b-a)(d-c)\int_a^b\int_c^d |D_2D_1f(t,s)|dsdt. \tag{2.4.1}$$

Proof. By following the proof of Theorem 2.3.3, we have the following identity

$$f(x,y) - \frac{1}{2}[f(x,c)+f(x,d)+f(a,y)+f(b,y)] + \frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$
$$= \frac{1}{4}A(D_2D_1f(x,y)), \tag{2.4.2}$$

for $(x,y) \in \Delta$. Integrating both sides of (2.4.2) over Δ, we get

$$\int_a^b\int_c^d f(t,s)dsdt - \frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt \;+(b-a)\int_c^d [f(a,s)+f(b,s)]ds\right]$$
$$+\frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$
$$= \frac{1}{4}\int_a^b\int_c^d A(D_2D_1f(t,s))dsdt. \tag{2.4.3}$$

Using the properties of modulus and integrals, we observe that

$$|A(D_2D_1f(x,y))| \leqslant \int_a^b\int_c^d |D_2D_1f(t,s)|dsdt. \tag{2.4.4}$$

From (2.4.3) and (2.4.4), we have

$$\left|\int_a^b\int_c^d f(t,s)dsdt - \frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt + (b-a)\int_c^d [f(a,s)+f(b,s)]ds\right]\right.$$
$$\left. +\frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right|$$
$$\leqslant \frac{1}{4}\int_a^b\int_c^d |A(D_2D_1f(t,s))|\,dsdt$$
$$\leqslant \frac{1}{4}(b-a)(d-c)\int_a^b\int_c^d |D_2D_1f(t,s)|dsdt,$$

which is the required inequality in (2.4.1) and the proof is complete.

Remark 2.4.1. From (2.4.2) it is easy to observe that the following inequality holds

$$\left|f(x,y)-\frac{1}{2}[f(x,c)+f(x,d)+f(a,y)+f(b,y)]\ +\frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right|$$

$$\leqslant \frac{1}{4}\int_a^b\int_c^d |D_2D_1f(t,s)|dsdt, \tag{2.4.5}$$

for $(x,y)\in\Delta$.

The next Theorem deals with the Trapezoid-type inequality investigated in [39].

Theorem 2.4.2. Let $f:\Delta\to\mathbb{R}$ be a continuous mapping on Δ, $D_2D_1f(x,y)$ exists on $(a,b)\times(c,d)$ and is bounded, then

$$\Bigg|\int_a^b\int_c^d f(t,s)dsdt-\frac{1}{2}\Bigg[(d-c)\int_a^b[f(t,c)+f(t,d)]dt$$

$$+(b-a)\int_c^d[f(a,s)+f(b,s)]ds\Bigg]$$

$$+\frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\Bigg|$$

$$\leqslant \frac{1}{16}\left\{(b-a)(d-c)\right\}^2\|D_2D_1f\|_\infty. \tag{2.4.6}$$

Proof. From the hypotheses, the following identity holds (see, Lemma 2.3.2):

$$\int_a^b\int_c^d p(x,t)q(y,s)D_2D_1f(t,s)dsdt$$

$$=(d-c)(b-a)f(x,y)-(d-c)\int_a^b f(t,y)dt$$

$$-(b-a)\int_c^d f(x,s)ds+\int_a^b\int_c^d f(t,s)dsdt, \tag{2.4.7}$$

for all $(x,y)\in\Delta$, where $p(x,t),\ q(y,s)$ are as given in Section 2.3. In (2.4.7) choose (i) $x=a,\ y=c$; (ii) $x=b,\ y=c$; (iii) $x=a,\ y=d$; and (iv) $x=b,\ y=d$ to obtain the following identities:

$$\int_a^b\int_c^d p(a,t)q(c,s)D_2D_1f(t,s)dsdt$$

$$=(d-c)(b-a)f(a,c)-(d-c)\int_a^b f(t,c)dt$$

$$-(b-a)\int_c^d f(a,s)ds+\int_a^b\int_c^d f(t,s)dsdt, \tag{2.4.8}$$

$$\int_a^b \int_c^d p(b,t)q(c,s)D_2D_1f(t,s)dsdt$$

$$= (d-c)(b-a)f(b,c) - (d-c)\int_a^b f(t,c)dt$$

$$-(b-a)\int_c^d f(b,s)ds + \int_a^b \int_c^d f(t,s)dsdt, \tag{2.4.9}$$

$$\int_a^b \int_c^d p(a,t)q(d,s)D_2D_1f(t,s)dsdt$$

$$= (d-c)(b-a)f(a,d) - (d-c)\int_a^b f(t,d)dt$$

$$-(b-a)\int_c^d f(a,s)ds + \int_a^b \int_c^d f(t,s)dsdt, \tag{2.4.10}$$

and

$$\int_a^b \int_c^d p(b,t)q(d,s)D_2D_1f(t,s)dsdt$$

$$= (d-c)(b-a)f(b,d) - (d-c)\int_a^b f(t,d)dt$$

$$-(b-a)\int_c^d f(b,s)ds + \int_a^b \int_c^d f(t,s)dsdt, \tag{2.4.11}$$

Adding (2.4.8)–(2.4.11) and dividing by 4, we have

$$\frac{1}{4}\int_a^b \int_c^d (p(a,t)+p(b,t))\,(q(c,s)+q(d,s))\,D_2D_1f(t,s)dsdt$$

$$= \int_a^b \int_c^d f(t,s)dsdt + \frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$

$$-\frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt + (b-a)\int_c^d [f(a,s)+f(b,s)]ds\right],$$

and as

$$p(a,t)+p(b,t) = 2t-(a+b), q(c,s)+q(d,s) = 2s-(c+d),$$

then we get the identity:

$$\int_a^b \int_c^d f(t,s)dsdt - \frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt + (b-a)\int_c^d [f(a,s)+f(b,s)]ds\right]$$

$$+\frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$

$$= \int_a^b \int_c^d \left(t - \frac{a+b}{2}\right)\left(s - \frac{c+d}{2}\right) D_2 D_1 f(t,s) ds dt. \tag{2.4.12}$$

Now, using the identity (2.4.12) and the properties of the integral, we get

$$\left| \int_a^b \int_c^d f(t,s) ds dt - \frac{1}{2}\left[(d-c)\int_a^b [f(t,c)+f(t,d)]dt + (b-a)\int_c^d [f(a,s)+f(b,s)]ds\right]\right.$$

$$\left. + \frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]\right|$$

$$\leqslant \int_a^b \int_c^d \left|t - \frac{a+b}{2}\right| \left|s - \frac{c+d}{2}\right| |D_2 D_1 f(t,s)| ds dt$$

$$\leqslant \frac{1}{16}\{(b-a)(d-c)\}^2 \|D_2 D_1 f\|_\infty.$$

Since a simple calculation gives,

$$\int_a^b \left|t - \frac{a+b}{2}\right| dt = \frac{(b-a)^2}{4}, \int_c^d \left|s - \frac{c+d}{2}\right| dt = \frac{(d-c)^2}{4},$$

the inequality (2.4.6) is thus obtained. The prof is complete.

In order to prove the next two Theorems we need the following integral identity proved in [6].

Lemma 2.4.1. Let $f : \Delta \to \mathbb{R}$ be a continuous mapping on Δ such that $D_2 f(a,\cdot), D_2 f(b,\cdot)$ are continuous on $[c,d]$, $D_1 f(\cdot,c), D_1 f(\cdot,d)$ are continuous on $[a,b]$ and $D_2 D_1 f(\cdot,\cdot)$ is continuous on Δ. Then we have the identity:

$$\int_a^b \int_c^d f(x,y) dy dx + (b-a)\int_c^d \left(y - \frac{c+d}{2}\right) f_2(y) dy + (d-c)\int_a^b \left(x - \frac{a+b}{2}\right) f_1(x) dx$$

$$= \frac{1}{4}(b-a)(d-c)[f(a,c)+f(a,d)+f(b,c)+f(b,d)]$$

$$+ \int_a^b \int_c^d \left(x - \frac{a+b}{2}\right)\left(y - \frac{c+d}{2}\right) D_2 D_1 f(x,y) dy dx, \tag{2.4.13}$$

where

$$f_1(x) = \frac{1}{2}[D_1 f(x,c) + D_1 f(x,d)], \tag{2.4.14}$$

for $x \in [a,b]$ and

$$f_2(y) = \frac{1}{2}[D_2 f(a,y) + D_2 f(b,y)], \tag{2.4.15}$$

for $y \in [c,d]$.

Proof. A simple integration by parts gives

$$\int_{\alpha}^{\beta} h(x)dx = \frac{h(\alpha)+h(\beta)}{2}(\beta-\alpha) - \int_{\alpha}^{\beta}\left(x-\frac{\alpha+\beta}{2}\right)h'(x)dx, \tag{2.4.16}$$

provided that $h : [\alpha,\beta] \to \mathbb{R}$ is absolutely continuous on $[\alpha,\beta]$. Using (2.4.15), we can write:

$$\int_a^b f(x,y)dx = (b-a)\frac{f(a,y)+f(b,y)}{2} - \int_a^b \left(x-\frac{a+b}{2}\right)D_1 f(x,y)dx, \tag{2.4.17}$$

for all $y \in [c,d]$. Integrating (2.4.17) over the interval $[c,d]$, we obtain

$$\int_c^d\left(\int_a^b f(x,y)dx\right)dy = \frac{1}{2}(b-a)\left[\int_c^d f(a,y)dy + \int_c^d f(b,y)dy\right]$$

$$-\int_c^d\left(\int_a^b\left(x-\frac{a+b}{2}\right)D_1 f(x,y)dx\right)dy.$$

Using x’s theorem, we can state:

$$\int_a^b\int_c^d f(x,y)dydx = \frac{1}{2}(b-a)\left[\int_c^d f(a,y)dy+ \int_c^d f(b,y)dy\right]$$

$$-\int_a^b\left(x-\frac{a+b}{2}\right)\left(\int_c^d D_1 f(x,y)dy\right)dx. \tag{2.4.18}$$

By the identity (2.4.16), we can also state:

$$\int_c^d f(a,y)dy = \frac{1}{2}[f(a,c)+f(a,d)](d-c) - \int_c^d\left(y-\frac{c+d}{2}\right)D_2 f(a,y)dy, \tag{2.4.19}$$

$$\int_c^d f(b,y)dy = \frac{1}{2}[f(b,c)+f(b,d)](d-c) - \int_c^d\left(y-\frac{c+d}{2}\right)D_2 f(b,y)dy, \tag{2.4.20}$$

and

$$\int_c^d D_1 f(x,y)dy = \frac{1}{2}[D_1 f(x,c)+D_1 f(x,d)](d-c)$$

$$-\int_c^d\left(y-\frac{c+d}{2}\right)D_2 D_1 f(x,y)dy. \tag{2.4.21}$$

Now, using (2.4.18) and (2.4.19)–(2.4.21), we have successively

$$\int_a^b\int_c^d f(x,y)dydx = \frac{1}{2}(b-a)\left[\frac{1}{2}[f(a,c)+f(a,d)](d-c) - \int_c^d\left(y-\frac{c+d}{2}\right)D_2 f(a,y)dy\right.$$

$$\left.+\frac{1}{2}[f(b,c)+f(b,d)](d-c) - \int_c^d\left(y-\frac{c+d}{2}\right)D_2 f(b,y)dy\right]$$

$$-\int_a^b\left(x-\frac{a+b}{2}\right)\left[\frac{1}{2}[D_1 f(x,c)+D_1 f(x,d)](d-c)\right.$$

$$-\int_c^d \left(y-\frac{c+d}{2}\right) D_2D_1f(x,y)dy\Big] dx$$

$$=\frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(d-c)$$

$$-\frac{1}{2}(b-a)\int_c^d \left(y-\frac{c+d}{2}\right) D_2f(a,y)dy-\frac{1}{2}(b-a)\int_c^d \left(y-\frac{c+d}{2}\right) D_2f(b,y)dy$$

$$-\frac{1}{2}(d-c)\int_a^b \left(x-\frac{a+b}{2}\right) D_1f(x,c)dx-\frac{1}{2}(d-c)\int_a^b \left(x-\frac{a+b}{2}\right) D_1f(x,d)dx$$

$$+\int_a^b\int_c^d \left(x-\frac{a+b}{2}\right)\left(y-\frac{c+d}{2}\right) D_2D_1f(x,y)dydx. \tag{2.4.22}$$

Rewriting (2.4.22), we get the desired identity in (2.4.13).

The Trapezoid-type inequalities given in [6] are embodied in the following theorems.

Theorem 2.4.3. Let $f,\ f_1,\ f_2$ be as in Lemma 2.4.1 and assume that $D_2D_1f(x,y)$ is bounded. Then

$$\Big|\int_a^b\int_c^d f(x,y)dydx+(b-a)\int_c^d \left(y-\frac{c+d}{2}\right) f_2(y)dy+(d-c)\int_a^b \left(x-\frac{a+b}{2}\right) f_1(x)dx$$

$$-\frac{1}{4}\ [f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(b-c)\Big|$$

$$\leqslant \frac{1}{16}\{(b-a)(d-c)\}^2\|D_2D_1f\|_\infty, \tag{2.4.23}$$

where $f_1(x)$ and $f_2(y)$ are given by (2.4.14) and (2.4.15).

Proof. Using the identity (2.4.13) and the properties of integral, we have

$$\Big|\int_a^b\int_c^d f(x,y)dydx+(b-a)\int_c^d \left(y-\frac{c+d}{2}\right) f_2(y)dy+(d-c)\int_a^b \left(x-\frac{a+b}{2}\right) f_1(x)dx$$

$$-\frac{1}{4}\ [f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(b-c)\Big|$$

$$\leqslant \int_a^b\int_c^d \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right| |D_2D_1f(x,y)|\,dydx$$

$$\leqslant \|D_2D_1f\|_\infty\int_a^b \left|x-\frac{a+b}{2}\right| dx\int_c^d \left|y-\frac{c+d}{2}\right| dy$$

$$=\|D_2D_1f\|_\infty\frac{(b-a)^2}{4}\frac{(d-c)^2}{4},$$

and the inequality (2.4.23) is proved.

Theorem 2.4.4. Let f, f_1, f_2 be as in Theorem 2.4.3 and assume that

$$\|f_1\|_\infty = \sup_{x\in[a,b]} |f_1(x)| < \infty, \quad \|f_2\|_\infty = \sup_{y\in[c,d]} |f_2(y)| < \infty.$$

Then

$$\left|\int_a^b\int_c^d f(x,y)dydx - \frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(b-c)\right|$$
$$\leqslant \frac{1}{4}(b-a)(d-c)\left[(b-a)\|f_1\|_\infty + (d-c)\|f_2\|_\infty + \frac{1}{4}(b-a)(d-c)\|D_2D_1f\|_\infty\right]. \quad (2.4.24)$$

Proof. As in the proof of Theorem 2.4.3, we have, by the identity (2.4.13) that

$$\left|\int_a^b\int_c^d f(x,y)dydx - \frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)](b-a)(b-c)\right|$$
$$\leqslant (b-a)\int_c^d \left|y-\frac{c+d}{2}\right| |f_2(y)|dy + (d-c)\int_a^b \left|x-\frac{a+b}{2}\right| |f_1(x)|dx$$
$$+\int_a^b\int_c^d \left|x-\frac{a+b}{2}\right|\left|y-\frac{c+d}{2}\right| |D_2D_1f(x,y)|\,dydx$$
$$\leqslant (b-a)\|f_2\|_\infty\frac{(d-c)^2}{4} + (d-c)\|f_1\|_\infty\frac{(b-a)^2}{4} + \|D_2D_1f\|_\infty\frac{(b-a)^2}{4}\frac{(d-c)^2}{4}$$
$$= \frac{1}{4}(b-a)(d-c)\left[(b-a)\|f_1\|_\infty + (d-c)\|f_2\|_\infty + \frac{1}{4}(b-a)(d-c)\|D_2D_1f\|_\infty\right].$$

Hence the proof is completed.

Remark 2.4.2. We note that, one can very easily obtain bounds on the right hand sides in (2.4.23) and (2.4.24) for $\|\cdot\|_p$ norm, $p \in [1,\infty)$. Here, we do not discuss the details.

2.5 Some multivariate Grüss-type integral inequalities

Our main goal in this section is to present some multivariate Grüss-type integral inequalities recently investigated by Pachpatte in [94,130].

Let $B = \prod_{i=1}^n [a_i, b_i]$ be a bounded domain in $\mathbb{R}^n$, the n-dimensional Euclidean space. For $x_i \in \mathbb{R}$, $x = (x_1,\dots,x_n)$ is a variable point in B and $dx = dx_1\cdots dx_n$. For any integrable function $u(x) : B \to \mathbb{R}$ we denote by $\int_B u(x)dx$ the n-fold integral $\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} u(x_1,\dots,x_n)dx_1\cdots dx_n$. For integrable functions $f,\ g : B \to \mathbb{R}$ on B and $p : B \to \mathbb{R}_+$ an integrable function on B such that $P = \int_B p(x)dx > 0$, we set

$$T(P,p,f,g;B) = \frac{1}{P}\int_B p(x)f(x)g(x)dx - \left(\frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx\right), \quad (2.5.1)$$

and assume that the integrals involved in (2.5.1) exist.

In the following theorems, we present the integral inequalities investigated in [130].

Theorem 2.5.1. Let $f,\ g : B \to \mathbb{R}$ be integrable functions on B and $p : B \to \mathbb{R}_+$ an integrable function on B such that $P = \int_B p(x)dx > 0$. Then

$$|T(P,p,f,g;B)| \leqslant \sqrt{T(P,p,f,f;B)}\sqrt{T(P,p,g,g;B)}, \tag{2.5.2}$$

and in addition if $\phi \leqslant f(x) \leqslant \Phi$, $\gamma \leqslant g(x) \leqslant \Gamma$ for each $x \in B$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are given real constants, then

$$|T(P,p,f,g;B)| \leqslant \frac{1}{4}(\Phi-\phi)(\Gamma-\gamma). \tag{2.5.3}$$

Proof. By direct computation, it is easy to observe that the following version of the Korkine's identity [79, p. 242] holds:

$$T(P,p,f,g;B) = \frac{1}{2P^2}\int_B\int_B p(x)p(y)(f(x)-f(y))(g(x)-g(y))dydx. \tag{2.5.4}$$

From (2.5.4), it is easy to observe that

$$T(P,p,f,f;B) = \frac{1}{P}\int_B p(x)f^2(x)dx - \left(\frac{1}{P}\int_B p(x)f(x)dx\right)^2. \tag{2.5.5}$$

Furthermore, by using the multivariate version of the Schwarz integral inequality, it is easy to observe that $T(P,p,f,f;B) \geqslant 0$. Similarly, we have

$$T(P,p,g,g;B) = \frac{1}{P}\int_B p(x)g^2(x)dx - \left(\frac{1}{P}\int_B p(x)g(x)dx\right)^2, \tag{2.5.6}$$

and $T(P,p,g,g;B) \geqslant 0$. From (2.5.4) and using the multivariate version of the Schwarz integral inequality, we have

$$\begin{aligned}
|T(P,p,f,g;B)|^2 &= \left\{\frac{1}{2P^2}\int_B\int_B p(x)p(y)(f(x)-f(y))(g(x)-g(y))dydx\right\}^2 \\
&\leqslant \left\{\frac{1}{2P^2}\int_B\int_B p(x)p(y)(f(x)-f(y))^2dydx\right\}\left\{\frac{1}{2P^2}\int_B\int_B p(x)p(y)(g(x)-g(y))^2dydx\right\} \\
&= \left\{\frac{1}{P}\int_B p(x)f^2(x)dx - \left(\frac{1}{P}\int_B p(x)f(x)dx\right)^2\right\} \\
&\quad \times \left\{\frac{1}{P}\int_B p(x)g^2(x)dx - \left(\frac{1}{P}\int_B p(x)g(x)dx\right)^2\right\} \\
&= T(P,p,f,f;B)T(P,p,g,g;B).
\end{aligned} \tag{2.5.7}$$

The desired inequality in (2.5.2) follows from (2.5.7).

It is easy to observe that the following identity also holds:

$$T(P,p,f,f;B) = \left(\Phi - \frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)f(x)dx - \phi\right)$$

$$-\frac{1}{P}\int_B p(x)\,(\Phi - f(x))\,(f(x)-\phi)\,dx. \tag{2.5.8}$$

Using the fact that $(\Phi - f(x))\,(f(x)-\phi) \geqslant 0$ in (2.5.8), we have

$$T(P,p,f,f;B) \leqslant \left(\Phi - \frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx - \phi\right). \tag{2.5.9}$$

Similarly, we have

$$T(P,p,g,g;B) \leqslant \left(\Gamma - \frac{1}{P}\int_B p(x)g(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx - \gamma\right). \tag{2.5.10}$$

Using (2.5.9) and (2.5.10) in (2.5.7), we get

$$|T(P,p,f,g;B)|^2 \leqslant \left(\Phi - \frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)f(x)dx - \phi\right)$$

$$\times\left(\Gamma - \frac{1}{P}\int_B p(x)g(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx - \gamma\right). \tag{2.5.11}$$

By using the elementary inequality $cd \leqslant \left(\frac{c+d}{2}\right)^2$; $c,\ d \in \mathbb{R}$, we observe that

$$\left(\Phi - \frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)f(x)dx - \phi\right) \leqslant \left(\frac{\Phi-\phi}{2}\right)^2, \tag{2.5.12}$$

$$\left(\Gamma - \frac{1}{P}\int_B p(x)g(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx - \gamma\right) \leqslant \left(\frac{\Gamma-\gamma}{2}\right)^2. \tag{2.5.13}$$

The required inequality in (2.5.3) follows from (2.5.11)–(2.5.13).

Remark 2.5.1. We note that the inequality established in (2.5.3) can be considered as a weighted multivariate generalization of the Grüss inequality. In the special case when $n=1$, from (2.5.3), we get the generalization of the Grüss inequality (3) given by Dragomir in [42].

Theorem 2.5.2. Let $f,\ g : B \to \mathbb{R}$ be integrable functions on B and $p : B \to \mathbb{R}_+$ an integrable function on B such that $P = \int_B p(x)dx > 0$. Then

$$|T(P,p,f,g;B)|$$

$$\leqslant \frac{1}{P}\int_B p(x)\left|\left(f(x) - \frac{1}{P}\int_B p(y)f(y)dy\right)\left(g(x) - \frac{1}{P}\int_B p(y)g(y)dy\right)\right|dx, \tag{2.5.14}$$

and in addition if $\phi \leqslant f(x) \leqslant \Phi$ for each $x \in B$, where $\phi,\ \Phi$ are real constants, then

$$|T(P,p,f,g;B)| \leqslant \frac{\Phi-\phi}{2}\sqrt{T(P,p,g,g;B)}. \tag{2.5.15}$$

Proof. In order to establish the inequality (2.5.14), we observe that

$$\frac{1}{P}\int_B p(x)\left(f(x)-\frac{1}{P}\int_B p(y)f(y)dy\right)\left(g(x)-\frac{1}{P}\int_B p(y)g(y)dy\right)dx$$

$$=\frac{1}{P}\int_B p(x)\left\{f(x)g(x)-\frac{1}{P}f(x)\int_B p(y)g(y)dy\right.$$

$$\left.-\frac{1}{P}g(x)\int_B p(y)f(y)dy+\frac{1}{P^2}\left(\int_B p(y)f(y)dy\right)\left(\int_B p(y)g(y)dy\right)\right\}dx$$

$$=\frac{1}{P}\int_B p(x)f(x)g(x)dx-\left(\frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx\right)$$

$$-\left(\frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx\right)+\frac{1}{P}\frac{1}{P^2}P\left(\int_B p(x)f(x)dx\right)\left(\int_B p(x)g(x)dx\right)$$

$$=\frac{1}{P}\int_B p(x)f(x)g(x)dx-\left(\frac{1}{P}\int_B p(x)f(x)dx\right)\left(\frac{1}{P}\int_B p(x)g(x)dx\right)$$

$$=T(P,p,f,g;B). \tag{2.5.16}$$

From (2.5.16) and using the properties of modulus, we get the desired inequality in (2.5.14). Following the proof of Theorem 2.5.1, we have $T(P,p,f,f;B)\geqslant 0$, $T(P,p,g,g;B)\geqslant 0$ and (2.5.7), (2.5.9), (2.5.12) hold. From (2.5.9) and (2.5.12), we get

$$T(P,p,f,f;B)\leqslant\frac{1}{4}(\Phi-\phi)^2. \tag{2.5.17}$$

The required inequality in (2.5.15) follows from (2.5.7) and (2.5.17). The proof is complete.

Remark 2.5.2. By taking $p(x)=1$ and hence $P=\prod_{i=1}^n(b_i-a_i)$ in (2.5.14), we get

$$|T(P,1,f,g;B)|\leqslant\frac{1}{P}\int_B\left|\left(f(x)-\frac{1}{P}\int_B f(y)dy\right)\left(g(x)-\frac{1}{P}\int_B g(y)dy\right)\right|dx. \tag{2.5.18}$$

The inequality (2.5.18) can be considered as the multivariate version of the integral inequality of the Grüss-type given by Dragomir and McAndrew in [34]. We note that the inequality (2.5.15) can be considered as a multivariate version of the Pre-Grüss inequality given by Matić, Pečarić and Ujević in [72].

Before giving the next result, we introduce the following notation used to simplify the details of presentation.

Let $D_i[a_i,b_i]=\{x_i : a_i<x_i<b_i\}$ for $i=1,\ldots,n$; $a_i,\ b_i\in\mathbb{R}$, $D=\prod_{i=1}^n D_i[a_i,b_i]$ and $\overline{D}$ be the closure of D. For any function $u:\overline{D}\to\mathbb{R}$, differentiable on D, we denote the first order partial derivatives by $\frac{\partial u(x)}{\partial x_i}$ and by $\int_D u(x)dx$ the n-fold integral. If

$$\left\|\frac{\partial u}{\partial x_i}\right\|_\infty=\sup_{x\in D}\left|\frac{\partial u(x)}{\partial x_i}\right|<\infty,$$

then we say that the partial derivatives $\frac{\partial u(x)}{\partial x_i}$ are bounded. For continuous functions p, q defined on $\overline{D}$ and differentiable on D and $w(x)$ a real-valued, nonnegative and integrable function for $x \in D$ and $\int_D w(x)dx > 0$ and $x_i,\ y_i \in D_i[a_i, b_i]$, we set

$$A(w,p,q;D) = \int_D w(x)p(x)q(x)dx$$

$$-\frac{1}{\int_D w(x)dx}\left(\int_D w(x)p(x)dx\right)\left(\int_D w(x)q(x)dx\right), \tag{2.5.19}$$

$$H(p,x_i,y_i) = \sum_{i=1}^{n}\left\|\frac{\partial p}{\partial x_i}\right\|_\infty |x_i - y_i|, \tag{2.5.20}$$

and assume that the integrals involved in (2.5.19) exist.

The Grüss-type integral inequalities established in [94] are given in the following theorem.

Theorem 2.5.3. Let $f(x),\ g(x)$ be real-valued continuous functions on $\overline{D}$ and differentiable on D, with partial derivatives $\frac{\partial f(x)}{\partial x_i}$, $\frac{\partial g(x)}{\partial x_i}$ being bounded. Let $w(x)$ be a real-valued, nonnegative and integrable function for $x \in D$ and $\int_D w(x)dx > 0$. Then

$$|A(w,f,g;D)| \leqslant \frac{1}{2\int_D w(x)dx}\int_D w(x)\left[|g(x)|\int_D H(f,x_i,y_i)w(y)dy\right.$$

$$\left. +|f(x)|\int_D H(g,x_i,y_i)w(y)dy\right]dx, \tag{2.5.21}$$

$$|A(w,f,g;D)| \leqslant \frac{1}{(\int_D w(x)dx)^2}\int_D w(x)\left(\int_D H(f,x_i,y_i)w(y)dy\right)$$

$$\times\left(\int_D H(g,x_i,y_i)w(y)dy\right)dx. \tag{2.5.22}$$

Proof. Let $x = (x_1,\dots,x_n) \in \overline{D}$, $y = (y_1,\dots,y_n) \in D$. From the n-dimensional version of the mean value theorem, we have (see [146, p.174])

$$f(x) - f(y) = \sum_{i=1}^{n}\frac{\partial f(c_1)}{\partial x_i}(x_i - y_i), \tag{2.5.23}$$

and

$$g(x) - g(y) = \sum_{i=1}^{n}\frac{\partial g(c_2)}{\partial x_i}(x_i - y_i), \tag{2.5.24}$$

where $c_i = (y_1 + \alpha_i(x_1 - y_1),\dots,y_n + \alpha_i(x_n - y_n))$ $(0 < \alpha_i < 1)$ for $i = 1,2$. Multiplying both sides of (2.5.23) and (2.5.24) by $g(x)$ and $f(x)$ respectively and adding, we get

$$2f(x)g(x) - g(x)f(y) - f(x)g(y)$$

$$= g(x) \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i) + f(x) \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i). \tag{2.5.25}$$

Multiplying both sides of (2.5.25) by $w(y)$ and integrating the resulting identity with respect to y over D, we have

$$2\left(\int_D w(y)dy\right) f(x)g(x) - g(x)\int_D w(y)f(y)dy - f(x)\int_D w(y)g(y)dy$$

$$= g(x)\int_D \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i)w(y)dy + f(x)\int_D \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i)w(y)dy. \tag{2.5.26}$$

Next, multiplying both sides of (2.5.26) by $w(x)$ and integrating the resulting identity with respect to x over D, we get

$$2\left(\int_D w(y)dy\right)\int_D w(x)f(x)g(x)dx$$

$$-\left(\int_D w(x)g(x)dx\right)\left(\int_D w(y)f(y)dy\right) - \left(\int_D w(x)f(x)dx\right)\left(\int_D w(y)g(y)dy\right)$$

$$= \int_D w(x)g(x)\left(\int_D \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i)w(y)dy\right)dx$$

$$+ \int_D w(x)f(x)\left(\int_D \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i)w(y)dy\right)dx. \tag{2.5.27}$$

Rewriting (2.5.27), using the notation given in (2.5.19) and the properties of modulus, we have

$$|A(w,f,g;D)| \leqslant \frac{1}{2\int_D w(x)dx}\left[\int_D w(x)|g(x)|\left(\int_D \sum_{i=1}^{n} \left|\frac{\partial f(c_1)}{\partial x_i}\right| |x_i - y_i| w(y)dy\right)dx\right.$$

$$\left. + \int_D w(x)|f(x)|\left(\int_D \sum_{i=1}^{n} \left|\frac{\partial g(c_2)}{\partial x_i}\right| |x_i - y_i| w(y)dy\right)dx\right]$$

$$\leqslant \frac{1}{2\int_D w(x)dx}\left[\int_D w(x)\left[|g(x)|\int_D \sum_{i=1}^{n} \left\|\frac{\partial f}{\partial x_i}\right\|_\infty |x_i - y_i| w(y)dy\right.\right.$$

$$\left.\left. + |f(x)|\int_D \sum_{i=1}^{n} \left\|\frac{\partial g}{\partial x_i}\right\|_\infty |x_i - y_i| w(y)dy\right]dx\right.$$

$$= \frac{1}{2\int_D w(x)dx}\int_D w(x)\left[|g(x)|\int_D H(f,x_i,y_i)w(y)dy + |f(x)|\int_D H(g,x_i,y_i)w(y)dy\right]dx.$$

This is the required inequality in (2.5.21).

Multiplying both sides of (2.5.23) and (2.5.24) by $w(y)$ and integrating the resulting identities with respect to y over D, we get

$$\left(\int_D w(y)dy\right) f(x) - \int_D w(y)f(y)dy = \int_D \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i)w(y)dy, \tag{2.5.28}$$

and

$$\left(\int_D w(y)dy\right) g(x) - \int_D w(y)g(y)dy = \int_D \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i)w(y)dy. \tag{2.5.29}$$

Multiplying the left hand sides and right hand sides of (2.5.28) and (2.5.29), we get

$$\begin{aligned}
&\left(\int_D w(y)dy\right)^2 f(x)g(x) - \left(\int_D w(y)dy\right) f(x) \int_D w(y)g(y)dy \\
&- \left(\int_D w(y)dy\right) g(x) \int_D w(y)f(y)dy + \left(\int_D w(y)f(y)dy\right)\left(\int_D w(y)g(y)dy\right) \\
&= \left(\int_D \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i)w(y)dy\right)\left(\int_D \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i)w(y)dy\right).
\end{aligned} \tag{2.5.30}$$

Multiplying both sides of (2.5.30) by $w(x)$ and integrating the resulting identity with respect to x over D and rewriting, we obtain

$$\begin{aligned}
A(w,f,g;D) = \frac{1}{\left(\int_D w(y)dy\right)^2} \int_D w(x) &\left(\int_D \sum_{i=1}^{n} \frac{\partial f(c_1)}{\partial x_i}(x_i - y_i)w(y)dy\right) \\
&\times \left(\int_D \sum_{i=1}^{n} \frac{\partial g(c_2)}{\partial x_i}(x_i - y_i)w(y)dy\right) dx.
\end{aligned} \tag{2.5.31}$$

From (2.5.31) and following the proof of inequality (2.5.21) with suitable modifications, we get the required inequality in (2.5.22). The proof is complete.

Remark 2.5.3. If we take $n = 1$ and $D = I = \{x : a < x < b\}$ in (2.5.21), then we get

$$\begin{aligned}
&\left| \int_a^b w(t)f(t)g(t)dt - \frac{1}{\int_a^b w(t)dt}\left(\int_a^b w(t)f(t)dt\right)\left(\int_a^b w(t)g(t)dt\right) \right| \\
&\leqslant \frac{1}{2\int_a^b w(t)dt} \int_a^b w(t) \left[|g(t)| \int_a^b \|f'\|_\infty |t-s| w(s)ds \right. \\
&\qquad \left. + |f(t)| \int_a^b \|g'\|_\infty |t-s| w(s)ds \right] dt.
\end{aligned} \tag{2.5.32}$$

Similarly, one can obtain the special version of (2.5.22). It is easy to see that the bound obtained on the right hand side in (2.5.32) (when $w(t) = 1$) is different from those given by Grüss in [61].

2.6 Multivariate Grüss-and Čebyšev-type discrete inequalities

In this section, we deal with certain multivariate Grüss- and Čebyšev-type discrete inequalities established by Pachpatte in [95,103,129].

Let $N_1 = \{1,2,\ldots,k+1\}$, $N_2 = \{1,2,\ldots,m+1\}$, $N_3 = \{1,2,\ldots,n+1\}$ for $k,m,n \in N$ and denote by $G = N_1 \times N_2$, $H = N_1 \times N_2 \times N_3$. For functions $h(x,y)$ and $e(x,y,z)$ defined respectively on G and H we define the operators $\Delta_1 h(x,y) = h(x+1,y) - h(x,y)$, $\Delta_2 h(x,y) = h(x,y+1) - h(x,y)$, $\Delta_2\Delta_1 h(x,y) = \Delta_2(\Delta_1 h(x,y))$ and $\Delta_1 e(x,y,z) = e(x+1,y,z) - e(x,y,z)$, $\Delta_2 e(x,y,z) = e(x,y+1,z) - e(x,y,z)$, $\Delta_3 e(x,y,z) = e(x,y,z+1) - e(x,y,z)$, $\Delta_2\Delta_1 e(x,y,z) = \Delta_2(\Delta_1 e(x,y,z))$, $\Delta_3\Delta_2\Delta_1 e(x,y,z) = \Delta_3(\Delta_2\Delta_1 e(x,y,z))$. Let $N_i[0,a_i] = \{0,1,2,\ldots,a_i\}$, $a_i \in N$, $i = 1,\ldots,n$ and $Q = \prod_{i=1}^{n} N_i[0,a_i]$. For a function $u(x) : Q \to \mathbb{R}$ we define the first order difference operators as $\Delta_1 u(x) = u(x_1+1,x_2,\ldots,x_n) - u(x), \ldots,$ $\Delta_n u(x) = u(x_1,\ldots,x_{n-1},x_n+1) - u(x)$ and denote the n-fold sum over Q with respect to the variable $y = (y_1,\ldots,y_n) \in Q$ by

$$\sum_y u(y) = \sum_{y_1=0}^{a_1-1} \cdots \sum_{y_n=0}^{a_n-1} u(y_1,\ldots,y_n).$$

Clearly $\sum_y u(y) = \sum_x u(x)$ for $x,\ y \in Q$. The notation

$$\sum_{t_i=y_i}^{x_i-1} \Delta_i u(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)$$

for $x_i,\ y_i \in \mathbb{N}_i[0,a_i]$; $i = 1,\ldots,n$, we mean for $i = 1$, it is $\sum_{t_1=y_1}^{x_1-1} \Delta_1 u(t_1,x_2,\ldots,x_n)$ and so on and for $i = n$, it is $\sum_{t_n=y_n}^{x_n-1} \Delta_n u(y_1,\ldots,y_{n-1},t_n)$. The functions $h(x,y),\ e(x,y,z)$ and $u(x)$ defined on G,H and Q respectively are said to be bounded if $\|h\|_\infty = \sup_{(x,y)\in G} |h(x,y)| < \infty$, $\|e\|_\infty = \sup_{(x,y,z)\in H} |e(x,y,z)| < \infty$ and $\|u\|_\infty = \sup_{x\in Q} |u(x)| < \infty$. We use the usual convention that the empty sum is taken to be zero. We give the following notation used to simplify the details of presentation:

$$A(\Delta_2\Delta_1 h(x,y)) = A[1,1;x,y;k,m;\Delta_2\Delta_1 h(s,t)]$$

$$= \sum_{s=1}^{x-1}\sum_{t=1}^{y-1} \Delta_2\Delta_1 h(s,t) - \sum_{s=1}^{x-1}\sum_{t=y}^{m} \Delta_2\Delta_1 h(s,t) - \sum_{s=x}^{k}\sum_{t=1}^{y-1} \Delta_2\Delta_1 h(s,t) + \sum_{s=x}^{k}\sum_{t=y}^{m} \Delta_2\Delta_1 h(s,t),$$

$$E(h(x,y)) = E[1,1;x,y;k+1,m+1;h]$$

$$= \frac{1}{2}\left[h(x,1) + h(x,m+1) + h(1,y) + h(k+1,y)\right]$$

$$-\frac{1}{4}\left[h(1,1) + h(1,m+1) + h(k+1,1) + h(k+1,m+1)\right],$$

$$B\left(\Delta_3\Delta_2\Delta_1 e(r,s,t)\right) = B\left[1,1,1;r,s,t;k,m,n;\Delta_3\Delta_2\Delta_1 e(u,v,w)\right]$$

$$= \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 e(u,v,w) - \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 e(u,v,w)$$

$$- \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 e(u,v,w) - \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 e(u,v,w)$$

$$+ \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 e(u,v,w) + \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 e(u,v,w)$$

$$+ \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 e(u,v,w) - \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 e(u,v,w),$$

$$L(e(r,s,t)) = L[1,1,1;r,s,t;k+1,m+1,n+1;e]$$

$$= \frac{1}{8}[e(1,1,1)+e(k+1,m+1,n+1)]$$

$$-\frac{1}{4}[e(r,1,1)+e(r,1,n+1)+e(r,m+1,1)+e(r,m+1,n+1)]$$

$$-\frac{1}{4}[e(k+1,s,n+1)+e(k+1,s,1)+e(1,s,n+1)+e(1,s,1)]$$

$$-\frac{1}{4}[e(k+1,m+1,t)+e(k+1,1,t)+e(1,m+1,t)+e(1,1,t)]$$

$$+\frac{1}{2}[e(1,s,t)+e(k+1,s,t)]+\frac{1}{2}[e(r,1,t)+e(r,m+1,t)]$$

$$+\frac{1}{2}[e(r,s,1)+e(r,s,n+1)],$$

and for $x \in Q$,

$$S(f,g,M;Q) = \frac{1}{M}\sum_{x} f(x)g(x) - \left(\frac{1}{M}\sum_{x} f(x)\right)\left(\frac{1}{M}\sum_{x} g(x)\right),$$

for some suitable functions $f,\ g$ and a constant M.

In the following Theorems we present the inequalities of the Grüss-and Čebyšev-type established in [103,129].

Theorem 2.6.1. Let $f,\ g : G \to \mathbb{R}$ be functions such that $\Delta_2\Delta_1 f(x,y)$, $\Delta_2\Delta_1 g(x,y)$ exist and bounded on G. Then

$$\left|\sum_{x=1}^{k}\sum_{y=1}^{m}\left[f(x,y)g(x,y)-\frac{1}{2}\left\{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))\right\}\right]\right|$$

$$\leqslant \frac{1}{8}km\sum_{x=1}^{k}\sum_{y=1}^{m}\left[|g(x,y)|\,\|\Delta_2\Delta_1 f\|_\infty+|f(x,y)|\|\Delta_2\Delta_1 g\|_\infty\right], \tag{2.6.1}$$

$$\left|\sum_{x=1}^{k}\sum_{y=1}^{m}[f(x,y)g(x,y)-\ \{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))\ \ -E(f(x,y))E(g(x,y))\}]\right|$$

$$\leqslant \frac{1}{16}(km)^2\,\|\Delta_2\Delta_1 f\|_\infty\,\|\Delta_2\Delta_1 g\|_\infty. \tag{2.6.2}$$

Proof. For $(x,y)\in G$ it is easy to observe that the following identities hold (see [86]):

$$f(x,y)=f(x,1)+f(1,y)-f(1,1)+\sum_{s=1}^{x-1}\sum_{t=1}^{y-1}\Delta_2\Delta_1 f(s,t),$$

$$f(x,y)=f(x,m+1)+f(1,y)-f(1,m+1)-\sum_{s=1}^{x-1}\sum_{t=y}^{m}\Delta_2\Delta_1 f(s,t),$$

$$f(x,y)=f(x,1)+f(k+1,y)-f(k+1,1)-\sum_{s=x}^{k}\sum_{t=1}^{y-1}\Delta_2\Delta_1 f(s,t),$$

$$f(x,y)=f(x,m+1)+f(k+1,y)-f(k+1,m+1)+\sum_{s=x}^{k}\sum_{t=y}^{m}\Delta_2\Delta_1 f(s,t).$$

Adding the above identities and rewriting, we have

$$f(x,y)-E(f(x,y))=\frac{1}{4}A\left(\Delta_2\Delta_1 f(x,y)\right), \tag{2.6.3}$$

for $(x,y)\in G$. Similarly, we have

$$g(x,y)-E(g(x,y))=\frac{1}{4}A(\Delta_2\Delta_1 g(x,y)), \tag{2.6.4}$$

for $(x,y)\in G$. Multiplying (2.6.3) by $g(x,y)$ and (2.6.4) by $f(x,y)$, adding the resulting identities, rewriting and then summing over G, we have

$$\sum_{x=1}^{k}\sum_{y=1}^{m}\left[f(x,y)g(x,y)-\frac{1}{2}\left\{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))\right\}\right]$$

$$=\frac{1}{8}\sum_{x=1}^{k}\sum_{y=1}^{m}\left[g(x,y)A\left(\Delta_2\Delta_1 f(x,y)\right)+f(x,y)A(\Delta_2\Delta_1 g(x,y))\right]. \tag{2.6.5}$$

From the properties of modulus and sums, it is easy to see that

$$|A(\Delta_2\Delta_1 f(x,y))| \leqslant \sum_{s=1}^{k}\sum_{t=1}^{m}|\Delta_2\Delta_1 f(s,t)| \leqslant \|\Delta_2\Delta_1 f\|_\infty (km), \tag{2.6.6}$$

$$|A(\Delta_2\Delta_1 g(x,y))| \leqslant \sum_{s=1}^{k}\sum_{t=1}^{m}|\Delta_2\Delta_1 g(s,t)| \leqslant \|\Delta_2\Delta_1 g\|_\infty (km). \tag{2.6.7}$$

From (2.6.5)–(2.6.7), we observe that

$$\begin{aligned}
&\left|\sum_{x=1}^{k}\sum_{y=1}^{m}\left[f(x,y)g(x,y)-\frac{1}{2}\left\{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))\right\}\right]\right| \\
&\leqslant \frac{1}{8}\sum_{x=1}^{k}\sum_{y=1}^{m}\left[|g(x,y)|\,|A(\Delta_2\Delta_1 f(x,y))|+|f(x,y)|\,|A(\Delta_2\Delta_1 g(x,y))|\right] \\
&\leqslant \frac{1}{8}km\sum_{x=1}^{k}\sum_{y=1}^{m}\left[|g(x,y)|\,\|\Delta_2\Delta_1 f\|_\infty+|f(x,y)|\|\Delta_2\Delta_1 g\|_\infty\right],
\end{aligned}$$

which is the required inequality in (2.6.1).

Multiplying the left hand sides and right hand sides of (2.6.3) and (2.6.4), we get

$$\begin{aligned}
&f(x,y)g(x,y)-\{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))-E(f(x,y))E(g(x,y))\} \\
&\qquad = \frac{1}{16}A(\Delta_2\Delta_1 f(x,y))A(\Delta_2\Delta_1 g(x,y)).
\end{aligned} \tag{2.6.8}$$

Summing both sides of (2.6.8) over G and using the properties of modulus, we have

$$\begin{aligned}
&\left|\sum_{x=1}^{k}\sum_{y=1}^{m}[f(x,y)g(x,y)-\{g(x,y)E(f(x,y))+f(x,y)E(g(x,y))-E(f(x,y))E(g(x,y))\}]\right| \\
&\qquad \leqslant \frac{1}{16}\sum_{x=1}^{k}\sum_{y=1}^{m}|A(\Delta_2\Delta_1 f(x,y))|\,|A(\Delta_2\Delta_1 g(x,y))|.
\end{aligned} \tag{2.6.9}$$

Now, using (2.6.6) and (2.6.7) in (2.6.9), we get (2.6.2). The proof is complete.

Theorem 2.6.2. Let $f,\ g : H \to \mathbb{R}$ be functions such that $\Delta_3\Delta_2\Delta_1 f(r,s,t)$, $\Delta_3\Delta_2\Delta_1 g(r,s,t)$ exist and bounded on H. Then

$$\begin{aligned}
&\left|\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\left[f(r,s,t)g(r,s,t)-\frac{1}{2}\left\{f(r,s,t)L(g(r,s,t))+g(r,s,t)L(f(r,s,t))\right\}\right]\right| \\
&\leqslant \frac{1}{16}kmn\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}\left[|g(r,s,t)|\|\Delta_3\Delta_2\Delta_1 f\|_\infty+|f(r,s,t)|\|\Delta_3\Delta_2\Delta_1 g\|_\infty\right],
\end{aligned} \tag{2.6.10}$$

$$\begin{aligned}
&\left|\sum_{r=1}^{k}\sum_{s=1}^{m}\sum_{t=1}^{n}[f(r,s,t)g(r,s,t)-\{f(r,s,t)L(g(r,s,t))\right. \\
&\qquad \left.+g(r,s,t)L(f(r,s,t))-L(f(r,s,t))L(g(r,s,t))\}]\right| \\
&\qquad \leqslant \frac{1}{64}(kmn)^2\|\Delta_3\Delta_2\Delta_1 f\|_\infty\,\|\Delta_3\Delta_2\Delta_1 g\|_\infty.
\end{aligned} \tag{2.6.11}$$

Proof. For $(r,s,t) \in H$, it is easy to observe that the following identities hold (see [83]):

$$f(r,s,t) = f(1,1,1) + f(1,s,t) + f(r,1,t) + f(r,s,1)$$
$$-f(1,1,t) - f(1,s,1) - f(r,1,1) + \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(1,1,n+1) + f(r,s,n+1) + f(1,s,t) + f(r,1,t)$$
$$-f(1,s,n+1) - f(r,1,n+1) - f(1,1,t) - \sum_{u=1}^{r-1}\sum_{v=1}^{s-1}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(r,m+1,t) + f(1,s,t) + f(1,m+1,1) + f(r,s,1)$$
$$-f(1,m+1,t) - f(r,m+1,1) - f(1,s,1) - \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(k+1,s,t) + f(r,1,t) + f(r,s,1) + f(k+1,1,1)$$
$$-f(k+1,1,t) - f(k+1,s,t) - f(r,1,1) - \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(r,m+1,t) + f(r,s,n+1) + f(r,m+1,n+1) + f(1,s,t)$$
$$-f(r,m+1,n+1) - f(1,m+1,t) - f(1,s,n+1) + \sum_{u=1}^{r-1}\sum_{v=s}^{m}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(r,m+1,t) + f(r,s,1) + f(k+1,s,t) + f(k+1,m,1)$$
$$-f(k+1,m+1,t) - f(k+1,s,1) - f(r,m+1,1) + \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=1}^{t-1} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(k+1,s,t) + f(k+1,1,n+1) + f(r,s,n+1) + f(r,1,t)$$
$$-f(k+1,1,n+1) - f(k+1,1,t) - f(r,1,n+1) + \sum_{u=r}^{k}\sum_{v=1}^{s-1}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 f(u,v,w),$$
$$f(r,s,t) = f(k+1,m+1,n+1) + f(k+1,s,t) + f(r,m+1,t) + f(r,s,n+1)$$
$$-f(k+1,m+1,t) - f(k+1,s,n+1) - f(r,m+1,n+1) - \sum_{u=r}^{k}\sum_{v=s}^{m}\sum_{w=t}^{n} \Delta_3\Delta_2\Delta_1 f(u,v,w).$$

Adding the above identities and rewriting, we have

$$f(r,s,t) - L(f(r,s,t)) = \frac{1}{8}B\left(\Delta_3\Delta_2\Delta_1 f(r,s,t)\right), \tag{2.6.12}$$

for $(r,s,t) \in H$. Similarly, we have

$$g(r,s,t) - L(g(r,s,t)) = \frac{1}{8}B\left(\Delta_3\Delta_2\Delta_1 g(r,s,t)\right), \tag{2.6.13}$$

for $(r,s,t) \in H$. From the properties of modulus and sums, we observe that

$$\left|B\left(\Delta_3\Delta_2\Delta_1 f(r,s,t)\right)\right| \leqslant \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} \left|\Delta_3\Delta_2\Delta_1 f(u,v,w)\right| \leqslant \|\Delta_3\Delta_2\Delta_1 f\|_\infty (kmn), \tag{2.6.14}$$

$$\left|B\left(\Delta_3\Delta_2\Delta_1 g(r,s,t)\right)\right| \leqslant \sum_{u=1}^{k}\sum_{v=1}^{m}\sum_{w=1}^{n} \left|\Delta_3\Delta_2\Delta_1 g(u,v,w)\right| \leqslant \|\Delta_3\Delta_2\Delta_1 g\|_\infty (kmn). \tag{2.6.15}$$

The rest of the proofs of (2.6.10) and (2.6.11) can be completed by closely looking at the proofs of (2.6.1) and (2.6.2) given in Theorem 2.6.1 with suitable modifications. We omit the further details.

The inequalities in the following Theorem are proved in [95].

Theorem 2.6.3. Let $f,\ g : Q \to \mathbb{R}$ be functions such that $\Delta_i f(x)$, $\Delta_i g(x)$ for $i = 1, \ldots, n$ exist and bounded on Q. Then

$$|S(f,g,M;Q)| \leqslant \frac{1}{2M^2} \sum_x \left[\sum_{i=1}^n \left[|g(x)| \|\Delta_i f\|_\infty + |f(x)| \|\Delta_i g\|_\infty \right] H_i(x) \right], \tag{2.6.16}$$

$$|S(f,g,M;Q)| \leqslant \frac{1}{2M^2} \sum_x \left[\sum_y \left[\sum_{i=1}^n \left[\|\Delta_i f\|_\infty \right] |x_i - y_i| \right] \left[\sum_{i=1}^n \left[\|\Delta_i g\|_\infty \right] |x_i - y_i| \right] \right], \tag{2.6.17}$$

where $M = \prod_{i=1}^n a_i$ and $H_i(x) = \sum_y |x_i - y_i|$.

Proof. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in Q, it is easy to observe that the following identities hold:

$$f(x) - f(y) = \sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i f(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\}, \tag{2.6.18}$$

$$g(x) - g(y) = \sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i g(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\}. \tag{2.6.19}$$

Multiplying both sides of (2.6.18) and (2.6.19) by $g(x)$ and $f(x)$ respectively and adding, we get

$$\begin{aligned} &2f(x)g(x) - g(x)f(y) - f(x)g(y) \\ &= g(x) \sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i f(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\} \\ &+ f(x) \sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i g(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\}. \end{aligned} \tag{2.6.20}$$

Summing both sides of (2.6.20) with respect to y over Q, using the fact that $M > 0$ and rewriting, we have

$$\begin{aligned} &f(x)g(x) - \frac{1}{2M} g(x) \sum_y f(y) - \frac{1}{2M} f(x) \sum_y g(y) \\ &= \frac{1}{2M} \left[g(x) \sum_y \left[\sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i f(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\} \right] \right. \\ &\left. + f(x) \sum_y \left[\sum_{i=1}^n \left\{ \sum_{t_i = y_i}^{x_i - 1} \Delta_i g(y_1, \ldots, y_{i-1}, t_i, x_{i+1}, \ldots, x_n) \right\} \right] \right]. \end{aligned} \tag{2.6.21}$$

Summing both sides of (2.6.21) with respect to x over Q, rewriting and using the properties of modulus, we have

$$\begin{aligned}
|S(f,g,M;Q)| &\leqslant \frac{1}{2M^2}\sum_x \left[|g(x)|\sum_y \left[\sum_{i=1}^n \left|\left\{\sum_{t_i=y_i}^{x_i-1} |\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right]\right. \\
&\qquad \left. +|f(x)|\sum_y \left[\sum_{i=1}^n \left|\left\{\sum_{t_i=y_i}^{x_i-1} |\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right]\right]. \\
&\leqslant \frac{1}{2M^2}\sum_x \left[|g(x)|\sum_y \left[\sum_{i=1}^n \|\Delta_i f\|_\infty \left|\sum_{t_i=y_i}^{x_i-1} 1\right|\right] +|f(x)|\sum_y \left[\sum_{i=1}^n \|\Delta_i g\|_\infty \left|\sum_{t_i=y_i}^{x_i-1} 1\right|\right]\right] \\
&= \frac{1}{2M^2}\sum_x \left[|g(x)|\sum_{i=1}^n \|\Delta_i f\|_\infty \sum_y |x_i-y_i| +|f(x)|\sum_{i=1}^n \|\Delta_i g\|_\infty \sum_y |x_i-y_i|\right] \\
&= \frac{1}{2M^2}\sum_x \left[\sum_{i=1}^n \left[|g(x)|\|\Delta_i f\|_\infty +|f(x)|\|\Delta_i g\|_\infty\right] H_i(x)\right],
\end{aligned}$$

which proves the inequality (2.6.16).

Multiplying the left hand sides and right hand sides of (2.6.18) and (2.6.19), we get

$$\begin{aligned}
& f(x)g(x)-g(x)f(y)-f(x)g(y)+f(y)g(y) \\
&= \left[\sum_{i=1}^n \left\{\sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right] \\
&\times \left[\sum_{i=1}^n \left\{\sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right].
\end{aligned} \tag{2.6.22}$$

Summing both sides of (2.6.22) with respect to y over Q and rewriting, we have

$$\begin{aligned}
& f(x)g(x)-\frac{1}{M}g(x)\sum_y f(y)-\frac{1}{M}f(x)\sum_y g(y)+\frac{1}{M}\sum_y f(y)g(y) \\
&= \frac{1}{M}\sum_y \left[\sum_{i=1}^n \left\{\sum_{t_i=y_i}^{x_i-1} \Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right] \\
&\times \left[\sum_{i=1}^n \left\{\sum_{t_i=y_i}^{x_i-1} \Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)\right\}\right].
\end{aligned} \tag{2.6.23}$$

Summing both sides of (2.6.23) with respect to x over Q, rewriting and using the properties of modulus, we have

$$\begin{aligned}
|S(f,g,M;Q)| &\leqslant \frac{1}{2M^2}\sum_x \left[\sum_y \left[\sum_{i=1}^n \left|\left\{\sum_{t_i=y_i}^{x_i-1} |\Delta_i f(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right]\right. \\
&\qquad \left.\times \left[\sum_{i=1}^n \left|\left\{\sum_{t_i=y_i}^{x_i-1} |\Delta_i g(y_1,\ldots,y_{i-1},t_i,x_{i+1},\ldots,x_n)|\right\}\right|\right]\right] \\
&\leqslant \frac{1}{2M^2}\sum_x \left[\sum_y \left[\sum_{i=1}^n \left[\|\Delta_i f\|_\infty\right]|x_i-y_i|\right]\left[\sum_{i=1}^n \left[\|\Delta_i g\|_\infty\right]|x_i-y_i|\right]\right],
\end{aligned}$$

which is the required inequality in (2.6.17). The proof is complete.

2.7 Applications

One of the main motivations for investigating different types of inequalities given in earlier sections was to apply them as tools in various applications. In this section we give applications of some of the inequalities and it is hoped that these inequalities will provide a fruitful source for future research.

2.7.1 *Some integral inequalities via Grüss inequality in inner product spaces*

In this subsection, we present some integral versions of Theorem 2.2.1 given by Dragomir and Gomm [54], that have potential for applications.
Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote $L^2_\rho(\Omega, \mathbb{K})$ ($\mathbb{K} = \mathbb{R}$, $\mathbb{C}$) the Hilbert space of all measurable functions $f : \Omega \to \mathbb{K}$ that are $2-\rho$-integrable on Ω, i.e., $\int_\Omega \rho(s)|f(s)|^2 d\mu(s) < \infty$, where $\rho : \Omega \to [0, \infty)$ is a given measurable function on Ω. The inner product $(\cdot, \cdot) : L^2_\rho(\Omega, \mathbb{K}) \times L^2_\rho(\Omega, \mathbb{K}) \to \mathbb{K}$ that generates the norm of $L^2_\rho(\Omega, \mathbb{K})$ is

$$(f, g)_\rho := \int_\Omega f(s)\overline{g(s)}\rho(s)d\mu(s). \tag{2.7.1}$$

The following proposition holds.

Proposition 2.7.1. Let $\phi,\ \gamma,\ \Phi,\ \Gamma \in \mathbb{K}$ and $h,\ f,\ g \in L^2_\rho(\Omega, \mathbb{K})$ be such that

$$\operatorname{Re}\left[(\Phi h(x) - f(x))\left(\overline{f(x)} - \overline{\phi}\,\overline{h(x)}\right)\right] \geqslant 0, \tag{2.7.2(a)}$$

$$\operatorname{Re}\left[(\Gamma h(x) - g(x))\left(\overline{g(x)} - \overline{\gamma}\,\overline{h(x)}\right)\right] \geqslant 0, \tag{2.7.2(b)}$$

for a.e. $x \in \Omega$ and

$$\int_\Omega |h(x)|^2 \rho(x)d\mu(x) = 1. \tag{2.7.3}$$

Then one has the inequality

$$\left|\int_\Omega \rho(x)f(x)\overline{g(x)}d\mu(x) - \left(\int_\Omega \rho(x)f(x)\overline{h(x)}d\mu(x)\right)\left(\int_\Omega \rho(x)h(x)\overline{g(x)}d\mu(x)\right)\right|$$

$$\leqslant \frac{1}{4}|\Phi - \phi||\Gamma - \gamma|, \tag{2.7.4}$$

and the constant $\frac{1}{4}$ in (2.7.4) is sharp.

Proof. Follows from Theorem 2.2.1 applied for the inner product (2.7.1), on taking into account that

$$\operatorname{Re}(\Phi h - f, f - \phi h)_\rho = \int_\Omega \rho(x) \operatorname{Re}\left[(\Phi h(x) - f(x))\left(\overline{f(x)} - \overline{\phi}\overline{h(x)}\right)\right] d\mu(x) \geqslant 0$$

and

$$\operatorname{Re}(\Gamma h - g, g - \gamma h)_\rho = \int_\Omega \rho(x) \operatorname{Re}\left[(\Gamma h(x) - g(x))\left(\overline{g(x)} - \overline{\gamma}\overline{h(x)}\right)\right] d\mu(x) \geqslant 0.$$

The details are omitted.

The following result may be stated as well:

Corollary 2.7.1. If $z,\ Z,\ t,\ T \in \mathbb{K},\ \rho \in L(\Omega, \mathbb{R})$ with $\int_\Omega \rho(x) d\mu(x) > 0$ and $f,\ g \in L^2_\rho(\Omega, \mathbb{K})$ are such that

$$\operatorname{Re}\left[(Z - f(x))\left(\overline{f(x)} - \overline{z}\right)\right] \geqslant 0, \tag{2.7.5(a)}$$

$$\operatorname{Re}\left[(T - g(x))\left(\overline{g(x)} - \overline{t}\right)\right] \geqslant 0, \tag{2.7.5(b)}$$

for a.e. $x \in \Omega$, then

$$\begin{aligned}&\left|\frac{1}{\int_\Omega \rho(x) d\mu(x)} \int_\Omega \rho(x) f(x) \overline{g(x)} d\mu(x)\right.\\ &\left.- \left(\frac{1}{\int_\Omega \rho(x) d\mu(x)} \int_\Omega \rho(x) f(x) d\mu(x)\right)\left(\frac{1}{\int_\Omega \rho(x) d\mu(x)} \int_\Omega \rho(x) \overline{g(x)} d\mu(x)\right)\right|\\ &\leqslant \frac{1}{4}|Z - z||T - t|.\end{aligned} \tag{2.7.6}$$

The constant $\frac{1}{4}$ in (2.7.6) is sharp.

Proof. Follows by Proposition 2.7.1 on choosing

$$h = \frac{1}{\left[\int_\Omega \rho(x) d\mu(x)\right]^{\frac{1}{2}}},$$

$$\Phi = Z\left[\int_\Omega \rho(x) d\mu(x)\right]^{\frac{1}{2}}, \quad \phi = z\left[\int_\Omega \rho(x) d\mu(x)\right]^{\frac{1}{2}}$$

$$\Gamma = T\left[\int_\Omega \rho(x) d\mu(x)\right]^{\frac{1}{2}}, \quad \gamma = t\left[\int_\Omega \rho(x) d\mu(x)\right]^{\frac{1}{2}}.$$

We omit the details.

As mentioned in [32], if $\rho : \Omega \subseteq \mathbb{R} \to [0,\infty)$ is a probability density function, i.e., $\int_\Omega \rho(t)dt = 1$, then $\rho^{\frac{1}{2}} \in L^2(\Omega,\mathbb{R})$ and obviously $\left\|\rho^{\frac{1}{2}}\right\|_2 = 1$. Consequently, if we assume that $f,\ g \in L^2(\Omega,\mathbb{R})$ and

$$a\rho^{\frac{1}{2}} \leqslant f \leqslant A\rho^{\frac{1}{2}}, b\rho^{\frac{1}{2}} \leqslant g \leqslant B\rho^{\frac{1}{2}},$$

a.e. on Ω, where $a,\ A,\ b,\ B$ are given real numbers, then by Proposition 2.7.1, one has the Grüss-type inequality

$$\left|\int_\Omega f(t)g(t)dt - \left(\int_\Omega f(t)\rho^{\frac{1}{2}}(t)dt\right)\left(\int_\Omega g(t)\rho^{\frac{1}{2}}(t)dt\right)\right| \leqslant \frac{1}{4}(A-a)(B-b). \tag{2.7.7}$$

The following particular inequalities are of interest.

1. If $f,\ g \in L^2(\mathbb{R},\mathbb{R})$ are such that

$$\frac{a}{\sqrt{\sigma}\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2} \leqslant f(x) \leqslant \frac{A}{\sqrt{\sigma}\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2},$$

$$\frac{b}{\sqrt{\sigma}\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2} \leqslant g(x) \leqslant \frac{B}{\sqrt{\sigma}\sqrt[4]{2\pi}} e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2},$$

for a.e. $x \in \mathbb{R}$, where $a,\ A,\ b,\ B \in \mathbb{R}$, $m \in \mathbb{R}$, $\sigma > 0$, then one has the following Normal-Grüss inequality

$$\left|\int_{-\infty}^{\infty} f(x)g(x)dx - \frac{1}{\sigma\sqrt{2\pi}}\left(\int_{-\infty}^{\infty} f(x)e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}dx\right)\left(\int_{-\infty}^{\infty} g(x)e^{-\frac{1}{4}\left(\frac{x-m}{\sigma}\right)^2}dx\right)\right|$$

$$\leqslant \frac{1}{4}(A-a)(B-b). \tag{2.7.8}$$

2. If $f,\ g \in L^2(\mathbb{R},\mathbb{R})$ are such that

$$\frac{a}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|} \leqslant f(x) \leqslant \frac{A}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|},$$

$$\frac{b}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|} \leqslant g(x) \leqslant \frac{B}{\sqrt{2\beta}} e^{-\left|\frac{x-\alpha}{2\beta}\right|},$$

for a.e. $x \in \mathbb{R}$, where $a,\ A,\ b,\ B \in \mathbb{R}$, $\alpha \in \mathbb{R}$, $\beta > 0$, then one has the following Laplace-Grüss inequality

$$\left|\int_{-\infty}^{\infty} f(x)g(x)dx - \frac{1}{2\beta}\left(\int_{-\infty}^{\infty} f(x)e^{-\left|\frac{x-\alpha}{2\beta}\right|}dx\right)\left(\int_{-\infty}^{\infty} g(x)e^{-\left|\frac{x-\alpha}{2\beta}\right|}dx\right)\right|$$

$$\leqslant \frac{1}{4}(A-a)(B-b). \tag{2.7.9}$$

3. If $f,\ g \in L^2([0,\infty),R)$ are such that

$$\frac{a}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}} \leqslant f(x) \leqslant \frac{A}{\sqrt{\Gamma(p)}} x^{\frac{p-1}{2}} e^{-\frac{x}{2}},$$

$$\frac{b}{\sqrt{\Gamma(p)}}x^{\frac{p-1}{2}}e^{-\frac{x}{2}} \leqslant g(x) \leqslant \frac{B}{\sqrt{\Gamma(p)}}x^{\frac{p-1}{2}}e^{-\frac{x}{2}},$$

for a.e. $x \in [0,\infty)$, where $a,\ A,\ b,\ B \in \mathbb{R}$, $p > 0$, then one has the following Gamma-Grüss inequality

$$\left|\int_0^\infty f(x)g(x)dx - \frac{1}{\sqrt{\Gamma(p)}}\left(\int_0^\infty f(x)x^{\frac{p-1}{2}}e^{-\frac{x}{2}}dx\right)\left(\int_0^\infty g(x)x^{\frac{p-1}{2}}e^{-\frac{x}{2}}dx\right)\right| \leqslant \frac{1}{4}(A-a)(B-b). \tag{2.7.10}$$

4. If $f,\ g \in L^2(x \in [0,1], R)$ are such that

$$\frac{a}{\sqrt{\mathrm{B}(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}} \leqslant f(x) \leqslant \frac{A}{\sqrt{\mathrm{B}(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}},$$

$$\frac{\overline{b}}{\sqrt{\mathrm{B}(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}} \leqslant g(x) \leqslant \frac{\overline{B}}{\sqrt{\mathrm{B}(p,q)}}x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}},$$

for a.e. $x \in [0,1]$ where $a,\ A,\ \overline{b},\ \overline{B} \in \mathbb{R}$, $p,\ q \in [1,\infty)$, then one has the following Beta-Grüss inequality

$$\left|\int_0^1 f(x)g(x)dx - \frac{1}{\sqrt{B(p,q)}}\left(\int_0^1 f(x)x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}dx\right)\left(\int_0^1 g(x)x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}dx\right)\right| \leqslant \frac{1}{4}(A-a)(\overline{B}-\overline{b}). \tag{2.7.11}$$

Finally, we note that Theorem 2.2.1 allows us to state some discrete versions of the Grüss-type inequalities for real and complex sequences, see [54]. Here we omit the details.

2.7.2 *Application to numerical integration*

In this section, we consider an application of Theorem 2.4.2 to numerical integration in connection with a general cubature formula given by Dragomir, Barnett and Pearce in [39]. First, by employing the identity (2.4.12), we present the perturbed version of Grüss inequality proved in [39], which may be useful in certain applications.

Theorem 2.7.1. Let $f,\ g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) with derivatives being bounded. Then

$$\begin{aligned}\Bigg|\frac{1}{b-a}\int_a^b f(x)g(x)dx &- \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \\ &+ \frac{1}{4}(f(b)-f(a))(g(b)-g(a))\Bigg| \\ \leqslant \frac{1}{2}&\left[\|f-f(a)\|_\infty\|g-g(a)\|_\infty + \|f(b)-f\|_\infty\|g(b)-g\|_\infty\right] \\ &+\frac{1}{16}(b-a)^2\|f'\|_\infty\|g'\|_\infty.\end{aligned} \tag{2.7.12}$$

Proof. Define the mapping $h:[a,b]^2\to\mathbb{R}$ given by

$$h(x,y)=(f(x)-f(y))(g(x)-g(y)),$$

and write the identity (2.4.12) for h, to get

$$\int_a^b\int_a^b h(x,y)dydx+\frac{1}{4}(b-a)^2[h(a,a)+h(a,b)+h(b,a)+h(b,b)]$$

$$=\frac{1}{2}(b-a)\int_a^b[h(s,a)+h(s,b)]ds+\frac{1}{2}(b-a)\int_a^b[h(a,s)+h(b,s)]ds$$

$$+\int_a^b\int_a^b\left(s-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)D_2D_1h(s,t)dtds. \tag{2.7.13}$$

We observe that

$$\frac{1}{4}[h(a,a)+h(a,b)+h(b,a)+h(b,b)]=\frac{1}{2}(f(b)-f(a))(g(b)-g(a)),$$

$$\frac{1}{2}\int_a^b[h(s,a)+h(s,b)]\,ds=\frac{1}{2}\int_a^b[h(a,s)+h(b,s)]ds$$

$$=\frac{1}{2}\int_a^b[(f(s)-f(a))(g(s)-g(a))+(f(b)-f(s))(g(b)-g(s))]\,ds,$$

$$D_2D_1h(x,y)=-f'(x)g'(y)-f'(y)g'(x),$$

and

$$\int_a^b\int_a^b\left(s-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)D_2D_1h(s,t)dtds$$

$$=-\int_a^b\int_a^b\left(s-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)[f'(s)g'(t)+f'(t)g'(s)]dtds$$

$$=-2\int_a^b\int_a^b\left(s-\frac{a+b}{2}\right)\left(t-\frac{a+b}{2}\right)f'(s)g'(t)dtds$$

$$=-2\int_a^b\left(s-\frac{a+b}{2}\right)f'(s)ds\int_a^b\left(t-\frac{a+b}{2}\right)g'(t)dt.$$

Consequently, by (2.7.13), we get

$$\int_a^b\int_a^b(f(x)-f(y))(g(x)-g(y))dydx+\frac{1}{2}(b-a)^2(f(b)-f(a))(g(b)-g(a))$$

$$=(b-a)\int_a^b[(f(s)-f(a))(g(s)-g(a))+(f(b)-f(s))(g(b)-g(s))]\,ds$$

$$-2\int_a^b\left(s-\frac{a+b}{2}\right)f'(s)ds\int_a^b\left(t-\frac{a+b}{2}\right)g'(t)dt. \tag{2.7.14}$$

Now, dividing by 2 and taking into account the fact that

$$\frac{1}{2}\int_a^b\int_a^b (f(x)-f(y))(g(x)-g(y))dydx = (b-a)\int_a^b f(x)g(x)dx - \int_a^b f(x)dx\int_a^b g(x)dx,$$

the identity (2.7.14) becomes

$$(b-a)\int_a^b f(x)g(x)dx - \int_a^b f(x)dx\int_a^b g(x)dx + \frac{1}{4}(b-a)^2(f(b)-f(a))(g(b)-g(a))$$

$$= \frac{1}{2}(b-a)\int_a^b [(f(s)-f(a))(g(s)-g(a)) + (f(b)-f(s))(g(b)-g(s))]ds$$

$$-\int_a^b \left(s-\frac{a+b}{2}\right) f'(s)ds\int_a^b \left(t-\frac{a+b}{2}\right) g'(t)dt. \tag{2.7.15}$$

Rewriting (2.7.15) and using the properties of modulus, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right)\right.$$

$$\left. + \frac{1}{4}(f(b)-f(a))(g(b)-g(a))\right|$$

$$\leqslant \frac{1}{2(b-a)}\left[(b-a)\|f-f(a)\|_\infty\|g-g(a)\|_\infty + (b-a)\|f(b)-f\|_\infty\|g(b)-g\|_\infty\right]$$

$$+\|f'\|_\infty\int_a^b \left|s-\frac{a+b}{2}\right| ds\|g'\|_\infty\int_a^b\left|s-\frac{a+b}{2}\right| ds. \tag{2.7.16}$$

A simple calculation gives

$$\int_a^b \left|s-\frac{a+b}{2}\right| ds = \frac{(b-a)^2}{4}. \tag{2.7.17}$$

Using (2.7.17) in (2.7.16), we deduce the desired inequality in (2.7.12). The proof is complete.

Consider the arbitrary division $I_n = a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ of $[a,b]$ and $J_m = c = y_0 < y_1 < \cdots < y_{m-1} < y_m = d$ of $[c,d]$, put $h_i := x_{i+1} - x_i$, $l_j := y_{j+1} - y_j$, $i = 0,1,\ldots,n-1$; $j = 0,1,\ldots,m-1$. Define the sum given by

$$C_T(f,I_n,J_m) := \sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{1}{2}h_i\int_{y_j}^{y_{j+1}}[f(x_i,t)+f(x_{i+1},t)]dt$$

$$+\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{1}{2}l_j\int_{x_j}^{x_{j+1}}\left[f(s,y_j)+f(s,y_{j+1})\right]ds$$

$$-\sum_{i=0}^{n-1}\sum_{j=0}^{m-1}\frac{1}{4}h_il_j\left[f(x_i,y_j)+f(x_i,y_{j+1})+f(x_{i+1},y_j)+f(x_{i+1},y_{j+1})\right]. \tag{2.7.18}$$

As an application of Theorem 2.4.2, in [39], the authors proved the following theorem.

Theorem 2.7.2. Let $f : [a,b] \times [c,d]$ be as in Theorem 2.4.2 and $I_n,\ J_m$ as above. Then we have the cubature formula

$$\int_a^b \int_c^d f(s,t)dtds = C_T(f,I_n,J_m) + R_T(f,I_n,J_m), \tag{2.7.19}$$

where the remainder term $R_T(f,I_n,J_m)$ satisfies the estimation

$$|R_T(f,I_n,J_m)| \leqslant \frac{1}{16}\|D_2D_1f\|_\infty \sum_{i=0}^{n-1} h_i^2 \sum_{j=0}^{m-1} l_i^2. \tag{2.7.20}$$

Proof. Apply Theorem 2.4.2 on $[x_i,x_{i+1}] \times [y_j,y_{j+1}]$ for $i = 0,1,\ldots,n-1$; $j = 0,1,\ldots,m-1$, to get

$$\begin{aligned}&\Bigg|\int_{x_i}^{x_{i+1}} \int_{yj}^{y_{j+1}} f(s,t)dtds \\ &- \left[\frac{1}{2}h_i \int_{y_j}^{y_{j+1}} \left[f(x_i,t) + f(x_{i+1},t)\right]dt + \frac{1}{2}l_j \int_{x_i}^{x_{i+1}} \left[f(s,y_j) + f(s,y_{j+1})\right]ds \right.\\ &\left. - \frac{1}{4}h_il_j\left[f(x_i,y_j) + f(x_i,y_{j+1}) + f(x_{i+1},y_j) + f(x_{i+1},y_{j+1})\right]\right]\Bigg| \\ &\leqslant \frac{1}{16}h_i^2l_i^2\|D_2D_1f\|_\infty. \end{aligned} \tag{2.7.21}$$

Summing both sides of (2.7.21) over i from 0 to $n-1$ and over j from 0 to $m-1$ and using the generalized triangle inequality, we deduce the desired inequality in (2.7.20).

2.7.3 *Approximation for the finite Fourier transform of two independent variables*

The Fourier transform has applications in a wide variety of fields in science and engineering. In this section, we present the inequality established by Hanna, Dragomir and Roumeliotis [66] for the error, in approximating the finite Fourier transform in two independent variables.

Let $\Delta = [a,b] \times [c,d]$ and $f : \Delta \to \mathbb{R}$ be a continuous mapping defined on Δ and $F(f)$ its finite Fourier transform. That is

$$F(f)(u,v;a,b,c,d) = \int_a^b \int_c^d f(x,y)e^{-2\pi i(ux+vy)}dydx, \tag{2.7.22}$$

$(u,v) \in \Delta$. For a function of one variable we use the notation

$$F(g)(u,a,b) = \int_a^b g(x)e^{-2\pi iux}dx.$$

The following inequality in approximating the finite Fourier transform (2.7.22) in terms of the exponential means was obtained in [66].

Theorem 2.7.3. Let $f : \Delta \to \mathbb{R}$ be an absolutely continuous mapping on Δ and assume that $D_2D_1f(x,y)$ exists on $(a,b)\times(c,d)$, then we have the inequality

$$|F(f)(u,v;a,b,c,d)-I_1-I_2+I_3| \leqslant \begin{cases} J_1; \\ J_2; \\ J_3; \end{cases} \tag{2.7.23}$$

for all $(u,v)\in\Delta$, where

$$I_1 := I_1(u,v;a,b,c,d) = E(u)\int_a^b F(f(s,\cdot))(v;c,d)ds,$$
$$I_2 := I_2(u,v;a,b,c,d) = E(v)\int_c^d F(f(\cdot,t))(u;a,b)dt,$$
$$I_3 := I_3(u,v;a,b,c,d) = E(u)E(v)\int_a^b\int_c^d f(s,t)dtds,$$

with

$$E(u) = E(-2\pi iub, -2\pi iua),$$
$$E(v) = E(-2\pi ivd, -2\pi ivc),$$

given that E is the exponential mean of complex numbers, that is

$$E(z,w) := \begin{cases} \dfrac{e^z-e^w}{z-w} & \text{if } z\neq w, \\ e^w & \text{if } z=w, \end{cases}$$

for $z,\ w\in C$, and

$$J_1 := J_1(a,b,c,d,\|D_2D_1f\|_\infty) = \frac{(b-a)^2(d-c)^2}{9}\|D_2D_1f\|_\infty,$$
$$\text{if } D_2D_1f(x,y)\in L_\infty(\Delta);$$
$$J_2 := J_2(a,b,c,d,\|D_2D_1f\|_p) = \left[\frac{2[(b-a)(d-c)]^{\frac{q+1}{2}}}{(q+1)(q+2)}\right]^{\frac{2}{q}}\|D_2D_1f\|_p,$$
$$\text{if } D_2D_1f(x,y)\in L_p(\Delta),\ \frac{1}{p}+\frac{1}{q}=1,\ \ p>1;$$
$$J_3 := J_3(a,b,c,d,\|D_2D_1f\|_1) = (b-a)(d-c)\|D_2D_1f\|_1,$$
$$\text{if } D_2D_1f(x,y)\in L_1(\Delta),$$

where

$$\|D_2D_1f\|_\infty = \sup_{(s,t)\in\Delta}|D_2D_1f(s,t)| < \infty,$$

$$\|D_2D_1f\|_p = \left(\int_a^b\int_c^d |D_2D_1f(s,t)|^p dtds\right)^{\frac{1}{p}} < \infty, \quad 1\leqslant p<\infty,$$

are the usual Lebesgue norms.

Proof. Using the identity obtained by Barnett and Dragomir in [8] (see, Lemma 2.3.2), we have

$$f(x,y) = \frac{1}{b-a}\int_a^b f(s,y)ds + \frac{1}{d-c}\int_c^d f(x,t)dt - \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dtds$$
$$+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d P(x,s)Q(y,t)D_2D_1f(s,t)dtds, \tag{2.7.24}$$

provided that f is continuous on Δ and

$$P(x,s) = \begin{cases} s-a, \ a \leqslant s \leqslant x, \\ s-b, \ x < s \leqslant b. \end{cases}$$

$$Q(y,t) = \begin{cases} t-c, & c \leqslant t \leqslant y, \\ t-d, & y < t \leqslant d. \end{cases}$$

If we replace $f(x,y)$ in (2.7.22) by its representation from (2.7.24), we get

$$F(f)(u,v;a,b,c,d)$$
$$= \int_a^b\int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a}\int_a^b f(s,y)ds\right)dydx + \int_a^b\int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c}\int_c^d f(x,t)dt\right)dydx$$
$$-\int_a^b\int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)}\int_a^b\int_c^d f(s,t)dtds\right)dydx + R(f,u,v;a,b,c,d), \tag{2.7.25}$$

where

$$R(f,u,v;a,b,c,d) = \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d \left(e^{-2\pi i(ux+vy)}\right)$$
$$\times\left(\int_a^b\int_c^d P(x,s)Q(y,t)D_2D_1f(s,t)dtds\right)dydx. \tag{2.7.26}$$

Let

$$I_1 = \int_a^b\int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{b-a}\int_a^b f(s,y)ds\right)dydx,$$

then

$$I_1 = \int_a^b \frac{e^{-2\pi iux}}{b-a}dx\left(\int_c^d e^{-2\pi ivy}\left(\int_a^b f(s,y)ds\right)dy\right)$$
$$= \frac{e^{-2\pi iub}-e^{-2\pi iua}}{-2\pi iu(b-a)}\int_a^b\left(\int_c^d e^{-2\pi ivy}f(s,y)dy\right)ds = E(u)\int_a^b F(f(s,\cdot))(v;c,d)ds.$$

In a similar fashion we obtain

$$I_2 = \int_a^b\int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{d-c}\int_c^d f(x,t)dt\right)dydx = E(v)\int_c^d F(f(\cdot,t))(u;a,b)dt$$

and

$$I_3 = \int_a^b \int_c^d \left(\frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)dtds \right) dydx$$

$$= \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(s,t)dtds \int_a^b \int_c^d e^{-2\pi iux} e^{-2\pi ivy} dydx$$

$$= E(u)E(v) \int_a^b \int_c^d f(s,t)dtds.$$

From (2.7.25) and using the properties of modulus, we have

$$|F(f)(u,v;a,b,c,d) - I_1 - I_2 + I_3|$$

$$= \left| \int_a^b \int_c^d \left(\int_a^b \int_c^d \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} P(x,s)Q(y,t)D_2D_1 f(s,t)dtds \right) dydx \right|$$

$$\leqslant \int_a^b \int_c^d \int_a^b \int_c^d \left| \frac{e^{-2\pi i(ux+vy)}}{(b-a)(d-c)} \right| |p(x,s)||Q(y,t)||D_2D_1 f(s,t)| dtdsdydz$$

$$= \int_a^b \int_c^d \int_a^b \int_c^d \frac{|p(x,s)||Q(y,t)|}{(b-a)(d-c)} |D_2D_1 f(s,t)| dtdsdydx. \tag{2.7.27}$$

Now, we observe that

$$\int_a^b \int_c^d \int_a^b \int_c^d |p(x,s)||Q(y,t)||D_2D_1 f(s,t)| dtdsdydx$$

$$\leqslant \|D_2D_1 f\|_\infty \left[\int_a^b \left(\int_a^b |p(x,s)|ds \right) dx \int_c^d \left(\int_c^d |Q(y,t)|dt \right) dy \right]$$

$$= \|D_2D_1 f\|_\infty \left[\int_a^b \left\{ \left. \frac{(s-a)^2}{2} \right|_a^x + \left. \frac{(b-s)^2}{2} \right|_x^b \right\} dx \int_c^d \left\{ \left. \frac{(t-c)^2}{2} \right|_c^y + \left. \frac{(d-t)^2}{2} \right|_y^d \right\} dy \right]$$

$$= \|D_2D_1 f\|_\infty \left[\left(\int_a^b \frac{(x-a)^2}{2} dx + \int_a^b \frac{(b-x)^2}{2} dx \right) \left(\int_c^d \frac{(y-c)^2}{2} dy + \int_c^d \frac{(d-y)^2}{2} dy \right) \right]$$

$$= \|D_2D_1 f\|_\infty \left[\frac{(b-a)^3}{3} \frac{(d-c)^3}{3} \right]. \tag{2.7.28}$$

Substituting (2.7.28) in (2.7.27), we obtain the first inequality in (2.7.28).

Applying Hölder's integral inequality for double integrals, we get

$$\int_a^b \int_c^d \int_a^b \int_c^d |p(x,s)||Q(y,t)||D_2D_1 f(s,t)| dtdsdydx$$

$$\leqslant \left(\int_a^b \int_c^d \int_a^b \int_c^d \{|p(x,s)||Q(y,s)|\}^q dt ds dy dx\right)^{\frac{1}{q}}$$

$$\times \left(\int_a^b \int_c^d \int_a^b \int_c^d |D_2D_1 f(s,t)|^p dt ds dy dx\right)^{\frac{1}{p}}$$

$$= \|D_2D_1 f\|_p \{(b-a)(d-c)\}^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left(\int_a^b |p(x,s)|^q ds\right) dx\right)^{\frac{1}{q}} \left(\int_c^d \left(\int_c^d |Q(y,t)|^q dt\right) dy\right)^{\frac{1}{q}}$$

$$= \|D_2D_1 f\|_p \{(b-a)(d-c)\}^{\frac{1}{p}}$$

$$\times \left(\int_a^b \left(\frac{(x-a)^{q+1}}{q+1} + \frac{(b-x)^{q+1}}{q+1}\right) dx\right)^{\frac{1}{q}} \left(\int_c^d \left(\frac{(y-c)^{q+1}}{q+1} + \frac{(d-y)^{q+1}}{q+1}\right) dy\right)^{\frac{1}{q}}$$

$$= \|D_2D_1 f\|_p \left[\frac{2^{\frac{2}{q}} (b-a)^{1+\frac{2}{q}} (d-c)^{1+\frac{2}{q}}}{\{(q+1)(q+2)\}^{\frac{2}{q}}}\right]. \tag{2.7.29}$$

Using (2.7.29) in (2.7.27), we get the second inequality in (2.7.23)

Finally, we obtain that

$$\int_a^b \int_c^d \int_a^b \int_c^d |p(x,s)||Q(y,t)||D_2D_1 f(s,t)| dt ds dy dx$$

$$\leqslant \sup_{(x,s)\in[a,b]^2} |p(x,s)| \sup_{(y,t)\in[c,d]^2} |Q(y,t)| \int_a^b \int_c^d \int_a^b \int_c^d |D_2D_1 f(s,t)| dt ds dy dx$$

$$= (b-a)(d-c) \int_a^b \int_c^d \int_a^b \int_c^d |D_2D_1 f(s,t)| dt ds dy dx$$

$$= \|D_2D_1 f\|_1 (b-a)^2 (d-c)^2. \tag{2.7.30}$$

Using (2.7.30) in (2.7.27), gives the final inequality in (2.7.23), where we have used the fact that

$$\max\{X,Y\} = \frac{x+y}{2} + \left|\frac{y-x}{2}\right|$$

The proof is complete.

2.8 Miscellaneous inequalities

2.8.1 *Dragomir [53]*

Let $(H,(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$) and $e \in H$, $\|e\|=1$. If γ, $\Gamma \in \mathbb{K}$ and x, $y \in H$ are such that

$$\operatorname{Re}\left(\Gamma e - \frac{x+y}{2}, \frac{x+y}{2} - \gamma e\right) \geqslant 0$$

or, equivalently,

$$\left\| \frac{x+y}{2} - \frac{\gamma+\Gamma}{2} e \right\| \leqslant \frac{1}{2}|\Gamma - \gamma|,$$

then, we have the inequality

$$\operatorname{Re}[(x,y) - (x,e)(e,y)] \leqslant \frac{1}{4}|\Gamma-\gamma|^2.$$

The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller constant.

2.8.2 *Ujević [153]*

Let $(X,(\cdot,\cdot))$ be a real inner product space and $\{e_i\}_1^n \subset X$, $(e_i,e_j)=\delta_{ij}$, where

$$\delta_{ij} = \begin{cases} 1 \text{ if } \ i=j, \\ 0 \text{ if } \ i \neq j. \end{cases}$$

If ϕ_i, γ_i, Φ_i, Γ_i, $i=1,2,\ldots,n$, are real numbers and x, $y \in X$ such that the conditions

$$\left(x - \sum_{i=1}^n \gamma_i e_i, \sum_{i=1}^n \Gamma_i e_i - x\right) \geqslant 0$$

and

$$\left(y - \sum_{i=1}^n \phi_i e_i, \sum_{i=1}^n \Phi_i e_i - y\right) \geqslant 0$$

hold, then we have the inequality

$$\left|(x,y) - \sum_{i=1}^n (x,e_i)(y,e_i)\right| \leqslant \frac{1}{4}\sqrt{\sum_{i=1}^n (\Phi_i - \phi_i)^2 \sum_{i=1}^n (\Gamma_i - \gamma_i)^2}.$$

The constant $\frac{1}{4}$ is the best possible.

2.8.3 *Dragomir [55]*

Let $(H,(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$). Let $\{e_i\}_{i\in I}$ be a family of orthonormal vectors in H, F a finite part of I, ϕ_i, Φ_i, γ_i, $\Gamma_i \in \mathbb{K}$, $i\in F$ and x, $y\in H$. If

$$\operatorname{Re}\left(\sum_{i=1}^{n}\Phi_i e_i - x, x-\sum_{i=1}^{n}\phi_i e_i\right)\geqslant 0,$$

$$\operatorname{Re}\left(\sum_{i=1}^{n}\Gamma_i e_i - y, y-\sum_{i=1}^{n}\gamma_i e_i\right)\geqslant 0,$$

or, equivalently,

$$\left\|x-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}e_i\right\|\leqslant \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

$$\left\|y-\sum_{i\in F}\frac{\Gamma_i+\gamma_i}{2}e_i\right\|\leqslant \frac{1}{2}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}},$$

hold, then we have the inequalities

$$\left|(x,y)-\sum_{i\in F}(x,e_i)(e_i,y)\right|\leqslant \frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}$$

$$-\left[\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i - x, x-\sum_{i\in F}\phi_i e_i\right)\right]^{\frac{1}{2}}\times\left[\operatorname{Re}\left(\sum_{i\in F}\Gamma_i e_i - y, y-\sum_{i\in F}\gamma_i e_i\right)\right]^{\frac{1}{2}}$$

$$\leqslant \frac{1}{4}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\left(\sum_{i\in F}|\Gamma_i-\gamma_i|^2\right)^{\frac{1}{2}}.$$

The constant $\frac{1}{4}$ is the best possible.

2.8.4 *Dragomir [55]*

Let $(H,(\cdot,\cdot))$ be an inner product space over K ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$). Let $\{e_i\}_{i\in I}$ be a family of orthonormal vectors in H, F a finite part of I and ϕ_i, $\Phi_i\in\mathbb{K}$, $i\in F$, x, $y\in H$ and $\lambda\in(0,1)$, such that either

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i-(\lambda x+(1-\lambda)y),\lambda x+(1-\lambda)y-\sum_{i\in F}\phi_i e_i\right)\geqslant 0,$$

or, equivalently,

$$\left\|\lambda x+(1-\lambda)y-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}e_i\right\|\leqslant \frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

holds. Then we have the inequality

$$\operatorname{Re}\left[(x,y)-\sum_{i\in F}(x,e_i)(e_i,y)\right]\leqslant\frac{1}{16}\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i-\phi_i|^2$$

$$-\frac{1}{4}\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-(\lambda x+(1-\lambda)y,e_i)\right|^2\leqslant\frac{1}{16}\frac{1}{\lambda(1-\lambda)}\sum_{i\in F}|\Phi_i-\phi_i|^2.$$

The constant $\frac{1}{16}$ is the best possible.

2.8.5 *Dragomir, Pečarić and Tepeš [56]*

Let $(H,(\cdot,\cdot))$ be an inner product space over $\mathbb{K}$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$). Let $\{e_i\}_{i\in I}$ be a family of orthonormal vectors in H, F a finite part of I and ϕ_i, $\Phi_i\in\mathbb{K}$, $i\in F$, x, $y\in H$ such that either the condition

$$\operatorname{Re}\left(\sum_{i\in F}\Phi_i e_i-x,x-\sum_{i\in F}\phi_i e_i\right)\geqslant 0,$$

or, equivalently,

$$\left\|x-\sum_{i\in F}\frac{\Phi_i+\phi_i}{2}e_i\right\|\leqslant\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}},$$

holds. Then we have

$$\left|(x,y)-\sum_{i\in F}(x,e_i)(e_i,y)\right|\leqslant\frac{1}{2}\left(\sum_{i\in F}|\Phi_i-\phi_i|^2\right)^{\frac{1}{2}}\|y\|-\sum_{i\in F}\left|\frac{\Phi_i+\phi_i}{2}-(x,e_i)\right||(y,e_i)|.$$

2.8.6 *Hanna, Dragomir and Roumeliotis [67]*

Let (Ω,Σ,μ) be a measure space consisting of a set Ω, Σ a σ-algebra of subsets of Ω and μ a countable additive and positive measure with values in $\mathbb{R}\cup\{\infty\}$ and $\rho:\Omega\to[0,\infty)$ be a μ-measurable function on Ω with $\int_\Omega p(s)d\mu(s)=1$. Denote by $L^2_\rho(\Omega,\mathbb{K})$ ($\mathbb{K}=\mathbb{R}$, $\mathbb{C}$) the Hilbert space of all measurable functions $f:\Omega\to K$ that are 2-ρ-integrable on Ω, i.e. $\int_\Omega\rho(s)|f(s)|^2d\mu(s)<\infty$. If f, $g\in L^2_\rho(\Omega,\mathbb{K})$ and there exist constants γ, $\Gamma\in\mathbb{K}$ such that either the condition

$$\operatorname{Re}\left[(\Gamma-f(s))\left(\overline{f(s)}-\overline{\gamma}\right)\right]\geqslant 0,$$

for μ-a.e., $s\in\Omega$ or equivalently

$$\left|f(s)-\frac{\gamma+\Gamma}{2}\right|\leqslant\frac{1}{2}|\Gamma-\gamma|,$$

for μ-a.e., $s \in \Omega$ holds, then

$$\left|\int_\Omega \rho(s) f(s)\overline{g(s)}d\mu(s) - \int_\Omega \rho(s) f(s) d\mu(s) \int_\Omega \rho(s)\overline{g(s)}d\mu(s)\right|$$

$$\leqslant \frac{1}{2}|\Gamma-\gamma| \left[\int_\Omega \rho(s)|g(s)|^2 d\mu(s) - \left|\int_\Omega \rho(s) g(s) d\mu(s)\right|^2\right]^{\frac{1}{2}}.$$

2.8.7 *Buşe, Cerone, Dragomir and Roumeliotis [12]*

Let $(H, \langle \cdot, \cdot \rangle)$ be a real or complex Hilbert space, $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable set and $\rho : \Omega \to [0,\infty)$ a Lebesgue measurable function with $\int_\Omega \rho(s)ds = 1$. Denote by $L_{2,\rho}(\Omega, H)$ the set of all Bochner measurable functions f on Ω such that $\|f\|_{2,\rho}^2 := \int_\Omega \rho(s)\|f(s)\|^2 ds < \infty$. Assume that $f,\ g \in L_{2,\rho}(\Omega, H)$ and there exist vectors $x,\ X,\ y,\ Y \in H$ such that

$$\int_\Omega \rho(t)\,\mathrm{Re}\langle X - f(t), f(t) - x\rangle dt \geqslant 0,$$
$$\int_\Omega \rho(t)\,\mathrm{Re}\langle Y - g(t), g(t) - y\rangle dt \geqslant 0,$$

or equivalently,

$$\int_\Omega \rho(t)\left\|f(t) - \frac{X+x}{2}\right\|^2 dt \leqslant \frac{1}{4}\|X-x\|^2,$$
$$\int_\Omega \rho(t)\left\|g(t) - \frac{Y+y}{2}\right\|^2 dt \leqslant \frac{1}{4}\|Y-y\|^2.$$

Then we have the inequality

$$\left|\int_\Omega \rho(t)\langle f(t), g(t)\rangle dt - \left\langle \int_\Omega \rho(t) f(t) dt, \int_\Omega \rho(t) g(t) dt \right\rangle\right| \leqslant \frac{1}{4}\|X-x\|\,\|Y-y\|$$

$$-\left[\int_\Omega \rho(t)\,\mathrm{Re}\langle X - f(t), f(t) - x\rangle dt \int_\Omega \rho(t)\,\mathrm{Re}\langle Y - g(t), g(t) - y\rangle dt\right]^{\frac{1}{2}} \leqslant \frac{1}{4}\|X-x\|\,\|Y-y\|.$$

The constant $\frac{1}{4}$ in both inequalities is sharp.

2.8.8 *Hanna, Dragomir and Cerone [62]*

Let $f,\ g : [a,b] \times [c,d] \to \mathbb{R}$ be two mappings such that

$$|f(x,y) - f(u,v)| \leqslant M_1|x-u|^{\alpha_1} + M_2|y-v|^{\alpha_2},$$

$$|g(x,y) - g(u,v)| \leqslant N_1|x-u|^{\beta_1} + N_2|y-v|^{\beta_2},$$

where M_1, M_2, N_1, N_2 are positive constants and α_1, α_2, β_1, β_2 are constants lying in $(0,1]$. Then we have the inequality:

$$\left|\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)g(x,y)dydx\right.$$

$$\left.-\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)dydx\right)\left(\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d g(x,y)dydx\right)\right|$$

$$\leqslant 4\left[M_1N_1\frac{(b-a)^{\alpha_1+\beta_1}}{(\alpha_1+\beta_1+1)(\alpha_1+\beta_1+2)}+M_1N_2\frac{2(b-a)^{\alpha_1}(d-c)^{\beta_2}}{(\alpha_1+1)(\alpha_1+2)(\beta_2+1)(\beta_2+2)}\right.$$

$$\left.+M_2N_1\frac{2(b-a)^{\beta_1}(d-c)^{\alpha_2}}{(\alpha_2+1)(\alpha_2+2)(\beta_1+1)(\beta_1+2)}+M_2N_2\frac{(d-c)^{\alpha_2+\beta_2}}{(\alpha_2+\beta_2+1)(\alpha_2+\beta_2+2)}\right].$$

2.8.9 *Pachpatte [94]*

Let the assumptions of Theorem 2.5.3 hold. Then

$$|A(w,f,g;D)|\leqslant\frac{1}{2\int_D w(x)dx}\int_D w(x)\left(H(f,x_i,y_i)H(g,x_i,y_i)w(y)dy\right)dx,$$

where A and H are as in Theorem 2.5.3.

2.8.10 *Pachpatte [94]*

Assume that the hypotheses of Theorem 2.6.3 hold. Let $w(x)$ be a real-valued nonnegative function defined on Q and $\sum_x w(x)>0$. Then

$$|P(w,f,g;Q)|\leqslant\frac{1}{2\sum_x w(x)}\sum_x w(x)\left[|g(x)|\sum_y E(f,x_i,y_i)w(y)+|f(x)|\sum_y E(g,x_i,y_i)w(y)\right],$$

$$|P(w,f,g;Q)|\leqslant\frac{1}{\left(\sum_x w(x)\right)^2}\sum_x w(x)\left(\sum_y E(f,x_i,y_i)w(y)\right)\left(\sum_y E(g,x_i,y_i)w(y)\right),$$

$$|P(w,f,g;Q)|\leqslant\frac{1}{2\sum_x w(x)}\sum_x w(x)\left(\sum_y E(f,x_i,y_i)E(g,x_i,y_i)w(y)\right),$$

where we have set the notations

$$P(w,p,q;Q)=\sum_x w(x)p(x)q(x)-\frac{1}{\sum_x w(x)}\left(\sum_x w(x)p(x)\right)\left(\sum_x w(x)q(x)\right),$$

$$E(p,x_i,y_i)=\sum_{i=1}^n\|\Delta_i p\|_\infty|x_i-y_i|,$$

for some functions $p,\ q:Q\to\mathbb{R}$.

2.8.11 *Pachpatte [130]*

Under the notations and definitions given in section 2.6, let $f,\ g : Q \to \mathbb{R}$ be summable functions on Q and $p : Q \to \mathbb{R}_+$ a summable function on Q such that $\overline{p} = \sum_x p(x) > 0$. Then

$$|F(\overline{P}, p, f, g; Q)| \leqslant \sqrt{F(\overline{P}, p, f, f; Q)}\sqrt{F(\overline{P}, p, g, g; Q)},$$

and in addition if $\phi \leqslant f(x) \leqslant \Phi$, $\gamma \leqslant g(x) \leqslant \Gamma$ for each $x \in Q$, where $\phi,\ \Phi,\ \gamma,\ \Gamma$ are given real constants, then

$$|F(\overline{P}, p, f, g; Q)| \leqslant \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma),$$

where

$$F(\overline{P}, p, f, g, Q) = \frac{1}{\overline{P}}\sum_x p(x) f(x) g(x) - \left(\frac{1}{\overline{P}}\sum_x p(x) f(x)\right)\left(\frac{1}{\overline{P}}\sum_x p(x) g(x)\right). \tag{o}$$

2.8.12 *Pachpatte [130]*

Under the notations and definitions given in Section 2.6, let $f,\ g : Q \to \mathbb{R}$ be summable functions on Q and $p : Q \to \mathbb{R}$ a summable function on Q such that $\overline{P} = \sum_x p(x) > 0$. Then

$$|F(\overline{P}, p, f, g; Q)| \leqslant \frac{1}{\overline{P}}\sum_x p(x)\left|\left(f(x) - \frac{1}{\overline{P}}\sum_y p(y) f(y)\right)\left(g(x) - \frac{1}{\overline{P}}\sum_y p(y) g(y)\right)\right|$$

and in addition if $\phi \leqslant f(x) \leqslant \Phi$ for each $x \in Q$, where $\phi,\ \Phi$ are given real constants, then

$$|F(\overline{P}, p, f, g; Q)| \leqslant \frac{\Phi - \phi}{2}\sqrt{F(\overline{P}, p, g, g; Q)},$$

where $\mathrm{F}\left(\overline{\mathrm{P}}, p, f, g; Q\right)$ is defined by (o).

2.9 Notes

The Grüss inequality has been generalized and extended over the last years in a number of ways. In [32], Dragomir investigated the Grüss type inequality in Theorem 2.2.1 in real or Complex inner product spaces. Lemmas 2.2.1 and 2.2.2 are due to Dragomir [53]. Theorem 2.2.2 provides a new proof of Theorem 2.2.1 by replacing the condition (2.2.5) by an equivalent but simpler assumption and is due to Dragomir [53]. Theorem 2.2.3 deals with the refinement of the inequality in Theorem 2.2.1 and is taken from Dragomir [53]. Theorem 2.2.4 is due to Dragomir, Pečarić and Tepeš [56], while Theorem 2.2.5 is due to Dragomir [43].

Lemmas 2.3.1 and 2.3.2 are respectively taken from Dragomir, Corone, Barnett and Roule-liotis [37] and Barnett and Dragomir [8] and the results presented in Theorems 2.3.1–2.3.4 are taken from Pachpatte [89,122,91,129]. Theorems 2.4.1 and 2.4.2 deal with the Trapezoid type inequalities and are taken respectively from Pachpatte [86], Dragomir, Barnett and Pearce [39], while Theorems 2.4.3 and 2.4.4 are due to Barnett and Dragomir [6]. The results presented in Sections 2.5 and 2.6 are due to Pachpatte and taken from [130,94,103,129,95]. The material included in Section 2.7 is devoted to the applications and adapted from Dragomir and Gomm [54], Dragomir, Barnett and Pearce [39] and Hanna, Dragomir and Roumeliotis [66]. Section 2.8 contains a few miscellanous inequalities investigated by various investigators.

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