Chapter 2

Stochastic Linear Pursuit-Evasion Game

2.1 Introduction

The 1950's saw the introduction of guided interceptor missiles and the launching of Sputnik I. Questions on pursuit and evasion were suddenly in everyone's mind. What is the best strategy to intercept a moving target? How can friendly planes best avert midair collisions? Thus the theory of Differential Games is permeated with the theory of military pursuit games. Dr. Rufus Isaacs, who was then with the Mathematics Department of the RAND Corporation realized that no one guidance scheme can be optimal against all types of evasion. An intelligent evader can deliberately maneuver to confuse the pursuer's predictions. Thus optimal pursuit and evasion must be considered equally.

Consider a stochastic linear pursuit-evasion game described by a linear stochastic differential equation of the form

$$\frac{d}{dt}x(t;\omega) = A(\omega)x(t;\omega) + B(\omega)u(t;\omega) - C(\omega)v(t;\omega) \tag{2.1.1}$$

for $t \geqslant 0$ and $\omega \in \Omega$, where

i) Ω is the supporting set of a complete probability measure space $(\Omega, \mathrm{A}, \mu)$;
ii) $x(t;\omega)$ is the unknown random n-dimensional state variable;
iii) $u(t;\omega)$ and $v(t;\omega)$ are the random control vectors;
and
iv) $A(\omega)$, $B(\omega)$, and $C(\omega)$ are random matrices of appropriate dimensions.

The problem is to choose a control $u_v(t;\omega)$, depending on the evader's control $v(t;\omega)$ such that

$$x\left(t_{u,v};\omega\right) \in M_\varepsilon \text{ for some } t_{u,v} \in \mathbb{R}_+$$

where M_ε is the terminal set to be defined in Section 2.3.

The object of this chapter is to prove the existence and uniqueness of a random solution, that is, a second order stochastic process, which satisfies equation (2.1.1) with probability one. In order to do this we integrate equation (2.1.1) with respect to t obtaining a vector stochastic integral equation of the form

$$\begin{aligned} x(t;\omega) &= x_0(\omega)e^{A(\omega)t} \\ &+ \int_0^t e^{A(\omega)(t-\tau)}\left[B(\omega)u(\tau;\omega) - C(\omega)v(\tau;\omega)\right]d\tau \end{aligned} \tag{2.1.2}$$

for $t \geqslant 0$ and $\omega \in \Omega$, with initial condition $x(0;\omega) = x_0(\omega)$. In the theory of stochastic integral equations the term $x_0(\omega)e^{A(\omega)t}$ is referred to as the *free stochastic term* or *free random vector* and $e^{A(\omega)(t-\tau)}$ as the *stochastic kernel*.

We will approach the question of existence and uniqueness of a random solution of equation (2.1.2) using the technique of admissibility theory introduced into the study of random integral equations by Tsokos [184]. To do this we must first define some topological spaces and state some results which are essential to this presentation.

2.2 Preliminaries and an Existence Theorem

We will be concerned with the space of random vectors in $L_2(\Omega, \mathrm{A}, \mu)$ where $L_2(\Omega, \mathrm{A}, \mu)$ denotes the set of all μ-equivalence classes of random vectors of the form $(x_1(\omega), \ldots, x_n(\omega)) = x(\omega)$ where for each $i = 1, 2, \ldots, n$, $x_i(\omega)$ is an element of $L_\infty(\Omega, \mathrm{A}, \mu)$. It is well known that $L_2(\Omega, \mathrm{A}, \mu)$ is a normed linear space over the real numbers with the usual definitions of component-wise addition and scalar multiplication with norm given by

$$\|x(\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} = \left\{\int_\Omega \left[x_1(\omega)^2 + x_2(\omega)^2 + \cdots + x_n(\omega)^2\right] d\mu(\omega)\right\}^{\frac{1}{2}}.$$

Definition 2.2.1. Let $C_c = C_c\left(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu)\right)$ denote the space for all continuous vector valued functions from $\mathbb{R}_+$ into $L_2(\Omega, \mathrm{A}, \mu)$, or second order stochastic processes on $\mathbb{R}_+$, with the topology of uniform convergence on every compact interval $[0, T]$, $T > 0$. That is, the sequence $x(t;\omega)_k$ converges to $x(t;\omega)$ in C_c if and only if

$$\lim_{k\to\infty}\left\{E\left|x(t;\omega)_k - x(t;\omega)\right|^2\right\}^{\frac{1}{2}} = \lim_{k\to\infty}\left\{\int_\Omega |x(t;\omega)_k - x(t;\omega)|^2 d\mu(\omega)\right\}^{\frac{1}{2}} = 0$$

uniformly on every interval $[0, T]$, $T > 0$.

Definition 2.2.1 simply says that the map $t \to x(t;\omega) = (x_1(t;\omega), x_2(t;\omega), \ldots, x_n(t;\omega))$ is continuous and that for each $t \in \mathbb{R}_+$ and each $i = 1, 2, \ldots, n$, $x_i(t;\omega) \in L_\infty(\Omega, \mathrm{A}, \mu)$. Thus for fixed $t \in \mathbb{R}_+$

$$\|x(t;\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} = \left\{ \int_\Omega \left[x_1(t;\omega)^2 + \cdots + x_n(t;\omega)^2\right] d\mu(\omega) \right\}^{\frac{1}{2}}.$$

$C_c\left(\mathbb{R}_+, L_2(\Omega,\mathrm{A},\mu)\right)$ is a linear space over the nonnegative real numbers with the usual definitions of addition and scalar multiplication for continuous functions. It should also be noted that C_c is locally convex with topology defined by the following family of semi-norms, Yoshida [207]

$$\left\{ \|x(t;\omega)\|_n : \|x(t;\omega)\|_n = \sup_{0\leqslant t\leqslant n} \left[\int_\Omega |x(t;\omega)|^2 d\mu(\omega) \right]^{\frac{1}{2}}, \quad n = 1,2,\ldots \right\}.$$

Let T denote a linear operator from the space $C_c\left(\mathbb{R}_+, L_2(\Omega,\mathrm{A},\mu)\right)$ into itself; and let B and D denote Banach spaces contained C_c.

Definition 2.2.2. The pair of Banach spaces (B,D) is called *admissible* with respect to the operator T if and only if $TB \subseteq D$.

Definition 2.2.3. The operator T is called *closed* if

$$x(t;\omega)_k \xrightarrow{B} x(t;\omega)$$

and

$$(Tx_k)(t;\omega) \xrightarrow{D} y(t;\omega)$$

imply that

$$(Tx)(t;\omega) = y(t;\omega).$$

Definition 2.2.4. The Banach space B is called *stronger* than the space $C_c(\mathbb{R}_+, L_2(\Omega,\mathrm{A},\mu))$ if every sequence which converges in B with respect to its norm also converges in C_c. The converse need not be true.

The following lemmas due to Tsokos [184] and Banach's fixed point theorem are the basic tools used in the following results.

Lemma 2.2.1. *Let T be a continuous operator from $C_c\left(\mathbb{R}_+, L_2\left(\Omega,\mathrm{A},\mu\right)\right)$ into itself. If B and D are Banach spaces stronger than C_c; and if the pair (B,D) is admissible with respect to T, then T is a continuous operator from B to D.*

Theorem 2.2.1 (Banach's Fixed Point Theorem). *If T is a contraction operator from a Banach space B into itself, then there exists a unique point $x^* \in B$ such that $T(x^*) = x^*$. That is, $x^* \in B$ is the unique fixed point of the operator T.*

Since T is a continuous linear operator from B to D, it is bounded in the sense that there exists a constant $M > 0$ such that

$$\|(Tx)(t;\omega)\|_D \leqslant M\|x(t;\omega)\|_B$$

for $x(t;\omega) \in B$. Thus we can define a norm for the operator T by

$$\|T\|_0 = \sup\left[\frac{\|(Tx)(t;\omega)\|_D}{\|x(t;\omega)\|_B} : \ x(t;\omega) \in B, \ \ \|x(t;\omega)\|_B \neq 0\right].$$

We are also guaranteed that

$$\|(Tx)(t;\omega)\|_D \leqslant \|T\|_0\|x(t;\omega)\|_B.$$

We can now state and prove a theorem on the existence and uniqueness of a random solution of a stochastic integral equation of which equation (2.2.1) is a special case.

2.2.1 *An Existence Theorem*

Consider a stochastic integral equation of the general form

$$x(t;\omega) = h(t,x(t;\omega)) + \int_0^t k(t,x(t;\omega);\omega)d\tau, \quad t \geqslant 0 \tag{2.2.1}$$

where

i) as usual $\Omega = \{\text{all } \omega\}$ is the supporting set of the complete probability measure space $(\Omega, \mathrm{A}, \mu)$ and $x(t;\omega)$ is the unknown n-dimensional vector-valued random function defined on $\mathbb{R}_+$;
ii) under appropriate conditions the stochastic kernel $k(\tau,x(t;\omega);\omega)$ is an n-dimensional vector-valued random function defined on $\mathbb{R}_+$;
and
iii) for each $t \in \mathbb{R}_+$ and each random vector $x(t;\omega)$, the stochastic free term $h(t,x(t;\omega))$ is an n-dimensional vector-valued random variable.

We now state an existence theorem.

Theorem 2.2.2. *Assume that equation (2.2.1) satisfies the following conditions:*

(i) $B \subseteq C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$ *and* $D \subseteq C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$ *are Banach spaces stronger than* $C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$;
(ii) *the pair* (B,D) *is admissible with respect to the operator* T *given by* $(Tx)(t;\omega) = \int_0^t x(t;\omega)d\tau$;

(iii) $k(t,x(t;\omega);\omega)$ *is a mapping from the set* $D_\rho = \{x(t;\omega) \in D : \|x(t;\omega)\|_D \leqslant \rho,$ $\rho \geqslant 0\}$ *into the space* B *such that* $\|k(t,x(t;\omega);\omega) - k(t,y(t;\omega);\omega)\|_B \leqslant \lambda \|x(t;\omega) - y(t;\omega)\|_D$ *for* $x(t;\omega)$ *and* $y(t;\omega)$ *in* D_ρ *and* $\lambda \geqslant 0$ *a constant;*
and

(iv) $x(t;\omega) \to h(t,x(t;\omega))$ *is a mapping from* D_ρ *into* D *such that* $\|h(t,x(t;\omega) - h(t,y(t;\omega))\|_D \leqslant \gamma \|x(t;\omega) - y(t;\omega)\|_D$ *for some* $\gamma \geqslant 0$.

Then there exists a unique random solution of equation (2.2.1) in D_ρ *provided that* $\gamma + \lambda M < 1$ *where* $M = \|T\|_0$ *and* $\|h(t,x(t;\omega))\|_D + M\,\|k(t,x(t;\omega);\omega)\|_B \leqslant \rho$.

The conditions on the above theorem can be weakened somewhat. We prove the following

Corollary 2.2.1. *Assume that equation (2.1.1) satisfies the conditions of Theorem 2.2.3. Then there exists a unique random solution if* $\gamma + \lambda M \leqslant 1$ *where* $M = \|T\|_0$ *and*

$$\|h,(t,x(t;\omega))\|_D + M\,\|k,(t,x(t;\omega))\|_B \leqslant \rho.$$

Proof. Note that the operator $(Tx)(t;\omega) = \int_0^t x(\tau;\omega)d\tau$ is continuous from B to D, hence bounded. We shall define a contraction mapping on D_ρ and then apply Banach's fixed point theorem. Define the operator U from D_ρ into D by

$$(Ux)(t;\omega) = h(t,x(t;\omega)) + \int_0^t k(\tau,x(\tau;\omega);\omega)d\tau.$$

To show inclusions consider any $x(t;\omega) \in D_\rho$.

$$\begin{aligned}\|(Ux)(t;\omega)\|_D &= \left\| h\left(t,x(t;\omega) + \int_0^t k(\tau,x(\tau;\omega);\omega)\,d\tau\right)\right\|_D \\ &\leqslant \|h(t,x(t;\omega))\|_D + \left\|\int_0^t k(\tau,x(\tau;\omega);\omega)\,d\tau\right\|_D \\ &\leqslant \|h(t,x(t;\omega))\|_D + M\,\|k(t,x(t;\omega);\omega)\|_B \leqslant \rho, \quad \text{by hypothesis.}\end{aligned}$$

Hence $(Ux)(t;\omega) \in D_\rho$ or $UD_\rho \subseteq D_\rho$.

Now let $x(t;\omega)$ and $y(t;\omega)$be elements of D_ρ. Since $(Ux)(t;\omega)$ and $(Uy)(t;\omega)$ are elements of the Banach space D, $[(Ux)(t;\omega) - (Uy)(t;\omega)] \in D$.

Thus,

$$
\begin{aligned}
&\|(Ux)(t;\omega)-(Uy)(t;\omega)\|_D \\
&= \left\| h(t,x(t;\omega)) + \int_0^t k(\tau,x(\tau;\omega);\omega)d\tau - h\left(t,y(t;\omega) - \int_0^t k(\tau,y(\tau;\omega);\omega)d\tau\right)\right\|_D \\
&= \left\| h(t,x(t;\omega)) - h(t,x(t;\omega)) + \int_0^t [k(\tau,x(\tau;\omega);\omega) - k(\tau,y(\tau;\omega);\omega)d\tau] \right\|_D \\
&\leqslant \|h(t,x(t;\omega)) - h(t,y(t;\omega))\|_D + \left\| \int_0^t [k(\tau,x(\tau;\omega);\omega) - k(\tau,y(\tau;\omega);\omega)d\tau] \right\|_D \\
&\leqslant \gamma\|x(t;\omega)-y(t;\omega)\|_D + \|T\|_0\|k(t,x(t;\omega);\omega) - k(t,y(t;\omega);\omega)\|_B \\
&\leqslant \gamma\|x(t;\omega)-y(t;\omega)\|_D + M\|k(t,x(t;\omega);\omega) - k(t,y(t;\omega);\omega)\|_B \\
&\leqslant \gamma\|x(t;\omega)-y(t;\omega)\|_D + M\lambda\|x(t;\omega)-y(t;\omega)\|_D \\
&= (\gamma+M\lambda)\|x(t;\omega)-y(t;\omega)\|_D.
\end{aligned}
$$

Thus we see that we need only to require that $(\gamma+M\lambda)\leqslant 1$ for the condition of the contraction mapping principle to be satisfied. Then, by Banach's fixed point theorem, there exists a unique point $x(t;\omega)\in D_\rho$ such that

$$
(Ux)(t;\omega) = h(t,x(t;\omega)) + \int_0^t k(\tau,x(\tau;\omega);\omega)\,d\tau = x(t;\omega). \qquad \square
$$

2.3 Existence of a Solution for a Stochastic Linear Pursuit-Evasion Game

2.3.1 *A General Stochastic Linear Pursuit-Evasion Game*

Consider a stochastic linear pursuit-evasion game described by a stochastic transition equation in (2.1.1). The problem is to choose controls $v(t;\omega)$ and $u_v(t;\omega)$ such that $x(t_{u,v};\omega)\in M_\varepsilon$ for some finite time $t_{u,v}$ where the terminal set M_ε is defined by

$$
M_\varepsilon = \left\{x(t;\omega); \|x(t;\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} \leqslant \varepsilon\right\}.
$$

As mentioned previously, we give only one transition equation. In case we have two objects, called the pursuer and evader, then we can consider $x(t;\omega)$ as the distance between them. The evader tries to maximize this distance or maximize the time until $\|x(t;\omega)\|\leqslant\varepsilon$ while the pursuer tries to minimize these conditions. Thus, by a simple transformation, a pursuit-evasion game becomes a contest to bring a point in n-dimensional space into an ε-ball about the origin. The pursuer, using $u(t;\omega)$, tries to minimize the time required while the evader, using $v(t;\omega)$, tries to maximize the time. If possible, he would like it to be infinite.

The state space of a differential game can be thought of as divided into two regions. In one region one player is able to force a win on the other; while in the other region the reverse happens. Isaacs uses the term *barrier* to define the boundary between the two regions. The physical interpretation is that if the initial state is outside the barrier, then the state can never be brought to the origin. That is, escape always occurs outside the barrier. From the control theory point of view, this represents an uncontrollable region. Inside the barrier, in the controllable region, capture always occurs.

In the deterministic setting Pontryagin [139], Pshenichnity [142], Sakawa [160], and other researchers have all given conditions which are sufficient for a linear differential game to be completed. We now consider conditions for completing the most general stochastization of a linear pursuit-evasion game.

The stochastic transition equation (2.2.1) is the most general formalization of a stochastic linear pursuit-evasion game in the sense that all the functions involved are stochastic. It is more general because the random function $x(t;\omega)$ appears on the right hand side. Physically this means that the object (s) being controlled have energy of their own. We may think, for example, of an incoming guided missile Dix [46]. The missile has its own guidance system; and its mission is to descend to a certain altitude over a given city before exploding. The pursuer (enemy in this case) is also sending control signals to the missile while our own forces (the evader) are trying to jam the signals as well as the onboard controls.

2.3.2 *A Special Case of Equation (2.2.1)*

Equation (2.1.1) is equivalent to a vector stochastic integral equation of the form

$$x(t;\omega) = x_0(\omega)e^{A(\omega)t} + \int_0^t e^{A(\omega)(t-\tau)}\left[B(\omega)u(\tau,\omega) - C(\omega)v(\tau;\omega)\right]d\tau, \quad t \geqslant 0 \quad (2.3.1)$$

for which we now give conditions for the existence and uniqueness of a random solution. Referring to equation (2.2.1) we can make the following identifications:

$$h(t,x(t;\omega)) = x_0(\omega)e^{A(\omega)t}$$
$$k(t,x(\tau;\omega);\omega) = e^{A(\omega)(t-\tau)}\left[B(\omega)u(\tau;\omega) - C(\omega)v(\tau;\omega)\right].$$

We note that conditions (ii) and (iii) under equation (2.2.1) are satisfied. In particular

ii) the stochastic kernel is an n-dimensional vector valued random function from $\mathbb{R}_+$ into $L_2(\Omega, \mathrm{A}, \mu)$;
and

iii) the stochastic free term $x_0(\omega)e^{A(\omega)t}$ is an n-dimensional vector-valued random variable, i.e. for each $t \in \mathbb{R}_+$, $x_0(\omega)e^{A(\omega)t} \in L_2(\Omega, \mathrm{A}, \mu)$.

Note that the Banach space $C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$ satisfies the definition of *stronger than itself*. Thus we can use the space $C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$ in place of both B and D in Theorem 2.2.3. Clearly the pair (C_c, C_c) is admissible with respect to T given by $(Tx)(t;\omega) = \int_0^t x(\tau;\omega)d\tau$. Condition (iii) of Theorem 2.2.3 is satisfied vacuously since $x(t;\omega)$ does not appear explicitly in the stochastic kernel. That is,

$$\|k(t, x(t;\omega);\omega) - k(t, y(t;\omega);\omega)\|_{C_c} = 0 \ \ \mu - \text{a.e.}$$

We proceed by proving a theorem concerning the existence and uniqueness of a random solution for equation (2.3.1) and hence (2.1.1).

Theorem 2.3.1. *Given any $\rho \geqslant 0$, define the set D_ρ by*

$$D_\rho = \{x(t;\omega) \in C_c : \|x(t;\omega)\|_{C_c} \leqslant \rho\}.$$

There exists a unique random solution of equation (2.3.2) provided that

(i) *the initial condition $x(0;\omega) = x_0(\omega) \in D_\rho$*
and
(ii) $\left|e^{A(\omega)t}\right| \leqslant 1$.

Proof. The proof of this theorem will consist of showing that all the conditions of Corollary 2.2.4 are satisfied.

1) The Banach space $C_c(\mathbb{R}_+, L_2(\Omega, \mathrm{A}, \mu))$ satisfies the definition of stronger than itself.
2) The pair (C_c, C_c) is admissible with respect to the operator T given by
$$(Tx)(t;\omega) = \int_0^t x(\tau;\omega)d\tau.$$
3) The stochastic kernel is a mapping from the set D into the space C_c such that $\|k(t, x(t;\omega);\omega) - k(t, y(t;\omega);\omega)\|_{C_c} = 0$ μ-a.e. for $x(t;\omega)$ and $y(t;\omega)$ in D_ρ. We just take $\lambda = 0$.
4) The stochastic free term is such that
$$\left\|x_0(\omega)e^{A(\omega)t} - y_0(\omega)e^{A(\omega)t}\right\|_{C_c} \leqslant \gamma \|x_0(\omega) - y_0(\omega)\|_{C_c} \text{ for some } \gamma \geqslant 0.$$

We just take $\gamma = \left|e^{A(\omega)t}\right|$. The conclusion then reduces to: There exists a unique random solution of equation (2.3.1) in D_ρ provided that $\gamma \leqslant 1$. We have assumed that $\left|e^{A(\omega)t}\right| \leqslant 1$; hence, the theorem is proven. □

In the next section of this study we shall consider the existence and uniqueness of a random solution of a stochastic linear pursuit-evasion game with deterministic controls.

2.4 The Solution of a Stochastic Linear Pursuit-Evasion Game With Nonrandom Controls

In this section we shall be concerned with stochastic pursuit-evasion games described by stochastic linear differential equations of the form

$$\frac{d}{dt}x(t;\omega) = A(\omega)x(t;\omega) + Bu(t) - Cv(t), \quad t \geqslant 0 \tag{2.4.1}$$

where

i) $\omega \in \Omega$, where Ω is the supporting set of a complete measure space $(\Omega, \mathrm{A}, \mu)$;
ii) $x(t;\omega)$ is the unknown $(n \times 1)$ random state variable;
iii) $u(t) \in E^r$ is the control vector of the pursuer, $v(t;\omega) \in E^s$ is the control vector of the evader; and
iv) $A(\omega)$ is a $(n \times n)$ random matrix and B and C are respectively $(n \times r)$ and $(n \times s)$.

It is immediately obvious that equation (2.4.1) is a special case of equation (2.1.1). This equation is still general in the sense that $x(t;\omega)$ appears on the right hand side; but here we consider deterministic control vectors. Physically this means that the controllers are attempting to control a randomly varying object with non-random controls. Thinking of an incoming missile, the path which it is following cannot be fitted to a deterministic trajectory. On the other hand if we are thinking of $x(t;\omega)$ as some function of the distance between a pursuer and an evader, then $x(t;\omega)$ could be random because either or both of the players are following random paths or because the players cannot measure the distance accurately.

Mathematically this means that the state is being affected by some stochastic process $w(t;\omega)$, but since the players cannot observe Ω, they seek optimal deterministic controls.

The purpose of this chapter is threefold. We will find the smallest max-min completion time for the game (2.4.1) as well as optimal controls for the pursuer and evader. Finally we will give sufficient conditions for completion of the game in a finite time.

2.4.1 *Preliminaries*

The above random differential system (2.4.1) can easily be reduced to the stochastic vector integral equation

$$x(t;\omega) = \Phi(t;\omega)x_0(\omega) + \int_0^t \Phi(t-\tau;\omega)[Bu(\tau) - Cv(\tau)]d\tau.$$

with initial conditions

$$x(0;\omega) = x_0(\omega)$$

where the matrix $\Phi(t;\omega)$ is given by $\Phi(t;\omega) = e^{A(\omega)(t)}$.

The problem is to choose controls $v(t)$ and $u_v(t)$ such that $x(t_{u,v};\omega) \in M_\varepsilon$ for some finite time $t_{u,v}$, where M_ε was defined in Section 2.2 as an ε-ball about the origin.

We shall consider the random solution $x(t;\omega)$ and the stochastic free term $\Phi(t;\omega)$ as functions of the real argument t with values in the space $L_2(\Omega, \mathrm{A}, \mu)$. The function $[Bu(t) - Cv(t)]$ is also a function of the real argument t whose values are in $L_2(\Omega, \mathrm{A}, \mu)$. The stochastic kernel $\Phi(t-\tau;\omega)$ is an essentially bounded function with respect to μ for every t and τ, $0 \leqslant \tau \leqslant t < \infty$, with values in $L_\infty(\Omega, \mathrm{A}, \mu)$. Thus the product $\Phi(t-\tau;\omega)[Bu(\tau) - Cv(\tau)]$ will always be in the space $L_2(\Omega, \mathrm{A}, \mu)$. We shall assume that the mapping

$$(t,\tau) \to \Phi(t-\tau;\omega)$$

from the set

$$\Delta = \{(t,\tau) : 0 \leqslant \tau \leqslant t < \infty\}$$

into $L_\infty(\Omega, \mathrm{A}, \mu)$ is continuous. That is,

$$\mu\text{-}\operatorname*{ess\,sup}_{\omega} |\Phi(t_n - \tau_n;\omega) - \Phi(t-\tau;\omega)| \to 0$$

as $n \to \infty$ whenever $(t_n, \tau_n) \to (t,\tau)$ as $n \to \infty$.

We shall define as *admissible controls* all measureable functions whose values belong (almost everywhere) to some given compact sets $U \subset E^r$ and $V \subset E^s$. $u(t) \in U$, $v(t) \in V$ for $t \geqslant 0$. Further, we shall assume that U is convex.

The terminal set, M_ε, is just an ε-ball about the zero element of $L_2(\Omega, \mathrm{A}, \mu)$. As mentioned previously, the problem is to choose admissible controls $v(t)$ and $u_v(t)$ such that

$$\Phi(t_{u,v};\omega)x_0(\omega) + \int_0^{t_{u,v}} \Phi(t_{u,v} - \tau)[Bu_v(\tau) - Cv(\tau)]\,d\tau \in M_\varepsilon \tag{2.4.2}$$

for some $t_{u,v} \in \mathbb{R}_+$.

Definition 2.4.1. The game (2.4.1) is said to be *completed from an initial point* $x(0;\omega) = x_0(\omega)$, if, no matter what control $v(t)$ the evader chooses, the pursuer can choose a control $u_v(t)$ such that $x(t;\omega) \in M_\varepsilon$ for some finite time t.

We shall define the functions $H_U(\eta)$ and $H_V(\xi)$ by

$$\begin{aligned} H_U(\eta) &= \sup_{u\in U} \eta u; \\ H_V(\xi) &= \sup_{v\in V} \xi v \end{aligned} \tag{2.4.3}$$

where η and ξ are arbitrary $(r \times 1)$ and $(s \times 1)$ vectors. Then there exist vectors $u_\eta \in U$ and $v_\xi \in V$ such that

$$\begin{aligned} H_U(\eta) &= \sup_{u \in U} \eta u = \eta u_\eta; \quad \text{and} \\ H_V(\xi) &= \sup_{v \in V} \xi v = \xi v_\xi. \end{aligned} \tag{2.4.4}$$

It can be shown that the function $H_U(\eta)\,[H_V(\xi)]$ defined by (2.4.3) is continuous with respect to $\eta[\xi]$. Furthermore, if $u_\eta\,\left[V_\xi\right]$ is uniquely determined in some neighborhood of $\eta[\xi]$, then $u_\eta\,\left[V_\xi\right]$ is continuous in that neighborhood.

For convenience we shall define the $(n \times r)$ and $(n \times s)$ matrices $K(t;\omega)$ and $L(t;\omega)$ by

$$\begin{aligned} K(t;\omega) &= \Phi(t;\omega)B; \\ L(t;\omega) &= \Phi(t;\omega)C. \end{aligned}$$

Equation (2.4.2) can now be rewritten as

$$\Phi(t_{u,v};\omega)\,x_0(\omega) + \int_0^{t_{u,v}} K(\tau;\omega)u_v(t_{u,v}-\tau)d\tau - \int_0^{t_{u,v}} L(\tau;\omega)v(t_{u,v}-\tau)d\tau \in M_\varepsilon \tag{2.4.5}$$

Theorem 2.4.1. *Given any admissible control $v(t)$, a necessary and sufficient condition for the existence of an admissible control $u_v(t)$ such that (2.3.5) holds for some finite time $t_{u,v} \geqslant 0$ is the existence of a $t \in \mathbb{R}_+$ such that*

$$-\varepsilon \leqslant \lambda\Phi(t;\omega)x_0(\omega) + \int_0^t H_U(\lambda K(\tau;\omega))d\tau - \int_0^t H_V(\lambda L(\tau;\omega))d\tau \tag{2.4.6}$$

for all $(1 \times n)$ vectors $\lambda(\omega) = \lambda$ such that $\|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)} = 1$.

Proof. Let λ be an arbitrary $(1 \times n)$ vector such that $\|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)} = 1$. Multiplying the left hand side of line (2.3.5) by $-\lambda$ on the left and applying Schwarz's inequality gives

$$-\lambda\Phi(t_{u,v};\omega)\,x_0(\omega) - \int_0^{t_{u,v}} \lambda K(\tau;\omega)u_v(t_{u,v}-\tau)d\tau + \int_0^{t_{u,v}} \lambda L(\tau;\omega)v(t_{u,v}-\tau)d\tau \leqslant \varepsilon.$$

Since the above inequality must hold for all $v(t) \in V$, it must hold for $\sup_{v\in V} \lambda L(t;\omega)v = H_V(\lambda L(t;\omega)) \geqslant \lambda L(t;\omega)v(t_{u,v}-t)$.

By definition, $H_U(\lambda K(t;\omega)) \geqslant \lambda K(t;\omega)u_v(t_{u,v}-t)$. Hence

$$\lambda\Phi(t_{u,v};\omega)\,x_0(\omega) + \int_0^{t_{u,v}} H_U(\lambda K(\tau;\omega))\,d\tau - \int_0^{t_{u,v}} H_V(\lambda L(\tau;\omega))d\tau \geqslant -\varepsilon.$$

Putting $t = t_{u,v}$ yields condition (2.4.6).

Now suppose that there is an admissible control $v(t)$ such that no admissible control $u_v(t)$ exists such that (2.4.5) holds for some finite time t. This means that the compact, convex set defined by

$$\left\{ \int_0^t k(\tau;\omega)u(t-\tau)d\tau : \;\; u(-\tau) \in U \right\}$$

does not intersect the compact sphere

$$-\Phi(t;\omega)x_0(\omega)+\int_0^t L(\tau;\omega)v(t-\tau)d\tau+M_\varepsilon.$$

Therefore, there is a vector $\lambda \in L_2(\Omega,\mathrm{A},\mu)$, $\|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)}=1$, such that

$$-\lambda\Phi(t;\omega)x_0(\omega)+\int_0^t \lambda L(\tau;\omega)v(t-\tau)d\tau+\lambda a>\int_0^t K(\tau;\omega)u(t-\tau)d\tau \tag{2.4.7}$$

for all $u(t)\in U$, $0\leqslant\tau\leqslant t<\infty$, and for all $a\in M_\varepsilon$. Since inequality (2.4.7) must hold for a $u(t)\in U$ such that

$$\lambda K(\tau;\omega)u(t-\tau)=H_U(\lambda K(\tau;\omega))=\sup_{u\in U}\lambda K(\tau;\omega)u$$

and for a vector $\alpha=-\varepsilon\lambda'\in M_\varepsilon$, and since

$$\int_0^t H_V\left(\lambda L(\tau;\omega)\right)d\tau\geqslant\int_0^t \lambda L(\tau;\omega)v(t-\tau)d\tau,$$

$$\lambda\left(-\varepsilon\lambda'\right)>\lambda\Phi(t;\omega)x_0(\omega)+\int_0^t H_U(\lambda K(\tau;\omega))d\tau-\int_0^t H_V\left(\lambda L(\tau;\omega)\right)d\tau$$

contradicting inequality (2.4.6) □

Corollary 2.4.1. *Given any admissible control $v(t)$, a necessary and sufficient condition for the existence of an admissible control $u_v(t)$ such that (2.4.5) holds for some finite time $t_{u,v}\geqslant 0$ is that there exists a $t\in\mathbb{R}_+$ such that*

$$\inf_{\lambda\in Q}\left[\lambda\Phi(t;\omega)x_0(\omega)+\int_0^t H_U(\lambda K(\tau;\omega))d\tau-\int_0^t H_V\left(\lambda L(\tau;\omega)\right)d\tau\right]\geqslant-\varepsilon$$

where Q is a set of $(1\times n)$ vectors $\lambda\in L_2(\Omega,\mathrm{A},\mu)$ such that $\|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)}=1$.

We shall denote by $u(t,\lambda)$ and $v(t,\lambda)$ the vectors $u\in U$ and $v\in V$ which maximize $\lambda K(t;\omega)u$ and $\lambda L(t;\omega)v$. That is,

$$H_U(\lambda K(t;\omega))=\sup_{u\in U}\lambda K(t;\omega)u=\lambda K(t;\omega)u(t,\lambda)$$

and

$$H_V(\lambda L(t;\omega))=\sup_{v\in V}\lambda L(t;\omega)v=\lambda L(t;\omega)v(t,\lambda).$$

Assume that for each $\lambda\in Q$, the controls $u(\tau,\lambda)$ and $v(\tau,\lambda)$ are uniquely determined for all $\tau\in[0,T]$ except on a set of measure zero. Then, see the remark following equation (2.4.4), the controls $u(\tau,\lambda)$ and $v(\tau,\lambda)$ are piecewise continuous on $[0,T]$.

The scalar function $F\left(t,\lambda;\omega,x_0(\omega)\right)$ will be defined by

$$\begin{aligned}&F\left(t,\lambda;\omega,x_0(\omega)\right)\\&=\lambda\Phi(t;\omega)x_0(\omega)+\int_0^t H_U(\lambda K(\tau;\omega))d\tau-\int_0^t H_V(\lambda L(\tau;\omega))d\tau \qquad (2.4.8)\\&=\lambda\Phi(t;\omega)x_0(\omega)+\lambda\int_0^t K(\tau;\omega)u(\tau,\lambda)(\lambda)d\tau-\lambda\int_0^t L(\tau;\omega)v(\tau,\lambda)d\tau.\end{aligned}$$

Lemma 2.4.1. *The gradient vector with respect to λ of the function $F(t,\lambda;\omega,x_0(\omega))$ is given by*

$$\operatorname{grad}_\lambda F(t,\lambda;\omega,x_0(\omega)) = x(t,\lambda;\omega,x_0(\omega))$$

where

$$x(t,\lambda;\omega,x_0(\omega)) = \Phi(t;\omega)x_0(\omega) + \int_0^t K(\tau;\omega)u(\tau,\lambda)d\tau - \int_0^t L(\tau;\omega)v(\tau,\lambda)d\tau. \quad (2.4.9)$$

Moreover $\operatorname{grad}_\lambda F(t,\lambda;\omega,x_0(\omega))$ *is continuous in t and λ.*

Proof. Let γ be an arbitrary $(1\times n)$ vector. Then, from the definition of $u(t,\lambda)$,

$$\begin{aligned} H_U((\lambda+\gamma)K(t;\omega)) - H_U(\lambda K(t;\omega)) &\geqslant (\lambda+\gamma)K(t;\omega)u(t,\lambda) - \lambda K(t;\omega)u(t,\lambda) \\ &= \gamma K(t;\omega)u(t,\lambda), \end{aligned}$$

and

$$\begin{aligned} H_U((\lambda+\gamma)K(t;\omega)) - H_U(\lambda K(t;\omega)) &\leqslant (\lambda+\gamma)K(t;\omega)u(t,\lambda+\gamma) - \lambda K(t;\omega)u(t,\lambda+\gamma) \\ &= \gamma K(t;\omega)u(t,\lambda+\gamma). \end{aligned}$$

Integrating with respect to t we get

$$\begin{aligned} \gamma\int_0^t K(\tau;\omega)u(\tau,\lambda)\,d\tau &\leqslant \int_0^t H_U((\lambda+\gamma)K(\tau;\omega))d\tau - \int_0^t H_U(\lambda K(t;\omega))d\tau \\ &\leqslant \int_0^t K(\tau;\omega)u(\tau,\lambda+\gamma). \end{aligned} \quad (2.4.10)$$

Let $t_1,t_2,\ldots,t_N$ $(0<t_1<t_2<\cdots<t_N<t)$ be the points where $u(t,\lambda)$ is not continuous and define the following subintervals of $[0,t]$:

$$\begin{aligned} I_0(\varepsilon) &= [0,\varepsilon) \\ I_i(\varepsilon) &= (t_i-\varepsilon, t_i+\varepsilon), \quad i=1,2,\ldots,N \\ I_{N+1}(\varepsilon) &= (t-\varepsilon,t] \\ I(\varepsilon) &= [0,t] - \bigcup_{i=0}^{N+1} I_i(\varepsilon). \end{aligned}$$

By the continuity of u, for sufficiently small $\varepsilon>0$ there exists a $\delta=\delta(\varepsilon)>0$ such that if $\|\gamma\|<\delta(\varepsilon)$ and $t\in I(\varepsilon)$, then $\|u(t,\lambda+\gamma)-u(t,\lambda)\|<\varepsilon$.
Since U is compact (closed and bounded), there is a $k>0$ such that $\|u(t,\lambda+\gamma)-u(t,\lambda)\|<k$ if $t\in\bigcup_{i=0}^{N+1} I_i(\varepsilon)$.
Therefore,

$$\int_0^t \|u(t,\lambda+\gamma)-u(t,\lambda)\|d\tau < \varepsilon t + 2\varepsilon(N+1)k. \quad (2.4.11)$$

Inequalities (2.4.10) and (2.4.11) imply that

$$\mathrm{grad}_\lambda \int_0^t \lambda K(\tau;\omega)u(\tau,\lambda)d\tau = \int_0^t K(\tau;\omega)u(\tau,\lambda)d\tau.$$

similarly,

$$\mathrm{grad}_\lambda \int_0^t \lambda L(\tau;\omega)v(\tau,\lambda)d\tau = \int_0^t L(\tau;\omega)v(\tau,\lambda)d\tau.$$

Hence (2.4.9) is proven. The continuity of $\mathrm{grad}_\lambda F(t,\lambda;\omega,x_0(\omega))$ is evident from the course of the proof. □

Since $F(t,\lambda;\omega,x_0(\omega))$ is continuous in λ and the set

$$Q = \left\{\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_n) \in L_2(\Omega,\mathrm{A},\mu) : \|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)} = 1\right\}$$

is compact. Thus, there is a $\lambda \in Q$ which attains the infimum of $F(t,\lambda;\omega,x_0(\omega))$. Let us denote it by λ_t. That is

$$\inf_{\lambda\in Q} F(t,\lambda;\omega,x_0(\omega)) = F(t,\lambda_t;\omega,x_0(\omega)).$$

For convenience, when the initial condition is $x_0(\omega)$, we will write $F_\omega(t,\lambda)$ and $x_\omega(t,\lambda)$ instead of $F\,(t,\lambda;\omega,x_0(\omega))$and $x\,(t,\lambda;\omega,x_0(\omega))$ respectively.

Lemma 2.4.2. *We have*

$$\inf_{\lambda\in Q} F_\omega(t,\lambda) = F_\omega(t,\lambda_t) = -\|x_\omega(t,\lambda_t)\|_{L_2(\Omega,\mathrm{A},\mu)} \tag{2.4.12}$$

where $x_\omega(t,\lambda_t)$ is given by equation (2.4.9).

Proof. Since $\min F_\omega(t,\lambda)$ is sought for $\|\lambda\|^2_{L_2(\Omega,\mathrm{A},\mu)} - 1 = 0$ and t is fixed, define

$$\overline{F}_\omega\,(t,\lambda,\theta) = F_\omega(t,\lambda) + \theta\left(\|\lambda\|^2_{L_2(\Omega,\mathrm{A},\mu)} - 1\right)$$

where θ is a Lagrange multiplier. Put

$$\frac{\partial}{\partial\lambda_i}\overline{F}_\omega = x_i(t,\lambda) + 2\theta\lambda_i = 0, \quad i = 1,2,\ldots,n$$

where $x_i(t,\lambda)$ and λ_i denote the ith components of $x_\omega(t,\lambda)$ and λ respectively. Solving we get

$$\lambda_i = \frac{x_i}{2\theta}.$$

$$\|\lambda\|^2_{L_2(\Omega,\mathrm{A},\mu)} = \int_\Omega \sum_{i=1}^n \left(\frac{x_i}{2\theta}\right)^2 d\mu(\omega) = 1.$$

$$\int_\Omega \sum_{i=1}^n x_1^2\, d\mu(\omega) = 4\theta^2.$$

$$\|x_\omega(t,\lambda)\|_{L_2(\Omega,\mathrm{A},\mu)} = 2\theta.$$

Hence,

$$\lambda_t = \frac{-x'_\omega(t,\lambda)}{\|x_\omega(t,\lambda)\|_{L_2(\Omega,\mathrm{A},\mu)}}. \tag{2.4.13}$$

Substituting line (2.4.13) into line (2.4.8) gives the desired result (2.4.12). □

Lemma 2.4.3. *Let us assume that for any time $t > 0$ and for $\lambda_1, \lambda_2 \in Q$,*

$$\|x_\omega(t,\lambda_1)\|_{L_2(\Omega,\mathrm{A},\mu)} = \|x_\omega(t,\lambda_2)\|_{L_2(\Omega,\mathrm{A},\mu)}$$

implies that $\lambda_1 = \lambda_2$. Then,

$$\frac{d}{dt}F_\omega(t,\lambda_t) = \lambda_t A(\omega)\Phi(t;\omega)x_0(\omega) + H_U(\lambda_t K(t;\omega)) - H_V(\lambda_t L(t;\omega)).$$

Proof. Let δ be an arbitrary real number. Since the matrix $\Phi(t;\omega) = e^{A(\omega)t}$, we see that

$$\Phi(t+\delta;\omega) = \Phi(t;\omega) + \int_t^{t+\delta} A(\omega)\Phi(\tau;\omega)d\tau.$$

Then,

$$\begin{aligned} F_\omega(t+\delta,\lambda) &= F_\omega(t,\lambda) \\ &+ \int_t^{t+\delta} [\lambda A(\omega)\Phi(\tau;\omega)x_0(\omega) + H_U(\lambda K(\tau;\omega)) - H_V(\lambda L(\tau;\omega))]d\tau. \end{aligned} \tag{2.4.14}$$

Now, by the definition of λ_t,

$$F_\omega(t,\lambda_{t+\delta}) \geqslant F_\omega(t,\lambda_t) = \inf_{\lambda\in Q} F_\omega(t,\lambda).$$

Thus, from (2.4.14), we get

$$\begin{aligned} F_\omega(t+\delta,\lambda_{t+\delta}) - F_\omega(t,\lambda_t) &\geqslant \int_t^{t+\delta} \lambda_{t+\delta}A(\omega)\Phi(\tau;\omega)x_0(\omega)d\tau \\ &+ \int_t^{t+\delta} [H_U(\lambda_{t+\delta}K(\tau;\omega)) - H_V(\lambda_{t+\delta}L(\tau;\omega))]\,d\tau. \end{aligned} \tag{2.4.15}$$

On the other hand, $F_\omega(t+\delta,\lambda_{t+\delta}) \leqslant F_\omega(t+\delta,\lambda_t)$ implies that

$$F_\omega(t+\delta,\lambda_{t+\delta}) - F_\omega(t,\lambda_t) \leqslant F_\omega(t+\delta,\lambda_t) - F_\omega(t,\lambda_t). \tag{2.4.16}$$

Since F is continuous in t, inequalities (2.4.15) and (2.4.16) show the continuity of F in t and λ_t. That is,

$$F_\omega(t+\delta,\lambda_{t+\delta}) \to F_\omega(t,\lambda_t) \quad \text{as } \delta \to 0. \tag{2.4.17}$$

From equation (2.4.12) it is clear that the assumption of Lemma 2.4.5 implies the uniqueness of the $\lambda \in Q$ which attains the infimum of $F_\omega(t,\lambda)$. It then follows from the continuity of F_Ω, line (2.4.17), that

$$\lambda_{t+\delta} \to \lambda_t \quad \text{as } \delta \to 0. \tag{2.4.18}$$

If $\delta > 0$, we get from inequalities (2.4.15) and (2.4.16)

$$\begin{aligned} &\frac{1}{\delta}\int_t^{t+\delta} [\lambda_{\tau+\delta}A(\omega)\Phi(\tau;\omega)x_0(\omega) + H_U(\lambda_{\tau+\delta}K(\tau;\omega)) - H_V(\lambda_{\tau+\delta}L(\tau;\omega))]d\tau \\ &\qquad \leqslant \frac{1}{\delta}[F_\omega(t+\delta,\lambda_{t+\delta}) - F_\omega(t,\lambda_t)] \\ &\qquad \leqslant \frac{1}{\delta}[F_\omega(t+\delta,\lambda_t) - F_\omega(t,\lambda_t)]. \end{aligned} \tag{2.4.19}$$

In view of (2.4.18) and the continuity of $H_U(\lambda K(t;\omega))$ and $H_V(\lambda L(t;\omega))$ in λ and t, it follows from inequality (2.4.19) that

$$\frac{d}{dt}F_\omega(t,\lambda_t) = \lambda_t A(\omega)\Phi(t;\omega)x_0(\omega) + H_U(\lambda_t K(t;\omega)) - H_V(\lambda_t L(t;\omega)).$$

If $\delta < 0$, the same result holds. Thus the lemma is proven. □

We are now in a position to give conditions under which the game (2.3.1) will have a finite maximum completion time.

2.4.2 *Completion of the Game*

Suppose that $\|x_0(\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} > \varepsilon$ and there is a time $t \in \mathbb{R}_+$ such that

$$\inf_{\lambda\in Q} F_\omega(t,\lambda) = F_\omega(t,\lambda_t) = -\varepsilon. \tag{2.4.20}$$

Theorem 2.4.2. *No matter what admissible control $v(t)$, $t \in \mathbb{R}_+$, the evader chooses, the game can be completed in a time not greater than t_0, where t_0 is the smallest nonnegative time satisfying (2.4.20). Furthermore, no matter what admissible control $u(t)$, $t \in \mathbb{R}_+$, the pursuer chooses, the evader can choose a control $v(t)$ such that the game cannot be completed in a time less than t_0.*

Proof. Given an arbitrary control $v(t) \in V$, $t \in \mathbb{R}_+$, we shall define the function

$$\begin{aligned} F_V(t,\lambda;\omega,x_0(\omega)) &= \lambda\Phi(t;\omega)x_0(\omega) \\ &+\lambda\int_0^t K(\tau;\omega)u(\tau;\lambda)\,d\tau - \lambda\int_0^t L(\tau;\omega)V(t-\tau)d\tau. \end{aligned} \tag{2.4.21}$$

From the definition of $v(t;\lambda)$ and equation (2.4.8) it is clear that

$$F_V(t,\lambda;\omega,x_0(\omega)) \geqslant F(t,\lambda;\omega,x_0(\omega))$$

for all $\lambda \in Q$. Hence,

$$\inf_{\lambda\in Q} F_v(t_0,\lambda;\omega,x_0(\omega)) \geqslant \inf_{\lambda\in Q} F_\omega(t_0,\lambda;\omega,x_0(\omega)) = -\varepsilon. \tag{2.4.22}$$

Let us also define the function

$$\begin{aligned} x_v(t,\lambda_t;\omega,x_0(\omega)) &= \Phi(t;\omega)x_0(\omega) \\ &+\lambda\int_0^t K(\tau;\omega)u(\tau;\lambda_\tau)\,d\tau - \lambda\int_0^t L(\tau;\omega)v(t-\tau)d\tau. \end{aligned} \tag{2.4.23}$$

where $\lambda_t \in Q$ attains the infimum of $F_v(t,\lambda;\omega,x_0(\omega))$ when t and $x_0(\omega)$ are fixed. Then, by Lemma 2.4.4,

$$\inf_{\lambda\in Q} F_V(t,\lambda;\omega,x_0(\omega)) = F_V(t,\lambda_t;\omega,x_0(\omega)) = -\|x_v(t,\lambda_t;\omega,x_0(\omega))\|_{L_2(\Omega,\mathrm{A},\mu)}. \tag{2.4.24}$$

Since $x_v(t,\lambda_t;\omega,x_0(\omega))$ is continuous in time t, and equations (2.4.22) and (2.4.24) imply that

$$-\left\|x_v\left(t_0,\lambda_{t_0};\omega,x_0(\omega)\right)\right\|_{L_2(\Omega,\mathrm{A},\mu)} \geqslant -\varepsilon,$$

there exists a time t^*, $0 \leqslant t^* \leqslant t_0$, such that

$$-\left\|x_v\left(t^*,\lambda_{t^*};\omega,x_0(\omega)\right)\right\|_{L_2(\Omega,\mathrm{A},\mu)} \geqslant -\varepsilon.$$

That is, the game can be completed in a time t^* which is not greater than t_0.
As in (2.4.21) we shall define another function $F_u(t,\lambda;\omega,x_0(\omega))$ by

$$F_u(t,\lambda;\omega,x_0(\omega)) = \lambda\Phi(t;\omega)x_0(\omega) + \lambda\int_0^t K(\tau;\omega)u(t-\tau)d\tau - \lambda\int_0^t L(\tau;\omega)u(\tau,\lambda)d\tau.$$

Now from the definition of $u(t,\lambda)$ and equation (2.4.8) we see that

$$F_u(t,\lambda;\omega,x_0(\omega)) \leqslant F(t,\lambda;\omega,x_0(\omega))$$

for all $\lambda \in Q$. Therefore,

$$\inf_{\lambda\in Q} F_u(t_0,\lambda;\omega,x_0(\omega)) \leqslant \inf_{\lambda\in Q} F(t_0,\lambda;\omega,x_0(\omega)) = -\varepsilon. \tag{2.4.25}$$

Following equation (2.4.23), let us define the function $x_u(t,\lambda_t;\omega,x_0(\omega))$ by

$$x_u(t,\lambda_t;\omega,x_0(\omega)) = \Phi(t;\omega)x_0(\omega) + \int_0^t K(\tau;\omega)u(t-\tau)d\tau - \int_0^t L(\tau;\omega)v(\tau,\lambda_\tau)\,d\tau$$

where $\lambda_t \in Q$ attains the infimum of $F_u(t,\lambda;\omega,x_0(\omega))$. Then again by Lemma 2.4.4,

$$\inf_{\lambda\in Q} F_u(t,\lambda;\omega,x_0(\omega)) - F_u(t,\lambda_t;\omega,x_0(\omega)) = -\left\|x_u(t,\lambda_t;\omega,x_0(\omega))\right\|_{L_2(\Omega,\mathrm{A},\mu)}. \tag{2.4.26}$$

Thus, by (2.4.25) and (2.4.26),

$$-\left\|x_u\left(t_0,\lambda_{t_0};\omega,x_0(\omega)\right)\right\|_{L_2(\Omega,\mathrm{A},\mu)} \leqslant -\varepsilon.$$

That is, the game cannot be completed in time less than t_0. Thus t_0 is the maximin completion time. □

The controls, $u(t) = u\left(t_0-t,\lambda_{t_0}\right)$ and $v(t) = v\left(t_0-t,\lambda_{t_0}\right)$ for $t \in [0,t_0]$, are optimal in the sense that the pursuer wants to complete the game as soon as possible and the evader wants to escape as long as possible. The time t_0 is the smallest maximin completion time of the game. When will a finite time t exist such that (3.3.1) holds?

Theorem 2.4.3. *If (i) the homogeneous stochastic differential equation*

$$\frac{d}{dt}x(t;\omega) = A(\omega)x(t;\omega) \tag{2.4.27}$$

is stochastically asymptotically stable; and (ii) $BU \supset CV$ where $BU = \{Bu : u \in U\}$ and $V = \{Cv; v \in V\}$ are subsets of E^n, then the game can be completed no matter what the initial condition $x_0(\omega) \in L_2(\Omega,\mathrm{A},\mu)$ may be.

Proof. Since $CV \subset BU$, whatever control $v(t) \in V$, $t \in \mathbb{R}_+$, the evader may choose, the pursuer can choose a control, such that

$$Bu(t) = Cv(t) \text{ for all } t \geqslant 0.$$

Since (2.4.27) is assumed to be stochastically asymptotically stable, there is a finite time t such that

$$\|x(t;\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} \leqslant \varepsilon.$$

Since $\Phi(t;\omega) = e^{A(\omega)t}$, $A(\omega)\Phi(t;\omega) = \Phi(t;\omega)A(\omega)$. That is, we can change the order of multiplication. Thus, the conclusion of Lemma 2.3.5 can be written as

$$\frac{d}{dt}F_\omega(t,\lambda_t) = \lambda_t\Phi(t;\omega)A(\omega)x_0(\omega) + \max_{\hat{u}\in BU}\lambda_t\Phi(t;\omega)\hat{u}(t) - \min_{\hat{v}\in CV}\lambda_t\Phi(t;\omega)\hat{v}(t) \quad (2.4.28)$$

$\square$

Theorem 2.4.4. *Assume that for any $t > 0$ and for any $\lambda_1,\ \lambda_2 \in Q$,*

$$\|x_\omega(t,\lambda_1)\|_{L_2(\Omega,\mathrm{A},\mu)} = \|x_\omega(t,\lambda_2)\|_{L_2(\Omega,\mathrm{A},\mu)}$$

implies that $\lambda_1 = \lambda_2$. If there exists a $\delta > 0$ such that

$$-A(\omega)x_0(\omega) + CV + M_\delta \subset BU; \quad (2.4.29)$$

and

$$\|\lambda_t\Phi(t;\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} \geqslant \delta \quad \textit{for all } t \in \mathbb{R}_+,$$

where $M_\delta = \left\{x(t;\omega) : \|x(t;\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} \leqslant \delta\right\}$, then the game starting from $x_0(\omega)$ can be completed.

Proof. Let $\gamma \in L_2(\Omega,\mathrm{A},\mu)$ be an arbitrary $(1\times n)$ vector such that $\|\gamma\|_{L_2(\Omega,\mathrm{A},\mu)} \geqslant \delta > 0$. Then

$$\max_{x(t;\omega)\in M_\delta} \gamma x(t;\omega) = \gamma x_\gamma(t;\omega) \geqslant \delta^2.$$

From relation (2.4.29), for arbitrary $x(t;\omega) \in M_\delta$ and $\hat{v}(t) \in CV$ there is a $\hat{u}(t) \in BU$ such that

$$-A(\omega)x_0(\omega) + \hat{v}(t) + x(t;\omega) = \hat{u}(t).$$

Hence, for all $\hat{v}(t) \in CV$ and for all γ such that $\|\gamma\|L_2(\Omega,\mathrm{A},\mu) \geqslant \delta$, there is a $\hat{u}(t) \in BU$ such that

$$\gamma(\hat{u}(y) - \hat{v}(t) + A(\omega)x_0(\omega)) \geqslant \delta^2 > 0.$$

The above inequality still holds for a $\hat{v}_\gamma$ such that

$$\gamma\hat{v}_\gamma(t) = \max_{\hat{v}(t)\varepsilon CV} \gamma\hat{v}(t).$$

Also

$$\gamma\hat{u}(t) \leqslant \gamma\hat{u}_\gamma = \max_{\hat{u}(t)\in BU} \gamma\hat{u}(t).$$

Hence, for all γ such that $\|\gamma\|_{L_2(\Omega,\mathrm{A},\mu)} \geqslant \delta > 0$,

$$\max_{\hat{u}(t)\varepsilon BU} \gamma\hat{u}(t) - \max_{\hat{v}(t)\in CV} \gamma\hat{v}(t) + \gamma A(\omega)x_0(w) \geqslant \delta^2.$$

Under the assumption of Theorem 2.4.8, Lemma 2.4.5 implies (2.4.28). Setting $\gamma = \lambda_t\Phi(t;\omega)$, we get

$$\frac{d}{dt}F_\omega(t,\lambda_t) \geqslant \delta^2 > 0 \text{ for all } t > 0.$$

Since $F_\omega(0,\lambda_0) = -\|x_0(\omega)\|_{L_2(\Omega,\mathrm{A},\mu)} < -\varepsilon < 0$, it is clear that the game which starts from $x_0(\omega)$ can be completed if $x_0(\omega)$ satisfies relation (2.4.29). □

In Theorem 2.4.6 we gave a condition such that the stochastic linear pursuit-evasion game (2.4.1) will have a maximin completion time. Then, in Theorems 2.4.7 and 2.4.8 we gave sufficient conditions for completion of the game no matter what the starting state is. We now give an interactive procedure for determining the minimum completion time and the optimal controls.

2.4.3 *The Optimal Controls*

Assuming that the game (2.4.1) with initial condition $x(0,\omega) = x_0(\omega)$ can be completed, we can find the minimum completion time t_0 and the vector λ_{t_0} satisfying condition (2.4.20) as follows. Choose $\varepsilon > 0$.

1. Set $\lambda_1 = \dfrac{-x_0'(\omega)}{\|x_0(\omega)\|_{L_2(\Omega,\mathrm{A},\mu)(\Omega,\mathrm{A},\mu)}}$ and them compute $F_\omega(t,\lambda_1)$ for $t \geqslant 0$ up to the time t_1 such that $F(t_1,\lambda_1) = -\varepsilon$. Clearly $t_1 \leqslant t_0$.

2. Let $F_\omega(t_i,\lambda_i) = -\varepsilon$, $i = 1,2,\ldots$, and find $\min_{\lambda\in Q} F_\omega(t_i,\lambda)$ using the gradient method of Lemma 2.3.3. Call it $F_\omega(t_i,\lambda_{i+1})$. That is,

$$\min_{\lambda\in Q} F_\omega(t_i,\lambda) = F_\omega(t_i,\lambda_{i+1}) \leqslant -\varepsilon.$$

3. Compute $F_\omega(t,\lambda_{i+1})$ for $t \geqslant t_i$ up to the time t_{i+1} such that $F_\omega(t_{i+1},\lambda_{i+1}) = -\varepsilon$. It is clear that

$$F_\omega(t,\lambda_{i+1}) \geqslant F_\omega(t,\lambda_t) \text{ for all } t \in [0,t_i+1].$$

4. Repeat steps 2 and3 above for $i = 2,3,\ldots$.

Since $t_i \leqslant t_{i+1} \leqslant t_0$ for all i, $\lim_{i\to\infty} t_i$ exists. Let us denote it by $t_0^* \leqslant t_0$. We have $F_\omega(t_i,\lambda_i) = -\varepsilon$ for all $i = 1,2,\ldots$ and $\lim_{i\to\infty} t_i = t_0^* \leqslant t_0$. Since $F_\omega(t,\lambda_t)$ is continuous in t, we get

$$F_\omega\left(t_0^*,\lambda_{t_0^*}\right) = -\varepsilon.$$

But t_0 is the smallest nonnegative time satisfying line (2.4.20). Thus, $t_0^* = t_0$. Also, $\lambda_{i+1} = \lambda_{t_i} \to \lambda_{t_0}$ from the left. If λ_t is not continuous at t_0, let $\lambda_{t_0}^-$ denote the limit from below. That is, $\lim_{\delta+0} \lambda_{t_0-\delta} = \lambda_{t_0}^-$. Thus the optimal controls are $u(t) = u\left(t_0 - t, \lambda_{t_0}^-\right)$, $v(t) = v\left(t_0 - t, \lambda_{t_0}^-\right)$ for all $t \in [0,t_0]$.

With the iterative procedure described above one can program the game for an electronic computer. It is first necessary to check if the game can indeed be completed. For this it is an easy matter to program the Corollary 2.4.2. That is, we must first check to see if there exists a finite time $t \in \mathbb{R}_+$ such that

$$\inf_{\lambda\in Q}\left[\lambda\Phi(t;\omega)x_0(\omega) + \int_0^{+} H_U(\lambda K(\tau;\omega))d\tau - \int_0^t H_V(\lambda L(\tau;\omega))d\tau\right] \geqslant -\varepsilon$$

where Q is the set of all $(1\times n)$ vectors λ such that $\|\lambda\|_{L_2(\Omega,\mathrm{A},\mu)} = 1$.

In this section we have considered stochastic linear differential games of the form

$$\frac{d}{dt}x(t;\omega) = A(\omega)x(t;\omega) + BU(t) - Cv(t), \quad t \geqslant 0$$

which is a special case of equation (2.1.1)? Here we have taken constant matrices B and C and control sets $U(t)$ and $V(t)$ which are compact subsets of Euclidean spaces. The method of investigation was to first reduce the problem to the existence of a random solution to the stochastic vector integral equation

$$x(t;\omega) = \Phi(t;\omega)x_0(\omega) + \int_0^t \Phi(t-\tau;\omega)[Bu(\tau) - Cv(\tau)]d\tau$$

where $\Phi(t;\omega) = e^{A(\omega)t}$.

We then proved several theorems on completion of the game. Theorem 2.4.1 and the Corollary 2.4.2 give necessary and sufficient conditions for the existence of a control for the pursuer so that he can force completion of the game in a finite time. No matter what controls that two players choose, Theorems 2.4.6 gives a condition sufficient to guarantee the completion of the game and also gives the minimum completion time. Theorem 2.4.7 gives conditions on the control sets, which are independent of the initial condition, which guarantee completion of the game; while Theorem 2.4.8 gives conditions on the control sets and the initial condition which force completion of the game.

Finally we presented an iterative procedure which can be used to find the minimum completion time mentioned in Theorem 2.4.6 and to find the optimal controls to force completion in this time.

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