Preface

The approximation of functions by algebraic polynomials, trigonometric polynomials, and splines, is not only an important topic of mathematical studies, but also provides powerful mathematical tools to such application areas as data representation, signal processing, non-parametric time-series analysis, computer-aided geometric design, numerical analysis, and solutions of differential equations. This book is an introduction to the mathematical analysis of such approximation, with a strong emphasis on explicit approximation formulations, corresponding error bounds and convergence results, as well as applications in quadrature. A mathematically rigorous approach is adopted throughout, and, apart from an assumed prerequisite knowledge of advanced calculus and linear algebra, the presentation of the material is self-contained.

The book is suitable for use as textbook for courses at both upper undergraduate and graduate level. Each of the ten chapters is concluded by a set of exercises, with a total of altogether 220 exercises, some of which are routine, whereas others are concerned with further development of the material.

The book evolved from lecture notes compiled by the author during approximately 25 years of teaching courses in Computational Mathematics and Approximation Theory at Department of Mathematical Science, Stellenbosch University. Several standard textbooks were consulted and used, some more extensively than others, during the preparation of the lecture notes and the resulting book, and include the following, as listed alphabetically according to author:

- N.I. Achieser, *Theory of Approximation* (translated by C.J. Hyman), Frederick Ungar Publishing Co., New York, 1956.
- H. Brass, *Quadraturverfahren*, Vandenhoek & Rupert, Göttingen, 1977.
- E.W. Cheney, *Introduction to Approximation Theory*, McGraw-Hill, New York, 1966.

- Ronald A. DeVore and George G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.
- Walter Gautschi, *Numerical Analysis: An Introduction*, Birkhäuser, Boston, 1997.
- Eugene Isaacson and Herbert Bishop Keller, *Analysis of Numerical Methods*, John Wiley & Sons, Inc., New York, 1966.
- Gunther Nürnberger, *Approximation by Spline Functions*, Springer, Berlin, 1989.
- M.J.D. Powell, *Approximation Theory and Methods,* Cambridge University Press, Cambridge, 1981.
- T.J. Rivlin, *An Introduction to the Approximation of Functions*, Blaisdell Publishing Co., Waltham, Mass, 1969.
- L.L. Schumaker, *Spline Functions: Basic Theory*, John Wiley & Sons, Inc., New York, 1981.

It should be pointed out that, whereas in some of the above-listed books concepts from Functional Analysis, like metric spaces, normed linear spaces and inner product spaces, as well as operators on those spaces, are assumed as prerequisite knowledge, the approach followed in this book is to develop any such concepts from first principles.
The contents of the respective chapters of the book can be summarized as follows.

- **Chapter 1: Polynomial Interpolation Formulas**
  The specific approximation method of polynomial interpolation is introduced, for which an existence and uniqueness theorem is then established by means of a Vandermonde matrix. Next, both the Lagrange and Newton interpolation formulas, as based on, respectively, the Lagrange fundamental polynomials and divided differences, are derived. Finally, an existence and uniqueness result, as well as a recursive computational method, are developed for Hermite interpolation, where also function derivatives are interpolated.
- **Chapter 2: Error Analysis for Polynomial Interpolation**
  A formulation in terms of a divided difference is established for the polynomial interpolation error, and a corresponding error bound is derived. Chebyshev polynomials are introduced, and shown to possess real zeros which yield interpolation points which minimize the error bound for polynomial interpolation.
- **Chapter 3: Polynomial Uniform Convergence**
  The concept of uniform convergence of a sequence of polynomial approximations to a given continuous function on a bounded interval is defined, and shown to apply in selected polynomial interpolation examples. It is then proved rigorously that, in contrast,

divergence to infinity occurs in the Runge example. Next, the Bernstein polynomials are introduced and their properties analyzed, by means of which it is then proved that the sequence of Bernstein polynomial approximations to any given continuous function on a bounded interval uniformly converges to that function, and thereby immediately yielding also the Weierstrass theorem, according to which a continuous function on a bounded interval can be uniformly approximated with arbitrary "closeness" by a polynomial. It is furthermore shown that a Bernstein polynomial approximation has remarkable shape-preservation properties, and, in addition, a convergence rate result is established for the case where the approximated function is continuously differentiable.

- **Chapter 4: Best Approximation**
  In the general setting of normed linear spaces, it is proved that, if the approximation set is a finite-dimensional subspace, then the existence of a best approximation is guaranteed. In addition, a uniqueness result for best approximation is established for those cases where the norm is generated by an inner product. The examples of best uniform polynomial approximation and best weighted L^2 polynomial approximation are highlighted as applications of the theoretical results.
- **Chapter 5: Approximation Operators**
  The notion of an approximation operator from a normed linear space to an approximation set is introduced, and examples from previous chapters are provided. The properties of linearity, exactness and boundedness with respect to approximation operators are discussed and analyzed, with particular attention devoted to operator norms, or Lebesgue constants. In addition, the Lebesgue inequality for bounding an approximation error with respect to the best approximation error is derived. Applications of the theory to particularly the polynomial interpolation operator are provided.
- **Chapter 6: Best Uniform Polynomial Approximation**
  It is proved that the equi-oscillation property of a polynomial approximation error is a necessary and sufficient condition for best uniform polynomial approximation, by means of which the uniqueness of a best uniform polynomial approximation is then established. The resulting best uniform polynomial approximation operator is shown in particular to satisfy a convergence rate which increases with the number of continuous derivatives of the approximated function, and by means of which, together with the Lebesgue inequality from Chapter 5, an interpolation error bound is obtained, from which it is then immediately seen that uniform convergence is obtained for the Runge

example of Chapter 3 if the uniformly distributed interpolation points are replaced by the Chebyshev points of Chapter 2. Finally, examples are provided of cases where explicit calculation of the best uniform polynomial approximation can be performed in a straightforward manner.

- **Chapter 7: Orthogonality**
  In the general setting of inner product spaces, it is proved that best approximation with respect to the norm generated by an inner product is achieved if and only if the approximation error is orthogonal to the approximation subspace, and properties of the corresponding best approximation operator are established. For those cases where the approximation subspace is finite-dimensional, a construction procedure for the best approximation, as based on the inversion of the corresponding Gram matrix, is derived, and it is moreover shown that the availability of an orthogonal basis yields an explicitly formulated best approximation. The Gram-Schmidt procedure for the construction of an orthogonal basis from any given basis is then derived by means of best approximation principles. For polynomials, an efficient three-term recursion formula for orthogonal polynomials with respect to any weighted inner product is obtained, and specialized to the important cases of Legendre polynomials and Chebyshev polynomials.
- **Chapter 8: Interpolatory Quadrature**
  After defining an interpolatory quadrature rule for the numerical approximation of a weighted integral as the (weighted) integral of a polynomial interpolant of the integrand, and introducing the concept of the polynomial exactness degree of a quadrature rule, it is shown that non-negative quadrature weights guarantee quadrature convergence for continuous integrands, with convergence rate increasing with the number of continuous derivatives of the integrand. Next, by choosing the underlying interpolation points as the zeros of a certain orthogonal polynomial, it is shown that optimal polynomial exactness is achieved, and thereby yielding the Gauss quadrature rules, for which the weights are then proved to be positive. The related Clenshaw-Curtis quadrature rule, as based on interpolation points obtained from extremal properties of Chebyshev polynomials, is shown to possess positive weights with explicit formulations. The rest of the chapter is devoted to Newton-Cotes quadrature, as based on uniformly distributed interpolation points. By employing results on polynomial interpolation from Chapters 1 and 2, explicit formulations in terms of Laplace coefficients of Newton-Cotes weights and error expressions are derived. In addition, compos-

ite Newton-Cotes quadrature rules, including the trapezoidal, midpoint and Simpson rules as special cases, are explicitly constructed and subjected to error and convergence analysis.

- **Chapter 9: Approximation of Periodic Functions**

  The trigonometric polynomials are introduced and analyzed, with the view to using them to approximate any given continuous periodic (with period $= 2\pi$) function. Next, as a Weierstrass-type theorem, it is proved that any continuous periodic function can be approximated with arbitrary "closeness" in the maximum norm by a trigonometric polynomial. The Fourier series operator is defined as the best L^2 approximation operator into the finite-dimensional space of trigonometric polynomials of a given degree, and explicitly calculated in terms of Fourier coefficients expressed as integrals. For those instances where these integrals are to be numerically approximated, the Euler-Maclaurin formula is derived, and shown to imply the remarkable efficiency of the trapezoidal rule when applied to the integral of a smooth periodic function over its full period, as in the Fourier coefficient case, and thereby yielding the discrete Fourier series operator. In the limit, the Fourier series operator yields the (infinite) Fourier series of a continuous periodic function, and for which L^2 convergence is immediately deduced. In order to investigate the uniform convergence of Fourier series, upper and lower bounds for the Lebesgue constant (with respect to the maximum norm) of the Fourier series operator are first derived, thereby showing that this Lebesgue constant grows to infinity at a logarithmic rate. After furthermore establishing two Jackson theorems on best approximation convergence rates, the Lebesgue inequality of Chapter 5 is then applied to prove the Dini-Lipschitz theorem, according to which Lipschitz continuity of a periodic function is a sufficient condition for the uniform convergence of its Fourier series, that is, the Fourier series converges pointwise to the function itself.

- **Chapter 10: Spline Approximation**

  Splines are introduced as piecewise polynomials of a given degree, and with breakpoints, or knots, at a finite number of specified points, together with the maximal smoothness requirement providing a proper extension of the corresponding polynomial space. Preliminary properties of splines are established, and it is shown that truncated powers provide a basis for spline spaces. Next, the compactly supported, and hence more efficient, *B*-spline basis is constructed and analyzed, with particular attention devoted to the case where the knots are placed at the integers, yielding the cardinal *B*-splines. The Schoenberg-Whitney theorem, which is an existence and uniqueness result for (non-local) spline interpolation, is then proved. In order to obtain a local spline approximation method, an explicit construction method is developed for spline quasi-interpolation, which combines the approximation properties of linearity, optimal polynomial exactness, and locality. In order to achieve, in addition, the property of interpolation, it is shown that such a local spline interpolation operator can be constructed explicitly by choosing the interpolation points as a specified subset of the spline knots. The Schoenberg operator is introduced, and shown to be a local spline approximation operator, which, for any fixed spline degree, and for the maximum spacing between spline knots tending to zero, possesses the uniform convergence property with respect to any continuous function on a bounded interval, and thereby establishing a Weierstrass-type theorem for splines. Similarly, sufficient conditions on the spline knot placement for the uniform convergence of the above-mentioned local spline interpolation operator are derived, as well as a corresponding convergence rate result by means of the Peano kernel theorem. Finally, the interpolatory quadrature rule obtained from the uniformly distributed knots case of this local spline interpolation operator is analyzed, and shown to yield a class of trapezoidal rules with endpoints corrections, which are precisely the classical Gregory rules of even order, after which results from Chapter 8 are employed to explicitly evaluate, in terms of Laplace coefficients, the weights and error expressions for these Gregory rules, and with particular attention devoted to the special case of the Lacroix rule.

Examples of courses that could be taught from this book are as follows:

- A one-semester mathematics course of "Mathematics of Approximation" can be taught from Chapters 1-8. If the course is oriented towards numerical methods and numerical analysis, then Chapters 9-10 can be adopted to replace Chapters 6-7.
- A one-year course of "Mathematics of Approximation" can be taught with first semester based on Chapters 1-7, and second semester on Chapters 8-10.
- A one-semester course "Polynomial and Spline Approximation" can be taught from Chapters 1-7 and Chapter 10 up to Section 10.6.
- A one-semester course of "Polynomial Approximation and Quadrature" can be taught from Chapters 1-8.
- A one-semester course of "Polynomial Approximation and Fourier series" can be taught from Chapters 1-7 and Chapter 9.

Material that could be regarded as optional in the courses listed above are Sections 6.5, 8.3 and 10.7; Section 9.3 from Theorem 9.3.3 onwards; as well as the proof of (3.1.8) in Example 3.1.3 of Section 3.1.

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