Chapter 2

Error Analysis For Polynomial Interpolation

As a continuation of Chapter 1, the notion of divided difference is applied to deduce the uniform error bound for polynomial interpolation for any given finite sample point set. In addition, an optimal sample point set, on which the minimum uniform error bound is achieved among all sample point sets with the same cardinality, is derived.

2.1 General error estimate

Let $[a,b]$ denote a bounded interval in $\mathbb{R}$, and suppose $f \in C[a,b]$, with $C[a,b]$ denoting the linear space of continuous functions $f : [a,b] \to \mathbb{R}$. For any non-negative integer n, let $\triangle_n := \{x_0, \ldots, x_n\}$ be a sequence of $n+1$ distinct points such that

$$\triangle_n \subset [a,b], \tag{2.1.1}$$

and, as in Theorem 1.1.2, denote by P_n^I the unique polynomial in π_n satisfying the interpolation conditions

$$P_n^I(x) = f(x), \quad x \in \triangle_n. \tag{2.1.2}$$

The corresponding polynomial interpolation error function is then defined by

$$E_n^I := f - P_n^I. \tag{2.1.3}$$

Hence $E_n^I \in C[a,b]$, with

$$E_n^I(x) = 0, \quad x \in \triangle_n. \tag{2.1.4}$$

The function E_n^I has the following explicit formulation in terms of a divided difference.

Theorem 2.1.1. *The error function E_n^I, as defined by* (2.1.3)*, satisfies, for any non-negative integer n,*

$$E_n^I(x) = \begin{cases} 0 & , x \in \triangle_n; \\ f[x,x_0,\ldots,x_n]Q_{n+1}(x) & , x \in [a,b] \setminus \triangle_n, \end{cases} \tag{2.1.5}$$

with $Q_{n+1} \in \pi_{n+1}$ *defined as in* (1.3.11)*, that is,*

$$Q_{n+1}(x) := \prod_{j=0}^{n}(x - x_j). \tag{2.1.6}$$

Proof. The first line of (2.1.5) has already been noted in (2.1.4).
Let $x \in [a,b] \setminus \triangle_n$ be fixed, and denote by P the unique interpolation polynomial in π_{n+1} such that

$$P(t) = f(t), \quad t \in \triangle_n \cup \{x\}. \tag{2.1.7}$$

It follows from (2.1.1) and (1.3.7) in Theorem 1.3.1 that

$$P(t) = P_n^I(t) + f[x_0,\ldots,x_n,x]Q_{n+1}(t), \tag{2.1.8}$$

with the polynomial Q_{n+1}^I defined as in (2.1.6). By setting $t = x$ in (2.1.8), and using (2.1.7), we obtain

$$f(x) = P_n^I(x) + f[x_0,\ldots,x_n,x]Q_{n+1}(x). \tag{2.1.9}$$

The second line of (2.1.5) is now a consequence of (2.1.8), (2.1.3), as well as the symmetry result of Theorem 1.3.6. ∎

In order to obtain a useful estimate for the error function E_n^I, we first prove the following property of divided differences.

Theorem 2.1.2. *For any non-negative integer n, let* $\triangle_n := \{x_0,\ldots,x_n\}$ *denote a sequence of* $n+1$ *distinct points in* $\mathbb{R}$*, and suppose* f *has n continuous derivatives in the smallest interval containing the points* $\{x_0,\ldots,x_n\}$*. Then the divided difference* $f[x_0,\ldots,x_n]$*, as defined by* (1.3.2)*,* (1.3.3)*, satisfies*

$$f[x_0,\ldots,x_n] = \frac{f^{(n)}(\xi)}{n!}, \tag{2.1.10}$$

for some point ξ *in the smallest interval containing the points* $\{x_0,\ldots,x_n\}$*.*

Proof. By applying (1.4.12), (1.4.13) in Theorem 1.4.3, and recalling the remark following the statement of Theorem 1.4.3, we deduce by means of the mean value theorem for integrals, together with (2.1.1), that there is a point ξ in the smallest interval containing the points $\{x_0,\ldots,x_n\}$ such that

$$\begin{aligned} f[x_0,\ldots,x_n] &= f^{(n)}(\xi)\int_0^{t_0}\cdots\int_0^{t_{n-1}} dt_n dt_{n-1}\ldots dt_1 \\ &= f^{(n)}(\xi)\int_0^{t_0}\cdots\int_0^{t_{n-2}} t_{n-1} dt_{n-1}\ldots dt_1 \end{aligned}$$

$$= \cdots = \frac{f^{(n)}(\xi)}{n!},$$

analogously to the final argument in the proof of Theorem 1.4.5. ■

We now combine Theorems 2.1.1 and 2.1.2, and use the fact that (2.1.6) implies

$$Q_{n+1}(x) = 0, \quad x \in \triangle_n, \tag{2.1.11}$$

to immediately deduce the following result, in which, as throughout the book, we adopt, for any non-negative integer m, the notation $C^m[a,b]$ to denote the linear space of functions $f : [a,b] \to \mathbb{R}$ such that $f^{(k)} \in C[a,b], k = 0,\ldots,m$, according to which $C^0[a,b] = C[a,b]$.

Theorem 2.1.3. *For a non-negative integer n, suppose $f \in C^{n+1}[a,b]$. Then, for any $x \in [a,b]$, there is a point $\xi \in (a,b)$ such that the error function E_n^I in* (2.1.3) *satisfies*

$$E_n^I(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} Q_{n+1}(x), \tag{2.1.12}$$

with the polynomial $Q_{n+1} \in \pi_{n+1}$ given by (2.1.6).

Next, for any function $g \in C[a,b]$, we introduce the notation

$$||g||_\infty := \max_{a \leqslant x \leqslant b} |g(x)|, \tag{2.1.13}$$

in terms of which the following interpolation error estimate holds.

Theorem 2.1.4. *The interpolation error function E_n^I in Theorem* 2.1.3 *satisfies the estimate*

$$||E_n^I||_\infty \leqslant \frac{||f^{(n+1)}||_\infty}{(n+1)!} ||Q_{n+1}||_\infty. \tag{2.1.14}$$

Proof. Let $x \in [a,b]$ be fixed. It follows from (2.1.12) in Theorem 2.1.3, together with (2.1.13), that

$$|E_n^I(x)| \leqslant \frac{||f^{(n+1)}||_\infty}{(n+1)!} ||Q_{n+1}||_\infty,$$

from which the desired estimate (2.1.14) then immediately follows. ■

Example 2.1.1. Consider the case $f(x) = \cos x$, and $[a,b] = [0, \frac{\pi}{2}]$.

(a) For $n = 2$, let

$$\triangle_2 = \{x_0, x_1, x_2\} := \{0, \frac{\pi}{4}, \frac{\pi}{2}\}. \tag{2.1.15}$$

Then, by using either of the interpolation formulas (1.2.5) or (1.3.17), we obtain

$$P_2^I(x) = \frac{8}{\pi^2}(1-\sqrt{2})x^2 + \frac{2}{\pi}(2\sqrt{2}-3)x + 1.$$

Moreover, the error estimate (2.1.14) yields

$$\max_{0\leqslant x\leqslant \frac{\pi}{2}} \left|\cos x - P_2^I(x)\right| \leqslant \frac{1}{3!}\left[\max_{0\leqslant x\leqslant \frac{\pi}{2}} |\sin x|\right] \max_{0\leqslant x\leqslant \frac{\pi}{2}} \left|x\left(x-\frac{\pi}{4}\right)\left(x-\frac{\pi}{2}\right)\right|$$

$$= \frac{1}{6}\,\frac{\sqrt{3}\pi^3}{288} = \frac{\sqrt{3}\pi^3}{1728} \approx 0.031. \tag{2.1.16}$$

(b) For $n=9$, let $\triangle_9 = \{x_0,\ldots,x_9\}$ denote any sequence of 10 distinct points in $[0,\frac{\pi}{2}]$. Then the corresponding interpolation polynomial P_9^I can be calculated by means of either (1.2.5) or (1.3.17), and the error estimate (2.1.14) gives

$$\max_{0\leqslant x\leqslant 2} \left|\cos x - P_9^I(x)\right| \leqslant \frac{1}{10!}\left[\max_{0\leqslant x\leqslant \frac{\pi}{2}} |\cos x|\right] \max_{0\leqslant x\leqslant \frac{\pi}{2}} \prod_{j=0}^{9} |x-x_j|$$

$$\leqslant \frac{1}{10!}\left(\frac{\pi}{2}\right)^{10} \approx 2.52\times 10^{-5}. \tag{2.1.17}$$

■

Observe that the upper bound on $||E_n^I||_\infty$, as given by the right hand side of (2.1.14), depends on f, n and $\triangle_n := \{x_0,\ldots,x_n\}$, with the dependence on $\triangle_n$ entirely restricted to the factor $||Q_{n+1}||_\infty$. Moreover, $||Q_{n+1}||_\infty$ is independent of f. We shall proceed in Section 2.2 to investigate the existence of a sequence $\triangle_n$ which minimizes $||Q_{n+1}||_\infty$.

2.2 The Chebyshev interpolation points

The Chebyshev polynomials $\{T_j:\ j=0,1,\ldots\}$ are defined recursively by

$$\left.\begin{aligned} T_0(x) &:= 1 \quad ; \quad T_1(x) := x \quad ; \\ T_{j+1}(x) &:= 2xT_j(x) - T_{j-1}(x), \quad j=1,2,\ldots. \end{aligned}\right\} \tag{2.2.1}$$

By using (2.2.1), we obtain

$$\left.\begin{aligned} &T_2(x) = 2x^2-1 \quad ; \quad T_3(x) = 4x^3-3x \quad ; \quad T_4(x) = 8x^4-8x^2+1; \\ &T_5(x) = 16x^5-20x^3+5x \quad ; \quad T_6(x) = 32x^6-48x^4+18x^2-1. \end{aligned}\right\} \tag{2.2.2}$$

The following properties are satisfied by the Chebyshev polynomials.

Theorem 2.2.1. *For $j\in\mathbb{N}$, the Chebyshev polynomial T_j, as defined in* (2.2.1)*, satisfies:*

(a) *T_j is a polynomial of degree j such that the leading coefficient in*

$$T_j(x) = \sum_{k=0}^{j} c_{j,k}x^k \tag{2.2.3}$$

is given by

$$c_{j,j} = 2^{j-1}; \tag{2.2.4}$$

(b)

$$T_j(x) = \cos(j \arccos x), \quad x \in [-1,1]; \tag{2.2.5}$$

(c)

$$|T_j(x)| \leqslant 1, \quad x \in [-1,1]; \tag{2.2.6}$$

(d)

$$T_j\left(\cos\left(\frac{j-k}{j}\,\pi\right)\right) = (-1)^{j-k}, \quad k = 0,\ldots,j; \tag{2.2.7}$$

(e)

$$T_j\left(\cos\left(\frac{2j-1-2k}{2j}\pi\right)\right) = 0, \quad k = 0,\ldots,j-1; \tag{2.2.8}$$

(f)

$$T_j(x) = 2^{j-1}\prod_{k=0}^{j-1}\left[x - \cos\left(\frac{2j-1-2k}{2j}\pi\right)\right], \quad x \in \mathbb{R}. \tag{2.2.9}$$

Proof. (a) The properties (2.2.3) and (2.2.4) follow inductively from the definition (2.2.1).
(b) Let the function sequence $\{g_j : \; j = 0,1,\ldots\}$ be defined by

$$g_j(x) := \; \cos(j \arccos x), \quad x \in [-1,1], \quad j = 0,1,\ldots, \tag{2.2.10}$$

and introduce the one-to-one mapping between the intervals $[0,\pi]$ and $[-1,1]$ as given by

$$x = \cos\theta, \quad \theta \in [0,\pi], \tag{2.2.11}$$

or equivalently,

$$\theta = \arccos x, \quad x \in [-1,1], \tag{2.2.12}$$

in terms of which (2.2.10) may be written as

$$g_j(x) = \cos(j\theta), \quad \theta \in [0,\pi], \quad j = 0,1,\ldots. \tag{2.2.13}$$

The trigonometric identity

$$\cos[(j+1)\theta] + \cos[(j-1)\theta] = 2(\cos\theta)\cos(j\theta),$$

together with (2.2.13) and (2.2.11), yields the identity

$$g_{j+1}(x) + g_{j-1}(x) = 2xg_j(x), \quad x \in [-1,1], \quad j = 1,2,\ldots,$$

and thus, by using also (2.2.13) for $j = 0$ and $j = 1$, as well as (2.2.11), we obtain

$$\left.\begin{aligned} g_0(x) \; &= \; 1; \quad g_1(x) \; = \; x; \\ g_{j+1}(x) \; &= \; 2xg_j(x) - g_{j-1}(x), \quad j = 1,2,\ldots, \end{aligned}\right\} x \in [-1,1]. \tag{2.2.14}$$

It follows from (2.2.1) and (2.2.14) that $g_j(x) = T_j(x), x \in [-1,1],\ j = 0,1,\ldots,$ which, together with (2.2.10), proves the formula (2.2.5).

(c) The property (2.2.6) is an immediate consequence of (2.2.5).

(d) For $j \in \mathbb{N}$ and $k = 0,\ldots,j$, we have, from (2.2.5),

$$\begin{aligned} T_j\left(\cos\left(\frac{j-k}{j}\pi\right)\right) &= \cos\left(j\arccos\left(\cos\left(\frac{j-k}{j}\pi\right)\right)\right) \\ &= \cos((j-k)\pi) = (-1)^{j-k}, \end{aligned}$$

which proves (2.2.7).

(e) Similarly, for $j \in \mathbb{N}$ and $k = 0,\ldots,j-1$, we deduce from (2.2.5) that

$$\begin{aligned} T_j\left(\cos\left(\frac{2j-1-2k}{2j}\pi\right)\right) &= \cos\left(j\arccos\left(\cos\left(\frac{2j-2k-1}{2j}\pi\right)\right)\right) \\ &= \cos\left(\left(j-k-\frac{1}{2}\right)\pi\right) = 0, \end{aligned}$$

and thereby proving (2.2.8).

(f) The explicit formulation (2.2.9) is an immediate consequence of (2.2.3), (2.2.4) and (2.2.8). ■

Observe from Theorem 2.2.1(f) that, for $j \in \mathbb{N}$, the Chebyshev polynomial T_j of degree j has precisely j distinct zeros in $(-1,1)$, with, more precisely,

$$T_j(t_{j,k}) = 0, \quad k = 0,\ldots,j-1, \tag{2.2.15}$$

where

$$t_{j,k} := \cos\left(\frac{2j-1-2k}{2j}\pi\right), \quad k = 0,\ldots,j-1, \tag{2.2.16}$$

and thus

$$-1 < t_{j,0} < t_{j,1} < \cdots < t_{j,j-1} < 1. \tag{2.2.17}$$

Moreover, according to Theorem 2.2.1(d), the Chebyshev polynomial T_j attains, for $j \in \mathbb{N}$, its maximum $(= 1)$ and minimum $(= -1)$ on $[-1,1]$ alternately, in the sense that

$$T_j(\xi_{j,k}) = (-1)^{j-k}, \quad k = 0,\ldots,j, \tag{2.2.18}$$

where

$$\xi_{j,k} := \cos\left(\frac{j-k}{j}\pi\right), \quad k = 0,\ldots,j, \tag{2.2.19}$$

and thus

$$-1 = \xi_{j,0} < \xi_{j,1} < \cdots < \xi_{j,j} = 1. \tag{2.2.20}$$

For any non-negative integer k, if $P(x) = \sum_{j=1}^{k} c_j x^j$, with leading coefficient $c_k = 1$, we say that P is a monic polynomial. The set of all monic polynomials in π_k will be denoted by the symbol $\widetilde{\pi}_k$. Observe from Theorem 2.2.1(a) that the normalized Chebyshev polynomials

$$\widetilde{T}_j := 2^{1-j} T_j, \quad j = 1, 2, \ldots, \tag{2.2.21}$$

are monic polynomials, that is,

$$\widetilde{T}_j \in \widetilde{\pi}_j, \quad j \in \mathbb{N}. \tag{2.2.22}$$

We shall rely on the following minimization property of $\widetilde{T}_j$.

Theorem 2.2.2. *For any $j \in \mathbb{N}$,*

$$\min_{P \in \widetilde{\pi}_j} \max_{-1 \leqslant x \leqslant 1} |P(x)| = \max_{-1 \leqslant x \leqslant 1} |\widetilde{T}_j(x)| = 2^{1-j}, \tag{2.2.23}$$

where $\widetilde{T}_j$ is the normalized Chebyshev polynomial defined by (2.2.21).

Proof. Let $j \in \mathbb{N}$. First, observe that (2.2.21), (2.2.6) and (2.2.7) imply the second equation in(2.2.23).

We use a proof by contradiction to prove the first equation in (2.2.23). Suppose therefore that there exists a polynomial $Q \in \widetilde{\pi}_j$ such that

$$\max_{-1 \leqslant x \leqslant 1} |Q(x)| < 2^{1-j}, \tag{2.2.24}$$

according to which $Q \neq \widetilde{T}_j$, and define the polynomial

$$R := (-1)^j (\widetilde{T}_j - Q), \tag{2.2.25}$$

for which it then follows that R is not the zero polynomial. Since $\widetilde{T}_j$ and Q are both monic polynomials in $\widetilde{\pi}_j$, it follows from (2.2.25) that

$$R \in \pi_{j-1}. \tag{2.2.26}$$

Now observe from (2.2.21) and (2.2.18) that

$$\widetilde{T}_j(\xi_{j,k}) = (-1)^{j-k} 2^{1-j}, \quad k = 0, \ldots, j, \tag{2.2.27}$$

where the sequence $\{\xi_{j,k} : k = 0, \ldots, j\}$ is given by (2.2.19), and satisfies (2.2.20).

By using (2.2.25), (2.2.27) and (2.2.24), we deduce that

$$R(\xi_{j,0}) = 2^{1-j} - (-1)^j Q(\xi_{j,0}) > 0;$$

$$R(\xi_{j,1}) = -2^{1-j} - (-1)^j Q(\xi_{j,1}) < 0,$$

and it follows from the intermediate value theorem that there is a point $\eta_1 \in (\xi_{j,0}, \xi_{j,1})$ such that $R(\eta_1) = 0$. Similarly it can be shown by means of (2.2.25), (2.2.27) and (2.2.24) that $R(\xi_{j,k})$ alternates in sign for $k = 1, \dots, j$, and that there consequently exist points $\eta_k \in (\xi_{j,k-1}, \xi_{j,k})$, $k = 2, \dots, j$, such that $R(\eta_k) = 0$, $k = 2, \dots, j$. Hence R has j distinct real zeros at $\{\eta_1, \dots, \eta_j\}$. Since also (2.2.26) holds, it follows that R must be the zero polynomial, which is a contradiction, and thereby concluding our proof of the first equation in (2.2.23). ■

We proceed to show how Theorem 2.2.2 can be used to minimize the factor

$$||Q_{n+1}||_\infty := \max_{a \leqslant x \leqslant b} |Q_{n+1}(x)| \tag{2.2.28}$$

in (2.1.14) with respect to the choice of the interpolation point sequence $\triangle_n := \{x_0, \dots, x_n\}$. To this end, we introduce the one-to-one mapping between the intervals $[-1, 1]$ and $[a, b]$ as given by

$$x = \frac{1}{2}(b-a)t + \frac{1}{2}(a+b), \quad t \in [-1, 1], \tag{2.2.29}$$

or equivalently,

$$t = \frac{2}{b-a}\left[x - \frac{1}{2}(a+b)\right], \quad x \in [a, b]. \tag{2.2.30}$$

Based on (2.2.15), (2.2.16) and (2.2.17), for $n \in \mathbb{N}$ and $j = n+1$, we now define the Chebyshev interpolation points

$$x_{n,j}^C := \frac{1}{2}(b-a)\cos\left(\frac{2n+1-2j}{2n+2}\pi\right) + \frac{1}{2}(a+b), \quad j = 0, \dots, n, \tag{2.2.31}$$

which then satisfy

$$a < x_{n,0}^C < x_{n,1}^C < \cdots < x_{n,n}^C < b. \tag{2.2.32}$$

Observe from (2.2.31) that the Chebyshev interpolation points are concentrated more densely towards the endpoints of the interval $[a, b]$. The following minimization property can now be proved by means of Theorem 2.2.2.

Theorem 2.2.3. *The factor* $||Q_{n+1}||_\infty$ *in the polynomial interpolation error estimate* (2.1.14) *of Theorem* 2.1.4 *is minimized by*

$$\min_{x_0, \dots, x_n \in [a,b]} ||Q_{n+1}||_\infty = \max_{a \leqslant x \leqslant b} \left| \prod_{j=0}^{n} (x - x_{n,j}^C) \right| = 2^{-n} \left(\frac{b-a}{2}\right)^{n+1}, \tag{2.2.33}$$

with $\{x_{n,j}^C : \ j = 0, \dots, n\}$ *denoting the Chebyshev interpolation points, as defined in* (2.2.31).

Proof. First, we use the one-to-one mapping (2.2.29), (2.2.30) between the intervals $[a,b]$ and $[-1,1]$ to deduce that

$$\begin{aligned}
&\min_{x_0,\ldots,x_n\in[a,b]}\ \max_{a\leqslant x\leqslant b}\left|\prod_{j=0}^{n}(x-x_j)\right| \\
&= \min_{x_0,\ldots,x_n\in[a,b]}\ \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}\frac{b-a}{2}\left[t-\frac{2}{b-a}\left(x_j-\frac{1}{2}(a+b)\right)\right]\right| \\
&= \left(\frac{b-a}{2}\right)^{n+1}\min_{t_0,\ldots,t_n\in[-1,1]}\ \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}(t-t_j)\right|. \qquad (2.2.34)
\end{aligned}$$

For the sequence $\{t_{n+1,j} : j=0,\ldots,n\}$ as defined by means of (2.2.16), it follows from Theorem 2.2.2, together with (2.2.21) and (2.2.9), that

$$\begin{aligned}
2^{-n} = \max_{-1\leqslant t\leqslant 1}\left|\widetilde{T}_{n+1}(t)\right| = \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}(t-t_{n+1,j})\right| &\geqslant \min_{t_0,\ldots,t_n\in[-1,1]}\ \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}(t-t_j)\right| \\
&\geqslant \min_{P\in\widetilde{\pi}_{n+1}}\ \max_{-1\leqslant t\leqslant 1}|P(t)| = 2^{-n},
\end{aligned}$$

and thus

$$\min_{t_0,\ldots,t_n\in[-1,1]}\ \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}(t-t_j)\right| = \max_{-1\leqslant t\leqslant 1}\left|\prod_{j=0}^{n}(t-t_{n+1,j})\right| = 2^{-n},$$

which, together with (2.2.34), and (2.1.6), yields the desired result (2.2.33). ■

By combining Theorems 2.1.4 and 2.2.3, we immediately derive the following optimal polynomial interpolation error estimate.

Theorem 2.2.4. *In Theorem* 2.1.3, *for any positive integer n, let the interpolation points be chosen as the Chebyshev interpolation points, that is,*

$$x_j = x^C_{n,j}, \quad j=0,\ldots,n, \qquad (2.2.35)$$

as defined by (2.2.31). *Then the error estimate*

$$||E^I_n||_\infty \leqslant \frac{1}{2^n(n+1)!}\left(\frac{b-a}{2}\right)^{n+1}||f^{(n+1)}||_\infty \qquad (2.2.36)$$

is satisfied.

Example 2.2.1. As in Example 2.1.1, we consider the case $f(x)=\cos x$, and $[a,b]=[0,\frac{\pi}{2}]$, in which case, for any $n\in\mathbb{N}$, the Chebyshev interpolation points are given, according to (2.2.31), by

$$x^C_{n,j} = \frac{\pi}{4}\left[\cos\left(\frac{2n+1-2j}{2n+2}\pi\right)+1\right], \quad j=0,\ldots,n, \qquad (2.2.37)$$

and the corresponding error estimate (2.2.36) is

$$\max_{0\leqslant x\leqslant\frac{\pi}{2}} \left|\cos x - P_n^I(x)\right| \leqslant \frac{1}{2^n(n+1)!}\left(\frac{\pi}{4}\right)^{n+1}. \tag{2.2.38}$$

(a) For $n=2$, it follows from (2.2.37) that

$$\{x_{2,0}^C, x_{2,1}^C, x_{2,2}^C\} = \left\{\frac{2-\sqrt{3}}{8}\pi, \frac{\pi}{4}, \frac{2+\sqrt{3}}{8}\pi\right\},$$

and (2.2.38) gives the estimate

$$\max_{0\leqslant x\leqslant\frac{\pi}{2}} \left|\cos x - P_2^I(x)\right| \leqslant \frac{1}{24}\left(\frac{\pi}{4}\right)^3 \approx 0.02,$$

which improves on the error estimate (2.1.16) in Example 2.1.1(a).

(b) For $n=9$, the formula (2.2.37) yields the Chebyshev interpolation points

$$x_{9,j}^C = \frac{\pi}{4}\left[\cos\left(\frac{19-2j}{20}\pi\right)+1\right], \quad j=0,\ldots,9,$$

and (2.2.38) gives the estimate

$$\max_{0\leqslant x\leqslant\frac{\pi}{2}} \left|\cos x - P_9^I(x)\right| \leqslant \frac{1}{2^9 10!}\left(\frac{\pi}{4}\right)^{10} \approx 4.81\times 10^{-11},$$

which is a considerable improvement on the error estimate (2.1.17) in Example 2.1.1(b).

■

2.3 Exercises

Exercise 2.1 For the function

$$f(x) = \frac{1}{\sqrt{x}},$$

find a point $\xi \in [\frac{1}{9}, 1]$, as guaranteed by Theorem 2.1.2, for which it holds that

$$f[\tfrac{1}{9}, \tfrac{1}{4}, 1] = \tfrac{1}{2} f''(\xi).$$

Exercise 2.2 Let

$$f(x) = \ln(x+2), \quad x\in[0,2],$$

and, for $n\in\{1,2\}$, denote by P_n^I the interpolation polynomial in π_n such that

$$P_n^I(x) = f(x), \quad x \in \triangle_n,$$

where

$$\triangle_1 := \{\tfrac{1}{2}, \tfrac{3}{2}\} \quad ; \quad \triangle_2 := \{\tfrac{1}{2}, 1, \tfrac{3}{2}\}.$$

For $n = 1$ and $n = 2$, calculate the polynomial P_n^I, as well as the interpolation error estimate (2.1.14) in Theorem 2.1.4, with $[a,b] = [0,2]$. Also, for $n = 1$ and $n = 2$, investigate the sharpness of these estimates by calculating the exact value of $||E_n^I||_\infty$.

Exercise 2.3 As a continuation of Exercise 2.2, let n be any positive integer, and suppose

$$\triangle_n := \{x_0, \ldots, x_n\} \subset [0,2]$$

is an arbitrary point sequence in $[0,2]$. Apply the interpolation error estimate (2.1.14) in Theorem 2.1.4 to show that

$$\max_{0 \leqslant x \leqslant 2} |\ln(x+2) - P_n^I(x)| \leqslant \frac{1}{n+1}, \qquad (*)$$

with P_n^I denoting the interpolation polynomial in π_n with respect to the interpolation point sequence $\triangle_n$.

Exercise 2.4 Calculate the Chebyshev polynomials T_7 and T_8, thereby extending the formulas in (2.2.2).

Exercise 2.5 Calculate, for $n = 1$ and $n = 2$, the sequences $\triangle_n^C$ defined by

$$\triangle_n^C := \{x_{n,0}^C, \ldots, x_{n,n}^C\}, \quad n \in \mathbb{N},$$

with $\{x_{n,0}^C, \ldots, x_{n,n}^C\}$ denoting the Chebyshev interpolation points, as given in (2.2.31), for the interval $[0,2]$.

Exercise 2.6 As a continuation of Exercise 2.5, repeat Exercises 2.2 and 2.3 with $\triangle_n$ replaced by $\triangle_n^C$, and with the interpolation error estimate (2.1.14) replaced by (2.2.36) in Theorem 2.2.4. In particular, obtain the analogue of the estimate $(*)$ in Exercise 2.3.

Exercise 2.7 As a continuation of Exercise 2.6, find, according to the error estimate obtained there, the smallest possible value of n for which it holds that

$$\max_{0 \leqslant x \leqslant 2} |\ln(x+2) - P_n^I(x)| < \frac{1}{100}.$$

Exercise 2.8 Apply Theorem 2.2.2 to obtain the minimum value

$$\min_{a,b,c \in \mathbb{R}} \max_{-1 \leqslant x \leqslant 1} |x^3 + ax^2 + bx + c|,$$

as well as the corresponding optimal values of the coefficients a, b and c.

Exercise 2.9 Prove that, for any fixed $j \in \mathbb{N}$, the sum of the coefficients of the Chebyshev polynomial T_j is equal to one.

[*Hint:* Use Theorem 2.2.1(b).]

Exercise 2.10 Prove that the Chebyshev polynomials $\{T_0, T_1, \ldots\}$ satisfy the condition

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_j(x) T_k(x) dx = 0, \quad \text{if } j \neq k.$$

[*Hint:* Apply the transformation (2.2.11), (2.2.12).]

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