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Six Recipes for Making Polyhedra

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This chapter includes six "recipes" for making polyhedra, devised by famous polyhedrachefs. Some recipes are for beginners, others are intermediate or advanced. You can use these recipes, or devise your own. Building models is fun, and will give you a deeper understanding of the chapters that follow.

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Constructing Polyhedra Without Being Told How To!

Marion Walter

Getting Started: How to Attach Polygons

Put some cut-out regular polygons on a table. Put a little glue on a flat tile, a plastic lid, or a piece of plastic, and spread out the glue a little so that you can dip a whole edge of a polygon into the glue.

Choose two polygons that you want to glue together along an edge, and dip one of these edges in the glue. Dip lightly; if polygons don't stick well it is usually because there is too much glue (Figure 2.1).

Hold the two edges together firmly. The joint will remain flexible but the polygons will stick together (Figure 2.2).

If you find later that you need extra glue on an edge of a polygon that you have already attached, you can (lightly) dip a toothpick or applicator stick in the glue to smear some along an edge.

What Shape Are You Going to Make?

It is most fun and most rewarding to make a shape you yourself create rather than following someone else's plans. How can you do this?

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Figure 2.1.

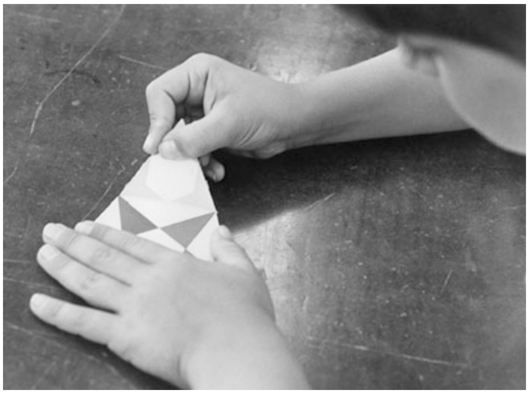

Figure 2.2.

Figure 2.3.

Figure 2.4.

Figure 2.5.

There are many ways to start. One way is to limit yourself to using only one or two different shapes — say triangles, or triangles and pentagons, or triangles and squares. What shapes can you make using triangles and only one pentagon? (See Figure 2.3).

The first shape the boy shown in Figure 2.4 made has a pentagon for its base and triangles for sides. It is called a pentagonal pyramid. Now make up another question of your own. What will your first shape look like? When you experiment freely, you may get a few surprises and you will learn a lot. For example, six triangles lie flat.

What a surprise: the shape in Figure 2.5 lies flat too! Notice that the twelve triangles that surround the hexagon help to make a bigger hexagon. The student shown in the photograph

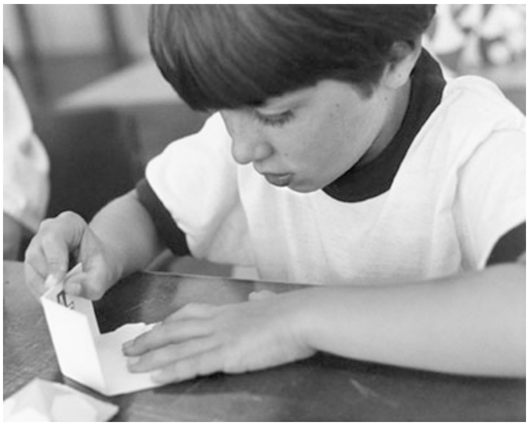

Figure 2.6.

Figure 2.8.

Figure 2.7.

Figure 2.9.

also had a surprise after she attached only six triangles to the hexagon. Do you think it will make a pyramid with a hexagonal base?

What shapes can you make with hexagons and squares? (See Figures 2.6 and 2.7).

Making shapes requires thinking ahead. Try to make a shape using only pentagons. What a relief: the two edges in Figure 2.8 really do seem to meet! How will the boy shown go on? Do the girls in Figures 2.9 and 2.10 seem to be making the same shape?

The shape in Figure 2.11 is made entirely of pentagons: how many of them were used? Turn it around and look at it. How many edges does it have? How many corners? How many edges meet at one corner? How many faces meet at a corner? This shape is a *dodecahedron*.

When you are experimenting, don't expect that your shape will always close! (Figure 2.12). Some shapes may have holes that you cannot fill with the shapes that we have; remember that we are using only regular polygons.

Shapes You Can Make with Triangles

The shape in Figure 2.13 is only one of the many you can make using just triangles. It is an *icosahedron*. Look at it from many sides. How many faces, edges, and corners does it have? Compare these numbers to the corresponding numbers you found for the dodecahedron.

In Figure 2.14 the girl is placing one five-sided pyramid over the base of another one. How many

Figure 2.10.

Figure 2.11.

Figure 2.12.

faces will this polyhedron have? What other shapes can you make with triangles?

A Note to the Teacher

Every problem leads to new observations and questions. For example, even the simple problem "Make all possible convex shapes using only equilateral triangles" is very rich in possibilities. These shapes are called *deltahedra*, after the triangular Greek letter Δ. Usually after some experimentation, students will discover the tetrahedron, the octahedron, the triangular and pentagonal bipyramids, and the icosahedron. Later the search also yields the 12-, 14-, and 16-sided deltahedra. Figure 2.15 shows a 14-sided deltahedron made of applicator sticks. Use sticks all of the same length. Some drugstores sell applicator sticks which are ideal; be sure to get the kind without cotton at each end. Hobby and craft stores often sell small-diameter wooden dowel rods which work well. Put a small amount of contact glue on the ends of the sticks and let it dry for about 15 minutes, until the glue is tacky. Then the sticks will join well and yet stay flexible. Don't be surprised if a cube or dodecahedron made of applicator sticks won't stand up, however. Unlike structures built entirely of triangles, these structures are nonrigid.

Figure 2.13.

Figure 2.14.

The observation that each deltahedron has an even number of faces leads to the question of why this should be so. The reason is straightforward once one sees it! Each triangle has three edges. If the shape has F faces, then there are $3F$ edges altogether. These $3F$ edges are glued in pairs, so there must be an even number of edges. Hence $3F$ and therefore F must be be even. Noticing that there exist 4-, 6-, 8-, 10-, 12-, 14-, 16-, and 20-sided deltahedra immediately sets off a search for an 18-sided one. Can an 18-sided deltahedron

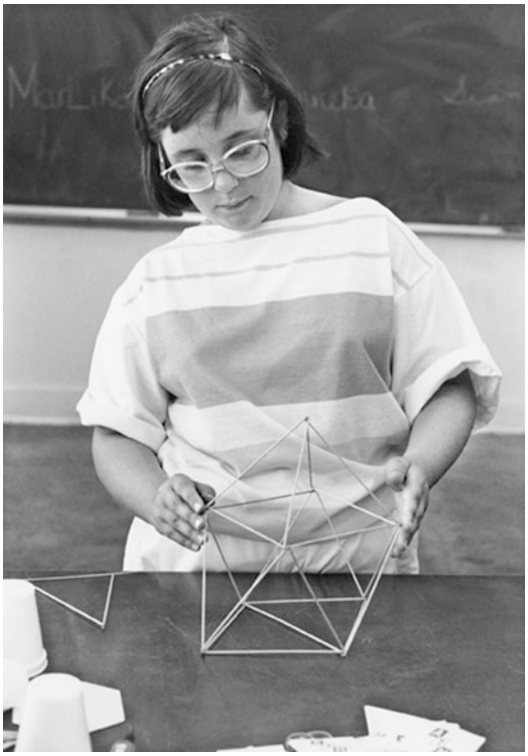

Figure 2.15.

be made? It was not until 1947 that the answer was proved to be *no*.

Looking at deltahedra is one thing; visualizing them without models is quite another. I found it difficult to close my eyes and visualize the 12-, 14-, 16-sided deltahedra. One day while I was looking at a cube made from applicator sticks and glue, I decided to pose problems by using the What-If-Not Strategy. The idea is that one starts with a situation, a theorem, a diagram, or in our case an object, lists as many of its attributes as one can, and then asks, “What if not?” For example, among the many attributes (not necessarily independent) of a cube that I had listed were the following:

1. All edges are equal.
2. All faces are squares.
3. The object is not rigid.
4. The top vertices are directly above the bottom ones.
5. Opposite faces are parallel.

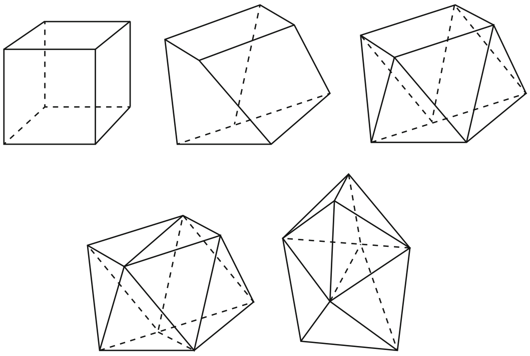

Figure 2.16.

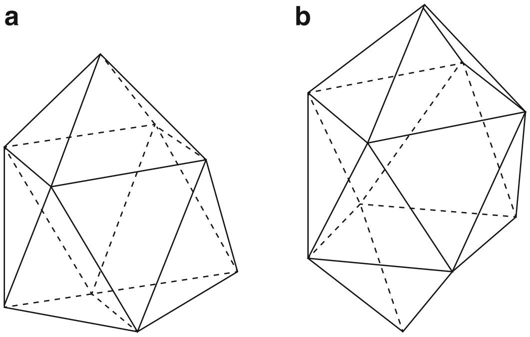

Figure 2.17.

Figure 2.18. Alice Shearer beginning construction of a model.

While working on attribute 4, I asked myself: "What if the top vertices were *not* directly above the bottom ones?" And because the contact glue gives movable joints, it was easy to give the top square a twist. As my twist approached 45°, I began to see an antiprism emerge. I attached sticks to complete the antiprism, but the shape wasn't rigid. The obvious thing to do to make it rigid was to add diagonals to the top and bottom squares. Since all the applicator sticks are of the same length, I had to squeeze the squares into "diamonds." The resulting shape was rigid- and was built of 12 equilateral triangles! (See Figure 2.16).

How else could I have made the antiprism rigid? I hastily removed the top diagonal, and added four sticks that meet above the square to form a square pyramid (Figure 2.17a.) Lo and behold, I had made a 14-sided deltahedron! From there it was a quick step to remove the bottom diagonal also, build another four-sided pyramid, and thus obtain the 16-sided deltahedron (Figure 2.17b).

Not only have these deltahedral "villains" now become friends, I see now that they are closely related to one another. One can also place the icosahedron in this family, since it is a pentagonal antiprism capped with two pentagonal pyramids. (Indeed the octahedron itself is an antiprism, and the tetrahedron can be viewed as an antiprism in which the two bases have shrunk to an edge. Two opposite edges may be considered degenerate polygons, which are here in antiprism orientation.) That leaves us only with the 6- and 10-sided deltahedra as "odd ones out," but they are both bipyramids and are easy to visualize.

Figure 2.19. Jane B. Phipps contemplating a polyhedron constructed from MATs.

A Word About Materials

Cardboard always works well; you should experiment with different weights. I prefer MATs, described in the next paragraph. All the polygons shown in these photographs are MATs. A glue used for carpets, such as Flexible Mold Compound – Mold It® is excellent, as is the English Copydex.

Adrien Pinel found that hexagonal cardboard beer mats (used in English pubs) were excellent for making polyhedra with holes and, when augmented by triangles and squares cut from the hexagons, became even more useful. It was not long before the Association of Teachers of Mathematics of Great Britain had regular polygons of three, four, five, six and eight sides produced from the same easy-to-glue material as the beer mats. They call them Mathematics Activity Tiles (MATs for short). They also produce rectangles and isosceles triangles. The polygons may be ordered separately or in two different kits: Kit A has 100 each of equilateral triangles, squares, pentagons, and hexagons, and Kit B has 200 each of triangles and squares and 50 each of pentagons, hexagons and octagons.

Constructing Pop-Up Polyhedra

Jean Pedersen

Required Materials

- One 22 × 28 inch piece of brightly colored heavyweight posterboard
- Six rubber bands
- One yard stick or meter stick
- One ballpoint pen
- One pair of scissors

General Instructions for Preparing the Pattern Pieces

Begin by drawing the pattern pieces on the posterboard as shown in Figure 2.20. Press hard with the ballpoint pen so that the posterboard will fold easily and accurately in the final assembly. Label the points indicated. Be certain to put the labels on what will become the cube (or octahedron) when the model is finished — not on the paper that surrounds it. Cut out the pattern pieces and snip the notches at A and B (but not the notches at C and D).

Constructing the Cube

1. Crease the pattern piece with square faces on all of the indicated fold lines, remembering that the unmarked side of the paper should be on the outside of the finished cube. Thus each individual fold along a marked line should hide that marked line from view.
2. Position the pattern piece so that it forms a cube with flaps opening from the top and the bottom, as shown in Figure 2.21.
3. Temporarily attach the two rectangles together inside the cube with paper clips. Then, with the cube still in its "up" position, cut through both thicknesses of paper at once to produce the notches at the positions which you already labeled C and D.
4. Connect three rubber bands together, as shown in Figure 2.22.
5. Slide one end-loop of this chain of rubber bands through the slot which you labeled A,

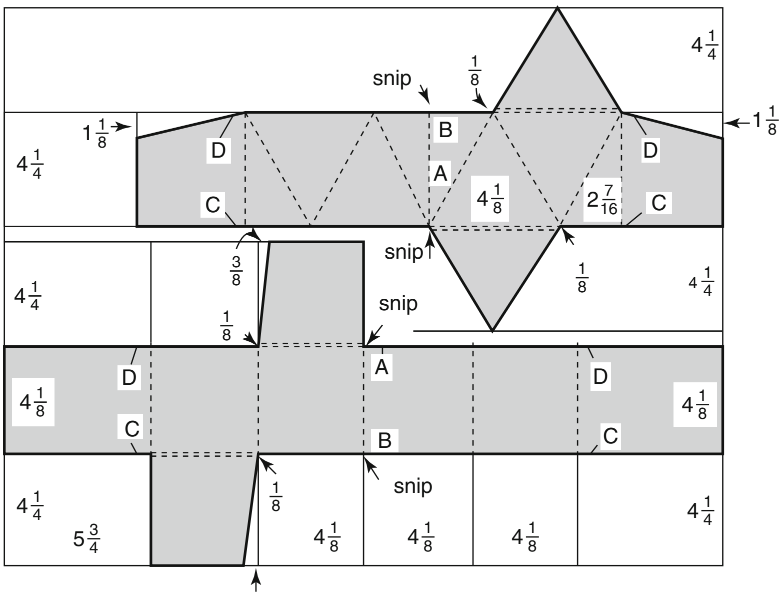

Figure 2.20.

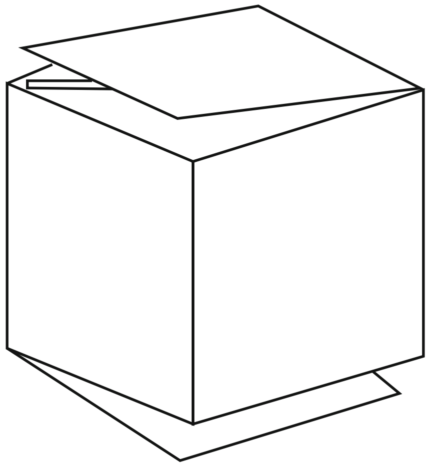

Figure 2.21.

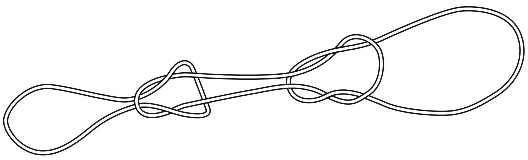

Figure 2.22.

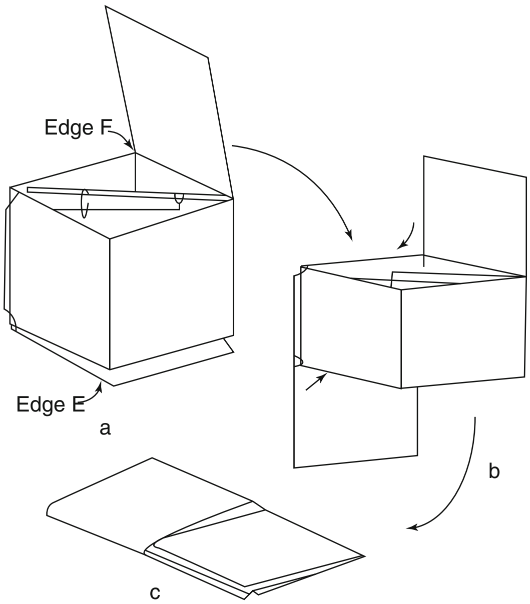

Figure 2.23.

and the other end-loop through the slot labeled B, leaving the knots on the outside of the cube.

6. Stretch the end loops of the rubber bands so that they hook into slots C and D, as shown in Figure 2.23. The bands must produce the right amount of tension for the model to work. If they are too tight the model will not go flat and if they are too loose the model won't pop up. You may need to do some experimenting to obtain the best arrangement.
7. Remove the paper clips when you are satisfied that the rubber bands are performing their function.
8. To flatten the model push the edges labeled E and F toward each other as shown in Figure 2.23b and wrap the flaps over the flattened portion as in Figure 2.23c.
9. Holding the flaps flat, toss the model into the air and watch it *pop up*. If you want it to make a louder noise when it snaps into position, glue an additional square onto each visible face of the cube in its "up" position. This also allows you to make the finished model very colorful.

Constructing the Octahedron

1. Crease on all the indicated fold lines so that the marked lines will be on the inside of the finished model.
2. Position the pattern piece so that it forms an octahedron with triangular flaps opening on the top and bottom, as shown in Figure 2.24a. Don't be discouraged by the complicated look of the illustration; the construction is so similar to the cube that once you have the pattern piece in hand, it becomes clear how to proceed.
3. Secure the quadrilaterals inside the octahedron with paper clips and cut through both thicknesses of paper to make the notches at C and D. Angle these cuts toward the center of the octahedron (so that the rubber bands will hook more securely). Gluing the quadrilaterals inside the model to each other in their proper position produces a sturdier model.

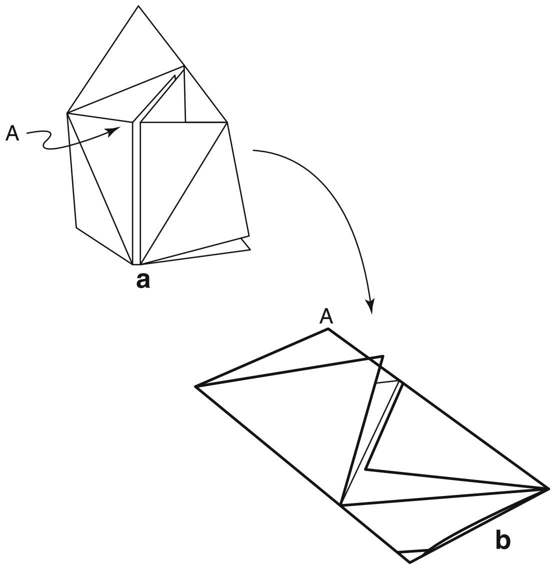

Figure 2.24.

4. Connect three rubber bands together, as shown in Figure 2.22.
5. Slide one loop-end of the rubber band arrangement through the slot A and the other loop-end through the slot B, leaving both knots on the outside of the octahedron.
6. Stretch the end loops of the rubber bands so that they hook into the slots at C and D. Some adjustment in the size of the rubber bands may be necessary, so experiment to find the best arrangement.
7. Remove the paper clips when you have a satisfactory arrangement of rubber bands.
8. To flatten the model put your fingers inside and pull at the vertices nearest A and D so that you are pulling those opposite faces away from each other until each is folded along an altitude of that triangular face. Then wrap the triangular flaps over the flattened portion so that it looks like Figure 2.24b.
9. Holding the triangular flaps flat, toss the model and watch it *pop up*. Just as with the cube, this model will make more noise if you glue an extra triangle on the exposed faces. Of course, if you use colored pieces the resulting model is more interesting.

A helpful hint:

If you store either the cube or the octahedron in its flattened position for several hours, or days, it may fail to pop up when tossed in the air. This is because the rubber bands lose their elasticity when stretched continuously for long periods of time. If the rubber bands have not begun to deteriorate, the model will behave normally as soon as you let the rubber bands contract for a short while.

The Great Stellated Dodecahedron

Magnus Wenninger

The great stellated dodecahedron (Figure 2.25) makes a lovely decoration or an interesting ornament for any time or place. It is very attractive, and when made as suggested here it is also very sturdy and rigid even though it is entirely hollow inside. The pattern to use for making this model is simply an isosceles triangle with base angles of 72 ° and a vertex angle of 36 °. The length of the base should be between 1 and 2 inches (between 2.5 and 5.0 cm). The angular measures just given will automatically make the equal sides of the triangle τ times longer than the base, where τ is 1.618034 (the golden section number). You will need 60 such triangles to complete one model. Very attractive results can be obtained by using different colors of index card, namely ten triangles of each of six different colors. Astonishingly beautiful results can be obtained by using glitter film with pressure sensitive adhesive backing to cover the index card.

Figure 2.25. Katherine Kirkpatrick studying models made in Magnus Wenninger's workshop.

Getting Started

Begin the work by first cutting all the glitter film triangles to exactly the same size. You can lay out a tessellated network of such triangles by marking the back or waxy side of a sheet of glitter film with a scoring instrument and then cutting out the triangles with scissors. Next peel off one corner of the waxy backing from the film and attach this to a piece of index card. Finally, remove the entire backing while you smooth out the film on the card. Now trim the card with scissors, leaving a border of card all around the film. A quarter inch or so is suitable (about 7 or 8 mm). Next trim the vertices of the triangle as suggested in Figure 2.26. You will now find it easy to bend or fold the card down along the edges of the film even without scoring the card. This edging of card serves as a tab for joining the triangles together. Use ordinary white paper glue (such as Elmer's Glue-All® for this purpose.

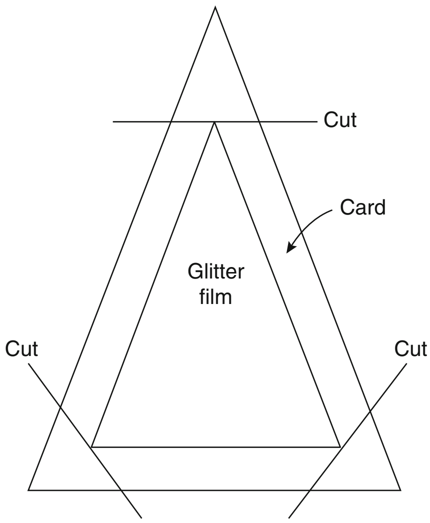

Figure 2.26.

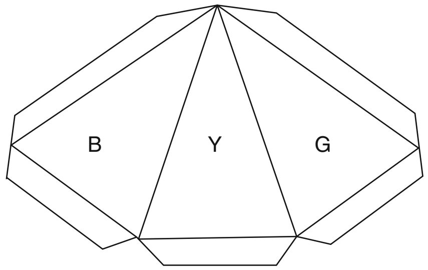

Figure 2.27.

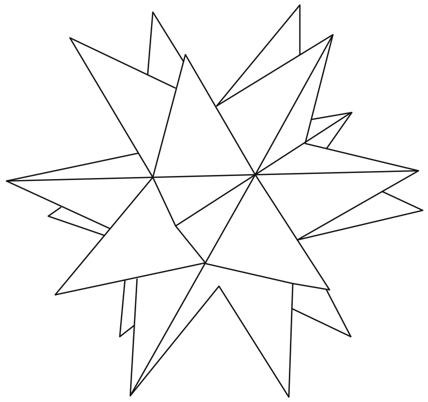

Figure 2.28.

Assembling the Model

Glue three triangles together as shown in Figure 2.27. Shape this part into a triangular pyramid without a base. This will then form one trihedral vertex of the great stellated dodecahedron. The color arrangement for ten vertices is as follows:

(1) B Y G	(6) B W G	Y = yellow or gold
(2) O B Y	(7) O W Y	B = blue
(3) R O B	(8) R W B	O = orange
(4) G R O	(9) G W O	R = red
(5) Y G R	(10) Y W R	G = green
		W = white or silver

The first five vertices or triangular pyramids are joined in a ring with the bottom edges of the middle Y, B, O, R, G of (1), (2), (3), (4), (5), forming an open pentagon. Then the next five parts are added to each edge of this pentagon, so that the W of (6) is glued to the Y of (1) and so on around. This completes half the model. You may find it a bit tricky to get the colors right at first, but the arrangement suggested here makes each star plane the same color. The triangles are star arms, so once you get started right it is not hard to continue.

The remaining ten vertices or parts have their colors in reverse order. They are the mirror image arrangement of the first ten. To make them, just read the color table in reverse order and from right to left. For example, vertex (11) will be R W Y, the reverse of (10) which is Y W R. And this is glued in place diametrically opposite to its counterpart on the model. Watching the colors of the star arms will help you get all the remaining parts in their proper places. As the model closes up it is helpful to use tweezers to get the tabs to adhere. The secret is to do only one pair of tabs at a time. On the last part glue one pair of tabs first. Then, when this has set firmly, put glue on the remaining two sets of tabs and close the triangular opening. The model now has sufficient rigidity so that the tabs will adhere by applying gentle pressure from the outside with your hands. An extra drop of glue at the base of each pyramid corner will provide extra strength where you may perceive a small opening remaining.

You should now see, if you have not already noticed this, that parallel star planes are the same color. Hence twelve star planes complete this model, two of each of the six colors. The twelve stars give this model its name: stellated dodecahedron (Figure 2.28). It is called "great" because it is the final stellation of the dodecahedron, truly a beautiful thing to behold!

"A thing of beauty is a joy forever."

Figure 2.29. Magnus Wenninger leading a workshop.

Creating Kaleidocycles and More

Doris Schattschneider

The transition from a flat pattern to a three-dimensional form can be fascinating to explore. Even the youngest child can shape a simple basket by cutting squares from the corners of a rectangular piece of construction paper and folding it up. But except for this well-known pattern learned as a preschool exercise, the two-dimensional pattern of an unfamiliar three-dimensional object often seems to yield little information about the object. Perhaps part of the reason for this is that we are rarely asked to imagine what shape will result from folding up a flat pattern. The following exercises provide hands-on exploration of some of the relations between flat nets and three-dimensional forms and provide an extra surprise in the creation of kinetic forms.

Folding Strips of Triangles

Begin by constructing strips of four connected congruent triangles like the one sketched in Figure 2.30. You should construct several different kinds of strips — those whose triangles are (1) equilateral triangles, (2) isosceles acute triangles, (3) isosceles right triangles, (4) isosceles obtuse triangles, (5) scalene triangles (a strip of acute triangles, or of right, or obtuse). *Note:* If you are in a hurry, use graph paper for rapid layout of the strips of congruent triangles. If more time is available, carry out rule-and-compass construction of the strips (brush up on the congruence theorems!).

Question: For each of the constructed nets, what three-dimensional shape will be formed when the net is folded along the common edges of its triangles?

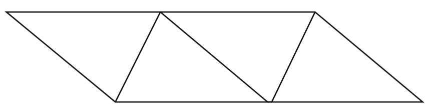

Figure 2.30.

First, guess answers to this question. Then score the connecting edges of the triangles (use a medium ballpoint pen held against a straight edge), cut out the strips of triangles, and fold each of them to see what happens. (All folds should be of the same type, folding the pattern back-to-back.)

After this, other strips of four triangles can be explored: for instance, a strip of four triangles, all acute, but not all congruent; a strip of triangles with some triangles right, others acute, and so forth. Exploring what happens when these nets are folded up leads to some natural questions:

1. When will four congruent triangles form a tetrahedron?
2. What must be true of four triangles if they are to form a tetrahedron?
3. Are there different flat nets (other than the strips of four triangles) that will fold up to make the same shapes as those formed by the strips of four triangles?

Kaleidocycles

Next, we will create and explore nets of connected strips of triangles. For ease and accuracy of construction, large paper and long (18 inch) rulers should be used. Graph paper can be purchased in size 17 × 22 inch, just right for two constructions. Drawing paper can easily be purchased in large sizes. Lay out each of the two grids shown in Figures 2.31a and 2.32. The grid in Figure 2.31a is made up of six connected vertical strips of congruent isosceles triangles that are characterized by the property that base equals altitude. The grid is easily laid out using graph paper; it is also easily constructed with ruler and compass because of the simple defining property of the triangles. (There is a grid of squares which underlies the triangular grid; this is shown in Figure 2.31b).

The grid in Figure 2.32 is made up of twelve connected vertical strips of isosceles right trian-

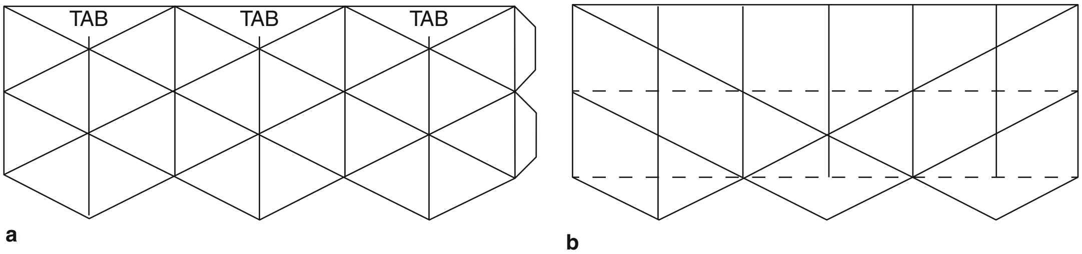

Figure 2.31.

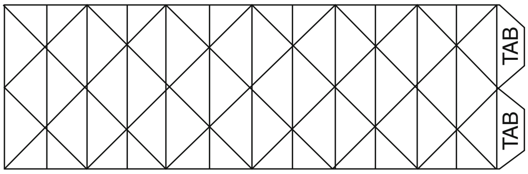

Figure 2.32.

gles, where the top and bottom triangles have been cut in half. This grid is obviously based on a grid of squares, and is easily laid out on graph paper or constructed with ruler and compass.

Before the grids are turned into three-dimensional objects, ask yourself the question that was asked earlier for the single strips. *Question:* From each of the constructed nets (as in Figures 2.31a and 2.32), what three-dimensional shape will be formed when they are folded along the common edges of the triangles?

Of course, you will need to use the earlier answers to the question in attempting to answer the question for the more complex nets. An auxiliary question that is worth asking is: Will all of the lines in the grid (which are common edges of pairs and triangles) play the same role in the three-dimensional form?

Now score all the lines in each grid (use a medium ballpoint pen), and cut out the nets around the outline (be sure to cut around the tabs). Fold the nets as follows:

1. Fold the net face-to-face (valley fold) on all vertical lines, including those to which the tabs are attached.
2. And fold the net back-to-back (mountain fold) on all diagonal lines.

Then cup the folded net in both hands, and gently squeeze it to encourage the top and the bottom to come together.

The net in Figure 2.31a should come together easily, with the half-triangles labeled as tabs completely covered. A chain of linked tetrahedra is formed. (Glue or tape the edges of the tetrahedra fitted over the tabs.) Holding the ends of the chain, bring the ends of the chain together, fitting the tabs at one end of the chain into the open edge at the other end of the chain. (If the chain does not come together easily, turn it until it does.) Glue or tape these last two edges to the tabs, completing the model.

The ring of six linked tetrahedra is a (carefully) crinkled torus (doughnut), and has the property that it can be *endlessly turned through its center hole*. Simply grasp the model in both hands and turn the tetrahedra inward, pushing the points through the center hole!

The Isoaxis

The net in Figure 2.32 will not come together to form a closed three-dimensional form, but rather it will form a (carefully) crinkled cylinder that will also turn through its center hold, changing its shape and appearing to "bloom" as it is turned. This form was discovered by graphic designer Wallace Walker, and is called Isoaxis®. Assemble Isoaxis as follows. Gently squeeze the scored and folded net so that it begins to curl and collapse along the fold lines. When fully collapsed, it will look like an accordion-folded paper with

Figure 2.33. Corraine Alves and Diana Weimer making kaleidocycles.

square cross section. (One method of achieving this state of the model is to begin at one end of the net, collapsing the net along the folds to form a square cross section, and holding the collapsed part between thumbs and forefingers, "gathering" the rest of the net into the collapsed state with the middle fingers.) The accordion-folded net should be pressed firmly; it is best if it can be pressed under a heavy object for 12 hours or more to set the folds fully. The two ends of the folded net are then joined (use tape or glue), matching tabs to the inside of opposite triangles. Join one tab at a time; the model will be tight, and so turn it through its center to join the second tab. To rotate this model, hold it in both hands and bring points to the center; push on the points. The crinkled cylinder will turn continuously through its center hole!

Further Exploration

There are many avenues for follow-up. A few are suggested below.

1. Explore the symmetry properties of the three-dimensional models. This can be enhanced by decorating the faces of the models to display various symmetries. One question that will need to be answered is: What faces in the flat net are adjacent (or become adjacent during rotation) in the three-dimensional forms?
2. Create other similar nets, varying the kinds of triangles chosen, and the number of triangles in the net. Fold in the same manner to see what three-dimensional forms result. A good challenge that can be met using only a knowledge of elementary geometry is: create other rings of tetrahedra having more tetrahedra, but such that the center hole in the ring is (in theory) a point, as is the case for the model in Figure 2.31. The model in Figure 2.31 has been called a "hexagonal kaleidocycle" by Walker and Schattschneider, because the center cross section of the assembled form is a regular hexagon.

Information on Construction Materials

The basic necessities for the above constructions are

- Paper
- 18-inch ruler
- Medium ballpoint pen
- Scissors
- White glue or tape

If the models are to be decorated, then coloring materials that will not weaken or warp the paper should be used. Since the models rotate, the paper chosen must not easily tear or break when bent repeatedly. Ordinary construction paper is not suitable. In addition, the paper should be heavy enough so that the three-dimensional models have suitable firmness. Medium-weight drawing paper, 100% rag, is excellent, and takes decoration well. Ordinary graph paper is too thin, but there are excellent heavier drafting and design papers that come in large sizes. The nets should not be made too small, or they become very difficult to put together and manipulate. A good size for the nets is 2.5 to 3 inches width for each "panel" of linked triangles for Figure 2.31, and 1.25 to 1.75 inches for each "panel" of linked triangles for Figure 2.32. The overall width of the nets should be in the range of 15 to 22 inches.

White glue seems to be best for assembling the models; in any case, the glue chosen should not warp the paper, nor should it be the "instant hold" variety since tabs need to be manipulated into place before the glue sets. If tape is used, then it must be the type used for hinging; ordinary clear plastic tape will break after a few turns of the model.

Giftwrap paper which has a pattern based on a square grid can be laminated to drawing paper to create a nicely decorated Isoaxis with an all-over pattern. Use a spray glue that will not be brittle when dry to affix the gift paper. The square grid of the pattern must be carefully followed for the lines of the net of Isoaxis.

The Rhombic Dodecahedron

Arthur L. Loeb

This is a recipe for constructing modules that generate the rhombic dodecahedron in two fundamentally different ways. The first construction stellates a cube with six square pyramids; the second stellates a regular octahedron with eight triangular pyramids.

The Pyramids

The first step is the construction of the sides of the pyramids. The square pyramids have an apex angle whose cosine equals $1/3$, while the triangular pyramids have an apex angle whose cosine equals $-1/3$. In order to produce mutually congruent dodecahedra by both methods, we construct the template shown in Figure 2.34 by the following steps:

1. Draw two mutually perpendicular lines. Call their intersection O.
2. Choose a point A, different from O, on one of the mutually perpendicular lines.
3. Draw a circle having radius equal to three times the distance OA, whose center is located on A.
4. Call the intersections of this circle with the extension of the line OA C and B, as shown. Call an intersection of the circle with the line perpendicular to OA D, as shown.
5. Connect C and D, as well as B and D.

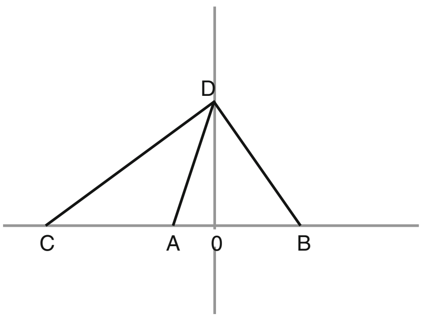

Figure 2.34.

The resulting template furnishes the following linear and angular dimensions:

- The length of the line segment CD is the edge length of the octahedron to be built.
- The length of the line segment BD is the edge length of the cube to be built.
- The triangle CAD is the shape of the sides of the triangular pyramids to be built.
- The triangle BAD is the shape of the sides of the square pyramids to be built.

Construction of the Polyhedra

1. Construct a regular octahedron whose edge length equals the length of line segment CD.
2. Construct a cube whose edge length equals the length of line segment BD.
3. Construct eight triangular pyramids whose bases are equilateral triangles having edge length equal to the length of line segment CD, and whose sides have the shape of triangle CAD.
4. Construct six square pyramids whose bases are square having edge length equal to the length of line segment BD, and whose sides have the shape of triangle BAD.

Juxtaposition of Polyhedra

Arrange the six square pyramids so that their square bases are in the configuration shown in Figure 2.35. Hinge them together so that they can rotate with respect to each other around their shared edges. When the pyramids are folded inward until their six apexes touch, they will form a cube congruent with the cube also constructed.

Arrange four of the triangular pyramids with their triangular bases in the configuration shown in Figure 2.36, and hinge them together as above. When folded in until their apices touch, they will form a regular tetrahedron. Repeat for the remaining four tetrahedra.

Place the six square pyramids around the cube, square faces joined to square faces. Place the

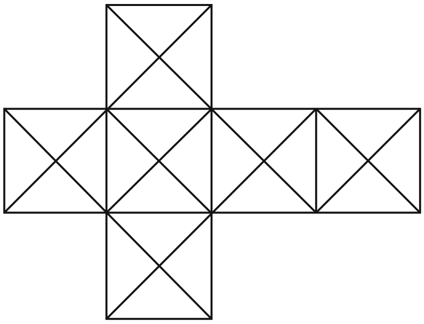

Figure 2.35.

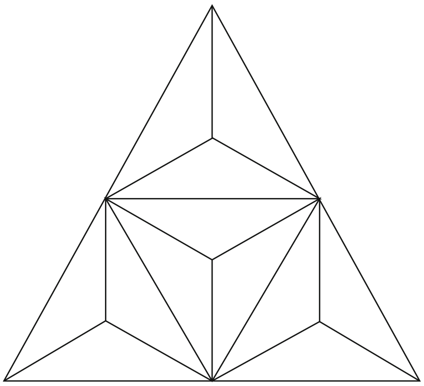

Figure 2.36.

eight triangular pyramids around the octahedron, with the equilateral triangles joined. The result should be two mutually congruent rhombic dodecahedra. Note that the cube edges constitute the shorter, the octahedron edges the longer, diagonals of the rhombic faces.

Place two square pyramids with their square bases joined. The result is an octahedron that is not regular, because its faces are not equilateral. Six of these irregular octahedra can be put together to form a rhombic dodecahedron. (*Note:* this would require *twelve* square pyramids rather than the six already constructed).

Space-Fillers

Of the polyhedra constructed, the following will fill space without interstitial spaces.

- Cube
- Rhombic dodecahedron
- Square pyramid
- Irregular octahedron
- Regular octahedron combined with eight triangular pyramids

Combinations of these (say cube in combination with square pyramids) are, of course, also possible.

A Note on Materials

Contributed by Jack Gray

Any useful polyhedral model is formed on the spectrum between "a rough sketch" and "a long-lasting work of art." The position on the spectrum is determined by the choices of materials, tools, and techniques as well as by the time and care used in the construction process. A rough sketch is always a valid precursor to a work of art. Expect to make a few mistakes on the sketch, and then try to conquer those in a second model.

Transparent tape is a good hinging material, while paper tape is thicker and more cumbersome. Use permanent tape, taking care to position it as follows. Place a strip of tape, sticky side up, on a flat surface. The strip should be longer by a good amount than the edge to be hinged. Weight down the ends of the tape so that it cannot move while you are connecting the polyedron to it.

Carefully lower the edge of the first polyhedron to the tape. Before letting it make contact, make sure that it is in the center of the width and the length of the tape. Make contact along the whole length of the edge.

Orient the second polyhedron to the first. Slide the second polyhedron down the face of the first until its edge touches the tape; then rotate it about that edge, so that contact with the tape is made along the entire edge. Trim off the excess tape with an X-Acto® knife or a single-edge razor blade.

Flip over the pair of joined polyhedra and inspect the tape hinge. Burnish it with your finger to complete contact along the full surface of the tape.

Figure 2.37. Arthur L. Loeb demonstrating his models.

Place another piece of tape of the same length on the flat surface. This will be used to tape the other side of the joined edges, creating a hinge that is equally strong on both sides. The tape should be weighted down as before, and care should be taken in centering the already-joined edges on the strip of tape before making contact. Let the joined faces lie flat against the tape to make contact along the full width of tape surface. Remove the excess tape and burnish as before.

If your sketch looks like a work of art, plan a finished model. Visit a local art store or hobby shop to examine the sheet materials that are available.

Various colored art papers, colored and transparent acetate, mirrored Mylar®, oak tag, and construction paper can be found. Fine rice papers are good for finishes, though they are too flexible for the body of such models. (In adding a surface finish of thin sheet material to your model, cut out each polygonal face so that it will not go across the hinge. Otherwise, the finish material will buckle when flexed.) Your experience making a sketch model will prepare you to pick materials "by feel."

Thicker sheet material like mat board and Plexiglas® need to have their edges mitered to half the dihedral angle between faces to prevent the thickness of the material from creating inaccuracies. Great care should be used in gluing such joints, so that glue does not spill onto the surfaces.

On a finished model, the hinging should be done with transparent polyester hinging tape. Cloth tape can be used on larger models.

Balloon Polyhedra

Erik Demaine, Martin Demaine, and Vi Hart

You have probably seen a long balloon twisted into a dog as in Figure 2.38. But did you know that the same balloon can be twisted into a regular octahedron like Figure 2.39? Here are two one-balloon constructions and their associated networks of edges and vertices (the technical term for such a network is a *graph*).

Balloon twisting offers a great platform for making, exploring, and learning about both polyhedra and graphs. Even the classic balloon dog can be thought of as a graph, with edges corresponding to balloon segments between the twists. Children of all ages can thus enjoy the physicality of the balloon medium while learning about mathematics.

Here we give a practical guide to making polyhedra and related geometric constructions with balloons, while briefly describing the mathematics and computer science related to balloon construction.

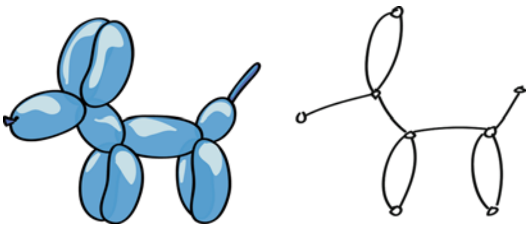

Figure 2.38. Classic dog (one balloon).

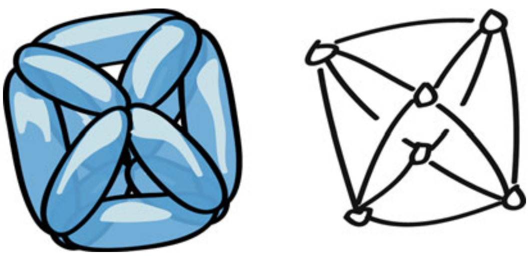

Figure 2.39. Octahedron (one balloon).

If Euler Were a Clown

What polyhedra can be made by twisting a single balloon, like the octahedron in Figure 2.39? The balloon must traverse every edge of the polyhedron in sequence, and can traverse each edge only once, though it can (and will) visit each vertex multiple times. Mathematicians will recognize this structure as an *Eulerian path*. We review the classic mathematics of this structure in the context of balloons by wondering: what is the simplest polyhedron that can be made from a single balloon?

The simplest (nonflat) polyhedron is a tetrahedron, but a tetrahedron cannot be made from a single balloon. Such a balloon would start at some vertex and end at some other vertex, but for every remaining vertex, each time the balloon enters the vertex it also exits the vertex. Thus, for a balloon twisting of a graph to possibly exist, every vertex except possibly two (the starting and ending vertices) must have an even number of edges incident to it, or even *degree*. But in the tetrahedron, all four vertices have odd degree. Thus the tetrahedron cannot be made from one balloon.

The four-dimensional analog of a tetrahedron (called the 4-*simplex*) consists of four tetrahedra glued together face-to-face. One 3D projection of a 4-simplex is a regular tetrahedron with all four vertices joined to a fifth vertex added in the center. Every vertex is thus joined to all four other vertices, giving it even degree. So the 4-simplex does not have the "too many odd-degree vertices" obstruction that the tetrahedron had. Indeed, Figure 2.40 shows a 4-simplex made from one balloon. Building one is somewhat difficult for practical reasons; we suggest you try it.

In fact, a single balloon can be twisted into any connected graph in which all vertices have even degree. The starting point is to traverse the graph naïvely: start the balloon at any vertex, route it along any incident edge, and keep going, at each step following any edge not already visited. Because the vertex degrees are all even, whenever the balloon enters a vertex, there is

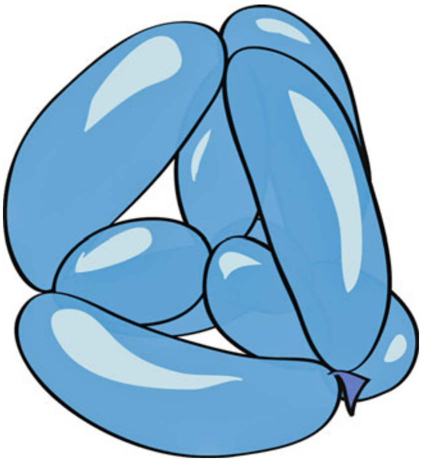

Figure 2.40. The 4-simplex, made from one balloon, is a puzzle to twist.

always another unvisited edge along which it can exit. The only exception is the starting vertex, where the balloon's initial exit left an odd number of unvisited edges. This vertex is the only place where the balloon might have to stop, and eventually this must happen because the balloon will run out of edges to visit. When this happens, the balloon forms a loop that visits some edges once, but possibly does not visit some edges at all.

Now consider just the graph of unvisited edges. It too has even degree at every vertex, because the balloon exited every vertex it entered, including the start vertex. Hence we can follow the algorithm again with a second balloon, and a third balloon, and so on, until the balloons cover all the polyhedron's edges (and no two balloons cover the same edge). Now we take any two balloons that visit the same vertex and merge them into one balloon by the simple switch shown in Figure 2.41. This also forms a loop, and visits all the same edges. Repeating this process, we end up with one balloon forming a loop that visits every edge exactly once.

Now we have a construction for twisting many graphs from a single balloon. We have seen one 4D polyhedron to which this construction applies, but what about 3D polyhedra? One example is the octahedron from Figure 2.39. The octahedron has six vertices, each of even degree 4. Therefore the general construction applies; Figure 2.42 shows a practical construction. The octahedron is actually the only Platonic solid twistable from a single balloon: for all the others, every vertex has odd degree (3 or 5).

Are there simpler 3D polyhedra than the octahedron that are twistable from one balloon? If we glue two tetrahedra together, we get a *triangular dipyramid*. The two apexes have odd degree 3, but the remaining vertices all have even degree 4. We know that a single balloon cannot tolerate more than two odd-degree vertices, but this polyhedron has just two, putting it right at the borderline of feasibility. Figure 2.43 shows that it indeed can be made from one balloon,

More generally, a single balloon can be twisted into any connected graph with exactly two vertices of odd degree. Just imagine adding an extra edge to the graph, connecting the two odd-degree vertices. This addition changes the degrees of the two odd-degree vertices by 1, making them even, and does not change the degree of any other vertices. Hence this modified graph has all vertices of even degree, so it can be twisted from a single balloon forming a loop. We can shift the loop of the balloon so that it starts and ends at one of the odd-degree vertices. Then we remove the added edge from both the graph and the balloon. We are left with a single balloon visiting every edge of the original graph exactly once. Naturally, the balloon starts at one odd-degree vertex and ends at the other.

Summarizing what we know, a one-balloon graph has at most two odd-degree vertices, and every connected graph with zero or two odd-degree vertices can indeed be twisted from one balloon. What about graphs with one odd-degree vertex? Don't worry about them; they don't exist. Euler showed that every graph has an even number of odd-degree vertices. To see why, imagine summing up the degrees of all the vertices. We can think of this sum as counting the edges of the graph, except that each edge gets counted twice, once from the vertex on either end. Therefore the sum is exactly twice the number of edges, which is an even number. The number of odd terms in

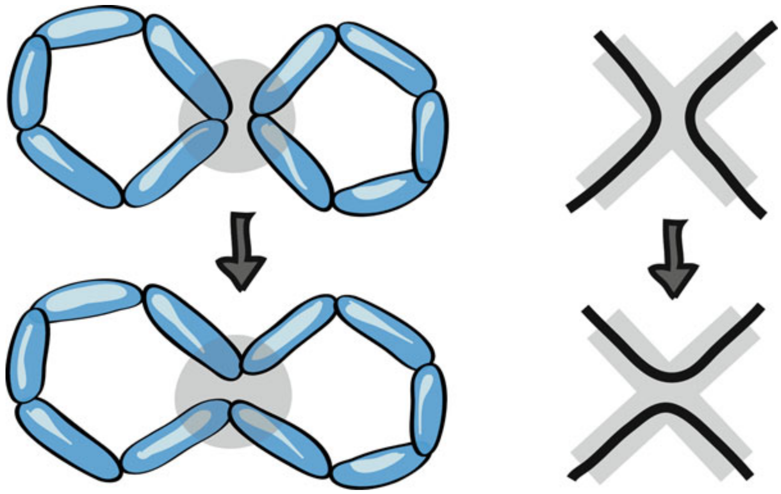

Figure 2.41. Joining two balloon loops (*top left*) into one balloon loop (*bottom right*) when the two loops visit a common vertex (*shaded*, and abstracted on the *right*).

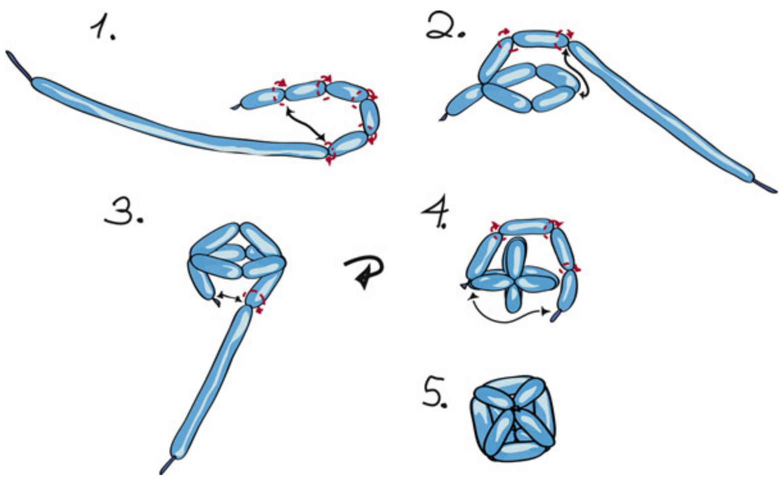

Figure 2.42. The octahedron is like the balloon dog of balloon polyhedra. With practice, it can be twisted quickly from one balloon.

the sum (the number of odd-degree vertices) must thus be even.

We conclude that we know all one-balloon graphs. In fact, we have just rediscovered the classic characterization of Eulerian paths, common in graph-theory textbooks, but in the context of balloons. If you are looking for a good challenge for making a polyhedron from one balloon, we recommend the cuboctahedron. Every vertex has degree 4, but there are twenty-four edges and sharper dihedral angles, making it difficult to twist except from especially narrow balloons.

Cheating with One Balloon

One trick for transforming graphs into one-balloon graphs is to double every edge: whenever two vertices are connected by an edge, add a second edge alongside it. This change doubles the degree of every vertex, so all resulting vertices

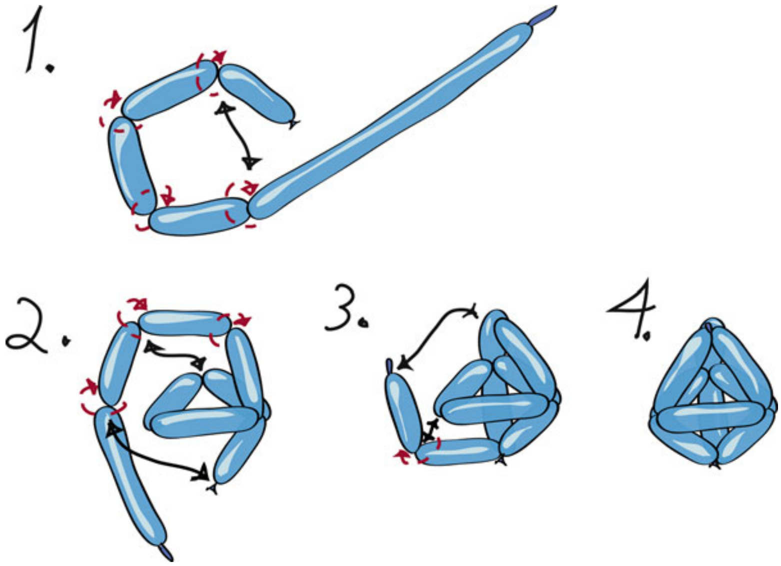

Figure 2.43. The triangular dipyramid is perhaps the simplest polyhedron twistable from one balloon, and it is easier than the octahedron.

Figure 2.44. Pop-twisting a tetrahedron from one balloon.

have even degree. Therefore we can make the doubled graph from one balloon. In other words, one balloon visits every edge in the original graph exactly twice instead of once.

In fact, we do not need to double every edge: we just need to double edges along paths connecting the odd-degree vertices in pairs, except for one pair of odd-degree vertices that we can leave alone. Minimizing the number of edges we double is the *Chinese postman problem*, a well-studied problem in computer science. Balloons offer a fun context for studying efficient algorithms for this problem.

After we have made a balloon with some of the edges doubled, we could *pop* the extra edges using a sharp object. The result is a more uniform aesthetic. For example, Figure 2.44 shows how a tetrahedron, which would normally require two balloons, can be made from one balloon with one popped segment. (We leave the construction of the tetrahedron with a doubled segment as an exercise for you. Remember to start and end at the two odd-degree vertices.) In practice, be careful to twist the ends of a segment extensively before popping it to prevent affecting the incident segments.

On the topic of balloon popping, a challenging type of puzzle is this: given an already-twisted balloon polyhedron, can you pop some of the balloon segments to make another desired graph?

This problem is known as *subgraph isomorphism*, and is among the family of computa-

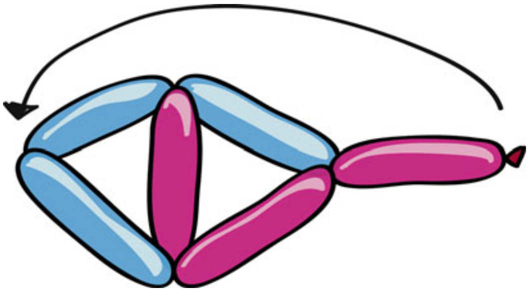

Figure 2.45. The tetrahedron is easy to twist from two balloons.

tionally intractable "NP-complete" problems, so there is likely no good algorithm to solve it.

Polyballoon Constructions

If we cannot make a graph with just one balloon, how many do we need? The minimum number of balloons that can make a particular graph, with each edge covered by exactly one segment, is the graph's *bloon number*.

There turns out to be a very simple formula for the bloon number: it is half the number of odd vertices (unless, of course, the graph has no odd vertices; then the bloon number is 1, not 0).

On the one hand, we cannot hope for fewer balloons: each odd-degree vertex must be the start or end of some balloon, so each balloon can "satisfy" only two odd-degree vertices. On the other hand, there is a construction with just this many balloons, using a simple construction similar to the arguments above. First, we add edges to the graph, connecting odd-degree vertices in disjoint pairs. The number of added edges is half the number of odd-degree vertices. The resulting graph has all vertices of even degree, because we added one edge incident to each vertex formerly of odd degree. Therefore it can be made from one balloon forming a loop. Now we remove all the edges we added. The number of removed edges, and hence the number of resulting balloons, is half the number of odd-degree vertices.

It now becomes an easy exercise to figure out how many balloons we need for our favorite polyhedron. For example, a tetrahedron requires only two balloons, as shown in Figure 2.45. An icosahedron requires six balloons, which can be made into identical two-triangle units as shown in Figure 2.46. The same unit can make a snub cube from twelve balloons, as shown in Figure 2.47, as well as a snub dodecahedron from thirty balloons (another exercise).

Note that, in both cases, there are two fundamentally different ways as to how the balloons can be assembled, right-handed and left-handed. This is a nice lesson on chirality (handedness).

For all Platonic and Archimedean solids, there is a construction out of balloons that

1. Uses the fewest possible balloons,
2. Uses balloons all of the same length, and
3. Preserves all or most of the symmetry of the polyhedron.

Achieving all of these properties together can be a fun puzzle. In fact, achieving just the first two properties is a computationally difficult problem: decomposing a graph into a desired number of equal-length balloons is a special case of "Holyer's problem" in graph theory, and it turns out to be among the family of difficult "NP-complete" problems, even for making polyhedra.

Tangles

Balloons can make many more graphs than just polyhedra. A simple extension is to look at disconnected graphs, made from multiple shapes. In *Orderly Tangles*, Alan Holden introduced *regular polylinks*, symmetric arrangements of identical regular polygons, which make a good subject for balloon twisting. Tangles have recently been explored with the aid of (freely available) computer software to find the right thicknesses of the pieces, which may be especially useful for determining the best-size balloons for twisting.

Perhaps the simplest example of a tangle is a model of the Borromean rings assembled from three rectangles, each lying in a coordinate plane. This model is fairly easy to construct from three balloons, as shown in Figure 2.48. Figure 2.49 shows a more challenging construction made

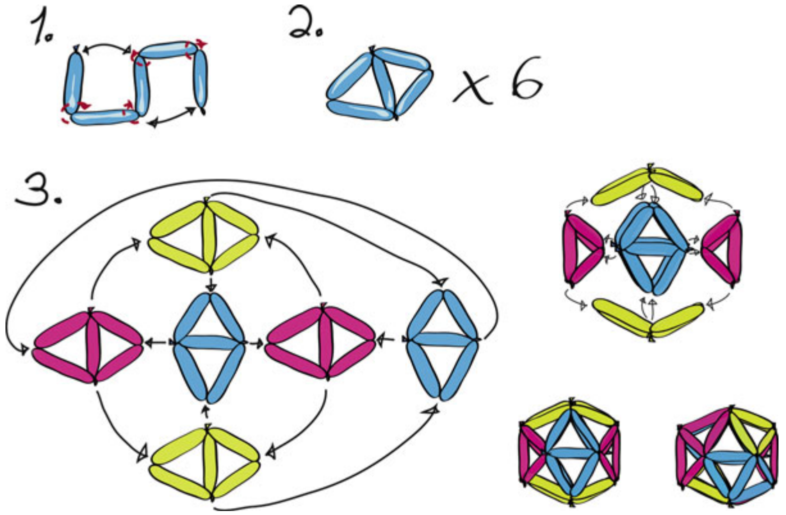

Figure 2.46. The icosahedron is a good example of joining several (six) identical balloon units. The tied balloon ends make it easy to attach vertices together.

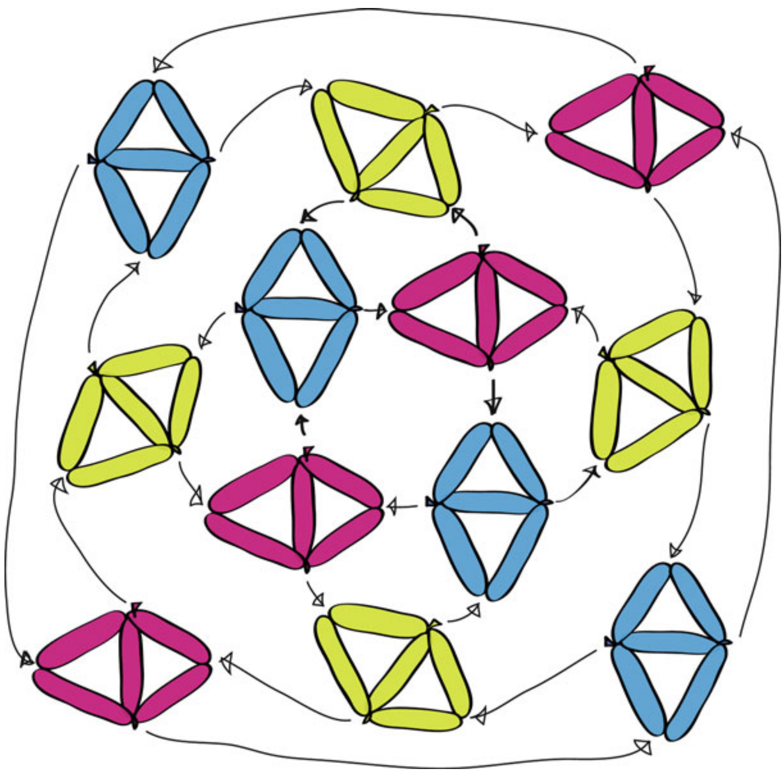

Figure 2.47. The snub cube is a bigger example of joining several (twelve) identical balloon units, recommended for groups of polyhedral balloon twisters. We leave the final form as a surprise.

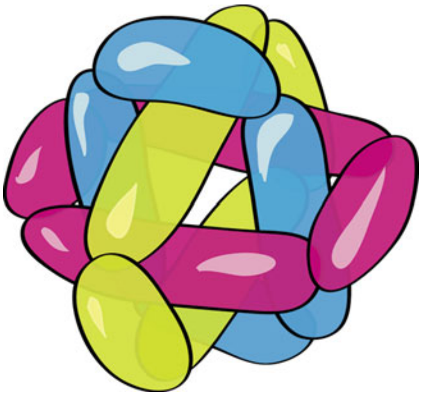

Figure 2.48. The Borromean rings are easy to make from three balloons.

from six squares; the difficulty is not twisting the squares, of course, but interlinking them in the correct over-under pattern. Harder still are the six-pentagon tangle and the four-triangle tangle.

A further generalization of regular polylinks are *polypolyhedra*, which allow the symmetrically arranged shapes to be Platonic solids in addition to regular polygons. Some examples such as the famous "five intersecting tetrahedra" have been around for many years, and recently even the subject of balloon twisting. But a thorough enumeration of all polypolyhedra is relatively recent. For a longer project, ideally with a group of mathematical balloon twisters, we recommend looking through this catalog of polypolyhedra.

Practical Guide for Twisting Balloon Polyhedra

Twisting balloons into polyhedra and related structures has been explored by several others. Several websites listed in the Notes for this chapter include video instructions. In the rest of this section, we give some practical tips for twisting your own balloons into polyhedra.

Long skinny balloons come in two main sizes: 160s (1-inch diameter, 60-inch length) and 260s (2 inches by 60 inches). For making one-balloon polyhedra, or complicated tangles, 160s are better. For most multiple-balloon polyhedra, the extra thickness of the 260s is better for stability. Of course, by not inflating all the way, or by attaching multiple balloons together, the balloon can reach any width-to-length ratio you like.

One of the biggest challenges is to get the balloon segments to be of equal length. Twisting a single balloon into the correct number of equal-length segments is something that just takes practice. If your lengths are off, you can always untwist and start again. When working with multiple balloons, it can be helpful to inflate them all to the same length before twisting them.

You may want to shorten the lengths of all the edges on a finished balloon structure, for example to get a tangle to fit together snugly. One way to do this is to grab all the edges at a vertex, and twist them all together. Performing such a twist at every vertex results in an aesthetically pleasing effect that accents the vertices.

Another problem you may run into is wanting to pass the end of a balloon through a hole that is narrower than the inflated balloon. If you can fit the deflated end of the balloon through, you can then squeeze the air through the deflated portion and inflate the other side. It is better, though, to figure out a twisting order that avoids this, and this is a puzzle in itself.

Enjoy your mathematical twists!

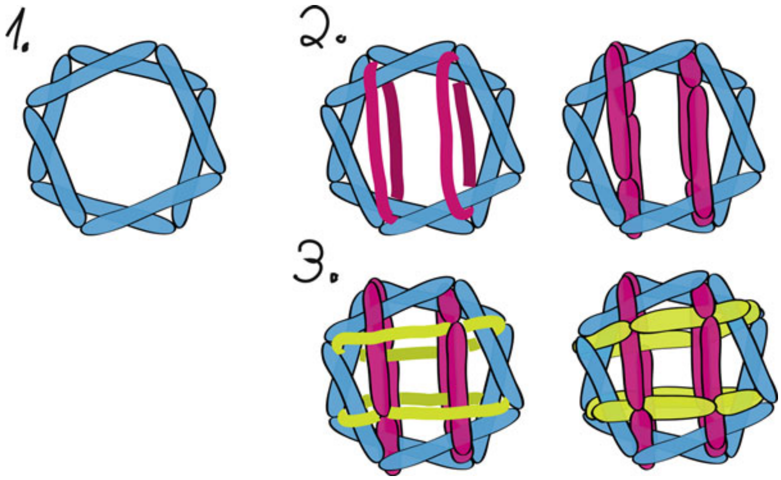

Figure 2.49. The six-square tangle is a good puzzle in assembling balloon tangles.

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