

Preface

Recent years have seen an explosion of research activities in a fascinating area called compressed sensing, compressive sensing, or compressive sampling. A Google Scholar search for articles containing one of these three terms in their title returned about 4,400 hits at the time this preface was written. The area of compressive sensing, at the intersection of mathematics, electrical engineering, computer science, and physics, takes its name from the premise that data acquisition and compression can be performed simultaneously. This is possible because many real-world signals are sparse, and even though they are acquired with seemingly too few measurements, exploiting sparsity enables one to solve the resulting underdetermined systems of linear equations. The reconstruction of sparse signals is not only feasible in theory, but efficient algorithms also exist to perform the reconstruction in practice. Moreover, involving randomness in the acquisition step enables one to utilize the minimal number of measurements. These realizations, together with their potential applications, have triggered the interest of the scientific community since around 2004. Some of the ingredients are of course much older than the advent of compressive sensing itself, and the underlying theory builds on various branches of mathematics. These branches include linear algebra, approximation theory, convex analysis, optimization, probability theory (in particular, random matrices), Banach space geometry, harmonic analysis, and graph theory. This book is a detailed and self-contained introduction to the rich and elegant mathematical theory of compressive sensing. It presents all the necessary background material without assuming any special prior knowledge—just basic analysis, linear algebra, and probability theory.

The perspective adopted here is definitely a mathematical one, only complemented at the beginning with a teaser on the strong potential for applications. Our taste partly dictated the choice of topics, which was limited by the need to keep this volume an introduction rather than an exhaustive treatise. However, the exposition is complete in the sense that we wanted every result to be fully proved for the material to become accessible to graduate students in mathematics as well as to engineers, computer scientists, and physicists. We have also made efforts to produce short and natural proofs that are often simplified versions of the ones found in the literature.

We both, independently, went through the process of rendering the foundations of compressive sensing understandable to students when we prepared lecture notes for courses given at Vanderbilt University, Drexel University, the University of Bonn, and ETH Zurich. This monograph is a further attempt to clarify the theory even more. Lecturers wishing to prepare a course based on it will find some hints at the end of Chap. 1.

The overall organization follows a path from simple to more complicated (so does the organization within chapters). This results in the structure of the book outlined below. The first chapter gives a brief introduction to the essentials of compressive sensing, describes some motivations and applications, and provides a detailed overview of the whole book. Chapters 2–6 treat the deterministic theory of compressive sensing. There, we cover the notion of sparsity, introduce basic algorithms, and analyze their performance based on various properties. Since the major breakthroughs rely on random matrices, we present the required tools from probability theory in Chaps. 7 and 8. Then Chaps. 9–12 deal with sparse recovery based on random matrices and with related topics. Chapter 13 looks into the use of lossless expanders and Chap. 14 covers recovery of random sparse signals with deterministic matrices. Finally, Chap. 15 examines some algorithms for ℓ_1 -minimization. The book concludes with three appendices which cover basic material from matrix analysis, convex analysis, and other miscellaneous topics.

Each chapter ends with a “Notes” section. This is the place where we provide useful tangential comments which would otherwise disrupt the flow of the text, such as relevant references, historical remarks, additional facts, or open questions. We have compiled a selection of exercises for each chapter. They give the reader an opportunity to work on the material and to establish further interesting results. For instance, snapshots on the related theory of low-rank matrix recovery appear as exercises throughout the book.

A variety of sparse recovery algorithms appear in this book, together with their theoretical analysis. A practitioner may wonder which algorithm to choose for a precise purpose. In general, all the algorithms should be relatively efficient, but determining which one performs best and/or fastest in the specific setup is a matter of numerical experiments. To avoid creating a bias towards any algorithm, we decided not to present numerical comparisons for the simple reason that running experiments in all possible setups is unfeasible. Nevertheless, some crude hints are given in the Notes section of Chap. 3.

It was a challenge to produce a monograph on a rapidly evolving field such as compressive sensing. Some developments in the area occurred during the writing process and forced us to make a number of revisions and additions. We believe that the current material represents a solid foundation for the mathematical theory of compressive sensing and that further developments will build on it rather than replace it. Of course, we cannot be totally confident in a prediction about a field moving so quickly, and maybe the material will require some update in some years.

Many researchers have influenced the picture of compressive sensing we paint in this book. We have tried to carefully cite their contributions. However, we are bound to have forgotten some important works, and we apologize to their authors for

that. Our vision benefited from various collaborations and discussions, in particular with (in alphabetical order) Nir Ailon, Akram Aldroubi, Ulaş Ayaz, Sören Bartels, Helmut Boelcskei, Petros Boufounos, Emmanuel Candès, Volkan Cevher, Albert Cohen, Ingrid Daubechies, Ron DeVore, Sjoerd Dirksen, Yonina Eldar, Jalal Fadili, Maryam Fazel, Hans Feichtinger, Massimo Fornasier, Rémi Gribonval, Karlheinz Gröchenig, Jarvis Haupt, Pawel Hitczenko, Franz Hlawatsch, Max Hügel, Mark Iwen, Maryia Kabanava, Felix Krahmer, Stefan Kunis, Gitta Kutyniok, Ming-Jun Lai, Ignace Loris, Shahar Mendelson, Alain Pajor, Götz Pfander, Alexander Powell, Justin Romberg, Karin Schnass, Christoph Schwab, Željka Stojanac, Jared Tanner, Georg Tauböck, Vladimir Temlyakov, Joel Tropp, Tino Ullrich, Pierre Vanderghenst, Roman Vershynin, Jan Vybiral, Rachel Ward, Hugo Woerdeman, Przemysław Wojtaszczyk, and Stefan Worm. We greatly acknowledge the help of several colleagues for proofreading and commenting parts of the manuscript. They are (in alphabetical order) David Aschenbrücker, Ulaş Ayaz, Bubacarr Bah, Sören Bartels, Jean-Luc Bouchot, Volkan Cevher, Christine DeMol, Sjoerd Dirksen, Massimo Fornasier, Rémi Gribonval, Karlheinz Gröchenig, Anders Hansen, Aicke Hinrichs, Pawel Hitczenko, Max Hügel, Mark Iwen, Maryia Kabanava, Emily King, Felix Krahmer, Guillaume Lecué, Ignace Loris, Arian Maleki, Michael Minner, Deanna Needell, Yaniv Plan, Alexander Powell, Omar Rivasplata, Rayan Saab, Željka Stojanac, Thomas Strohmer, Joel Tropp, Tino Ullrich, Jan Vybiral, Rachel Ward, and Hugo Woerdeman. We are grateful to Richard Baraniuk, Michael Lustig, Jared Tanner, and Shreyas Vasanawala for generously supplying us with figures for our book. We thank our host institutions for their support and the excellent working environment they provided during the preparation of our project: Vanderbilt University, Université Pierre et Marie Curie, and Drexel University for Simon Foucart; the Hausdorff Center for Mathematics and the Institute for Numerical Simulation at the University of Bonn for Holger Rauhut. Parts of the book were written during research visits of Holger Rauhut at Université Pierre et Marie Curie, at ETH Zurich, and at the Institute for Mathematics and Its Applications at the University of Minnesota. Simon Foucart acknowledges the hospitality of the Hausdorff Center for Mathematics during his many visits to Bonn. We are grateful to the Numerical Harmonic Analysis Group (NuHAG) at the University of Vienna for allowing us to use their online BibTeX database for managing the references. Simon Foucart acknowledges the financial support from the NSF (National Science Foundation) under the grant DMS-1120622 and Holger Rauhut acknowledges the financial support from the WWTF (Wiener Wissenschafts-, Forschungs- und Technologie-Fonds) through the project SPORTS (MA07-004) as well as the European Research Council through the Starting Grant StG 258926.

Finally, we hope that the readers enjoy their time studying this book and that the efforts they invest in learning compressive sensing will be worthwhile.

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