Chapter 2
Quantile-Based Reliability Concepts

Abstract There are several functions in reliability theory used to describe the patterns of failure in different mechanisms or systems as a function of age. The functional forms of many of these concepts characterize the life distribution and therefore enable the identification of the appropriate model. In this chapter, we discuss these basic concepts, first using the distribution function approach and then introduce their analogues in terms of quantile functions. Various important concepts introduced here include the hazard rate, mean residual life, variance residual life, percentile residual life, coefficient of variation of residual life, and their counterparts in reversed time. The expressions for all these functions for standard life distributions are given in the form of tables to facilitate easy reference. Formulas for the determination of the distribution from these functions, their characteristic properties and characterization theorems for different life distributions by relationships between various functions are reviewed. Many of the quantile functions in the literature do not have closed-form expressions for their distributions, and they have to be evaluated numerically. This renders analytic manipulation of these reliability functions based on the distribution function rather difficult. Accordingly, we introduce equivalent definitions and properties of the traditional concepts in terms of quantile functions. This leads to hazard quantile function, mean residual quantile function and so on. The interrelationships between these functions are presented along with characterizations. Various examples given in the sequel illustrate how the quantile based reliability functions can be found directly from the quantile functions of life distributions. Expressions of such functions for standard life distributions can also be read from the tables provided in each case.

2.1 Concepts Based on Distribution Functions

The notion of reliability, in the statistical sense, is the probability that an equipment or unit will perform the required function, under conditions specified for its operation, for a given period of time. In Sect. 1.3, we defined life distributions and

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gave several examples of such distributions used in the literature under different contexts. When a unit does not perform its intended function, we say that it has failed. This can happen in different forms such as mechanical breakdown, decrease in performance below an assigned level, defective performance, and so on. The primary concern in reliability theory is to understand the patterns in which failures occur, for different mechanisms and under varying operating environments, as a function of age. Accordingly, several concepts have been developed that help in evaluating the effect of age, based on the distribution function of the lifetime random variable X and its residual life X_t introduced earlier in Sect. 1.2. In this section, we present some key concepts and their properties as background material for later discussions using the quantile functions as the basic fabric.

2.1.1 Hazard Rate Function

The hazard rate of X is defined as

$$h(x) = \lim_{\delta \downarrow 0} \frac{P(x \le X < x+\delta | X > x)}{\delta} \tag{2.1}$$

so that $\delta h(x)$ is approximately the conditional probability that a unit will fail in the next small interval of time δ, given that the unit has survived age x. When $F(x)$ is absolutely continuous with probability density function $f(x)$, (2.1) reduces to

$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d \log \bar{F}(x)}{dx} \tag{2.2}$$

for all x for which $\bar{F}(x) > 0$. Treated as a function of age x, the hazard rate function is also referred to as the failure rate function, instantaneous death rate, force of mortality, and intensity function in other areas of study like survival analysis, actuarial science, biosciences, demography and extreme value theory. The origin of hazard rate can be traced back to the 'force of mortality' used in connection with the construction of life tables as models of human mortality.

Integrating (2.2) over $(0,x)$ and using $F(0) = 0$, we get

$$\bar{F}(x) = \exp\left\{ -\int_0^x h(t)dt \right\}. \tag{2.3}$$

The inversion formula in (2.3) is often used to characterize life distributions in terms of the functional form of $h(x)$, that could be postulated from the physical properties of the failure rate patterns. While postulating the form of $h(x)$, the following theorem is helpful in the choice of $h(x)$.

Theorem 2.1 (Marshall and Olkin [412]). *A necessary and sufficient condition that a function $h(x)$ is the hazard rate of a distribution is that*

(i) $h(x) \geq 0$;
(ii) $\int_0^x h(t)dt < \infty$ *for some* $x > 0$;
(iii) $\int_0^\infty h(t)dt = \infty$;
(iv) $\int_0^x h(t)dt = \infty$ *implies* $h(y) = \infty$ *for every* $y > x$.

Example 2.1. The exponential power model has survival function

$$\bar{F}(x) = \exp[-(e^{(\lambda x)^\alpha} - 1)], \quad x > 0.$$

The probability density function is

$$f(x) = -\frac{d\bar{F}(x)}{dx} = \lambda^\alpha \alpha x^{\alpha-1} \exp[-(e^{(\lambda x)^\alpha} - 1)] e^{(\lambda x)^\alpha},$$

and so

$$h(x) = \frac{f(x)}{\bar{F}(x)} = \alpha\lambda^\alpha x^{\alpha-1} \exp[(\lambda x)^\alpha].$$

Example 2.2. Let $X_1, X_2, \dots, X_n$ be independent random variables and $Z = \min(X_1, X_2, \dots, X_n)$. Then,

$$P(Z > x) = P(X_1 > x, X_2 > x, \dots, X_n > x)$$

or

$$\bar{F}_Z(x) = \bar{F}_{X_1}(x) \dots \bar{F}_{X_n}(x).$$

Logarithmic differentiation leads to

$$h_z(x) = h_{X_1}(x) + \cdots + h_{X_n}(x). \tag{2.4}$$

The above model constitutes a series system with n independent components having life distributions $F_{X_1}, \dots, F_{X_n}$. This is more general than the system illustrated in Sect. 1.5, since the components here are not identically distributed. Formula (2.4) could be employed to construct new life distributions from standard ones.

Hjorth [272] chose X_1 to be Pareto II (see Table 1.1) and X_2 to be Rayleigh with

$$\bar{F}_{X_2}(x) = \exp\left(-\frac{1}{2}\alpha x^2\right), \quad x > 0,$$

to obtain the model

$$\bar{F}(x) = e^{-\frac{\alpha x^2}{2}} (1 + \theta x)^{-\beta}.$$

The resulting hazard rate is

$$h(x) = \alpha x + \beta\theta(1+\theta x)^{-1}.$$

In a similar manner, Jaisingh et al. [291] considered a three-component model consisting of exponential, Pareto II and Weibull (see Table 1.1) to produce the model

$$\bar{F}(x) = \beta^{\theta}(x+\beta)^{-\theta}\exp[-\alpha x - \lambda^{-1}\delta x^{\lambda}], \quad x > 0,$$

with corresponding hazard rate

$$h(x) = \alpha + \theta(x+\beta)^{-1} + \delta\lambda x^{\lambda-1}.$$

Further examples of similar models can be seen in Wang [577] and Jiang and Murthy [294]. The linear failure rate distribution and quadratic failure rate distribution (Gore et al. [223]) with respective hazard functions

$$h_1(x) = a + bx$$

and

$$h_2(x) = a + bx + cx^2$$

can also be interpreted in the same manner, although they have been derived independently without such assumptions. Some of these distributions also figure in the context of additive hazards models considered in Nair and Sankaran [446].

The hazard rate functions of various distributions, their analysis with extensive references are given in Lai and Xie [368]. We have presented in Table 2.1 the expressions for $h(x)$ of the life distributions given in Table 1.1. It may be noticed that all the distributions in Example 2.2 do not have closed-form expressions for $Q(u)$ and therefore do not form part of Table 1.1

2.1.2 *Mean Residual Life Function*

Another important notion is based on the residual life introduced earlier in Sect. 1.2. For a unit which has survived until x, the lifetime remaining to it is $(X - x|X > x)$ with survival function (1.3)

$$\bar{F}_x(t) = \frac{\bar{F}(x+t)}{\bar{F}(x)}.$$

Table 2.1 Hazard rate functions of distributions in Table 1.1

No.	Distribution	Hazard rate
1	Exponential	λ
2	Weibull	$\lambda\sigma^{-\lambda}x^{\lambda-1}$
3	Pareto II	$c(x+\alpha)^{-1}$
4	Rescaled beta	$c(R-x)^{-1}$
5	Half-logistic	$e^{\frac{x}{\sigma}}[\sigma(1+e^{\frac{x}{\sigma}})]^{-1}$
6	Power	$\beta x^{\beta-1}(\alpha^{\beta}-x^{\beta})^{-1}$
7	Pareto I	αx^{-1}
8	Burr XII	$kcx^{c-1}(1+x^{c})^{-1}$
9	Gompertz	BC^{x}
10	Log logistic	$\beta\alpha^{\beta}x^{\beta-1}(1+\alpha^{\beta}x^{\beta})$
11	Exponential geometric	$\lambda(1-pe^{-\lambda x})^{-1}$
12	Generalized Weibull	$\alpha x^{\alpha-1}(\beta^{\alpha}-\lambda x^{\alpha})^{-1}$
13	Exponentiated Weibull	$\dfrac{\lambda\theta(\frac{x}{\sigma})^{\lambda-1}[1-\exp(-\frac{x}{\sigma})^{\lambda}]^{\theta-1}\exp[-(\frac{x}{\sigma})^{\lambda}]}{1-[1-\exp(-\frac{x}{\sigma})^{\lambda}]^{\theta}}$
14	Generalized exponential	$\dfrac{\theta(1-e^{-\frac{x}{\sigma}})^{\theta-1}e^{-\frac{x}{\sigma}}}{\sigma[1-(1-e^{-\frac{x}{\sigma}})^{\theta}]}$
15	Extended Weibull	$\lambda\sigma^{-\lambda}x^{\lambda-1}[1-(1-\theta)e^{-(\frac{x}{\sigma})^{\lambda}}]^{-1}$
16	Inverse Weibull	$\dfrac{\lambda\sigma^{\lambda}x^{-\lambda-1}e^{-(\frac{x}{\sigma})^{\lambda}}}{[1-e^{-(\frac{\sigma}{x})^{\lambda}}]}$
17	Generalized Pareto	$\dfrac{a+1}{ax+b}$
18	Exponential power	$\alpha\lambda^{\alpha}x^{\alpha-1}\exp[-(\lambda x)^{\alpha}]$
19	Modified Weibull	$\alpha\lambda(\frac{x}{\sigma})^{\lambda-1}\exp[\frac{x}{\sigma}]^{\lambda}$
20	Log Weibull	$\frac{k\rho}{1+\rho x}[\log(1+\rho x)]^{k-1}$
21	Dimitrakopoulou et al.	$\alpha\lambda\beta x^{\beta-1}(1+\lambda x^{\beta})^{\alpha-1}$
22	Logistic exponential	$\dfrac{ke^{\lambda(x+\theta)}(e^{\lambda(x+\theta)}-1)^{k-1}}{1+(e^{\lambda(x+\theta)}-1)^{k}}$
23	Kus	$\dfrac{\beta\lambda e^{-\beta x}\exp[\lambda e^{-\beta x}]}{1-\exp[\lambda e^{-\beta x}]}$
24	Greenwich	$ax(b^{2}+x^{2})^{-1}$
25	Generalized half-logistic	$\frac{(1-kx)^{-1}}{1+(1-kx)^{1/k}}$

The expected value of the distribution $\bar{F}_x(t)$ is called the mean residual life function and is denoted by $m(x)$. Thus, when $E(X) < \infty$,

$$m(x) = \int_0^\infty \frac{\bar{F}(t+x)}{\bar{F}(x)} dt = \frac{1}{\bar{F}(x)} \int_x^\infty \bar{F}(t) dt \tag{2.5}$$

for all x for which $\bar{F}(x) > 0$. When $\bar{F}(x)$ has a density $f(x)$, we have

$$m(x) = \frac{1}{\bar{F}(x)} \int_x^\infty (t-x) f(t) dt. \tag{2.6}$$

Like the hazard rate function, the origin of mean residual life also traces back to the life table function 'expectation of life', used by actuaries. It is called the mean excess function in actuarial science.

Differentiating (2.5) with respect to x and rearranging the terms, the identity

$$h(x) = \frac{1 + m'(x)}{m(x)} \tag{2.7}$$

results. The function $m(x)$ determines the distribution of X uniquely by virtue of the formula

$$\bar{F}(x) = \frac{\mu}{m(x)} \exp\left\{ -\int_0^x \frac{dt}{m(t)} \right\}. \tag{2.8}$$

Both the hazard function and the mean residual life function are conditional on the survival until x. The former provides information in an infinitesimal interval after x, while the latter contributes to the entire interval $[x, \infty)$. For further comparison of the two measures, we refer to Muth [436]. Guess and Proschan [228] and Nanda et al. [458] have both reviewed the basic results and various applications of the mean residual life function and associated orderings and properties.

Example 2.3. Consider the Weibull distribution with survival function

$$\bar{F}(x) = \exp\left\{ -\left(\frac{x}{\sigma}\right)^{\frac{1}{2}} \right\}, \quad x > 0.$$

Then,

$$\begin{aligned} m(x) &= \frac{1}{\bar{F}(x)} \int_x^\infty e^{-(\frac{t}{\sigma})^{\frac{1}{2}}} dt \\ &= 2e^{(\frac{x}{\sigma})^{\frac{1}{2}}} \int_{x^{\frac{1}{2}}}^\infty y e^{-\sigma^{-\frac{1}{2}} y} dy \quad (\text{with } y = x^{\frac{1}{2}}) \\ &= 2\sigma^{\frac{1}{2}} (\sigma^{\frac{1}{2}} + x^{\frac{1}{2}}). \end{aligned}$$

Table 2.2 Mean residual life functions of some distributions

Distribution	$m(x)$
Exponential	λ^{-1}
Power	$\dfrac{(\beta+1)\alpha^{\beta}(1-x)+x^{\beta+1}-1}{(\beta+1)(\alpha^{\beta}-x^{\beta})}$
Pareto II	$\dfrac{x+\alpha}{c-1}$
Rescaled beta	$\dfrac{R-x}{R+1}$
Pareto	$(\alpha-1)^{-1}x$
Half-logistic	$\sigma(1+e^{\frac{x}{\sigma}})\log(1+e^{-\frac{x}{\sigma}})$
Exponential geometric	$-(\lambda p)^{-1}e^{\lambda x}(1-pe^{-\lambda x})\log(1-pe^{-\lambda x})$
Exponential geometric extension	$-\{\beta(1-\theta)\}^{-1}e^{\beta x}[1-(1-\theta)e^{-\beta x}]$ $\log[1-(1-\theta)e^{-\beta x}]$
Adamidis et al. [17]	$(\bar{F}(x)=\theta e^{-\beta x}[1-(1-\theta)e^{-\beta x}]^{-1},\ x>0)$

Example 2.4. Let X be distributed as exponential geometric with

$$\bar{F}(x)=(1-p)e^{-\lambda x}(1-pe^{-\lambda x})^{-1}.$$

Then,

$$\begin{aligned} m(x) &= \frac{1-pe^{-\lambda x}}{(1-p)e^{-\lambda x}}\int_x^{\infty}\frac{(1-p)e^{-\lambda x}}{1-pe^{-\lambda x}}dx \\ &= -(\lambda p)^{-1}e^{-\lambda x}(1-pe^{-\lambda x})\log(1-pe^{-\lambda x}). \end{aligned}$$

Further examples of mean residual life functions are presented in Table 2.2.

Not every function can be the mean residual life function of a life distribution. The following theorem helps to conclude whether a given function can represent a mean residual life.

Theorem 2.2 (Guess and Proschan [228] and Nanda et al. [458]). *A necessary and sufficient condition for $m(x)$ to be a mean residual life function is that*

(i) $m(x)$ *has range* $[0,\infty)$ *for all* $x\geq 0$*;*
(ii) $m(0)=\mu>0$*;*
(iii) $m(x)$ *is right continuous;*
(iv) $m(x)+x$ *is increasing;*
(v) when there exists an x_0 *satisfying* $\lim_{x\downarrow x_0}m(x)=0$*, then* $m(x)=0$ *holds for* x *in* $[x_0,\infty)$*. If there is no* x_0 *for which the above limit is 0, then* $\int_0^{\infty}\frac{dx}{m(x)}=\infty$*.*

The formula in (2.7) makes it easy to find the hazard function when $m(x)$ is given. However, the problem is with the converse. When $h(x)$ is known, the differential equation resulting from (2.7) in $m(x)$ is difficult to solve for most distributions. Hence, efforts were put in to finding simpler relationships between $m(x)$ and $h(x)$ satisfied by distributions. The price paid for simplicity in such cases is the limitation to the range of applicability. Starting with individual distributions like gamma and negative binomial (Osaki and Li [475]), the work in this direction progressed to characterization of Pearson family (Nair and Sankaran [442]), the exponential family (Consul [155]), mixtures of distributions (Abraham and Nair [12]), generalized Pearson system (Sankaran et al. [516]) and other generalizations (Gupta and Bradley [238]). The general relationship

$$E(C(X)|X > x) = \mu_C + \sigma_C h(x)g(x), \tag{2.9}$$

for some $g(x)$ and a measurable function $C(x)$ with $\mu_C = EC(X)$ and $\sigma_C^2 = V(C(X))$, is seen to hold for the class of distributions satisfying

$$\frac{f'(x)}{f(x)} = \frac{\mu_C - C(x) - g'(x)}{\sigma_C g(x)}$$

and conversely (Nair and Sudheesh [449]). The special case $C(X) = X$ gives the necessary relationship in terms of $m(x)$ and $h(x)$. Apart from providing such relationships for a wider class of distributions, (2.9) was employed to develop lower bound to the variance that compares favourably with the well-known Cramer–Rao and Chapman–Robbins inequalities. Details can be seen in Nair and Sudheesh [449, 450] and also in the references in Nair and Sankaran [442]. In an alternative approach, Nair and Sankaran [443] viewed the mean residual life function as the expectation of the conditional distribution of residual life given age, arising from the joint distribution of age and residual life in renewal theory.

2.1.3 Variance Residual Life Function

For a lifetime random variable X with $E(X^2) < \infty$, the variance residual life function is defined as

$$\begin{aligned} \sigma^2(x) &= V(X - x|X > x) = V(X|X > x) \\ &= E(X - x^2|X > x) - m^2(x) \\ &= \frac{1}{\bar{F}(x)} \int_x^\infty (t-x)^2 f(t)dt - m^2(x). \end{aligned} \tag{2.10}$$

The integral on the right side can be simplified by integration by parts as

$$\int_x^\infty (t-x)^2 f(t)dt = 2\int_x^\infty (t-x)\bar{F}(t)dt = 2\int_x^\infty \int_u^\infty \bar{F}(t)\, dt\, du$$

and therefore

$$\sigma^2(x) = 2[\bar{F}(x)]^{-1} \int_x^\infty \int_u^\infty \bar{F}(t)\, dt\, du - m^2(x). \tag{2.11}$$

Apart from the usual meaning as a measure of spread of the residual life distribution and its role in finding the variance of the sample mean residual life, $\sigma^2(x)$ has some other important applications in reliability analysis. Launer [377], who introduced this concept, used it to distinguish life distributions based on its monotonic properties while Gupta and Kirmani [243, 245] considered characterizations using $\sigma^2(x)$ (see also Gupta [234] and Gupta et al. [246]). Gupta [234] established that

$$\frac{d}{dx}\sigma^2(x) = h(x)(\sigma^2(x) - m^2(x)). \tag{2.12}$$

It follows from (2.3) that (Abouammoh et al. [8])

$$\bar{F}(x) = \exp\left[-\int_0^x \frac{\frac{d\sigma^2(t)}{dt}}{\sigma^2(t) - m^2(t)}\right]. \tag{2.13}$$

Equation (2.13) makes it clear that both $\sigma^2(x)$ and $m(x)$ are required to retrieve $F(x)$. With the variance and mean in place, the coefficient of variation of residual life becomes

$$C(x) = \frac{\sigma(x)}{m(x)}. \tag{2.14}$$

Gupta [234] showed that

$$\frac{d}{dx}\sigma^2(x) = m(x)(1 + m'(x))(C^2(x) - 1)$$

which arises from (2.12), (2.14) and (2.7). In a later work, Gupta and Kirmani [243] found

$$m(x) = (1 + C^2(x))^{-1}\left\{\int_0^x C^2(t)dt + m(1 + C^2(0)) - x\right\}. \tag{2.15}$$

Since $m(x)$ characterizes $F(x)$ and $m(x)$ is expressed uniquely in terms of $C(x)$ by (2.15), it is evident that $C(x)$ also determines $F(x)$ uniquely. They also showed that if two life distributions $F(x)$ and $G(x)$ have the same means and equal residual coefficient of variation for all x, then $F = G$. This was further strengthened in Gupta and Kirmani [243] by the conditions $m_F(x_0) = m_G(x_0)$ for some $x_0 \geq 0$ and $\sigma_F^2(x) = \sigma_G^2(x)$, for the equality of $F(x)$ and $G(x)$ for all x.

Unlike the hazard and mean residual life functions, there is no direct formula that expresses $F(x)$ in terms of $\sigma^2(x)$ only. This brings in the importance of characterizing specific distributions or families by the functional form of $\sigma^2(x)$. The works of Dallas [166], Adatia et al. [19], Koicheva [351], Ghitany et al. [213], Navarro et al. [465] and El-Arishi [184] all belong to this category. Most of these results are subsumed in the general formula given in Nair and Sudheesh [451], which states that if (2.9) holds, then

$$V(C(X)|X>x) = \sigma_C E[C'(X)g(X)|X>x] + (\mu_C - M(x))(M(x)-C(x)), \quad (2.16)$$

where $M(x) = E(C(X)|X>x)$. Conversely, if there exists a measurable function $C(x)$ for which $C'(x) \neq 0$ for all $x>0$, satisfying (2.15), then (2.9) holds and

$$\frac{f'(x)}{f(x)} = \frac{\mu_C - C(x) - g'(x)}{\sigma_C g(x)}.$$

When $C(X) = X$, the implication to $\sigma^2(x)$ from (2.16) is obvious.

Example 2.5. The generalized Pareto distribution has survival function

$$\bar{F}(x) = \left(1+\frac{ax}{b}\right)^{-\frac{a+1}{a}}, \; x>0; b>0, a>-1. \quad (2.17)$$

The form of this distribution is quite amenable to deriving several characterizations based on reliability functions. It consists of three distributions, viz., the exponential ($a \to 0$), the Pareto II when $a = (C^{-1}-1)$ and $b = a\alpha$, and the rescaled beta when $a = -(1+C^{-1})$ and $b = Ra$; see Table 1.1 for details. All the three constituent distributions are important models in reliability on their own accord. For the model in (2.17), we have

$$m(x) = \left(1+\frac{ax}{b}\right)^{\frac{a+1}{a}} \int_x^\infty \left(1+\frac{at}{b}\right)^{-\frac{a+1}{a}} dt = ax+b.$$

Hence, a linear mean residual life function characterizes the generalized Pareto model of which, the exponential has $m(x) = b$ ($a=0$), Pareto II has $a>0$ so that $m(x)$ is increasing, and rescaled beta has $-1<a<0$ giving a decreasing mean residual life.

Notice also that the hazard rate of (2.16) is

$$h(x) = \frac{(a+1)}{(ax+b)}.$$

Hence,

$$m(x)h(x) = \text{constant},$$

a relationship that affords another characterization; see Mukherjee and Roy [431] for further details.

Again, we have

$$\sigma^2(x) = 2\left(1+\frac{ax}{b}\right)^{\frac{a+1}{a}} \int_x^\infty \int_u^\infty \left(1+\frac{at}{b}\right)^{-\frac{a+1}{a}} dt\,du$$

from (2.11). After simplification, we obtain

$$\sigma^2(x) = \frac{a+1}{1-a} b^2 (ax+b)^2.$$

Thus, we have the identity

$$\sigma^2(x) = Km^2(x). \tag{2.18}$$

Conversely, if we assume (2.18), upon substituting it in (2.13), we get

$$m'(x) = \frac{k-1}{k+1}$$

which implies that $m(x)$ is linear and X is distributed as generalized Pareto.

2.1.4 Percentile Residual Life Function

The mean and variance of residual life are popular measures in lifelength analysis with potential applications in other fields of study. However, there are instances like censored data, or observations from heavily skewed distributions in which the empirical counterparts of the two functions are difficult to compute. Moreover, the other limitations that were described in Chap. 1 in connection with the use of conventional moments are also true for $m(x)$ and $\sigma^2(x)$. An alternative in such cases is the percentile residual life function first studied by Haines and Singpurwalla [258]; see also Launer [376].

For any $0 < \alpha < 1$, the αth percentile residual life function is the αth percentile of the residual life distribution of X. Thus, recalling from (2.4) the expression for the survival function of the residual life, the αth percentile residual life function, denoted by $p_\alpha(x)$, is

$$\begin{aligned} p_\alpha(x) &= F_x^{-1}(\alpha) \\ &= \inf\{x | F_x(t) \geq \alpha\} \\ &= \inf\left\{x | 1 - \frac{\bar{F}(x+t)}{\bar{F}(x)} \geq \alpha\right\} \end{aligned}$$

$$= \inf\{y|\bar{F}(y) \le (1-\alpha)\bar{F}(x)\} - x$$
$$= F^{-1}(1-(1-\alpha)\bar{F}(x)) - x. \quad (2.19)$$

Thus, $p_\alpha(x)$ can be expressed in terms of the baseline distribution function $F(x)$. From (2.19), it is clear that $p_\alpha(x)$ is a solution of the functional equation

$$F(p_\alpha(x)+x) = 1-(1-\alpha)\bar{F}(x) = \alpha + (1-\alpha)F(x). \quad (2.20)$$

We interpret $p_\alpha(x)$ as the age that will be survived, on the average, by $100(1-\alpha)\%$ of units that have lived beyond age x.

Example 2.6. The exponential distribution $\bar{F}(x) = e^{-x}$, $x > 0$, has $p_\alpha(x)$ defined by (2.20) as

$$1 - e^{-(x+p_\alpha)} = 1-(1-\alpha)e^{-x}$$

which simplifies to

$$p_\alpha = -\log(1-\alpha)$$

which is a constant, independent of x for any choice of α in $(0,1)$. On the other hand, choosing (Song and Cho [545])

$$\bar{F}(x) = e^{-x}(1+\theta \sin x),\ x \ge 0, |\theta| < 2^{-\frac{1}{2}},$$

and $\alpha = 1 - e^{-2\pi}$, (2.20) yields

$$\begin{aligned}\bar{F}(p_\alpha(x)+x) &= 1 - e^{-2\pi}e^{-x}(1+\theta\sin x)\\ &= 1 - e^{-(2\pi+x)}(1+\theta\sin(x+2\pi))\\ &= \bar{F}(2\pi+x).\end{aligned}$$

Since F is continuous and strictly increasing, we get

$$p_\alpha(x) = 2\pi$$

which is the same as that of the exponential when $\alpha = 1-e^{-2\pi}$.

It is clear from the above example that the percentile residual life function does not determine $F(x)$ uniquely. Thus, the problem of searching conditions for characterizing distributions in terms of $p_\alpha(x)$ has received the attention of many researchers like Schmittlein and Morrison [523], Arnold and Brockett [38], Gupta and Langford [247], Joe [300], Song and Cho [545], Lillo [399] and Lin [402]. A comprehensive solution was offered by Gupta and Langford [247] (see also Joe [300]) who identified (2.20) as a particular case of the Schroder functional equation

$$R(\phi(t)) = uR(t), \quad 0 \leq t < \infty, \tag{2.21}$$

where $0 < u < 1$ and $\phi(t)$ is a continuous and strictly increasing function on $[0,\infty]$ satisfying $\phi(t) > t$ for all t. The general solution of (2.22) is

$$R(t) = R_0(t)K(\log R_0(t)),$$

where $K(\cdot)$ is a periodic function with period $-\log u$ and $R_0(t)$ is a particular solution of (2.21) which is positive, continuous and strictly decreasing such that $R(0) = 1$. Thus, there is no unique solution to (2.20). Song and Cho [545] (correcting a result of Arnold and Brockett [38]) proved that if F is continuous and strictly increasing and if for $0 < \alpha_1 < \alpha_2 < 1$, $\frac{\log(1-\alpha_1)}{\log(1-\alpha_2)}$ is irrational, then F is uniquely determined by $p_{\alpha_1}(x)$ and $p_{\alpha_2}(x)$. More general results due to Lin [402] are the following:

1. If $F(x)$ and $G(x)$ are continuous distributions on $[0,\infty)$ such that $F(0) = G(0) = u_0$ in $[0,1)$, then for a fixed number α, $p_{\alpha,F}(x) = p_{\alpha,G}(x)$ if and only if

$$\bar{F}(x) = \bar{G}K_1(-\log \bar{G}(x)) \quad 0 \leq x < r,$$
$$\bar{G}(x) = \bar{F}K_2(-\log \bar{F}(x)) \quad 0 \leq x < r,$$

   where K_i, $i = 1,2$, are periodic functions with the same period $1-\alpha$ and r is the common right extremity of the supports of F and G;
2. For real numbers α_i in $(0,1)$ such that $\frac{\log(1-\alpha_1)}{\log(1-\alpha_2)}$ is irrational and $p_{\alpha_i,F}(x) = p_{\alpha_i,G}(x)$, $i = 1,2$, we have $F(x) = G(x)$.

2.2 Reliability Functions in Reversed Time

2.2.1 *Reversed Hazard Rate*

In this section, we consider functions similar to those explained in Sect. 2.1 but are conditioned on the event $X \leq x$, that is, the unit is assumed to have a lifetime less than or equal to x. The primary notion in this connection is the reversed hazard rate $\lambda(x)$ given by

$$\lim_{\Delta \to 0} \frac{P(x-\Delta < X \leq x | X \leq x)}{\Delta} \tag{2.22}$$

and hence

$$\Delta\lambda(x) = P(x-\Delta < X < x | X \leq x) + o(\Delta).$$

Thus, for all x for which $F(x) > 0$,

$$\lambda(x) = \frac{d}{dx}\log F(x) = \frac{f(x)}{F(x)}.$$

Hence, the probability that a unit with life X which has survived age $x-\Delta$ will fail in the next small interval of time Δ given that it will not survive age x is $\Delta\lambda(x)$. Introduced by Keilson and Sumita [323], the function $\lambda(x)$ has been used in various contexts such as estimation and modelling for left censored data, stochastic orderings, characterization of distributions, and in developing repair and maintenance strategies. Block et al. [111] have shown that there does not exist a non-negative random variable having increasing or constant reversed hazard rate function. If X has support (a,b), $-\infty \le a < b \le \infty$, then $\lambda_{-X}(x) = h_X(-x)$ where x, $-x \in (a,b)$ which is a duality property that justifies the adjective 'reversed' associated with $\lambda(x)$. Finkelstein [198] observed that for possible application of $\lambda(x)$ in reliability studies, the above duality property is not relevant. Moreover, with the upper extremity of the interval of support being usually infinity, the properties of the reversed hazard rate for non-negative random variables cannot be formally observed from the corresponding properties of $h(x)$. Like $h(x)$, we can use $\lambda(x)$ also to recover the distribution of X by means of the relation

$$F(x) = \exp\left\{-\int_x^\infty \lambda(t)dt\right\} \tag{2.23}$$

obtained by integrating (2.22) over (x,∞).

Example 2.7. The generalized exponential distribution with (see Table 1.1)

$$F(x) = (1-e^{-\lambda x})^\theta,\ x > 0;\ \lambda,\theta > 0,$$

has

$$f(x) = \lambda\theta(1-e^{-\lambda x})^{\theta-1}e^{-\lambda x},$$

and so

$$\lambda(x) = \theta\lambda(e^{\lambda x}-1)^{-1}.$$

Example 2.8. Let $X_1, X_2, \ldots, X_n$ be independent random variables and $W = \max(X_1, X_2, \ldots, X_n)$. Then,

$$P(W \le x) = P(X_1 \le x, X_2 \le x, \ldots, X_n \le x).$$

Logarithmic differentiation leads to

$$\lambda_W(x) = \lambda_{X_1}(x) + \lambda_{X_2}(x) + \cdots + \lambda_{X_n}(x).$$

Table 2.3 Reversed hazard rate functions of some life distributions

Distribution	$F(x)$	$\lambda(x)$
Power	$(\frac{x}{\alpha})^{\beta}$, $0 \le x \le \alpha$	βx^{-1}
Reciprocal exponential	$\exp(-\frac{\lambda}{x})$, $x > 0$, $\lambda > 0$	λx^{-2}
Reciprocal Lomax	$(1 + \frac{1}{\alpha x})^{-c}$, $x > 0$	$\frac{c}{x(1+\alpha x)}$
Reciprocal Weibull	$\exp[-(\frac{1}{\sigma x})^{\lambda}]$, $x > 0$	$\frac{\lambda}{\sigma^{\lambda} x^{\lambda+1}}$
Reciprocal beta	$(1 - \frac{1}{Rx})^{c}$, $\frac{1}{R} < x < \infty$	$\frac{C}{x(Rx-1)}$
Reciprocal Gompertz	$\exp[\frac{-B(C^{x^{-1}}-1)}{\log C}]$	$\frac{B}{x^2} C^{\frac{1}{x}}$
Generalized exponential	$(1 - e^{-\lambda x})^{\theta}$, $x > 0$	$\frac{\theta\lambda}{e^{\lambda x} - 1}$
Burr	$(1 + x^{-C})^{-k}$, $x > 0$	$\frac{kc}{x(1+x^{C})}$
Generalized power	$(1 - x^{-\beta})^{\theta}$, $x > 1$	$\frac{\beta\theta}{x(x^{\beta}-1)}$
Negative Weibull	$\exp[-\theta(x^{-\beta-1})]$	$\frac{\theta\beta}{x^{\beta+1}}$

This model constitutes a parallel system with n independent components with life distribution functions $F_{X_1}, \ldots, F_{X_n}$.

A review of the main results and applications of $\lambda(x)$ are given in Nair and Asha [439]. For many of the distributions in Table 1.1, the reversed hazard functions are complicated, though they can be obtained as in the above example. A useful result that enables one to get models with simple expressions for $\lambda(x)$ is the following.

Theorem 2.3. *For a non-negative random variable X with hazard rate $h(x)$, its reciprocal $\frac{1}{X}$ has reversed hazard rate $\lambda^*(x)$ that satisfies*

$$h(x) = \frac{1}{x^2}\lambda^*\left(\frac{1}{x}\right)$$

or

$$\lambda^*(x) = \frac{1}{x^2} h\left(\frac{1}{x}\right).$$

Table 2.3 contains some distributions belonging to the above category.

2.2.2 Reversed Mean Residual Life

The random variable $(x - X | X \le x)$ is called the inactivity time or reversed residual life of X. It represents the time elapsed since the failure of a unit given that its lifetime is at most x. We can write the distribution function of the reversed residual life as

$$F_x(t) = P((x - X) \leq t | X \leq x)$$
$$= \frac{F(x) - F(x-t)}{F(x)}$$

with corresponding density function

$$f_x(t) = \frac{f(x-t)}{F(x)}.$$

Accordingly, the mean inactivity time (reversed mean residual life) becomes

$$r(x) = \int_0^x \frac{t f(x-t)}{F(x)} dt = \frac{1}{F(x)} \int_0^x F(t) dt \tag{2.24}$$

and

$$r(x)F(x) = \int_0^x F(t) dt.$$

Differentiating with respect to x and using the definition of $\lambda(x)$, we obtain

$$\lambda(x) = \frac{1 - r'(x)}{r(x)}. \tag{2.25}$$

Hence, from (2.23), we get

$$F(x) = \exp\left\{ -\int_x^{\infty} \frac{1 - r'(t)}{r(t)} dt \right\}. \tag{2.26}$$

As in the case of the mean residual life function, for a chosen function $r(x)$ to be a reversed mean residual life function, the following conditions have to be satisfied.

Theorem 2.4 (Finkelstein [198]). *A function $r(x)$ is a reversed mean residual life of a non-negative random variable X if and only if*

(i) $r(x) \geq 0$ *for all* $x > 0$, *with* $r(0) = 0$;

(ii) $r'(x) < 1$;

(iii) $\int_0^{\infty} \frac{1 - r'(t)}{r(t)} dt = \infty$;

(iv) $\int_x^{\infty} \frac{1 - r'(t)}{r(t)} dt < \infty$ *for* $x > 0$.

2.2.3 Some Other Functions

Kundu and Nanda [360] discussed the properties of the reversed variance residual life function

$$\begin{aligned} v(x) &= V(x-X|X \leq x) \\ &= E((x-X)^2|X \leq x) - r^2(x) \\ &= \frac{2}{F(x)} \int_0^x \int_0^u F(t)\, dt\, du - r^2(x) \end{aligned} \tag{2.27}$$

and also the corresponding coefficient of variation given by

$$a(x) = \frac{[v(x)]^{\frac{1}{2}}}{r(x)}.$$

They obtained the identity

$$\frac{dv(x)}{dx} = \lambda(x) r^2(x)[1 - a^2(x)]$$

and used it to characterize the distribution

$$F(x) = \frac{[\{(2b-\mu)C^2 - \mu\} + (1-C^2)x]^{\frac{2C^2}{1-C^2}}}{(b-\mu)(1+C^2)}, \quad \frac{\mu + (\mu - 2b)C^2}{1-C^2} < x < b,$$

by the property $a(x) = C$.

Example 2.9. In the case of the power distribution with

$$\begin{aligned} F(x) &= \left(\frac{x}{\alpha}\right)^{\beta}, \ 0 \leq x \leq \alpha, \\ f(x) &= \alpha^{-\beta} \beta x^{\beta-1}, \end{aligned}$$

we have $\lambda(x) = \beta x^{-1}$. Again, $r(x) = (\beta+1)x$, and so

$$r(x)\lambda(x) = \beta(\beta+1)^{-1}, \ \text{ a constant.}$$

Upon using

$$\int_0^x \int_0^u F(t) dt du = \frac{2}{\alpha^{\beta}} \frac{x^{\beta+2}}{(\beta+1)(\beta+2)}$$

and (2.27), we obtain

$$v(x) = \frac{\beta x^2}{(\beta+2)(\beta+1)^2} = \frac{\beta}{\beta+2} r^2(x).$$

One can prove that all these are characterizations, with the help of (2.23), (2.26), (2.25) and (2.27). Note the similarity between the above and those of the generalized Pareto distribution in Example 2.3.

The reversed percentile residual life $q_\alpha(x)$, for $0 < \alpha < 1$, is defined as (Nair and Vineshkumar [453])

$$\begin{aligned} q_\alpha(x) &= F_x^{-1}(\alpha) = \inf[t|F_x(t) \geq \alpha] \\ &= \inf[t|F(x-t) \leq (1-\alpha)F(x)] \\ &= x - F^{-1}[(1-\alpha)F(x)]. \end{aligned}$$

The functional equation that solves for $q_\alpha(x)$ is

$$F(x - q_\alpha(x)) = (1-\alpha)F(x). \tag{2.28}$$

By obtaining a solution of the form

$$F(x) = G(x)K(-\log(x)),$$

where $K(\cdot)$ is a periodic function with period $-\log(1-\alpha)$ and $G(x)$ is a particular solution, Nair and Vineshkumar [453] concluded that $F(x)$ is uniquely determined by two percentile functions $q_\alpha(x)$ and $q_\beta(x)$, with $\frac{\log(1-\alpha)}{\log(1-\beta)}$ being irrational. They also showed that $q_\alpha(x)$ and $\lambda(x)$ are related through

$$q'_\alpha(x) = 1 - \frac{\lambda(x)}{\lambda(x - q_\alpha(x))}.$$

2.3 Hazard Quantile Function

We have seen several distribution functions for which the corresponding quantile functions cannot be obtained in explicit algebraic form. In practice, the solution of $F(x) = u$ is obtained numerically. Similarly, there are quantile functions that do not permit closed-form expressions for $F(x)$. Hence, the reliability functions introduced in the last two sections and their properties are of limited use for algebraic manipulations and analysis. In view of this, we need translation of the definitions and properties in terms of quantile functions. This approach will facilitate all forms of analysis with the same scope and strength as in the distribution function approach. In addition, they offer new results and opportunities by way of models and methods of analysis. The main source of the discussions in the rest of this chapter is Nair and Sankaran [444]. We assume that $F(x)$ is continuous and strictly increasing so that all quantile related functions are well defined.

Setting $x = Q(u)$ in (2.2) and using the relationship

$$f(Q(u)) = [q(u)]^{-1},$$

we have the definition of the hazard quantile function as

$$H(u) = h(Q(u)) = [(1-u)q(u)]^{-1}. \tag{2.29}$$

In this definition, $H(u)$ is interpreted as the conditional probability of the failure of a unit in the next small interval of time given the survival of the unit at $100(1-u)\%$ point of the distribution. Gilchrist [215] refers to (2.29) as the p-hazard (with p taking the place of u in our notation) and points out some forms of hazard functions. From (2.29), we have

$$q(u) = [(1-u)H(u)]^{-1} \tag{2.30}$$

and so

$$Q(u) = \int_0^u \frac{dp}{(1-p)H(p)}. \tag{2.31}$$

The last two equations can be employed for the unique determination of the distribution of X as illustrated in the following examples.

Example 2.10. Taking

$$Q(u) = u^{\theta+1}(1+\theta(1-u)), \quad \theta > 0,$$

we have

$$q(u) = u^{\theta}[1+\theta(\theta+1)(1-u)]$$

and so

$$H(u) = [(1-u)u^{\theta}(1+\theta(\theta+1)(1-u))]^{-1}.$$

Note that there is no analytic solution for $x = Q(u)$ that gives $F(x)$ in terms of x.

Example 2.11. Given the hazard quantile function of a distribution as

$$H(u) = \frac{a+1}{b}(1-u)^{\frac{a}{a+1}},$$

from (2.30), we have

$$q(u) = \frac{b}{a+1}(1-u)^{-\frac{a}{a+1}-1},$$

and so from (2.31), we obtain

$$Q(u) = \frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1],$$

the quantile function of the generalized Pareto distribution.

The hazard quantile functions that characterize the life distributions in Table 1.1 are presented in Table 2.4. More examples are available in Chap. 3 wherein we discuss new models.

Example 2.12. Suppose we are given the hazard quantile function of a distribution as

$$H(u) = \frac{1}{2}\left(\frac{1+u}{1-u}\right)^{k+1} \quad \text{for } k > 0.$$

Then, from (2.30), we have the quantile density function as

$$q(u) = \frac{1}{(1-u)H(u)} = 2\frac{(1-u)^k}{(1+u)^{k+1}}.$$

So, from (2.31), we obtain the quantile function of the distribution as

$$Q(u) = \int_0^u q(p)dp = 2\int_0^u \frac{(1-p)^k}{(1+p)^{k+1}}dp = \frac{1}{k}\left\{1 - \left(\frac{1-u}{1+u}\right)^k\right\},$$

which is the quantile function of the generalized half-logistic distribution as presented in Table 1.1.

The application of hazard quantile functions is not limited to the appraisal of the mechanism of failures in a specific failure time model. It can also provide the identification of the model in a given data situation by means of characterization theorems. The characterization problems discussed earlier and elsewhere in the distribution function approach automatically hold for quantile functions under the transformation $x = Q(u)$. Other than these, we can find new characterizations exclusively in the quantile set-up, which is illustrated in the following theorem.

Table 2.4 Hazard quantile functions of distributions in Table 1.1

Distribution	$H(u)$
Exponential	λ
Weibull	$\lambda\sigma^{-1}(-\log(1-u))^{1-\frac{1}{\lambda}}$
Pareto II	$c\alpha^{-1}(1-u)^{1/c}$
Rescaled beta	$cR^{-1}(1-u)^{-\frac{1}{c}}$
Half-logistic	$(2\sigma)^{-1}(1+u)$
Power	$\beta\alpha^{-1}(1-u)^{-1}u^{1-\frac{1}{\beta}}$
Pareto	$\alpha\sigma^{-1}(1-u)^{\frac{1}{\alpha}}$
Burr XII	$ck(1-u)^{\frac{1}{k}}[(1-u)^{-\frac{1}{k}}-1]^{1-\frac{1}{c}}$
Loglogistic	$\alpha\beta(1-u)^{\frac{1}{\beta}}u^{1-\frac{1}{\beta}}$
Exponential geometric	$\lambda(1-p)^{-1}(1-pu)$
Generalized Weibull	$\alpha\beta^{-1}(1-u)^{-\lambda}[1-\frac{(1-u)^{\lambda}}{\lambda}]^{1-\frac{1}{\alpha}}$
Exponentiated Weibull	$\lambda\theta(1-u^{\frac{1}{\theta}})(-\log(1-u^{\frac{1}{\theta}})^{1-\frac{1}{\lambda}}$ $(1-u)^{-1}u^{1-\frac{1}{\theta}}$
Generalized exponential	$\sigma^{-1}\theta(1-u^{\frac{1}{\theta}})(1-u)^{-1}u^{1-\frac{1}{\theta}}$
Exponential power	$\lambda\alpha[1+\log(1-u)]$ $[-\log(1+\log(1-u))]^{1-\frac{1}{\alpha}}$
Modified Weibull extension	$\lambda\sigma^{-1}[\alpha\sigma-\log(1-u)]$ $[\log(1-\frac{\log(1-u)}{\alpha\sigma})]^{1-\frac{1}{\lambda}}$
Log Weibull	$\rho k\exp[-(-\log(1-u))^{\frac{1}{k}}]$ $[-\log(1-u)]^{1-\frac{1}{k}}$
Greenwich	$ab^{-1}(1-u)^{-\frac{2}{a}}[(1-u)^{\frac{2}{a}}-1]^{\frac{1}{2}}$
Extended Weibull	$\frac{\lambda}{\theta\sigma}[\log\frac{\theta+(1-\theta)(1-u)}{1-u}]^{1-\frac{1}{\lambda}}$ $(\theta+(1-\theta)(1-u))$
Inverse Weibull	$\lambda\sigma^{-1}u(1-u)^{-1}(-\log u)^{\frac{1}{\lambda}+1}$
Generalized Pareto	$b^{-1}(a+1)(1-u)^{\frac{a}{a+1}}$
Generalized half-logistic	$\frac{1}{2}\left(\frac{1+u}{1-u}\right)^{k+1}$

Theorem 2.5. *Let X be a non-negative random variable with absolutely continuous distribution function $F(x)$ and quantile function $Q(u)$. Then, the hazard quantile function of X is of the linear form*

$$H(u) = a + bu,\ a > 0, \tag{2.32}$$

for all $0 < u < 1$, if and only if

$$Q(u) = \log\left(\frac{a+bu}{a(1-u)}\right)^{\frac{1}{a+b}}. \tag{2.33}$$

Proof. When X has its quantile function as in (2.33), we have

$$q(u) = [(1-u)(a+bu)]^{-1}$$

and so

$$H(u) = a + bu.$$

To prove the sufficiency, we obtain from (2.32) that

$$q(u) = [(1-u)(a+bu)]^{-1},$$

and upon integrating it from 0 to u, we get

$$Q(u) = (a+b)^{-1}\log\left(\frac{a+bu}{1-u}\right) + C.$$

Setting $u = 0$, the condition $Q(0) = 0$ readily gives $C = -\frac{\log a}{a+b}$, and consequently (2.33) holds. We notice that (2.33) represents a family of distributions with some well-known models as particular cases. For the special case when $b = 0$, $a = \lambda^{-1}$, $\lambda > 0$, we have the exponential distribution; for the case when $a = b = (2\sigma)^{-1} > 0$, we have the half-logistic distribution; the case $a = \lambda(1-p)^{-1} > 0$, $b = -\frac{p\lambda}{1-p} = -pa < 0$ corresponds to the exponential geometric, and finally $a = \frac{1}{\alpha} > 0$, $b = -\frac{1}{\alpha} = -1 < 0$ leads to Pareto II distribution with parameter $(\alpha, 1)$. $\square$

The above characterization theorem can be used to identify the adequacy of the model for a given data in the following manner. Using (2.29), we obtain the empirical version of $H(u)$ as

$$\bar{H}(u) = [(1-u)\bar{q}(u)]^{-1}.$$

If the points $(u, \bar{H}(u))$ plotted on a graph for different values of u in $(0,1)$ lie approximately on a straight line, it suggests the distribution in (2.33). The specific member of the model is identified on the basis of the estimates of a and b derived from the plot, for example.

The family of distributions in (2.33) will be referred to in the sequel as the linear hazard quantile family. It is easy to invert (2.33) to obtain the distribution function as

$$F(x) = \frac{1 - e^{-(a+b)x}}{1 + \frac{b}{a}e^{-(a+b)x}} \quad x > 0,\ a > 0.$$

2.4 Mean Residual Quantile Function

Recall that the mean residual life function is defined as

$$m(x) = \frac{1}{\bar{F}(x)} \int_x^\infty t f(t) dt - x.$$

In terms of quantiles, the mean residual quantile function is thus given by

$$\begin{aligned} M(u) = m(Q(u)) &= \frac{1}{1-u} \int_u^1 Q(p) dp - Q(u) \\ &= (1-u)^{-1} \int_u^1 [Q(p) - Q(u)] dp. \end{aligned} \tag{2.34}$$

The same expression can also be obtained from (1.4). Also

$$M(u) = (1-u)^{-1} \int_u^1 (1-p) q(p) dp. \tag{2.35}$$

We interpret $M(u)$ as the average remaining life beyond the $100(1-u)\%$ of the distribution.

Equivalence of (2.34) and (2.35) is readily verified by integrating by parts the RHS of (2.35). From (2.35) and the definition of the hazard quantile function $H(u)$ in (2.29), we have

$$M(u) = (1-u)^{-1} \int_u^1 \frac{dp}{H(p)}. \tag{2.36}$$

Differentiating (2.35) with respect to u, we obtain

$$(1-u)q(u) = M(u) - (1-u)M'(u)$$

or

$$[H(u)]^{-1} = M(u) - (1-u)M'(u), \tag{2.37}$$

where $M'(u)$ is the derivative of $M(u)$ with respect to u. The last two equations determine $M(u)$ from $H(u)$ and vice versa. Finally, the distribution of X is recovered from $M(u)$ when (2.37) is inserted into (2.31) as

Table 2.5 Mean residual quantile functions

Distribution	$M(u)$
Exponential	λ^{-1}
Pareto II rescaled	$\frac{\alpha}{c-1}(1-u)^{-\frac{1}{c}}$
Beta	$\frac{R}{c+1}(1-u)^{\frac{1}{c}}$
Half-logistic	$\frac{2\sigma}{1-u}\log\frac{2}{1+u}$
Exponential geometric	$\frac{1-p}{\lambda p(1-u)}\log\frac{1-pu}{1-p}$
Power	$\frac{\alpha}{1-u}[1-u^{\frac{1}{\beta}}-(\beta+1)^{-1}(1-u^{1+\frac{1}{\beta}})]$
Generalized Pareto	$b(1-u)^{-\frac{a}{a+1}}$

$$Q(u) = \int_0^u \frac{M(p)-(1-p)M'(p)}{1-p}dp$$

$$= \int_0^u \frac{M(p)}{1-p}dp - M(u) + \mu \quad \text{since } M(0)=\mu. \tag{2.38}$$

Since the quantile density function also specifies the distribution, a simpler formula is

$$q(u) = (1-u)^{-1}M(u) - M'(u). \tag{2.39}$$

Example 2.13. Let

$$Q(u) = u^{\theta+1}[1+\theta(1-u)], \quad \theta > 0.$$

Then, we have

$$q(u) = u^{\theta}[1+\theta(\theta+1)(1-u)]$$

and so

$$M(u) = \int_u^1 (p^{\theta}-p^{\theta+1})[1+\theta(\theta+1)(1-p)]\,dp$$

$$= \frac{3+\theta(2\theta+3)}{(\theta+1)(\theta+2)(\theta+3)} - \frac{u^{\theta+1}}{(\theta+1)(\theta+2)}$$

$$\times\Bigg[\{(1+(1-u)(1+\theta))(1+\theta(\theta+1)(1-u))\}$$

$$-\frac{\theta(\theta+1)}{\theta+3}u(2+(1-u)(1+\theta))\Bigg].$$

The expressions for $M(u)$ for most distributions are quite complicated involving special functions. A few simple cases are presented in Table 2.5.

We now prove a characterization result by the functional form of $M(u)$.

Theorem 2.6. *A lifetime random variable X has the linear hazard quantile distribution in (2.33) if and only if, for all $0 < u < 1$,*

$$M(u) = \frac{1}{b(1-u)} \log\left(\frac{a+b}{a+bu}\right). \tag{2.40}$$

Proof. From Theorem 2.5, we use the expression for $q(u)$ to write

$$M(u) = \frac{1}{1-u} \int_u^1 (1-p)[(1-p)(a+bp)]^{-1} dp$$

which leads to (2.40) and the 'only if' part. Conversely, from (2.40), we have

$$M'u) = \frac{1}{b}\left[\frac{1}{(1-u)^2} \log \frac{a+b}{a+bu} - \frac{b}{(1-u)(a+bu)}\right]$$

and so (2.37) yields

$$H(u) = a + bu.$$

Hence, by Theorem 2.4, X has its quantile function as (2.33). The special cases are slightly different from Theorem 2.4 and so need enumeration. Firstly, for the Pareto II case with parameter $(\alpha, 1)$, the mean does not exist and so it is not a member of the class for which (2.40) is true.

Secondly, the case of the exponential distribution needs the evaluation of $M(u)$ as a limit when $b \to 0$ using L'Hospital rule. In fact,

$$\begin{aligned} M(u) &= \lim_{b\to 0} \frac{1}{b(1-u)}[\log(a+b) - \log(a+bu)] \\ &= \lim_{b\to 0} \frac{1}{1-u}\left[\frac{1}{a+b} - \frac{u}{a+bu}\right] \\ &= \frac{1}{a}\,. \end{aligned}$$

The other cases are as in Theorem 2.4 and this completes the proof. □

Another characterization is, as in the distribution function approach, from the relationship between $M(u)$ and $H(u)$.

Theorem 2.7. *The relationship*

$$M(u) = (1-u)^{-1}[A + B \log H(u)], \tag{2.41}$$

for all $0 < u < 1$, holds for a lifetime random variable X if and only if it has linear hazard quantile distribution in (2.33).

Proof. First, we assume X has distribution specified by (2.33). Then,

$$\begin{aligned} M(u) &= \frac{1}{b(1-u)}[\log(a+b) - \log(a+bu)] \\ &= (1-u)^{-1}[A + B\log H(u)], \end{aligned}$$

where $A = b^{-1}\log(a+b)$ and $B = -b^{-1}$. Conversely, let (2.41) hold. Then

$$\int_u^1 (1-p)q(p)\,dp = A + B\log H(u)$$

which yields, on differentiation,

$$-(1-u)q(u) = \frac{B}{H(u)}H'(u)$$

or

$$-\frac{1}{H(u)} = \frac{BH'(u)}{H(u)}$$

giving $H(u)$ as a linear function. This completes the proof. □

2.5 Residual Variance Quantile Function

From the definition of the variance residual life function in (2.10), the corresponding quantile-based function takes on the form

$$\begin{aligned} V(u) = \sigma^2(Q(u)) &= (1-u)^{-1}\int_u^1 Q^2(p)dp - [(1-u)^{-1}\int_u^1 Q(p)dp]^2 \\ &= (1-u)^{-1}\int_u^1 Q^2(p)dp - (M(u)+Q(u))^2. \end{aligned} \tag{2.42}$$

Since

$$(1-u)(M(u)+Q(u)) = \int_u^1 Q(p)dp, \tag{2.43}$$

upon differentiating (2.43), we obtain

$$(1-u)(M'(u)+Q'(u)) - (M(u)+Q(u)) = -Q(u)$$

giving

$$M(u) = (1-u)(M'(u) + Q'(u)). \tag{2.44}$$

Also, from (2.42), we have

$$(1-u)V(u) = \int_u^1 Q^2(p)dp - (1-u)(M(u)+Q(u))^2.$$

Differentiating this and on using (2.44), we obtain

$$\begin{aligned}(1-u)V'(u) - V(u) &= Q^2(u) - 2(1-u)(M(u)+Q(u))(M'(u)+Q'(u)) \\ &\quad +(M(u)+Q(u))^2 \\ &= -M^2(u),\end{aligned}$$

where $V'(u)$ is the derivative of $V(u)$ with respect to u. Thus, the mean residual quantile function is determined from the residual variance quantile function as

$$M^2(u) = V(u) - (1-u)V'(u). \tag{2.45}$$

Integrating this over $(u,1)$, we get

$$V(u) = (1-u)^{-1}\int_u^1 M^2(p)\,dp. \tag{2.46}$$

Equation (2.46) expresses the fact that the residual variance quantile function is determined from the mean residual quantile function. In view of the characterization of the distribution by $M(u)$, it follows from (2.45) and (2.46) that $V(u)$ characterizes the life distribution. This result is stronger than the one currently available in the literature and mentioned earlier in Sect. 2.1.3. Notice also that both $h(x)$ and $m(x)$ are present in the identity (2.12), but (2.46) needs only the knowledge of $M(u)$ to derive $V(u)$. With simple forms of $M(u)$ being not available for many common distributions, and even where they are available, the integral in (2.46) leads to no closed-form solutions, characterizations are rare to find.

The coefficient of variation $C^*(u)$ is defined in the quantile formulation as

$$C^{*2}(u) = \frac{V(u)}{M^2(u)}. \tag{2.47}$$

Then, we have

$$\begin{aligned}\frac{1}{C^{*2}(u)} &= \frac{V(u)-(1-u)V'(u)}{V(u)} \\ &= 1-(1-u)\frac{V'(u)}{V(u)}\end{aligned}$$

or

$$\frac{d \log V(u)}{du} = (1-u)^{-1}(1-(C^*(u))^{-2}). \tag{2.48}$$

Example 2.14. For the generalized Pareto distribution, we have

$$Q(u) = \frac{b}{a}[(1-u)^{-\frac{a}{a+1}} - 1],$$
$$M(u) = b(1-u)^{-\frac{a}{a+1}}.$$

Hence, from (2.46), we find

$$V(u) = (1-u)^{-1}\int_u^1 b^2(1-p)^{-\frac{2a}{a+1}}\,dp$$
$$= \frac{1+a}{1-a}b^2(1-u)^{-\frac{2a}{a+1}};$$

also, we have

$$V(u) = KM^2(u), \quad K = \frac{1+a}{1-a}.$$

Compare these with the results in Example 2.5 when the definitions based on the distribution function are applied to the corresponding functions. The coefficient of variation in this case is $C(u) = K^{\frac{1}{2}}$, a constant.

2.6 Other Quantile Functions

We briefly mention some other quantile functions required for the discussions in the sequel. The αth percentile residual quantile function is obtained from (2.19) as

$$P_\alpha(u) = p_\alpha(Q(u)) = Q[1-(1-\alpha)(1-u)] - Q(u). \tag{2.49}$$

Lillo [399] has pointed out that $Q(u)$ is uniquely determined from the knowledge of $P_\alpha(u)$ and the quantile function in an interval, viz.,

$$G(u) = Q(u), \quad 0 \le u \le B_1 = W_\alpha(0),$$

where $G(u)$ is a continuous increasing function defined on $[0, W_\alpha(0))$, satisfying $G(0) = 0$ and $W_\alpha(u) = Q[1-(1-\alpha)(1-u)]$.

Table 2.6 Reversed hazard quantile functions of distributions in Table 2.3

Distribution	$Q(u)$	$\Lambda(u)$
Power	$\alpha u^{\frac{1}{\beta}}$	$\beta(\alpha u^{\frac{1}{\beta}})^{-1}$
Reciprocal exponential	$-\frac{1}{\lambda}(\log u)^{-1}$	$\lambda^{-1}(\log u)^2$
Reciprocal beta	$[R(1-u^{\frac{1}{c}})]^{-1}$	$Rc(1-u^{\frac{1}{c}})^2 u^{-\frac{1}{c}}$
Reciprocal Pareto II (Lomax)	$[\alpha(u^{-\frac{1}{c}}-1)]^{-1}$	$c\alpha(1-u^{\frac{1}{c}})^2 u^{-\frac{1}{c}}$
Reciprocal Weibull	$\sigma(-\log u)^{\frac{1}{\lambda}}$	$\sigma\lambda(-\log u)^{1+\frac{1}{\lambda}}$
Generalized exponential	$-\lambda^{-1}\log(1-u^{\frac{1}{\theta}})$	$\lambda(1-u^{\frac{1}{\theta}})u^{-\frac{1}{\theta}}$
Burr	$(u^{-\frac{1}{k}}-1)^{-\frac{1}{c}}$	$ck(u^{\frac{1}{k}}-1)^{1+\frac{1}{c}}u^{\frac{1}{k}}$
Generalized power	$(1-u^{\frac{1}{\theta}})^{-\frac{1}{\beta}}$	$\theta\beta(1-u^{\frac{1}{\theta}})^{1+\frac{1}{\beta}}u^{-\frac{1}{\theta}}$
Negative Weibull	$(1-\log u^{\frac{1}{\theta}})^{-\frac{1}{\beta}}$	$\beta\theta(1-\log u^{\frac{1}{\theta}})^{1+\frac{1}{\beta}}$

Various reliability functions in reversed time can also be defined in a manner similar to those in Sect. 2.2. Since the algebra is almost parallel, we give only the relevant results.

The reversed hazard quantile function is

$$\Lambda(u) = \lambda(Q(u)) = [uq(u)]^{-1},$$

and it determines the distribution through the formula

$$Q(u) = \int_0^u [p\Lambda(p)]^{-1} dp. \tag{2.50}$$

The reversed hazard quantile functions of some distributions are presented in Table 2.6.

Similarly, the reversed mean residual quantile function is given by

$$\begin{aligned} R(u) = r(Q(u)) &= u^{-1}\int_0^u [Q(u)-Q(p)]dp \\ &= u^{-1}\int_0^u pq(p)dp. \end{aligned} \tag{2.51}$$

Furthermore, we have

$$Q(u) = R(u) + \int_0^u p^{-1}R(p)dp, \tag{2.52}$$

$$\begin{aligned} [\Lambda(u)]^{-1} &= R(u) + uR'(u), \\ R(u) &= u^{-1}\int_0^u [\Lambda(p)]^{-1}dp, \\ H(u) &= (1-u)^{-1}u\Lambda(u), \\ (1-u)M(u) &= \mu + uR(u) - Q(u). \end{aligned}$$

The reversed variance residual quantile function given by

$$D(u) = u^{-1} \int_0^u Q^2(p)dp - (Q(u) - R(u))^2 \tag{2.53}$$

satisfies the relation

$$R^2(u) = D(u) + uD'(u),$$

and so

$$D(u) = u^{-1} \int_0^u R^2(p)dp.$$

where $D'(u)$ is the derivative of $D(u)$ with respect to u.

Example 2.15. The one-parameter family with

$$Q(u) = u^{1+\theta}(\theta u + 1 - \theta), \quad 0 \le u \le 1;\ 0 < \theta \le 1,$$

has its quantile density function as

$$q(u) = u^{\theta}(\theta u + (\theta u + (\theta u + 1 - \theta)(1 + \theta)).$$

Hence,

$$\Lambda(u) = [u^{\theta+1}(1 + \theta^2 + \theta(\theta+2)u)]$$

and so

$$\begin{aligned} R(u) &= \int_0^u p\, q(p)dp \\ &= \frac{u^{\theta+1}}{(\theta+2)(\theta+3)}[(1-\theta^2)(3+\theta) + (\theta+2)^2 u]. \end{aligned}$$

Two other important concepts of interest in quantile-based reliability theory are the total time on test transforms and the *L*-moments of residual life. These will be discussed separately in Chaps. 5 and 6, respectively.

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