Chapter 2
Graphs

2.1 Basic Definitions

The most common and useful structure for encoding discrete information is the *graph*, an abstract structure which is designed to record relationships between objects. For simple undirected graphs, the following definition suffices. A *graph* G is a pair $G = (V, \sim)$, where $V = V(G)$ is the vertex set and $\sim$ is an irreflexive, symmetric relation on $V(G)$, called *adjacency*. We let $E(G)$ denote the edge set, that is, the set of unordered pairs of adjacent vertices of G. If it is more convenient, we will indicate a graph $G = (V, \sim)$ by specifying its vertex and edge set, $G = (V, E)$.

For example, Fig. 2.1 illustrates the graph whose vertex set is the set of twelve numbers $V = \{2, 3, \ldots, 13\}$ in which two numbers are said to be adjacent if they have a common prime divisor. The utility of graphs is their ability to facilitate the discovery or description of properties of the defining relation, and to this end, a large body of common vocabulary has been developed. For instance, a subset I of the vertex set V of a graph is called *independent* if no edge of G has two endpoints in I. Clearly, independent sets of the divisor graph will have significance in number theory.

If $G = (V, \sim)$ with edge set E, the edge corresponding to adjacent vertices a and b is denoted variously by $(a, b) = \{a, b\} = ab = ba$. Vertices a and b are called the *endpoints* (or *endvertices*) of the edge ab. The number of vertices adjacent to a given vertex a is called the *valence* of a and denoted by $\mathrm{val}(a)$. Note that many graph theorists use *degree* of a vertex a instead of $\mathrm{val}(a)$.

We say that an edge is *incident* to its endpoints. Since every edge is incident to two endpoints, summing over the vertex valences yields twice the number of edges. This simple but useful observation is called the *handshaking lemma*:

$$\sum_{a \in V} \mathrm{val}(a) = 2|\mathrm{E}|.$$

If the vertices of graph G are ordered, then G can be conveniently encoded by its *adjacency matrix*, a $|V| \times |V|$ matrix of zeros and ones in which there is a 1 in position (i, j) if and only if the ith and jth vertices, in the given ordering, are

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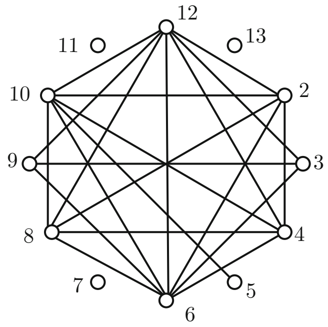


Fig. 2.1 The divisor graph on $\{2, 3, \ldots, 13\}$

adjacent. Also of interest is the *incidence matrix*, which requires an ordering of both the vertices and the edges. It is an $|E| \times |V|$ matrix in which there is a 1 in position (i, j) if and only if the ith edge is incident to the jth vertex.

It is natural to represent a graph with a diagram by using a "dot" to signify each vertex and a (possibly curved) line segment connecting the two dots a and b if $a \sim b$. The larger the number of vertices and more complex the adjacency relation, the less will be our ability to visually gather useful information from such a figure. Even with only 12 vertices, the drawing of the divisor graph in Fig. 2.1 does not reveal to us very much of the structure. A graph is a purely abstract concept, and its representation as a diagram leaves a great deal of freedom, so perhaps a more judicious placement of the vertices would make apparent to us some property the present drawing obscures.

2.2 Examples of Graphs

The simplest examples of graphs need no figures. They are the *discrete graphs*, graphs in which no pair of vertices is adjacent. The discrete graph on n vertices is no more or less complicated than a set of n vertices. At the other extreme, we have the *complete graph*, K_n, on n vertices, in which every pair of vertices is adjacent:

$$K_n = (\{v_1, v_2, \ldots, v_n\}, \{(v_i, v_j) \mid i < j; i, j = 1, \ldots, n\}).$$

Although the complete graphs pictured in Fig. 2.2 have a certain charm, it is mainly derived from the symmetrical placement of the vertices and the pleasant pattern of the completely irrelevant edge crossings. Adjacency in the complete graph also gives no information beyond the set of vertices.

Nevertheless, the complete and discrete graphs become important as *subgraphs.* A subgraph of a graph $G = (V, E)$ is a graph $G' = (V', E')$ such that $V' \subseteq V$, $E' \subseteq E$. Every graph will have many subgraphs which are discrete, but finding subgraphs which are complete graphs is, quite literally, a hard problem; see, for example, [29].

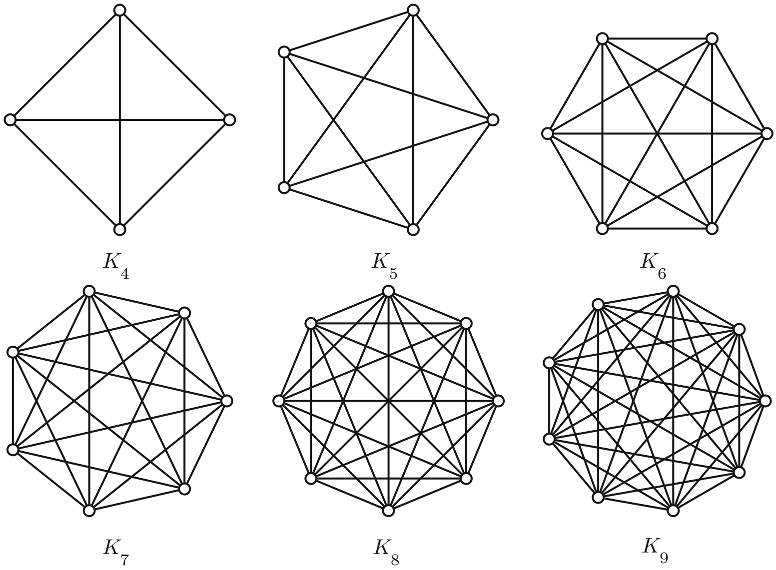


Fig. 2.2 Complete graphs $K_n, n = 4, 5, \ldots, 9$

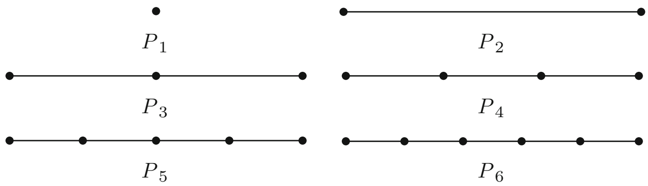


Fig. 2.3 Paths $P_n, n = 1, 2, \ldots, 6$

A subgraph G' of G is said to be *induced* if every pair of vertices in V' which is adjacent in G is also adjacent in G'. An independent set in a graph is an induced discrete subgraph, and finding independent sets in a graph is just as easy, and just as hard, as finding complete subgraphs.

2.2.1 Paths

The *path* on n vertices, $P_n = (V, E)$, is defined by

$$V = \{v_1, v_2, \ldots, v_n\} \qquad E = \{v_i v_{i+1} \mid i = 1, \ldots, n-1\}.$$

As with an edge, the vertices v_1 and v_n in the above example are called *endvertices* (Fig. 2.3) of P_n. The vertices which are not endvertices are called *internal*.

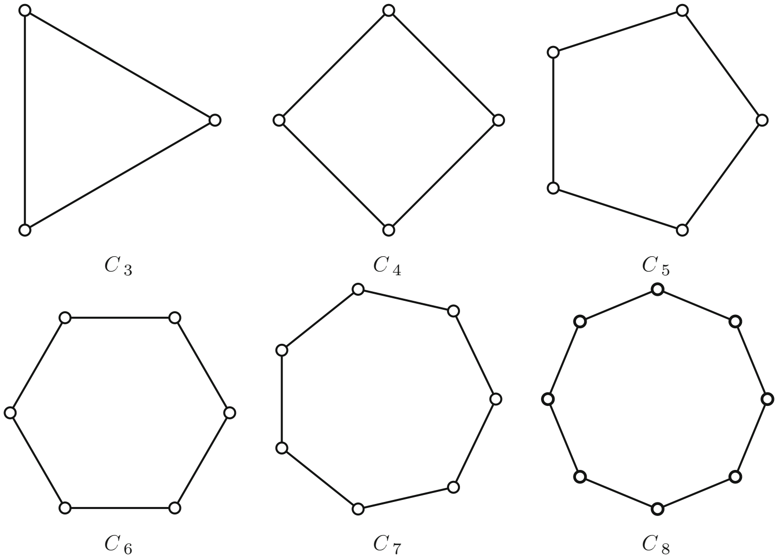


Fig. 2.4 Small cycles $C_n, n = 3, 4, \ldots, 8$

While not trivial, the adjacency relation is still very simple, just a simple ordering of the vertices, so as with the complete and discrete graphs, the main interest of P_n lies in its role as a subgraph. The most important concept in graph theory is defined via subgraphs which are paths: If each pair of vertices in a graph G is the endvertices of some path, then we say the graph G is *connected*. A graph which is not connected is said to be *disconnected*. A disconnected graph may be partitioned into maximal connected subgraphs, called *connected components*. The graph in Fig. 2.1 has three connected components which consist of a single vertex. Does it have any others? See Exercise 2.3.

Paths are also used to define the *distance* between the two vertices v and u as the fewest number of edges in any path in G connecting u and v.

2.2.2 Cycles

The cycle of length n, $C_n = (V, E)$, is defined by

$$V = \{v_1, v_2, \ldots, v_n\}, \quad E = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}.$$

See Fig. 2.4, and again, the adjacency relation is very simple, and these graphs derive their main interest as subgraphs. A graph with no subgraph which is a cycle is said to be an *acyclic graph*. In an acyclic graph, two vertices can be endvertices of at most one path. A connected acyclic graph is called a *tree*, and an acyclic graph is also called a *forest*, since it is the union of its connected components, each of which is a tree.

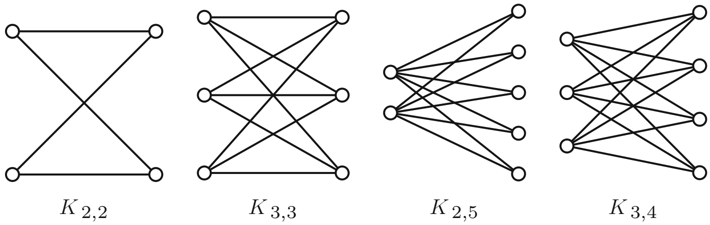


Fig. 2.5 Small complete bipartite graphs $K_{m,n}$

A subgraph $G' \leq G$ is said to be a *spanning* subgraph of G if $V(G') = V(G)$. It is not hard to show that a graph is connected if and only if it contains a spanning tree. Obviously, a graph with $|V|$ vertices and $|E|$ edges contains $2^{|V|}$- induced subgraphs and $2^{|E|}$ spanning subgraphs. These numbers indicate that it is sometimes difficult to find subgraphs of a particular kind in a large graph. Nevertheless, finding a spanning tree in a graph is computationally simple. By contrast, finding a spanning cycle, called a *Hamilton cycle*, is intractable. A graph is called *Hamiltonian graph* if it contains a Hamilton cycle as subgraph. To determine if a graph G is Hamiltonian, we have to provide a one-to-one correspondence f between the vertices of C_n and those of G which preserves adjacency.

2.2.3 Complete Bipartite Graphs and Multipartite Graphs

The *complete bipartite graph*, $K_{m,n} = (V, E)$, is defined by

$$V = \{a_1, a_2, \dots, a_m, b_1, b_2, \dots b_n\} \quad E = \{a_i b_j \mid i = 1, \dots, m; j = 1, \dots, n\}.$$

See Fig. 2.5. Alternatively, we can define a complete bipartite graph as a graph whose vertex set is partitioned into two sets (the as and bs in the example above), and two vertices are adjacent if they are in different sets of the bipartition. This naturally generalizes to any partition of the set V. The *complete multipartite graph*, $K_{n_1,n_2,\dots,n_p} = (V, E)$, is defined by (Fig. 2.6)

$$V = \{a_{ij} \mid 1 \leq i \leq p; 1 \leq j \leq n_i\}$$
$$E = \{a_{ij} a_{kl} \mid i \neq k\}.$$

A complete multipartite graph has a vertex set partitioned into p sets, and two vertices are adjacent if they belong to different sets of the partition. For the graph $K_{n_1,n_2,\dots,n_p}$ with $n_1 = n_2 = \dots = n_p = n$, the more economical notation $K_{p(n)}$ is occasionally used.

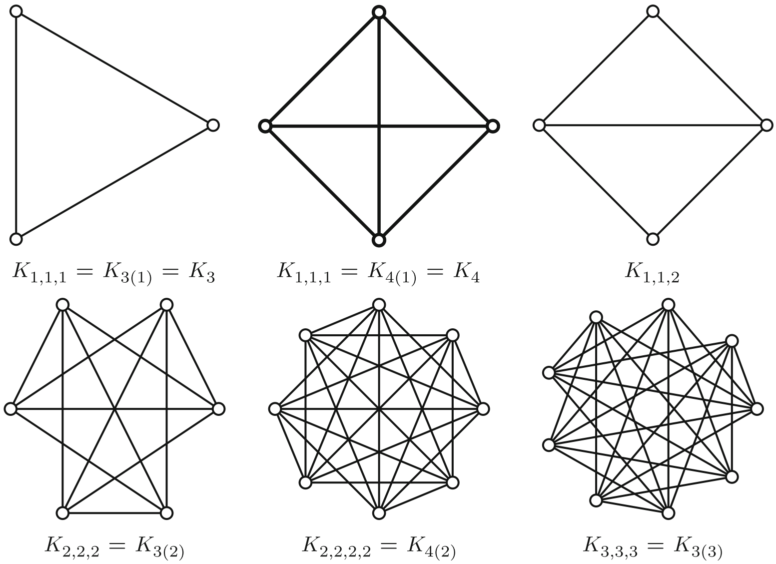


Fig. 2.6 Small complete multipartite graphs

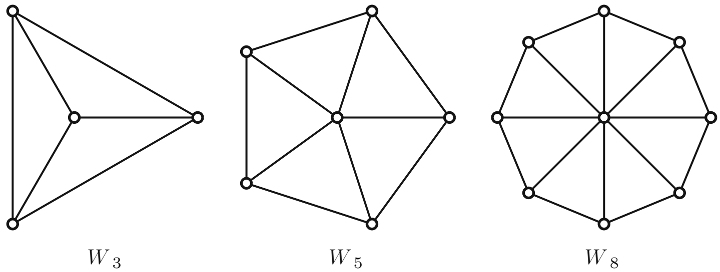


Fig. 2.7 Wheel graphs W_n, sometimes called *pyramid graphs*

2.2.4 Wheel Graphs

If you append to the vertex set of the cycle graph C_n a new vertex c and specify that the new vertex is adjacent to all the vertices u_i of the cycle, the graph created, $W_n = (V, E)$

$$V = \{c, u_1, \ldots, u_n\} \quad E = \{cu_i, u_i u_{i+1} \mid i = 1, \ldots, n\},$$

indices modulo n, is called the *wheel graph*; see Fig. 2.7. The adjacency relation on the wheel graph indicates that the bijection on the vertices defined by $f(c) = c$ and $f(u_i) = u_{i+1}$ takes adjacent pairs to adjacent pairs or, equivalently, defines a bijection on the edge set E. This is an example of a graph *automorphism*.

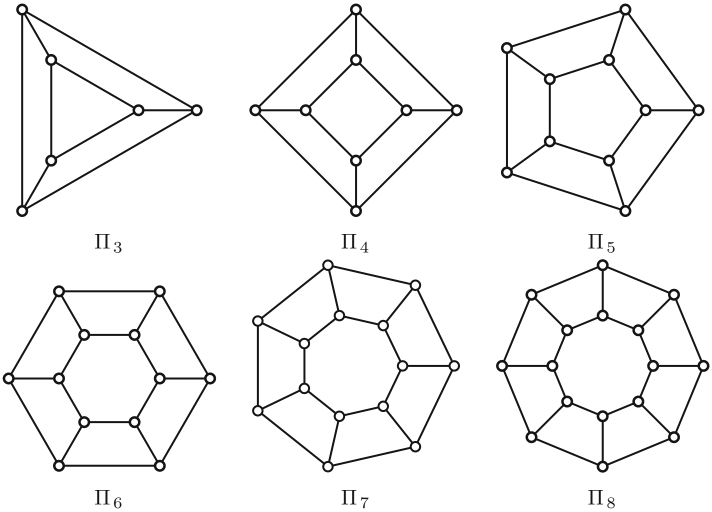


Fig. 2.8 Prisms $\Pi_n, n = 3, 4, ..., 8$

The symmetric drawing of the wheel graph W_n suggests a three-dimensional object, specifically a three-dimensional pyramid with an n-sided base, viewed from above. In the same way, any polyhedron made up of vertices, edges, and faces gives rise to a graph by simply retaining the adjacency information between the vertices and edges and forgetting the faces. This graph is called the *skeleton* or *1-skeleton* of the polyhedron. So the 1-skeleton of an n-sided pyramid is the wheel graph W_n. For this reason, the wheel graph is known also as the *pyramid graph*.

This brings us to several more interesting examples of graphs.

2.2.5 Prism Graphs

A regular prism is a polyhedron with two parallel opposite faces, called bases, that are congruent regular polygons. All the other faces, called lateral faces, are squares formed by the straight lines through corresponding vertices of the bases. $\Pi_n = (V, E)$, the n-sided *prism graph*, is the skeleton of a prism whose base is an n-gon:

$$V = \{u_1, \dots, u_n, v_1, \dots, v_n\} \quad E = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_i \mid i = 1, \dots, n\}.$$

with indices modulo n; see Fig. 2.8.

2.2.6 Antiprism Graphs

A regular antiprism of order n is a polyhedron with two regular n-gons as bases and $2n$ equilateral triangles as side faces. Its 1-skeleton is the graph $A_n = (V, E)$, the $n-$sided *antiprism graph*,

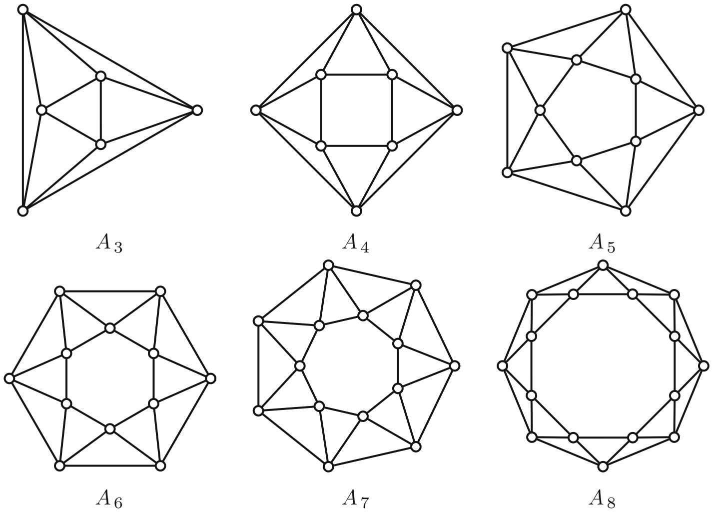


Fig. 2.9 Antiprism graphs $A_n, n = 3, 4, \dots, 8$

$$V = \{u_1, \dots, u_n, v_1, \dots, v_n\} \quad E = \{u_i u_{i+1}, v_i v_{i+1}, u_i v_{i+1}, u_i v_i \mid i = 1, \dots, n\},$$

with indices modulo n; see Fig. 2.9. Regular three-dimensional prisms and antiprisms may be constructed as follows. Take two identical regular n-gons in the plane. Translate one vertically out of the plane and, for the antiprism, rotate the other by π/n in the plane. The vertical translation is continued until the nearest neighbors between the two polygons have distance equal to the side length of the n-gons. The three-dimensional solid determined by the points of these two n-gons has symmetries generated by rotations and reflections. It may be expected that the graph, whose vertices are not held rigidly in place by the solid and in fact are not actually locations at all, will have other purely combinatorial automorphisms; however, we will see later that the automorphisms of the prism and antiprism graphs are exactly those which arise from the symmetries of the associated highly symmetric solid.

2.2.7 Platonic and Archimedean Graphs

The regular pyramids, prisms, and antiprisms each in general have two classes of faces, the bases on the one hand and the side faces on the other. For small cases, however, the side faces become indistinguishable from the bases, and new symmetries occur. The regular triangular pyramid, the regular quadrilateral prism, and the regular triangular antiprism become the regular tetrahedron, cube, and octahedron, respectively. These solids are completely regular in the sense that there is a symmetry between each pair of vertices, each pair of edges, and each

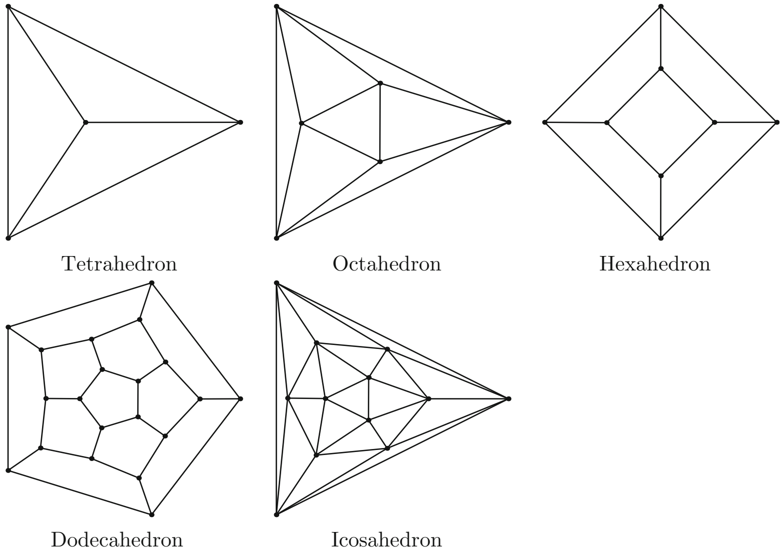


Fig. 2.10 Five platonic graphs: *tetrahedron, octahedron, hexahedron (cube), dodecahedron, and icosahedron*

pair of faces. There are two other solids with this degree of symmetry, namely, the dodecahedron and the icosahedron, and these five *Platonic solids* give us the five *Platonic graphs*; see Fig. 2.10. If we require symmetry between each pair of vertices but only require symmetry between pairs of regular polygonal faces of the same type, then the resulting solids are the Archimedean solids and give rise to the *Archimedean graphs*; see Fig. 2.11.

Platonic and Archimedean solids have a long and rich history and are well studied [22].

2.2.8 Polyhedral Graphs

Directly using the 1-skeleton is only one way of generating a graph from a polyhedron. More generally, we can consider the collection of all the vertices, edges, and faces in the solid and consider the incidences between them. The following example uses the cube graph to show how we can associate a graph to any convex polyhedron. The cube has 8 vertices,

$$1, 2, 3, 4, 5, 6, 7, 8,$$

12 edges,

$$a, b, c, d, e, f, g, h, i, j, k, l,$$

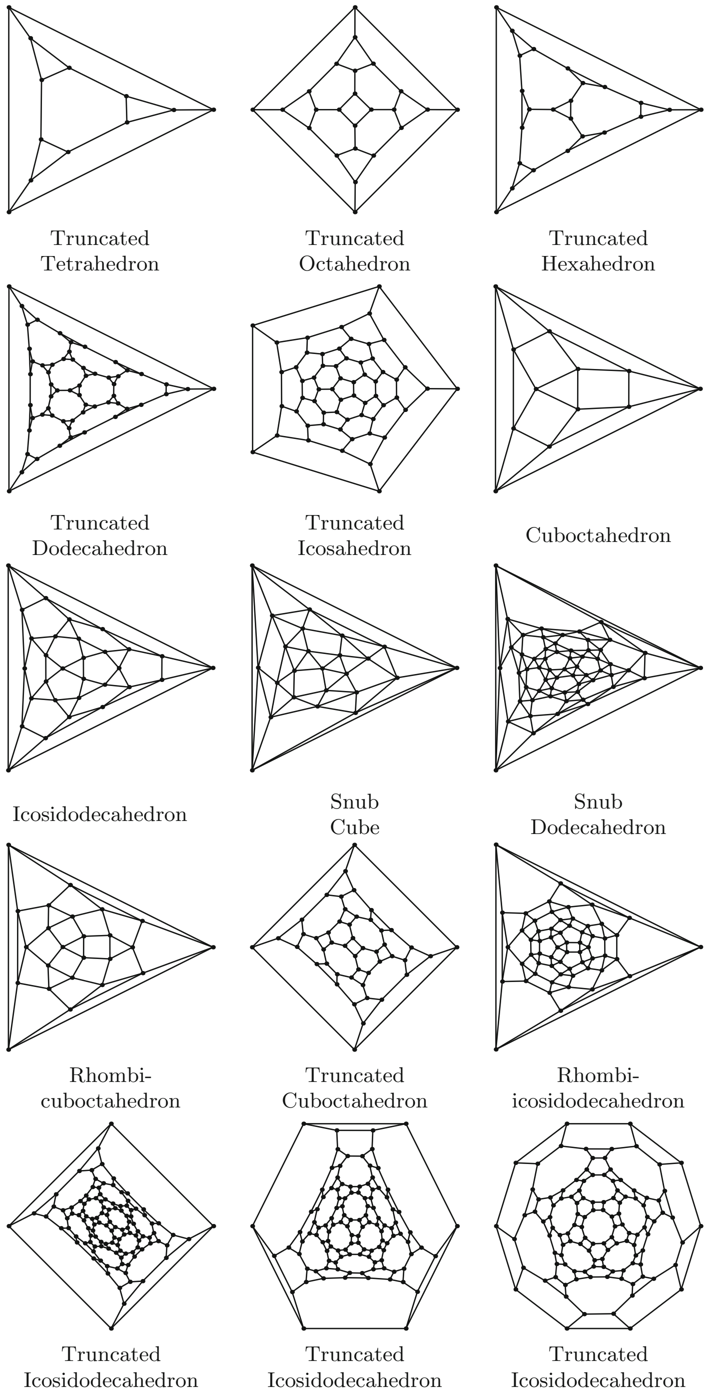


Fig. 2.11 Thirteen Archimedean graphs (The last one is shown in three different forms)

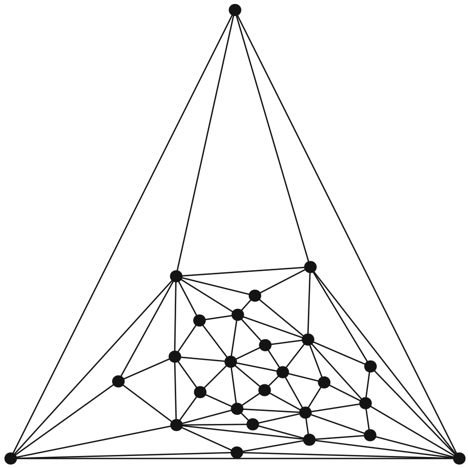


Fig. 2.12 The incidence graph of the cube as drawn by VEGA

and 6 faces,

$$A, B, C, D, E, F.$$

Define a graph on $8+12+6 = 26$ vertices with the property that two elements x and y are adjacent in the graph if and only if they are incident on the cube. So each edge is incident to two faces as well as its two endpoints, and each face is incident to the vertices and edges on its boundary. In Fig. 2.12, the 12 vertices corresponding to the edges of the cube are easily distinguished from the 8 vertices corresponding to the vertices of the cube or the six corresponding to the faces of the cube by their valence: Edge vertices have valence 4, vertex vertices have valence 6, and face vertices have valence 8.

2.2.9 Generalized Petersen Graphs

All classes of graphs considered so far have arisen naturally from geometry or illustrate relations so regularly that they really do not require the full generality of graph theory. The next example, possibly the most celebrated graph, is a true native to the subject. Clearly related to the simple prism graphs, it regularly appears in statements of theorems in graph theory as an exceptional case. It is the *Petersen graph*, $\mathrm{GP}(5, 2)$; see Fig. 2.13 which gives the classic drawing of the Petersen graph, the drawing which inspired the following generalization.

For a positive integer $n \geq 3$ and $0 < r < n/2$, *the generalized Petersen graph* $\mathrm{GP}(n, r)$ has a vertex set $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ and edges of the form $u_i v_i$, $u_i u_{i+1}$, $v_i v_{i+r}$ for $i \in \{1, \ldots, n\}$ with indices modulo n, ($i \in \mathbb{Z}_n$). In the diagrams, the vertices u_i form a cycle on the outside connected by the edges $u_i v_i$ to the vertices v_i arranged compatibly on the inside, where the n edges $v_i v_{i+r}$ form a pentagram in the case of the classic Petersen graph and form one or several cycles in the general case depending on whether r and n have a common divisor.

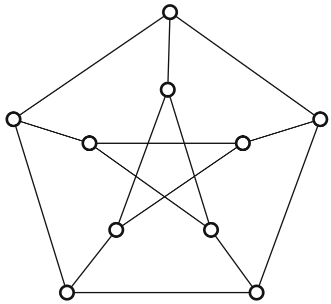


Fig. 2.13 The Petersen graph GP(5, 2)

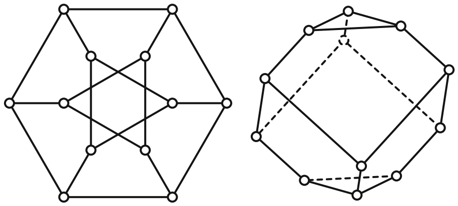


Fig. 2.14 The Dürer graph GP(6, 2) obtained as the skeleton of a truncated cube

All vertices in $\mathrm{GP}(n, r)$ are of degree 3. The values of r are restricted to avoid, in the case $r \equiv 0 \pmod{n}$, a vertex adjacent to itself or, for even n, the case $r \equiv n/2 \pmod{n}$ which would have the vertices v_i having valence of only 2. The other values of r, $n/2 < r < n$ produce duplicates since $v_i v_{i+r} = v_{i+r} v_i = v_{i+r} v_{i+r-r} = v_{i'} v_{i'-r}$; thus, we have that $\mathrm{GP}(n, r) = \mathrm{GP}(n, n-r)$.

If $r = 1$, then $\mathrm{GP}(n, 1)$ is the same as the prism graph $\Pi(n)$, so $n = 5$ is the first interesting case, and $\mathrm{GP}(5, 2)$ is the only nonprism example. For $n = 6$, we also have a single nonprism example, the so-called *Dürer graph*; see Fig. 2.14 which gets its name from a solid, at first glance a truncated cube resting on one of the two triangular faces which are produced when two antipodal corners of the cube have been cut away. This mysterious solid appears in the famous medieval engraving "Melancholia I" by the Nürnberg artist Albrecht Dürer.

Up until now, each of our graph examples could be distinguished from one another by the numbers of vertices and edges. For the generalized Petersen graphs, this is no longer the case. For fixed n, each graph $\mathrm{GP}(n, r)$ has $2n$ vertices and $3n$ edges, and each vertex is of valence 3. So the question arises as to whether they have distinct graph structures. We say two graphs are *isomorphic* if there is a bijection between the vertex sets which preserves the property of adjacency.

For each of $n = 7$ and $n = 8$, there are two generalized Petersen graphs on our list; see Fig. 2.15. For $\mathrm{GP}(8, 2)$, the interior figure is two 4-cycles, while for $\mathrm{GP}(8, 3)$, it is a single 8-cycle. Moreover, since we can easily check that $\mathrm{GP}(8, 3)$ has no 4-cycles, $\mathrm{GP}(8, 2)$ cannot be isomorphic to $\mathrm{GP}(8, 3)$. For $n = 7$, both graphs have several 7-cycles, and the situation is less obvious. It is not hard to show that there is no isomorphism which sends the vertices of the outer 7-cycle of $\mathrm{GP}(7, 2)$ to the outer cycle of $\mathrm{GP}(7, 3)$, so let us consider an isomorphism which sends the vertices of the outer 7-cycle of $\mathrm{GP}(7, 2)$ to the inner 7-cycle of $\mathrm{GP}(7, 3)$ and vice versa.

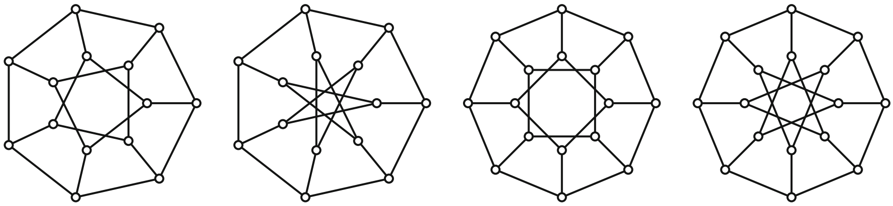


Fig. 2.15 Generalized Petersen graphs $\mathrm{GP}(n, r)$ for $n = 7$ and $n = 8$

We will consider the general case. Suppose that there is an isomorphism f between $\mathrm{GP}(n, r)$ and $\mathrm{GP}(n, s)$:

$$\mathrm{GP}(n, r) = (\{a_i, b_i \mid i \in \mathbb{Z}_n\}, \{a_i a_{i+1}, a_i b_i, b_i b_{i+r} \mid i \in \mathbb{Z}_n\})$$
$$\mathrm{GP}(n, s) = (\{c_i, d_i \mid i \in \mathbb{Z}_n\}, \{c_i c_{i+1}, c_i d_i, d_i d_{i+s} \mid i \in \mathbb{Z}_n\})$$

which interchanges the inner and outer n-cycles. Without loss of generality, we may say $f(a_0) = d_0$. So, using the ring edges, $f(a_1) = d_{\pm s}$, say $f(a_1) = d_s$, and then inductively $f(a_i) = d_{is}$ for all i, and using the spoke edges, $f(b_i) = c_{is}$ for all i, and in particular $f(b_{i+r}) = c_{(i+r)s}$ and, since f is an isomorphism, c_{is} and $c_{(i+r)s}$ must be adjacent, so $c_{(i+r)s} = c_{is\pm 1}$. Thus, adjacency is preserved if and only if $rs \equiv \pm 1 \pmod{n}$.

So, in particular, $\mathrm{GP}(7, 2)$ is isomorphic to $\mathrm{GP}(7, 3)$ by a ring-swapping isomorphism, and we write $\mathrm{GP}(7, 2) \cong \mathrm{GP}(7, 3)$.

For $n = 9$ and $n = 10$, there are three examples each of the generalized Petersen graphs; see Fig. 2.16, and since $2 \cdot 4 = 8 \equiv -1 \pmod{9}$, we have $\mathrm{GP}(9, 2) \cong \mathrm{GP}(9, 4)$. In 2009, Staton and Steimle [93] proved the following result:

Theorem 2.1. *For $2 \leq r, s \leq n - 2$ with $gcd(n, r) = gcd(n, s) = 1$, the generalized Petersen graphs $\mathrm{GP}(n, r)$ and $\mathrm{GP}(n, s)$ are isomorphic if and only if either $r \equiv \pm s \pmod{n}$ or $r \cdot s \equiv \pm 1 \pmod{n}$.*

So, in other words, the only way two generalized Petersen graphs with connected inner rings can be isomorphic is either by an isomorphism which preserves the outer ring or one which exchanges the inner and outer rings. For more details, see [9, 75].

2.2.10 Cages

The next collection of examples may also be regarded as generalizations of the Petersen graph. The Petersen graph has many 5-cycles, not simply the outer and inner 5-cycles of the standard diagram. Moreover, it is easy to see that $\mathrm{GP}(5, 2)$ has no shorter cycles. There are smaller graphs than the Petersen graph with no 3-cycles

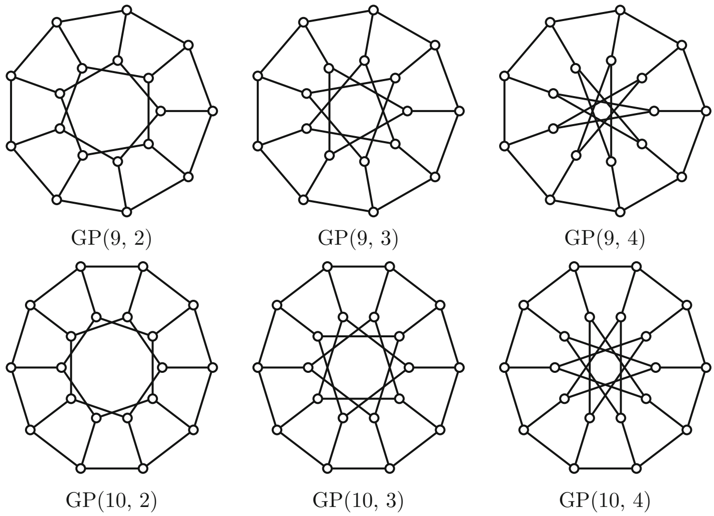


Fig. 2.16 More generalized Petersen graphs GP(n, r)

and no 4-cycles, but those also have few cycles of any kind since they are unions of simple cycles, trees, etc., and they avoid multiple cycles by having low valence. The Petersen graph, however, has many cycles since every vertex has valence 3. The cages generalize this property.

The length of the shortest cycle in a graph is called the *girth* of the graph. The girth of a graph without cycles, a tree or forest, is defined to be infinite, so the girth of a simple graph is at least 3.

The girth depends only on the isomorphism class of the graph, that is, it is a *graph invariant*. Computing the girth involves solving a nontrivial optimization problem. For the example graphs presented so far, the complete graphs have girth three, and the complete bipartite graphs have girth four (or infinity). It can be shown that the graphs arising as the 1-skeleta of the three-dimensional polyhedra illustrated all have girth equal to the number of edges in the smallest facial cycle, although this is not true in general. The girth of GP$(5, 2)$, as remarked above, is 5.

A graph is said to be a *g-cage* if it is trivalent, has girth g, and there exists no trivalent graph with girth g having fewer vertices. Note that the definition does not preclude there being more than one cage of a particular size. The complete graph K_4 is the unique 3-cage, and the complete bipartite graph $K_{3,3}$ is the only 4-cage. The Petersen graph GP$(5, 2)$ is the only 5-cage.

The previous examples were defined by a predetermined structure, so we could simply list examples. The g-cages are defined by the graph theoretic properties which they must satisfy, so it is neither clear which graphs belong on the list nor, given such a list, whether the graphs on the list do indeed belong on it. A complete structure theorem for g-cages is unknown, although many examples have been computed. At least we may establish a lower bound on the number of vertices a g-cage must have:

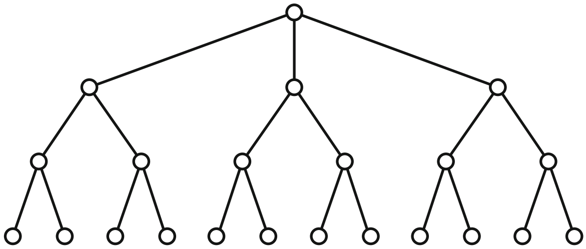


Fig. 2.17 A level 4 binary tree with a root of valence 3

Theorem 2.2. *A $2k$ cage must have at least $2^{k+1} - 2$ vertices. A $2k + 1$ cage must have at least $3 \cdot 2^k - 2$ vertices.*

Proof. To establish a lower bound on the number of vertices of a g-cage, we start with a single vertex, list its three neighbors, each of those has two neighbors, etc. For the case $g = 2k + 1$ up to the k-level, each neighbor set gives rise to two new unrecorded vertices, creating a *binary tree* whose root vertex is of valence 3; see Fig. 2.17. Edges between vertices on the same level are only allowed on level k, so we have at least $1 + 3 + 3 \cdot 2 + 3 \cdot 4 + \cdots + 3 \cdot 2^{k-1}$ vertices, yielding the desired bound. For even girth, there is a similar construction; see Exercise 2.18. □

It turns out that the 6-cage is the Heawood graph; see Chap. 5, Fig. 5.32. The 7-cage has 24 vertices and is depicted in Chap. 3, Fig. 3.35. The 8-cage is known as the Cremona–Richmond graph; see Fig. 5.28; however, graph theorists prefer to call it the Tutte 8-cage. We will learn more about the Heawood graph and the Tutte 8-cage later and the relationship of the former to projective planes and the latter to the *hexagrammum mysticum* of Pascal.

It is interesting that the 9-cages were not found until quite recently. The search for 9-cages involved a lot of computer checking, and the result came as a surprise. There are 18 nonisomorphic 9-cages. All smaller cages have regular structure and are unique. However, the 9-cages do not show any apparent structure; they are computed in [13].

Balaban found one of the three 10-cages which is shown in Fig. 2.18. It is perhaps of interest to note that the 10-cages were known, see [76], before all the 9-cages were computed. The reason is simply that the gap between the easily proven lower bound and the actual size of the cage is larger for the 9-cage than for the 10-cage. By Theorem 2.2, there is no trivalent graph of girth 9 on fewer than 46 vertices and there is no such graph of girth 10 on fewer than 62 vertices. Since the 9-cage has 58 vertices [13] and the 10-cage has 70 vertices, the respective gaps are 12 for the 9-cage and only 8 for the 10-cage. For a survey on cages, see [108] where Wong states an interesting conjecture.

Conjecture 2.3. Every g-cage with g even is bipartite.

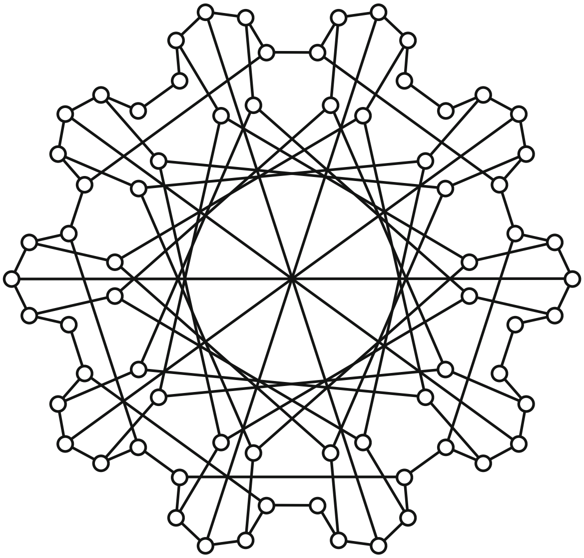


Fig. 2.18 Balaban's 10-cage

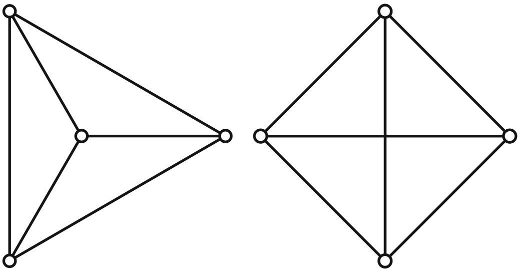


Fig. 2.19 A plane and a nonplane embedding of K_4

2.2.11 *Planar Graphs*

A graph that can be drawn in the plane so that edges intersect only at vertices is called planar. By a *drawing* of a graph G, we mean a representation of G in the plane such that vertices are represented by distinct points in the plane and edges by (curved) line segments connecting their endpoints. We will be more precise in Sect. 2.6.6. A drawing without edge crossings is called a *plane embedding* of the graph. Clearly, any tree can be drawn without edge crossings. Let G be a connected planar graph and consider a plane embedding of it. Such a drawing subdivides the plane into regions, one of which is unbounded. To avoid this special case, it is better to consider an embedding into the sphere, in which case we call the regions the *faces* of G. For example, in the plane embedding of K_4 in Fig. 2.19, we count four faces, namely, three triangles and the infinite outer face. Often, when we have a plane embedding of a graph and we count faces, we are implicitly regarding the plane as part of a large sphere and the exterior region then counts as one face.

Given a connected planar graph G on n vertices, together with a plane (sphere) embedding, choose a spanning tree T in G. T has $n-1$ edges. The plane drawing of T, induced by the plane drawing of G, has only one face. Inserting an additional edge of $E(G)-E(T)$ will divide this region into two. Inductively, we get one more face by inserting an additional edge. We started out with n vertices, $n-1$ edges, and 1 face. We add $e-(n-1)$ edges to get f faces, so $e-(n-1) = f-1$ or $n-e+f = 2$. The alternating sum $n-e+f$ is called the *Euler characteristic*. It is a property of the surface in which the graph is embedded, and we say that the Euler characteristic of the sphere equals 2.

2.3 Regularity

2.3.1 Regular Graphs

Cycles, complete graphs, prisms, and antiprisms are all examples of *regular* graphs. A graph is k-*regular* or k-*valent* if all of its vertices have valence k. Cycles are 2-valent, and, conversely, 2-valent graphs are collections of disjoint cycles. Even graphs which are 1-valent graphs are not without interest. A 1-valent graph is a disconnected set of edges. One way of studying a large graph is to partition the edge set into spanning 1-valent graphs, called 1-*factors*. A spanning subgraph which is k-valent is called a k-factor. By contrast to the simple structure of 1-valent and 2-valent graphs, 3-valent graphs exist in much greater variety. Please note the important difference between the way the word "regular" is used in the contexts of geometry and graph theory. In geometry, a regular figure has a high degree of symmetry. The regular examples we have seen so far are irregular in the sense of being atypical. A regular graph typically has no symmetry at all; see Fig. 2.20 in which the outside 12-cycle has been rather haphazardly connected to interior cycles of various sizes.

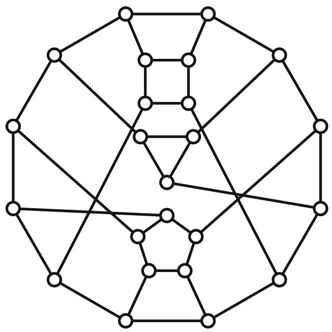


Fig. 2.20 A 3-regular but asymmetric graph

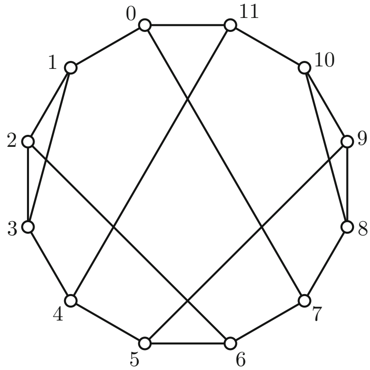


Fig. 2.21 LCF code: $[-5, 2, 4, -2, -5, 4, -4, 5, 2, -4, -2, 5]$

2.3.2 Cubic Graphs and LCF Notation

Three-valent graphs are also called *cubic*. Recall that a graph is Hamiltonian if it has a spanning cycle. If a cubic graph is Hamiltonian, we can draw it as a $|V|$-cycle with inserted chords, which leads to a convenient notation, the LCF notation, named for developers Lederberg, Coxeter, and Frucht. See [20, 33].

Given the Hamilton cycle, all we have to do to specify the graph is to list the lengths of chords measured in jumps when we traverse the vertices along the Hamilton cycle. Such a list is called the *LCF notation*. For instance, K_4 can be described by $[2, 2, 2, 2]$. $K_{3,3}$ is $[3, 3, 3, 3, 3, 3]$, and the cube Q_3 is $[3, 5, 3, 5, 3, 5, 3, 5]$ or $[3, -3, 3, -3, 3, -3, 3, -3]$ if we let a negative jump denote a chord measured in the opposite direction. We can also use exponent notation in order to shorten repeated subsequences. Here is an equivalent shorthand notation for the above examples: $\mathrm{LCF}(K_4) = [2^4], \mathrm{LCF}(K_{3,3}) = [3^6], \mathrm{LCF}(Q_3) = [(3, -3)^4]$.

Example 2.4. The graph G with

$$\mathrm{LCF}(G) = [-5, 2, 4, -2, -5, 4, -4, 5, 2, -4, -2, 5]$$

is depicted in Fig. 2.21.

2.3.3 Regularity and Bipartite Graphs

In Sect. 2.2.3, we examined the complete bipartite graphs. In general, a graph $G = (V, E)$ is *bipartite* if V can be partitioned into two nonempty sets V_1 and V_2 such that each edge has one of its endvertices in V_1, the other in V_2. Note that if G is connected and bipartite, the *bipartition* of the vertex set is uniquely determined, namely, two vertices are in the same set of the bipartition if and only if their distance in G is even. For disconnected graphs, bipartiteness clearly implies bipartiteness of

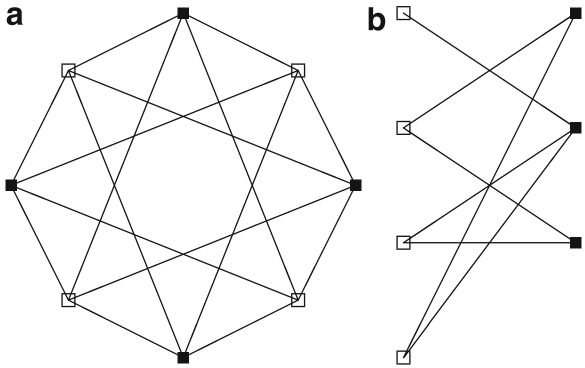


Fig. 2.22 Bipartite graphs with *black* and *white* bipartition

each of its connected components. There are several standard ways to indicate in a diagram that a graph is bipartite, for example, to arrange the vertex sets V_1 and V_2 on two different lines; see Fig. 2.22b. Except for very small graphs, a better method to indicate the bipartition is to color the vertices, say, black and white. It is easy to visually check if every edge has one black and one white endpoint; see, for example, Fig. 2.22a.

We have already seen that the incidences of a configuration can be encoded as a graph, see Fig. 1.12, and in this graph, the points and the lines form a vertex partition, so naturally bipartite graphs are of particular interest, and in particular regular bipartite graphs.

If every vertex of the bipartite graph $G = (V_1 \cup V_2, E)$ has valence k, then $k|V_1| = k|V_2|$, so unless $k = 0$, we have $|V_1| = |V_2|$, i.e. V must be partitioned into two sets of equal cardinality. In particular, $|V|$ must be even. For $k = 1$, we get a set of mutually nonincident edges. The following graph theoretic result was first formulated and proved in terms of configurations by Steinitz in his Ph.D. dissertation, [94].

Theorem 2.5. *Every k-valent bipartite graph G can be written as the edge disjoint union of k 1-factors.*

Proof. We use induction on k. For $k = 1$, there is nothing to show. We assume $k > 1$ and want to show that a k-valent bipartite graph G contains a 1-factor F. We then use the induction hypothesis on $G - F$ to obtain the desired decomposition of the edge set.

To construct a 1-factor, select mutually nonincident edges until every edge not yet selected is incident with at least one of the edges selected so far. Let us call this maximal set of mutually nonincident edges M. If M is not spanning, let v be a vertex not covered by M and consider the set A of all paths starting at v, then using an edge of M, an edge not in M, then an edge in M, etc. We can find a set of mutually nonincident edges which is of larger cardinality than M if there is at least one path in A that ends at another uncovered vertex u by removing from M all edges of M on this $u - v$-path and adding its edges not in M. If A does not contain such

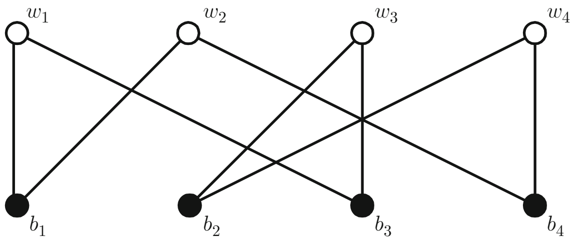


Fig. 2.23 The graph from Example 2.6

a path, then the subgraph of G induced by A, $I(A)$, has only one vertex, namely, u, which is not covered by M. Thus, the vertices of $I(A)$ in the same bipartition as u have valence k, and there is one vertex fewer in the other bipartition class, a contradiction. □

This theorem enables us to encode regular bipartite graphs on $2n$ vertices by k permutations of the set $\{1, 2, \ldots, n\}$. Given n black vertices $B = \{b_1, \ldots, b_n\}$ and n white vertices, $W = \{w_1, \ldots, w_n\}$ and a k-regular graph on these $2n$ vertices with a bipartition respecting the colors. Suppose the edges are decomposed into k 1-factors. For each 1-factor, let the black endpoints adjacent to $w_1, \ldots, w_n$ be $b_{i_1}, \ldots, b_{i_n}$ respectively, then set the permutation of $1, \ldots, n$ corresponding to the 1-factor to be $i_1, \ldots, i_n$.

Example 2.6. Suppose we have the simple 8-cycle of Fig. 2.23 with vertex set $\{w_1, w_2, w_3, w_4, b_1, b_2, b_3, b_4\}$ and edge set

$$\{(w_1, b_1), (b_1, w_2), (w_2, b_4), (b_4, w_4), (w_4, b_2), (b_2, w_3), (w_3, b_3), (b_3, w_1)\}.$$

There is a unique partition of the edges into two 1-factors:

$$\{(w_1, b_1), (w_2, b_4), (w_3, b_3), (w_4, b_2)\} \cup \{(b_3, w_1), (b_1, w_2), (b_2, w_3), (b_4, w_4)\}$$

and the two permutations

$$\begin{matrix} 1 & 4 & 3 & 2 \\ 3 & 1 & 2 & 4 \end{matrix}$$

encode the graph. If we wish to augment the 8-cycle to a 3-valent bipartite graph, we need to add another 1-factor. Not any one factor will do, however, since many will correspond to sets of edges some of which we already have. The permutations we can allow must not have the same value in the ith position as either of the previous two; in other words, if we add the permutation as a row in the array above, there must be distinct elements in each column. So,

$$\begin{matrix} 1 & 4 & 3 & 2 \\ 3 & 1 & 2 & 4 \\ 2 & 3 & 4 & 1 \\ 4 & 2 & 1 & 3 \end{matrix}$$

would be one way to complete the example to a complete bipartite graph.

In general, k permutations on n symbols give rise to a k-valent bipartite (simple) graph, provided that distinct permutations move a symbol to distinct symbols.

2.3.4 Semiregular Bipartite Graphs

If we want to relax the condition of regularity to allow unequal bipartition, we can at least require that all vertices of the same color have the same valence. In this case, we call the bipartite graph *semiregular*. We write $G = (V_1 \cup V_2; k_1, k_2)$ to indicate bipartition and vertex valences. Prescribing the size of the bipartition imposes restrictions on the values of k_1 and k_2. Certainly, if G is simple, the k_1 and k_2 are bounded by $|V_2|$ and $|V_1|$ respectively. Moreover, a semiregular bipartite graph $G = (V_1 \cup V_2; k_1, k_2)$ must satisfy

$$|V_1|k_1 = |V_2|k_2.$$

Given $|V_1|$ and $|V_2|$, we might ask for all possible values of k_1 and k_2 so that a semiregular bipartite $G = (V_1 \cup V_2; k_1, k_2)$ exists. $|V_1| = 5$ and $|V_2| = 3$, for example, allow the only possible solution $k_1 = 3$ and $k_2 = 5$, yielding the complete bipartite graph $K_{3,5}$ as the unique connected structure satisfying the requirements.

Given k_1 and k_2, we may ask for the smallest vertex set on which there is a semiregular bipartite graph with the prescribed regularity. Again, we get as a unique answer the complete bipartite graph.

It is not difficult to show, see Exercise 2.32, that the obvious necessary conditions on the parameters, namely, $|V_1|k_1 = |V_2|k_2$, $k_1 \leq |V_2|$ and $k_2 \leq |V_1|$, are also sufficient for the existence of a simple semiregular bipartite graph $G = (V_1 \cup V_2; k_1, k_2)$.

The situation for constructibility changes drastically if we add as extra requirement that G must have girth larger than 4. This is not an arbitrary condition. A quadrilateral in the incidence graph corresponds to two distinct lines having two distinct points in common.

To construct a graph $G = (V_1 \cup V_2; k_1, k_2)$ of girth larger than 4, we need to insure that all k_1 neighbors of a vertex in $|V_1|$ have disjoint sets of $k_2 - 1$ neighbors in $|V_1|$, and we get the necessary condition $|V_1| \geq 1 + k_1(k_2 - 1)$. By symmetry, we have the corresponding requirement on the size of $|V_2|$, namely, $|V_2| \geq 1 + k_2(k_1 - 1)$. Unfortunately, these obvious necessary conditions are not sufficient to ensure the existence of G. According to Gropp [38], there does not exist any 5-valent bipartite graph on 44 vertices of girth larger than 6. The smallest parameter set satisfying the necessary conditions, but for which the existence of a bipartite semiregular graph is not known, is $|V_1| = 30$, $|V_2| = 20$, $k_1 = 4$, $k_2 = 6$. [38] gives several more examples.

2.3.5 Permutations

We have seen how permutations are useful to construct regular bipartite graphs, so we would like to recall a few facts about them. A *permutation* on the set V is a bijection $\mathbf{p}$ of V onto itself, $\mathbf{p} : V \to V$. The set of all permutations on V

is denoted by $\mathrm{Sym}(V)$. Usually, we consider permutations of the "standard set" $V = \{1, 2, \ldots, n\}$. In this case, we write $\mathrm{Sym}(V) = \mathrm{Sym}(n) = S_n$ and call it the *symmetric group*.

Example 2.7. Each row of the configuration table of the Pappus configuration, see Fig. 1.9, is a permutation of the nine points:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 7 & 2 & 1 & 6 & 9 & 3 & 4 & 5 \\ 6 & 1 & 8 & 9 & 7 & 3 & 4 & 5 & 2 \end{array}$$

The ith column gives the images of vertex i under the three permutations. Since the images are distinct, the three rows define three 1-factors of a regular bipartite graph which is, in fact, the incidence graph of the Pappus configuration.

Since a permutation maps V onto itself, it may be composed with itself, and listing the successive images of elements in cyclic order gives us the cycle notation for a permutation:

$$\mathbf{a}_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9)$$

$$\mathbf{a}_2 = (184)(273)(569)$$

$$\mathbf{a}_3 = (163857492)$$

A permutation, such as $\mathbf{a}_3$, which consists of a single cycle is called *cyclic permutation*. Permutation $\mathbf{a}_2$ has all cycles of the same length, and such a permutation is called *polycyclic* or *semiregular*. Permutation $\mathbf{a}_1$, the *identity permutation*, is polycyclic in the trivial sense that all cycles are of length 1. Each of its cycles specifies element x for which $\mathbf{a}(x) = x$ called a *fixed point* of the permutation. Let $\mathrm{Fix}(\mathbf{p})$ denote $\mathrm{Fix}(\mathbf{p}) = \{x \in V | \mathbf{p}(x) = x\}$ and let $\mathrm{fix}(\mathbf{p}) = |\mathrm{Fix}(\mathbf{p})|$. Hence, $\mathrm{fix}(\mathbf{a}_1) = 9, \mathrm{fix}(\mathbf{a}_2) = \mathrm{fix}(\mathbf{a}_3) = 0$. A fixed-point free permutation is also called a *derangement*. The set of derangements over V is denoted by $\mathrm{Der}(V)$. If seen as a subset of S_n, it is denoted by $\mathrm{Der}(n)$.

A permutation whose longest cycle has length 2 is called an *involution*. If, in addition, it has no fixed points, it is called a *fixed-point free involution*. The *order* of the permutation $\mathbf{p} \in \mathrm{Sym}(n)$ is the least integer k, such that $\mathbf{p}^k$ is identity. The number n is called the *degree* of permutation $\mathbf{p}$.

Each permutation can be depicted in graphic form with each element $x \in V$ represented by a vertex and with vertices x and $\mathbf{p}(x)$ adjacent. Such a graph does not encode the directions of the cycles, so it is more common to draw an arrow from x to $\mathbf{p}(x)$. This is a *directed graph*. The graph so drawn is not in general a simple graph. Any fixed point will give an element adjacent to itself, and every cycle of length two will give two vertices joined by two arrows, one in each direction. See Fig. 2.24. The connected components of the graph of the permutation are called *orbits*.

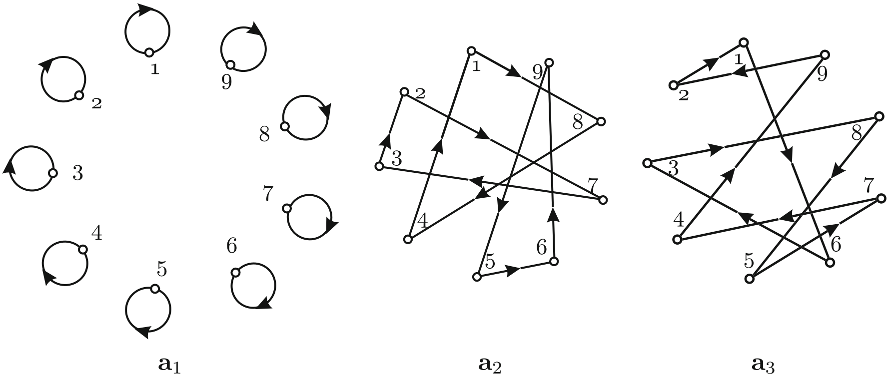


Fig. 2.24 Permutations $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

2.3.6 Directed Graphs and Multigraphs

A graph which allows for loops and multiple edges is often called a *multigraph*. In order to accomplish a mathematical description of multigraphs, we consider two disjoint sets V and E, the vertex set, and the edge set, as well as a function that assigns each edge $e \in E$ a subset of V consisting of at least one and most two vertices which are, as before, called the *endvertices* of e. A *loop* is an edge whose image set has a single element, and two edges assigned to the same set are said to be *parallel*. If a direction is required, each edge is assigned an ordered pair of two vertices.

Let $S \subseteq V \times E$ denote the collection of vertex–edge incident pairs. We have $s \in S$ if and only if $s = (v, e)$ and e is an edge one of whose end-vertices is v. S is called the set of *semiedges* or *arcs*.

The most general definition that we will use defines a graph G as a quadruple (V, S, i, r) such that V and S are sets, i is a map $i : S \to V$ that assigns each arc $s \in S$ its end-vertex $i(s) \in V$, and $r : S \to S$ is an involution $r^2 = 1$ mapping each arc to its *opposite arc*. In this model, the set of edges E is given as the set of orbits of r. If r is allowed to have fixed points, the corresponding orbits have a single element and the corresponding edge is called a *half-edge*. Structures with half edges are sometimes called *pregraphs*.

If not clear from the context, for any graph X, we will use the sets $V(X)$, $E(X)$, $S(X)$, the adjacency relation $\sim_X$, the mapping i_X, and the involution r_X. Note that our definition of graph isomorphism was only given for simple graphs. In Exercise 2.50, we discuss this notion for general graphs.

2.4 Operations on Graphs

We shall now describe several operations on graphs that can be used to generate new, large graphs from old, simple ones.

For the following list of operations, the reader is encouraged to take pencil and paper and produce several drawings combining examples from Sect. 2.2.

2.4.1 Graph Complement

The *graph complement* $Y = X^{\mathrm{c}}$ has $V(Y) = V(X)$ and $x \sim y$ in Y if an only if x is not adjacent to y in X.

2.4.2 Graph Union

We define the *graph union* $X \cup Y$ as the disjoint union of two graphs. So the vertex set is the disjoint union of $V(X)$ and $V(Y)$, and two vertices are adjacent if they are adjacent in X or adjacent in Y. Even if X and Y are connected, $X \cup Y$ is disconnected, having X and Y as connected components. On the other hand, a connected graph cannot be written as the graph union of any proper subgraphs.

If two graphs are isomorphic, we write $2X$ for $X \cup X$, $3X$ for $X \cup X \cup X$, etc.

2.4.3 Graph Join, Cone, and Suspension

The *graph join* $X * Y$ of graphs X and Y can be defined in terms of graph union and graph complement:

$$X * Y = (X^{\mathrm{c}} \cup Y^{\mathrm{c}})^{\mathrm{c}}.$$

Joining X to a single vertex, the *apex*, is called *coning*. We denote by $C(X)$ the *cone* over X. This operation generalizes to a k-fold cone $C^{(k)}(X)$ in which k new vertices are introduced. A twofold cone is known as *suspension*. Finally, $K_{m,n} = K_m^c * K_n^c$. (see Exercise 2.63.)

2.4.4 One-Point Union and Connectivity

Given a graph X with vertex u and graph Y with vertex v, the *one-point union* of X and Y with respect to u and v, $X \cup_{u,v} Y$, is obtained from the disjoint union by identifying the vertices u and v. So $G = X \cup_{u,v} Y$ then $|V(G)| = |V(X)| + |V(Y)| - 1$. Every path in $X \cup_{u,v} Y$ from a vertex in X to a vertex in Y must pass through the identified vertex. It is a *cut vertex*.

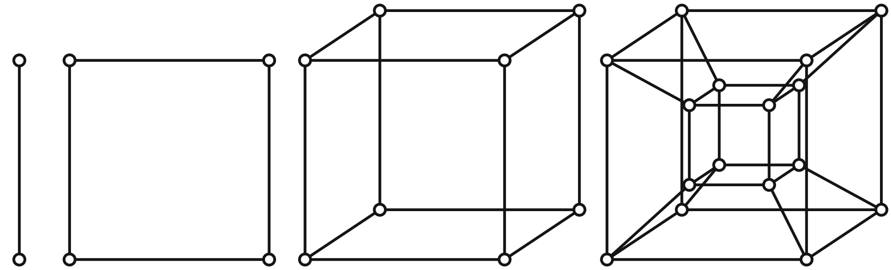


Fig. 2.25 Small hypercube graphs $Q_n; n = 1, 2, 3, 4$

If a graph G cannot be written as a disjoint union of subgraphs nor as a one-point union of subgraphs, it is said to be 2-connected, and between any pair of vertices, there must be two internally disjoint paths.

In general, a graph is called *n-connected* if it contains n internally disjoint paths between any pair of its vertices. The *connectivity* of a graph X is the largest k for which X is k-connected. Connectivity is a graph invariant. The connectivity of C_n, for example, is 2, P_n has connectivity 1, while K_n has connectivity $n-1$. Note that in an n-connected graph, every vertex must have valence at least n.

2.4.5 Cartesian Product

Let X and Y be any two simple graphs. The *Cartesian product,* $X \Box Y$, has vertex set

$$V(X \Box Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \Box Y)$ are adjacent if and only if either $x = x'$ and $y \sim y'$ or $x \sim x'$ and $y = y'$.

Clearly, $C_4 = K_2 \Box K_2$ and the hexahedron or cube Q_3 is the threefold Cartesian product of K_2 with itself, i.e., $Q_3 = C_4 \Box K_2$.

The prism Π_n, for example, can be expressed as the Cartesian product of a cycle of length n and the complete graph on 2 vertices, $\Pi_n = C_n \Box K_2$.

Since $\Box$ is associative (see Exercise 2.57), we can consider the Cartesian product of several factors. Taking n factors equal to K_2, we obtain the *hypercube graph* Q_n. $Q_n = K_2 \Box K_2 \Box \cdots \Box K_2$. Small hypercube graphs are depicted in Fig. 2.25. Note that Q_2 is used as the symbol $\Box$ to denote the Cartesian product.

In Exercise 2.58, an alternate definition of Q_n is given.

2.4.6 Tensor Product

Let X and Y be any two simple graphs. The *tensor product* $X \times Y$ has vertex set

$$V(X \times Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \times Y)$ are adjacent if and only if $x \sim x'$ and $y \sim y'$. The tensor product of two K_2's is the disjoint union of two edges and is used as symbol $\times$ for the tensor product.

2.4.7 Strong Product

Let X and Y be any two simple graphs. The *strong product* $X \boxtimes Y$ has the vertex set

$$V(X \boxtimes Y) = V(X) \times V(Y).$$

Vertices (x, y) and (x', y') from $V(X \boxtimes Y)$ are adjacent if and only if either $x \sim x'$ and $y \sim y'$ or $x = x'$ and $y \sim y'$ or $x \sim x'$ and $y = y'$. Again, the strong product of two K_2's is used as multiplication symbol.

2.4.8 Line Graph

For any simple graph X, let $L(X)$ denote the graph whose vertex set $V(L(X))$ is $E(X)$ and two vertices e and e' from $V(L(X))$ are adjacent if and only if e and e' are incident (as edges of X) with a common vertex of X. The line graph of K_4, for example, is the octahedron graph.

2.4.9 Subdivision Graph

The *subdivision graph* $S(X)$ has $V(S(X)) = V(X) \cup E(X)$, and two vertices x and e of $S(X)$ are adjacent if and only if $x \in V(X)$ and $e \in E(X)$ and x is incident to e in X. A drawing of the subdivision graph is obtained from a drawing of the graph by inserting one new vertex in the interior of each edge.

2.4.10 Graph Square

For a given graph X, its *square* X^2 is a graph on the same vertex set with two vertices adjacent if and only if they are at distance at most 2 in X. Each vertex in X^2 is contained in a clique of size $\deg_X(v) + 1$.

The *pure square* $X^2 - E(X)$ is a graph on the same vertex set as X, with two vertices adjacent if and only if they are at distance 2 in X.

2.5 Graph Colorings

2.5.1 Vertex Colorings

A mapping c from $V(X)$ to any finite set of colors C is called a *vertex coloring* if no two adjacent vertices are assigned the same color of C. The smallest number of colors needed for a (proper) vertex coloring of a graph G is called a *chromatic number* of a graph and is denoted by $\chi(G)$.

Example 2.8. It is not hard to see that the chromatic number of a cycle C_n is 2 if n is even and 3 if n is odd.

Example 2.9. Clearly, the chromatic number of the tetrahedron graph is 4. Since the octahedron graph is $K_{2,2,2}$, its chromatic number is 3. The cube is bipartite; therefore, it has a chromatic number equal to 2. Since the dodecahedron graph contains an odd cycle, its chromatic number is at least 3. It is not hard to find a proper 3-coloring of $\mathrm{GP}(10,2)$. We leave the determination of the chromatic number of the icosahedron to the exercises.

The study of colorings of graphs constitutes an important branch of graph theory. The problem of determining the exact upper bound on the chromatic number of planar graphs was an outstanding open problem in graph theory for over a 100 years until it was solved by the aid of a computer in 1976 by Appel and Haken [1], yielding the *four color theorem*. Clearly, colorings of graphs have played an important role in the development of topological graph theory.

Theorem 2.10 (Brooks). *Let G be a connected graph. The chromatic number of G is less than or equal to the maximum valence of any vertex in G unless G is complete or an odd cycle.*

Proof. Let $\Delta(G)$ denote the maximum valence of any vertex in G. If $\Delta \leq 2$ then G is K_2, a path or a cycle, and we know that paths and even cycles are bipartite. We now assume $\Delta \geq 3$ and proceed by induction on the number of vertices of G. Clearly, a graph on 4 vertices which is not complete is 3-colorable. Let G be a graph on more than 4 vertices and $\Delta(G) \geq 3$. For any vertex v in G, note that $G - v$ has maximum valence at most Δ and cannot have $K_{\Delta+1}$ as a component; hence, $G - v$ is Δ-colorable by the induction hypothesis. If v is a vertex of valence less than Δ, then any Δ coloring of the components of $G - v$ easily extends to a Δ-coloring of G. So if G is not Δ-colorable, every vertex of G must have valence Δ. Moreover, in every Δ-coloring of $G-v$, all colors are used coloring the neighbors of v; otherwise, we could extend the coloring to G using the missing color. Fix a coloring of $G - v$ that does not extend to G and consider the subgraph $X_{i,j}$ induced by the vertices colored with color i or color j. The two neighbors of v colored i and j must be in the same component $X'_{i,j}$ of $X_{i,j}$ because, otherwise, we could interchange the roles of i and j in one of these components to obtain a Δ-coloring of $G - v$ where two neighbors of v have the same color. Consider now a path in $X'_{i,j}$ connecting two neighbors of v. Let u be the closest to v along this path so that u has valence larger than 2 in $X'_{i,j}$. Then u can be recolored, but this recoloring disconnects the $X_{i,j}$ so that the neighbors of v are in different parts, contradicting our earlier observation. Therefore, $X'_{i,j}$ is simply a path. In two such paths, $X'_{i,j}$ and $X'_{i,k}$, $i \neq j \neq k$ can only intersect in an endpoint, namely, the neighbor of v with color i, since any interior intersection point could be recolored leading as before to disconnect $X'_{i,j}$. If all neighbors of v are pairwise adjacent, G is complete and there is nothing to show, so assume without loss of generality that given a Δ-coloring of $G-v$, the neighbors of v colored i and j are not adjacent so that $X'_{i,j} = v_i u ... v_j$ is a path of length at least 3. We now change colors i and k on the path $X'_{i,k}$. In this new coloring, the vertex u is both on $X'_{i,j}$ and on $X'_{j,k}$ contradicting the established fact that these paths only intersect at their endpoint v_j. □

In Example 2.2.8, we obtained a graph together with a vertex coloring from a geometric object. Conversely, we will see in Chap. 5 that a graph together with a proper vertex coloring with k colors is sufficient to describe a geometric object, specifically a rank k incidence structure.

2.5.2 Edge Colorings

A mapping $c : E(G) \to C$ from the edge set $E(G)$ to some finite set C is called an (admissible or proper) edge coloring if for any two incident edges g and f we have $c(g) \neq c(f)$. The least number of colors needed to properly color the edges of g is called the *chromatic index* and is denoted by $\chi'(G)$.

Clearly, the maximum valence Δ is a lower bound for the chromatic index. Any edge coloring problem can be translated into a vertex coloring problem on $L(G)$; the line graph of G and Brooks' Theorem 2.10 provides an upper bound by observing that $\Delta(L(G)) \leq 2\Delta(G)-2$. However, this bound is not tight; in fact, the difference between upper and lower bound in terms of Δ is surprisingly small.

Theorem 2.11 (Vizing). *The chromatic index of a simple graph G satisfies the following inequalities:*

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1.$$

Proof. The first inequality is trivial, so the second one is the only thing we have to prove. We use induction on the number of edges. Consider a graph G with maximum degree Δ. If G has fewer than Δ edges, there is nothing to show. Assume that, by induction hypothesis, $G-e$ has a $\Delta+1$-coloring for every edge e, but none of these colorings can be extended to a coloring of G.

Consider an edge $e = vw_0$ and fix a $\Delta + 1$-coloring c_0 of $G_0 = G - vw_0$. Since the maximum degree of any vertex is Δ and c_0 is a $\Delta + 1$-coloring, there is at least one color missing at every vertex. If the same color is missing from both v and w_0, we are done. Let α be the color missing at v with respect to c_0 and β the color missing at w_0. So there is an edge of color β incident with v. We call a path whose edges are alternately colored α and β an $\alpha - \beta$-path and observe that any $\alpha - \beta$-path starting at w_0 must end in v; otherwise, we could exchange α and β along this path and extend c_0 to a coloring of G by coloring vw_0 with α after that switch.

Now choose a maximal sequence $w_0, w_1, \ldots, w_k$ of distinct neighbors of v such that the color of the edge xw_i is missing at w_{i-1}. For each of the graphs $G_i = G - vw_i$, we define a coloring c_i derived from c_{i-1} by coloring the edge v, w_{i-1}, which does not exist in G_{i-1}, by the color of vw_i in c_{i-1}. All these colorings differ only in edges incident to v, but the set of colors used for edges incident to v is the same for all these colorings.

Let β be a color missing at w_k with respect to the coloring c_0 (and subsequently with respect to c_k). By the maximality of k, there is an index $i \in \{1, \ldots, k\}$ such that $c_0(vw_i) = \beta$. An $\alpha - \beta$-path, P, from w_k with respect to c_k must end in v; in fact, it must end with the edge vw_{i-1}. With respect to the coloring, c_0 β is missing at w_{i-1}. Let P' be an $\alpha - \beta$-path with respect to c_{i-1} starting at w_{i-1} in G_{i-1}. P and P' are identical except for edges incident to v, so P' contains w_k. Since there is no β edge at w_k with respect to c_{i-1}, P' ends in w_k, contradicting the assumption that c_{i-1} cannot be extended to G. $\square$

2.6 From Geometry to Graphs and Back

There are numerous paths leading from geometry to graphs and back. We have already met the skeleta of polyhedra as a rich source of interesting graphs. Here, we mention some more of such interesting connections. But first, let us recall the concept of *metric space*. This structure lies somewhere between geometry and topology. It captures those properties of usual Euclidean space that measure distance between any two points in space.

2.6.1 Metric Space and Distance Function

A set M together with a function $d : M \times M \to \mathbb{R}$ is called a *metric space* if the following are true:

1. $d(x, y) \geq 0$ for any two points $x, y \in M$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$ for any two points $x, y \in M$.
3. (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$ for any three points $x, y, z \in M$.

The function d is called the *distance function* of M.

Example 2.12. The Euclidean plane, $\mathbb{R}^2 = \{(x, y) | x, y \in \mathbb{R}\}$, is a metric space for $d((x, y), (x', y')) = \sqrt{(x - x')^2 + (y - y')^2}$

Given a metric space, we define a *closed ball* $B(x, r)$ with center $x \in M$ and radius $r > 0$ as follows:

$$B(x, r) = \{y \in M | d(x, y) \leq r\}.$$

2.6.2 Distances in Graphs

In a connected graph G, we define the *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ to be the length of the shortest path between u and v. Clearly, d_G defines a metric space on the vertex set $V(G)$. This metric space is usually described by the *distance matrix* $D(G)$ with entry $D_{i,j} = d_G(v_i, v_j)$ for a given ordering $v_1, v_2, ..., v_n$ of the vertices of G. For an arbitrary vertex $v \in V(G)$, we define the *distance sequence* $d_{G,v} = (1, d_1, d_2, \ldots)$ where d_k denotes the number of vertices at distance k from v. Usually, we only consider $d_k > 0$.

Example 2.13. Prism graphs are but one example of graphs in which every vertex has the same distance sequence because for any pair of vertices u and v, there is an automorphism mapping u to v. For instance, for Π_3, we have $d_{(\Pi_3,v)} = (1, 3, 2)$. Similarly, we get $d_{(\Pi_4,v)} = (1, 3, 3, 1), d_{(\Pi_5,v)} = (1, 3, 4, 2)$.

2.6.3 Intersection Graphs

Given a family of sets $\mathcal{B} = \{B_1, B_2, \ldots, B_n\}$, we may define its *intersection graph*. The vertex set is $\mathcal{B}$, and two vertices are adjacent if and only if the corresponding sets have nonempty intersection. We note that there is a variation to this construction, namely, we may construct a general graph by putting $|B_i \cap B_j|$ edges between B_i and B_j.

Example 2.14. Consider the following seven sets in the plane: the three sides of a regular triangle, the three heights, and the inscribed circle. It is not hard to see that the corresponding intersection graph is K_7.

Intersection graphs are universal in the sense that any graph can be represented as an intersection graph. However, by selecting various geometric objects as sets, we get interesting families of graphs. For instance, the so-called *interval graphs* are intersection graphs of finite families of line segments in the $\mathbb{R}^1$ line.

2.6.4 Intersection Graphs of a Family of Balls

Given a set of n points $V = \{v_1, v_2, \ldots, v_n\}$ in some metric space and a positive number $r > 0$, we may draw n closed balls $B_i := B(v_i, r), i = 1, 2, \ldots, n$, each

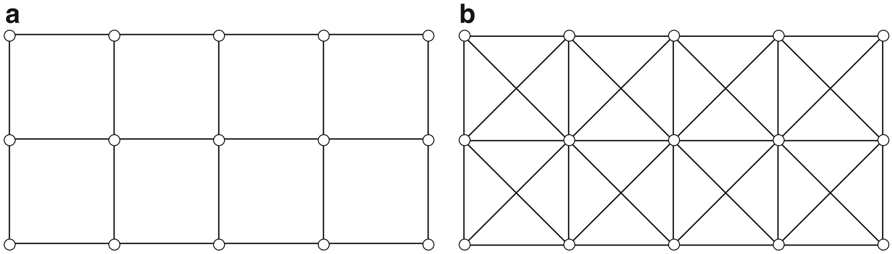


Fig. 2.26 The grid graph $Gr(3,5) = P_3 \square P_5$ and $P_3 \boxtimes P_5$ as unit sphere graphs

ball B_i centered at v_i and having radius r. Define a graph $G(V, r)$ as follows: The vertices are the n selected points. Two vertices v_i and v_j are adjacent if and only if the corresponding balls intersect, i.e., if $B_i \cap B_j \neq \emptyset$. The radius r will be called the *unit* and the graph a *unit sphere graph*.

Here are some specific examples:

Example 2.15. Let us select the following points in the Euclidean plane: (x, y), $x \in \{1, 2, \ldots, a\}$, $y \in \{1, 2, \ldots, b\}$. Hence, $n = ab$. Let $r = 0.5$. The unit sphere graph is the well-known $a \times b$ grid graph $Gr(a, b)$, which we can simply describe as the Cartesian product of the paths P_a and P_b. Figure 2.26a shows the case for $a = 3$ and $b = 5$. If r is increased, there is no change in the structure of the graph until $r = \sqrt{2}/2$, when the diagonals of the 4-cycles appear; see Fig. 2.26b.

It would be interesting and useful to characterize the unit sphere graphs in $\mathbb{R}^2$ and $\mathbb{R}^3$. For instance, all platonic graphs arise as unit sphere graphs in $\mathbb{R}^3$. One has to take the vertices of the corresponding platonic solid and radius r to be one half of the edge length.

Example 2.16. In order to obtain the cube graph Q_3, one can take

$$V = \{(0,0,0), (0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$$

and $r = 1/2$.

This example shows that the cube graph can be described by a careful choice of 8 points in some metric space. There is another approach to this construction. It involves convex sets.

2.6.5 *Convex Sets*

A set of points $K \subseteq \mathbb{R}^3$ is *convex,* if for any two points $x, y \in K$, every point z on the line segment from x to y belongs to K. For any set $S \subseteq \mathbb{R}^3$, we can find the smallest convex set $S \subseteq \text{conv}(S) \subseteq \mathbb{R}^3$, called the *convex closure* or *convex hull* of S.

Example 2.17. The convex closure of the set

$$V = \{(0,0,0),(0,0,1),(0,1,0),(0,1,1),(1,0,0),(1,0,1),(1,1,0),(1,1,1)\}$$

is a cube.

This gives us another general mechanism for constructing graphs from simple geometric objects:

$$\text{Finite set } S \to \text{conv}\,(S) \to \text{skeleton}$$

Starting with a finite set of points in $\mathbb{R}^3$, its convex closure is a convex polyhedron whose 1-skeleton is a graph.

The intersection graph of the seven projective lines B_i of the Fano configuration, see Sect. 1.1.1 of Chap. 1, is K_7 and does not capture the whole combinatorial structure of the configuration. Taking in addition to the sets B_i also all the sets $C_{i,j} = B_i \cap B_j$ that are not empty, the resulting intersection graph captures all the combinatorial structure of the Fano plane. Deleting all edges $B_i B_j$ yields a cubic bipartite graph on 14 vertices still containing all combinatorial information about the Fano configuration. The vertices labeled C may be considered the vertices of the configuration, while the B's are the lines. Edges of the graph indicate what point is on what line or which line goes through which point.

2.6.6 Representations and Drawings of Graphs

Let G be a graph and let S be a set, and let $\mathcal{P}(S)$ denote the *power set* of S, that is, the set of all subsets of S. A pair of mappings

$$\rho_V : V(G) \to S, \qquad \rho_E : E(G) \to \mathcal{P}(S)$$

is called a *graph representation* or an S-representation of the graph G if $\rho_V(v) \in \rho_E(e)$ provided v is incident with e. If there is no fear for confusion, we omit the subscripts of ρ since the argument determines which mapping is considered. We only consider representations for which no pair of vertices is mapped to the same element of S.

Sometimes, we only specify ρ_V and have no need for ρ_E. In such a case, we may tacitly assume that for each $e = uv \in E(G)$, we have $\rho_E(uv) := \{\rho_V(u), \rho_V(v)\}$.

If S is a vector space, the representation is called a *vector representation*. If S is a metric space, the representation is called a *metric representation.* In a metric representation, we define the *length* of each edge $e = uv$ relative to representation ρ as $||e||_\rho = d(\rho(u), \rho(v)) > 0$. In a simple graph G, the length of each edge is strictly positive.

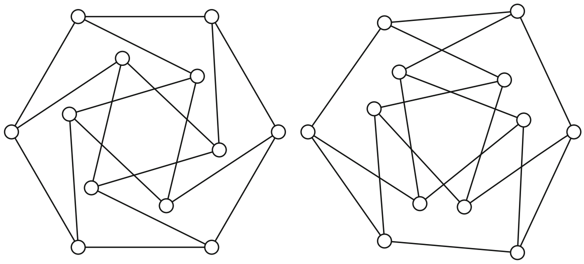


Fig. 2.27 Each generalized Petersen graph is a unit distance graph. In particular, this is true for the Dürer graph GP(6, 2)

$S = \mathbb{R}^n$ is of particular importance to us because real n-space is both a metric and a vector space. An $\mathbb{R}^2$ representation is called *planar* and an $\mathbb{R}^3$ representation is called *spacial representation.* In both cases, we define $\rho_E(uv) := conv(\rho(u), \rho(v))$. Each edge is therefore represented as the line segment connecting the two represented vertices. Such a representation is called *graph drawing.*

Each figure depicting a graph in this book has now a formal description as a graph drawing defined above. We have to define when two drawings are equal (or equivalent). Obviously, we may consider two drawings that differ by an isometry equivalent. But we may also neglect the difference in scale. This means, for instance, we can always set the barycenter to be the origin and set the shortest edge length to be 1 to obtain a "standard" drawing. We define the *energy* of a drawing to be the sum of the lengths of all line segments representing the edges.

The *dilation coefficient* is the quotient between the longest and shortest edge of the drawing.

Graph drawings with dilation coefficient 1 are known as *unit distance graphs*.

2.6.7 Generalized Petersen Graphs as Unit Distance Graphs

All generalized Petersen graphs GP(n, k) can be drawn in the plane as unit distance graphs. We embed the outer rim as a regular polygon with side length 1. We also embed the inner rim as a collection of star polygons of side length 1. If $k = 1$, the inner polygon is congruent with the outer polygon. Translating one polygon by a unit vector yields the appropriate coordinates for the representation. Note that every prism graph Π_n can be drawn in the plane as a unit distance graph. If $k \neq 1$, the radius of the inner rim is different from the radius of the outer rim. This means that if the radii differ by less than 1, we can rotate the inner rim so that the distance between the two adjacent vertices along a spoke becomes 1. The vertices of the outer rim are given the coordinates $\rho(v_i) = (R\cos(\mathrm{i}\pi/n), R\sin(\mathrm{i}\pi/n))$, and the vertices in the inner rim are given the coordinates $\rho(u_i) = (r\cos(\phi + \mathrm{i}\pi/n), r\sin(\phi + \mathrm{i}\pi/n)$, where $R = 1/(2\sin(\pi/n)), r = 1/(2\sin(r\pi/n))$, and $\phi = \arccos((R^2 + r^2 - 1)/(2rR))$. This method works if $R - r < 1$. In particular, the case of the Dürer graph GP(6, 2) is shown in Fig. 2.27.

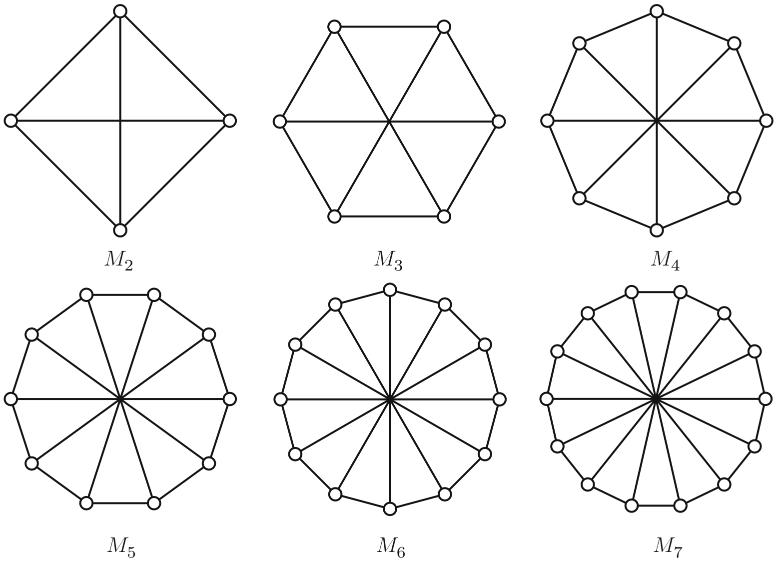


Fig. 2.28 Small Möbius ladders M_n

2.7 Exercises

Exercise 2.1. Consider a regular polygon with n sides in the plane. Use it to define a graph X_n whose vertex set consists of vertices of the polygon and two vertices are adjacent if and only if they belong to the same edge of your polygon. Prove that X_n is isomorphic to the cycle C_n.

Exercise 2.2. Find all independent sets of size greater than 3 in the divisor graph on $\{2, 3, \ldots, 13\}$.

Exercise 2.3. How many connected components does the common divisor graph of Fig. 2.1 have?

Exercise 2.4. The *Möbius ladder* M_n is obtained from the cycle C_{2n} by adding n main diagonals:

$$V = \{v_1, v_2, \ldots, v_{2n}\}$$

$$E = \{v_1v_2, v_2v_3, \ldots, v_{2n-1}v_{2n}, v_{2n}v_1, v_1v_{n+1}, v_2v_{n+2}, \ldots, v_nv_{2n}\}.$$

Prove that the Möbius ladder M_n can be obtained from the prism graph Π_n by deleting and reattaching only two edges (Fig. 2.28).

Exercise 2.5. Draw all nonisomorphic trees on n vertices for $n = 1, 2, 3, 4, 5$.

Exercise 2.6. Show that a graph is a forest if and only if each of its connected components is a tree.

Exercise 2.7. Prove the following "two out of three" theorem for trees.

Let G be a graph on n vertices. Then, any two of the following conditions imply the third:

- G is acyclic.
- G is connected.
- G has $n-1$ edges.

Exercise 2.8. A *fullerene* is a trivalent convex polyhedron whose faces are only pentagons and hexagons. We also call its skeleton by the same name. Prove that the smallest fullerene has 20 vertices.

Exercise 2.9. Prove that any fullerene (see Exercise 2.8) has exactly 12 pentagons.

Exercise 2.10. Prove that there are no fullerenes (see Exercise 2.8) on 22 vertices.

Exercise 2.11. Find all fullerenes (see Exercise 2.8) among the generalized Petersen graphs.

Exercise 2.12. Find all fullerenes (see Exercise 2.8) among the Platonic and Archimedean graphs.

Exercise 2.13. Consider the vertices and edges of the tetrahedron graph. Say that a vertex v is *across* from any edge which is part of a 3-cycle not containing v. Further, say that two edges are *across* from one another if they do not belong to any common 3-cycle. Define a graph on the ten vertices and edges of the tetrahedron where adjacency is determined by across. Show that this graph is isomorphic to the Petersen graph.

Exercise 2.14. Prove directly that the generalized Petersen graphs $\mathrm{GP}(7,2)$ and $\mathrm{GP}(7,3)$ are isomorphic.

Exercise 2.15. Prove that there is an isomorphism between $\mathrm{GP}(n,r)$ and $\mathrm{GP}(n,s)$ preserving the outer n-gon if and only if $r \equiv \pm s \pmod{n}$.

Exercise 2.16. Prove that the generalized Petersen graphs $\mathrm{GP}(8,2)$ and $\mathrm{GP}(8,3)$ are not isomorphic.

Exercise 2.17. Decide whether each of the generalized Petersen graphs pictured in Figs. 2.15 and 2.16 is planar.

Exercise 2.18. Complete the argument in Theorem 2.2 in the case of even girth. Note that the base of the construction does not have to be a single vertex.

Exercise 2.19. What is the girth of the graph in Exercise 2.34?

Exercise 2.20. Determine the size of the smallest cubic bipartite graph of girth larger than 4 and construct an example.

Exercise 2.21. Prove that the girth of the Petersen graph $\mathrm{GP}(5,2)$ is 5.

Exercise 2.22. Determine all generalized Petersen graphs $\mathrm{GP}(n,r)$ of girth 5.

Exercise 2.23. Prove the following result. A graph is bipartite if and only if it contains no cycles of odd length.

Exercise 2.24. Show that if G has $c+1$ nonempty bipartite connected components, there are 2^c bipartitions of the vertex set.

Exercise 2.25. Write an LCF code for the Dürer graph.

Exercise 2.26. Write an LCF code for K_4.

Exercise 2.27. Show that M_n (defined in Exercise 2.4) admits a description via LCF notation. Show that it is isomorphic to the graph $[(n)^{2n}]$.

Exercise 2.28. Show that the Heawood graph admits a description via LCF notation. Show that it is isomorphic to the graph $[(5,-5)^7]$.

Exercise 2.29. Prove that the Heawood graph has no cycles of length less than 6.

Exercise 2.30. Show that the Tutte 8-cage is isomorphic to the graph

$$[(-7, 9, 13, -13, -9, 7)^5].$$

Exercise 2.31. Show that the Balaban 10-cage is Hamiltonian and find an LCF notation for it.

Exercise 2.32. Given the parameters $|V_1|, |V_2|, k_1, k_2$, satisfying $|V_1|k_1 = |V_2|k_2$, $k_1 \leq |V_2|$, and $k_2 \leq |V_1|$, construct a semiregular bipartite graph $G = (V_1 \cup V_2; k_1, k_2)$. Hint: Let the ith vertex of V_1 be adjacent to vertices $\{i, i+1, \ldots, i+k_1-1 \pmod{|V_2|}\}$.

Exercise 2.33. Formulate and prove a structure theorem analogous to Theorem 2.5 for semiregular bipartite graphs.

Exercise 2.34. Here is a table for the Fano configuration:

1	1	1	2	2	3	3
2	4	6	4	5	4	5
3	5	7	6	7	7	6

Draw the corresponding regular bipartite graph and rewrite the table to reflect the partition into 1-factors.

Exercise 2.35. Prove that the Möbius ladder M_n is bipartite if and only if n is odd.

Exercise 2.36. Redraw the polyhedral graph of the cube, Fig. 2.12, coloring the vertices to indicate the tripartition. Make your drawing as symmetrical as possible.

Exercise 2.37. Write down all permutations in $\mathrm{Sym}(3)$.

Exercise 2.38. Write down all involutions in $\mathrm{Sym}(4)$.

Exercise 2.39. Write down all fixed-point free permutations in $\mathrm{Sym}(5)$. In other words, determine the set $\mathrm{Der}(5)$.

Exercise 2.40. Write down all fixed-point free involutions in Sym(4) and in Sym(5).

Exercise 2.41. Determine the number of semiregular permutations in Sym(2006).

Exercise 2.42. Let p be a prime. Show that the number of semiregular permutations in Sym(p) equals $(p-1)!$

Exercise* 2.43. Determine the number of semiregular permutations in Sym(n).

Exercise* 2.44. Determine the number of derangements in Der(n).

Exercise* 2.45. Determine the number of involutions in Sym(n).

Exercise* 2.46. Determine the number of fixed-point free involutions in Sym(n).

Exercise 2.47. Express the order of a permutation in terms of its cycle structure.

Exercise 2.48. The definition of a cycle C_n in Sect. 2.2.2 applies to cycles with $n \geq 3$. Define C_1 (the loop) and C_2 as general graphs (see 2.3.6).

Exercise 2.49. Prove that there is—up to isomorphism—only one cubic graph on 4 vertices with 3 loops.

Exercise 2.50. Define the notion of isomorphism for general graphs and pregraphs.

Exercise 2.51. Show that the wheel graph W_n is isomorphic to the cone over C_n.

Exercise 2.52. Show that $K_{2,2,2}$ is isomorphic to the suspension over C_4.

Exercise 2.53. Show that K_{n+1} is isomorphic to the cone over K_n.

Exercise 2.54. Show that the prism graph Π_n is isomorphic to the Cartesian product $K_2 \square C_n$.

Exercise 2.55. Show that C_4 can be expressed using only single vertex graphs and the operations of graph union $\cup$ and graph join $*$.

Exercise 2.56. Show that a graph G can be expressed using only single vertex graphs and the operations of $\cup$ and $*$ if and only if G has no induced subgraph isomorphic to P_4.

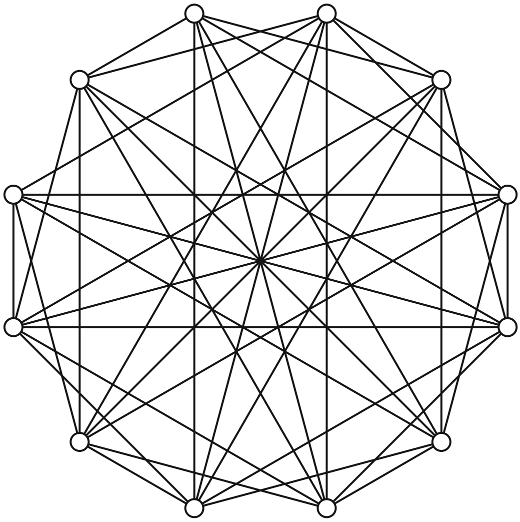
Exercise 2.57. Show that the Cartesian product $\square$ is associative.

Exercise 2.58. We defined Q_n, the hypercube in dimension n, as Cartesian product of n factors equal to K_2. Show that Q_n may also be defined as follows: The vertex set of Q_n consists of n-tuples of 0's and 1's. Two vertices are adjacent if they differ in exactly one coordinate.

Exercise 2.59. Show that $G_1 \square G_2$ is connected if and only if both G_1 and G_2 are connected.

Exercise 2.60. Show that if G_1 and G_2 are both connected, then $G_1 \square G_2$ is 2-connected.

Fig. 2.29 $(\mathrm{GP}(6,2)^2)^c$



Exercise 2.61. Given the valence of a vertex $v_1 \in V(G_1)$ and the valence of $v_2 \in V(G_2)$, what can you say about the valence of $(v_1, v_2) \in V(G_1 \square G_2)$?

Exercise 2.62. Prove that the strong product of any two paths is a unit sphere graph.

Exercise 2.63. Show that $K_{m,n} = K_n^c * K_m^c$.

Exercise 2.64. Show that K_4, $K_{2,2,2}$, Q_3, and $\mathrm{GP}(10,2)$ are four out of the five Platonic graphs. Design a graph theoretical method for constructing the missing icosahedron graph.

Exercise 2.65. Show that the graph in Fig. 2.29 is indeed the complement of the square of the Dürer graph.

Exercise 2.66. Show that all hypercube graphs are Hamiltonian.

Exercise 2.67. A graph is called a *benzenoid graph* if it can be obtained by selecting a connected subset of hexagons in an infinite planar hexagonal lattice (representing graphene). Show that all benzenoid graphs can be described as unit sphere graphs in the plane.

Note that in theoretical chemistry, a benzenoid graph is sometimes defined in various slightly different ways. Benzenoid graphs represent molecules of polyhexes, i.e., polycyclic aromatic hydrocarbons. Vertices correspond to carbon atoms, edges to the carbon–carbon bonds, while hydrogen atoms are not shown.

Exercise 2.68. Let $K(X)$ denote the number of 1-factors in a graph G. Show that benzene has two 1-factors: $K(B_1) = 2$.

Exercise 2.69. Prove that a bipartite graph with bipartition sets of unequal size has no 1-factor. Use this result to show that $K(B_5) = 0$. Note that in chemistry, a 1-

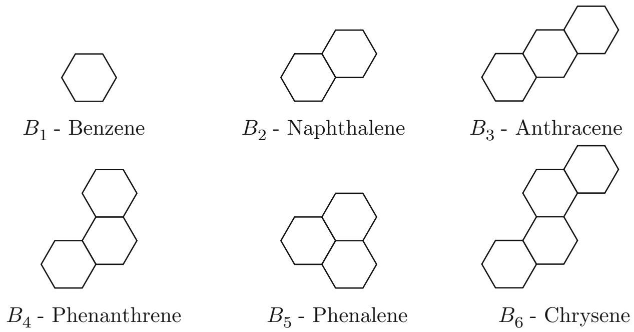
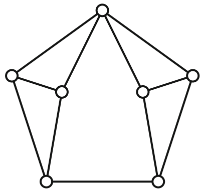


Fig. 2.30 Small benzenoid graphs

Fig. 2.31 The Moser graph



factor is called a *Kekulé structure*. It is known that polyhex hydrocarbons without Kekulé structures are extremely unstable, for instance, phenalene.

Exercise 2.70. Determine the number of Kekulé structures $K(B_n)$ for the benzenoid graphs in Fig. 2.30.

Exercise 2.71. Prove that every graph is an intersection graph of some family of sets. In particular, determine a family of sets whose intersection graph is isomorphic to $\mathrm{GP}(5,2)$.

Exercise 2.72. Determine the coordinates for the vertices of a 4-valent convex polyhedron whose skeleton is isomorphic to the cubeoctahedron graph.

Exercise 2.73. Determine the chromatic number for each Archimedean graph.

Exercise 2.74. Find the minimal dilation coefficient of any planar representation of the Moser graph in Fig. 2.31.

Exercise 2.75. It is easy to verify that K_4 is not a unit distance graph in the plane. Consider a drawing of K_4 in the plane with only two distinct edge lengths. How many such nonisomorphic drawings are there? (Hint: there are six). Compute the dilation coefficient for all such drawings.

Exercise 2.76. Show that there exists a spatial drawing of Q_3 with dilation coefficient 1. Is it unique?

Exercise* 2.47. Find a planar drawing of a generalized Petersen graph $\mathrm{GP}(n,r)$ with dilation coefficient 1.

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