Chapter 2
Single Delay Case

In this chapter we consider the class of retarded type linear systems with one delay. There are several reasons for restricting our attention to this class before proceeding to more general ones. First, from a methodological point of view, it seems that dealing with single-delay systems simplifies the understanding of basic concepts and creates a firm basis for developing the concepts for more general cases. Second, for the case of single delay we often obtain more complete results than in a more general setting. Finally, results for the single-delay case are not as cumbersome as those for the more general classes of time-delay systems.

We introduce the fundamental matrix of such a system and provide an explicit expression for the solution of an initial value problem. Exponential stability conditions, both in terms of characteristic eigenvalues of the system and in terms of Lyapunov functionals, are presented. The general scheme for the computation of quadratic functionals with prescribed time derivatives along the solutions of a time-delay system is explained in detail. It is demonstrated that the functionals are defined by special matrix-valued functions. We show that these matrix-valued functions are natural counterparts of the classical Lyapunov matrices that appear in the computation of Lyapunov quadratic forms for a delay-free linear system; therefore, they are known as Lyapunov matrices for a time-delay system. A substantial part of the chapter is devoted to an analysis of the basic properties of the Lyapunov matrices. Then, Lyapunov functionals that admit various quadratic lower and upper bounds are introduced. They are called functionals of the complete type. Finally, we make use of complete type functionals to derive exponential estimates of the solutions of time-delay systems, robustness bounds for perturbed systems, evaluation of quadratic performance indices, and computation of critical values of system parameters.

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2.1 Preliminaries

We consider a retarded type time-delay system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0x(t) + A_1x(t-h), \quad t \geq 0. \tag{2.1}$$

Here A_0, A_1 are given real $n \times n$ matrices, and h is a positive time delay.

Let $\varphi : [-h, 0] \to R^n$ be an initial function. We assume that the function belongs to the space, $PC([-h,0], R^n)$, of piecewise continuous functions defined on the segment $[-h,0]$. Let $x(t,\varphi)$ stand for the solution of system (2.1) under the initial condition

$$x(\theta,\varphi) = \varphi(\theta), \quad \theta \in [-h,0],$$

and let $x_t(\varphi)$ denote the restriction of the solution to the segment $[t-h,t]$

$$x_t(\varphi) : \theta \to x(t+\theta,\varphi), \quad \theta \in [-h,0].$$

We omit the argument φ in these notations and write $x(t)$ and x_t instead of $x(t,\varphi)$ and $x_t(\varphi)$ where no confusion may arise.

Recall that the Euclidean norm is used for vectors and the induced matrix norm for matrices. For elements of the space $PC([-h,0], R^n)$ we use the uniform norm

$$\|\varphi\|_h = \sup_{\theta \in [-h,0]} \|\varphi(\theta)\|.$$

The principal objective of this section is to present an explicit expression for the solutions of system (2.1) in terms of their initial functions.

2.1.1 Fundamental Matrix

The key element needed to derive this expression is the fundamental matrix of the system. In many respects, the fundamental matrix plays the same role for system (2.1) as the matrix exponent does for linear delay-free systems.

Definition 2.1 ([3]). It is said that the $n \times n$ matrix $K(t)$ is the fundamental matrix of system (2.1) if it satisfies the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = K(t)A_0 + K(t-h)A_1, \quad t \geq 0, \tag{2.2}$$

and $K(t) = 0_{n\times n}$ for $t < 0$, $K(0) = I$. Here I is the identity matrix.

Remark 2.1. The fundamental matrix also satisfies the matrix equation

$$\frac{\mathrm{d}}{\mathrm{d}t}K(t) = A_0K(t) + A_1K(t-h), \quad t \geq 0.$$

This does not mean that matrix $K(t)$ commutes individually with the coefficient matrices A_k, $k = 0, 1$.

Proof. To verify this remark, it is sufficient to compare the Laplace image of the fundamental matrix as a solution of matrix Eq. (2.2) with that of the matrix equation given in the remark. □

2.1.2 Cauchy Formula

Now we are ready to present the main result of this section.

Theorem 2.1 ([3]). *Given an initial function $\varphi \in PC([-h,0],R^n)$, the following equality holds:*

$$x(t,\varphi) = K(t)\varphi(0) + \int_{-h}^{0} K(t-\theta-h)A_1\varphi(\theta)\mathrm{d}\theta, \quad t \geq 0. \tag{2.3}$$

This expression for $x(t,\varphi)$ is known as the Cauchy formula.

Proof. Let $t > 0$ and assume that $\xi \in (0,t)$; then

$$\begin{aligned}\frac{\partial}{\partial \xi}\left[K(t-\xi)x(\xi,\varphi)\right] = &-\left[K(t-\xi)A_0 + K(t-\xi-h)A_1\right]x(\xi,\varphi) \\ &+K(t-\xi)\left[A_0x(\xi,\varphi) + A_1x(\xi-h,\varphi)\right].\end{aligned}$$

Integrating the last equality by ξ from 0 to t we obtain that

$$x(t,\varphi) - K(t)\varphi(0) = -\int_0^t K(t-\xi-h)A_1x(\xi,\varphi)\mathrm{d}\xi + \int_0^t K(t-\xi)A_1x(\xi-h,\varphi)\mathrm{d}\xi.$$

The second integral on the right-hand side of the preceding equality can be transformed as follows:

$$\int_0^t K(t-\xi)A_1x(\xi-h,\varphi)\mathrm{d}\xi = \langle \theta = \xi - h \rangle = \int_{-h}^{t-h} K(t-\theta-h)A_1x(\theta,\varphi)\mathrm{d}\theta.$$

Since the matrix $K(\theta) = 0_{n\times n}$ for $\theta \in [-h, 0)$, the upper limit of the integral on the right-hand side can be increased up to t,

$$\int_{-h}^{t-h} K(t-\theta-h)A_1 x(\theta,\varphi)\mathrm{d}\theta = \int_{-h}^{t} K(t-\theta-h)A_1 x(\theta,\varphi)\mathrm{d}\theta.$$

As a result we arrive at the equality

$$x(t,\varphi) = K(t)\varphi(0) + \int_{-h}^{0} K(t-\theta-h)A_1 x(\theta,\varphi)\mathrm{d}\theta, \quad t \geq 0.$$

As $x(\theta,\varphi) = \varphi(\theta)$ for $\theta \in [-h, 0]$, the preceding equality coincides with (2.3). □

2.2 Exponential Stability

We now introduce the stability concept that will be used for system (2.1) in the remainder of this part of the book.

Definition 2.2 ([3]). System (2.1) is said to be exponentially stable if there exist $\gamma \geq 1$ and $\sigma > 0$ such that any solution $x(t,\varphi)$ of the system satisfies the inequality

$$\|x(t,\varphi)\| \leq \gamma \mathrm{e}^{-\sigma t} \|\varphi\|_h, \quad t \geq 0. \tag{2.4}$$

Remark 2.2. It is well known that the exponential stability of system (2.1) is equivalent to the asymptotic stability of the system; see [3].

Definition 2.3. A complex number s_0 is said to be an eigenvalue of system (2.1) if it is a root of the characteristic function,

$$f(s) = \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right),$$

of the system. The set

$$\Lambda = \{s \mid f(s) = 0\}$$

is known as the spectrum of the system.

The next statement shows that the property of exponential stability depends on the location of the spectrum of system (2.1).

Theorem 2.2 ([3]). *System (2.1) is exponentially stable if and only if the spectrum of the system lies in the open left half-plane of the complex plane,*

$$Re(s_0) < 0, \quad s_0 \in \Lambda.$$

The following result is a simplified version of the Krasovskii theorem 1.8. It provides sufficient conditions for the exponential stability of system (2.1).

Theorem 2.3. *System (2.1) is exponentially stable if there exists a functional*

$$v : PC([-h,0],R^n) \to R$$

such that the following conditions hold.

1. *For some positive* α_1, α_2

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

2. *For some* $\beta > 0$ *the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) \leq -\beta \|x(t)\|^2, \quad t \geq 0,$$

holds along the solutions of the system.

2.3 Problem Formulation

Motivated by Theorem 2.3 we address in this section the construction of quadratic functionals that satisfy the theorem conditions. Our approach is based on one of the principal ideas of the direct Lyapunov method, which can be formulated in our case as follows. First, select a time derivative, and then compute the functional whose time derivative along the solution of system (2.1) coincides with the selected one. Since the system is linear and time invariant, it seems natural to start with the case where the time derivative is a quadratic form. One technical assumption needed at the beginning, and which will be dropped later, is that system (2.1) is exponentially stable.

Problem 2.1. Let system (2.1) be exponentially stable. Given a quadratic form $w(x) = x^T W x$, find a functional $v_0(\varphi)$, defined on $PC([-h,0],R^n)$, such that along the solutions of the system the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0(x_t) = -x^T(t) W x(t), \quad t \geq 0. \tag{2.5}$$

2.4 Delay-Free Case

Here we would like to explain, using the case of a delay-free system, some principal ideas behind our approach to the solution of Problem 2.1. Let us consider an exponentially stable system of the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = Ax. \tag{2.6}$$

By $x(t,x_0)$ we denote the solution of the system with a given initial condition $x(0,x_0) = x_0$. If the initial condition is not important, then we will use the shorthand notation $x(t)$ instead of $x(t,x_0)$.

Given a quadratic form $w(x) = x^T W x$, we are looking for a Lyapunov function $v(x)$ such that the following equality holds:

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}v(x)\right|_{(2.6)} = -w(x). \tag{2.7}$$

Control theory textbooks, see [24, 30, 71], teach us that the desired Lyapunov function is also a quadratic form, $v(x) = x^T V x$. The function satisfies the preceding equality if matrix V is a solution of the classical Lyapunov matrix equation

$$A^T V + VA = -W. \tag{2.8}$$

The evident simplicity of the construction of the Lyapunov function $v(x)$ is achieved due to the valuable information that the function is a quadratic form. Let us assume now that this information is not available. This is exactly the situation that we have now in Problem 2.1. The question is: Is it possible to reveal the form of the Lyapunov function during the construction of $v(x)$? To answer this question, we address ourselves to the Eq. (2.7). First, we observe that if we substitute into (2.7) a solution of system (2.6), then the equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}v(x(t)) = -w(x(t)).$$

It is evident that the preceding equation defines the function $v(x(t))$ up to an additive constant. To define the constant correctly, we recall that the desired Lyapunov function $v(x)$ should be equal to zero for $x = 0$, $v(0) = 0$. Now we integrate the last equality on the segment $[0,T]$, where $T > 0$,

$$v(x(T,x_0)) - v(x_0) = -\int_0^T x^T(t,x_0) W x(t,x_0)\mathrm{d}t.$$

Since system (2.6) is exponentially stable, $x(T,x_0) \to 0$ and $v(x(T,x_0)) \to 0$, as $T \to \infty$. At the limit we obtain the equality

$$v(x_0) = \int_0^\infty x^T(t,x_0)Wx(t,x_0)\mathrm{d}t. \tag{2.9}$$

The improper integral on the right-hand side of the preceding equality converges due to exponential stability of system (2.6). It is well known that

$$x(t,x_0) = \mathrm{e}^{At}x_0.$$

Replacing $x(t,x_0)$ in (2.9) by this expression we arrive at the equality

$$v(x_0) = x_0^T \left(\int_0^\infty \mathrm{e}^{A^T t} W \mathrm{e}^{At} \mathrm{d}t \right) x_0,$$

which demonstrates that the Lyapunov function is a quadratic form, $v(x) = x^T V x$, with the matrix

$$V = \int_0^\infty \mathrm{e}^{A^T t} W \mathrm{e}^{At} \mathrm{d}t. \tag{2.10}$$

It is a matter of simple calculation to verify that the matrix satisfies Eq. (2.8).

We summarize now the essential elements of the presented construction process. First, the exponential stability assumption allows us to justify formula (2.9). Second, the explicit expression of the solutions of system (2.6) allows us to clarify the form of the desired function. Finally, we see that there is no need to evaluate the improper integral (2.10) to compute the matrix since the matrix Eq. (2.8) serves the same purpose.

It will be shown that this approach to the construction of Lyapunov functions can be extended to the case of time-delay systems in the sense that it is possible to compute a Lyapunov functional that solves Problem 2.1.

2.5 Computation of $v_0(\varphi)$

In this section an important step toward the construction of quadratic functionals that satisfy Theorem 2.3 will be made. That is, we derive an explicit formula for a functional that solves Problem 2.1. We also introduce a Lyapunov matrix that defines the functional.

Equation (2.5) defines the functional $v_0(\varphi)$ up to an additive constant. It follows from the first condition of Theorem 2.3 that this additive constant should be set in such a way that for the trivial initial function $0_h \in PC([-h,0],R^n)$, $v_0(0_h)=0$. Integrating Eq. (2.5) from $t=0$ to $t=T>0$ we obtain

$$v_0(x_T(\varphi)) - v_0(\varphi) = -\int_0^T x(t,\varphi)Wx(t,\varphi)\mathrm{d}t.$$

Since system (2.1) is exponentially stable, $x_T(\varphi) \to 0_h$ as $T \to \infty$, and we arrive at the expression

$$v_0(\varphi) = \int_0^\infty x^T(t,\varphi)Wx(t,\varphi)\mathrm{d}t, \quad \varphi \in PC([-h,0],R^n).$$

The exponential stability of system (2.1) implies that the improper integral on the right-hand side of the preceding equality is well defined. If we replace $x(t,\varphi)$ under the integral sign by Cauchy formula (2.3), then

$$\begin{aligned} v_0(\varphi) = & \int_0^\infty \left[K(t)\varphi(0) + \int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W \\ & \times \left[K(t)\varphi(0) + \int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t \\ = & \, \varphi^T(0)\left[\int_0^\infty K^T(t)WK(t)\mathrm{d}t\right]\varphi(0) \\ & + 2\varphi^T(0)\int_0^\infty K^T(t)W\left[\int_{-h}^0 K(t-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta\right]\mathrm{d}t \\ & + \int_0^\infty \left[\int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W\left[\int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t. \end{aligned}$$

Here for the first time we encounter the matrix

$$U(\tau) = \int_0^\infty K^T(t)WK(t+\tau)\mathrm{d}t, \tag{2.11}$$

which will play a crucial role in the following study. Since the columns of the fundamental matrix $K(t)$ are solutions of (2.1) with specific initial conditions, it is easy to verify that the matrix admits an upper exponential estimate of the form

$$\|K(t)\| \leq \gamma e^{-\sigma t}, \quad t \geq 0. \tag{2.12}$$

This estimate guarantees that the matrix $U(\tau)$ is well defined for $\tau \in R$.

Lemma 2.1. *Given $\tau_0 \in R$, the improper integral (2.11) converges absolutely and uniformly with respect to $\tau \in [\tau_0, \infty)$.*

Proof. Given $\tau_0 \in R$, it follows directly from (2.12) that

$$\left\|K^T(t)WK(t+\tau)\right\| \leq \gamma^2 \|W\| e^{-\sigma(2t+\tau)}, \quad t \geq 0.$$

Now, let $\tau \in [\tau_0, \infty)$; then the inequality

$$\int_0^\infty \left\|K^T(t)WK(t+\tau)\right\| dt \leq \frac{\gamma^2}{2\sigma} \|W\| e^{-\sigma\tau_0}$$

proves the statement. □

We will demonstrate now that matrix (2.11) allows us to present the functional $v_0(\varphi)$ in a form more suitable for subsequent analysis. To begin with, we observe that the first term of the functional can be written as

$$R_1 = \varphi^T(0) \left[\int_0^\infty K^T(t)WK(t)dt\right] \varphi(0) = \varphi^T(0)U(0)\varphi(0).$$

Lemma 2.1 justifies the following change in the integration order in the second term:

$$\begin{aligned} R_2 &= 2\varphi^T(0) \int_0^\infty K^T(t)W \left[\int_{-h}^0 K(t-h-\theta)A_1\varphi(\theta)d\theta\right] dt \\ &= 2\varphi^T(0) \int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)d\theta. \end{aligned}$$

To present the last term in a similar form, we consider the integral

$$J = \int_0^\infty K^T(t-\tau_1)WK(t-\tau_2)dt,$$

where τ_1 and τ_2 are two nonnegative constants. This integral can be transformed as follows:

$$J = \int_0^\infty K^T(t-\tau_1)WK(t-\tau_2)\mathrm{d}t = \int_{-\tau_1}^\infty K^T(\xi)WK(\xi+\tau_1-\tau_2)\mathrm{d}\xi$$
$$= \int_{-\tau_1}^0 K^T(\xi)WK(\xi+\tau_1-\tau_2)\mathrm{d}\xi + U(\tau_1-\tau_2).$$

Since $K(\xi)=0_{n\times n}$, for $\xi\in[-\tau_1,0)$, the first summand on the right-hand side of the preceding line of equalities disappears, and we obtain

$$J = U(\tau_1-\tau_2).$$

Now, based on Lemma 2.1, we present the last term in the form

$$R_3 = \int_0^\infty \left[\int_{-h}^0 K(t-h-\theta_1)A_1\varphi(\theta_1)\mathrm{d}\theta_1\right]^T W \left[\int_{-h}^0 K(t-h-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}t$$
$$= \int_{-h}^0 \varphi^T(\theta_1)A_1^T \left[\int_{-h}^0 \left(\int_0^\infty K^T(t-h-\theta_1)WK(t-h-\theta_2)\mathrm{d}t\right)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$
$$= \int_{-h}^0 \varphi^T(\theta_1)A_1^T \left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1.$$

These transformations show that the quadratic functional $v_0(\varphi)$ can be written as

$$v_0(\varphi) = \varphi^T(0)U(0)\varphi(0) + 2\varphi^T(0)\int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta$$
$$+ \int_{-h}^0 \varphi^T(\theta_1)A_1^T\left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1. \qquad (2.13)$$

Observe that all the terms on the right-hand side of (2.13) depend on the matrix $U(\tau)$. This is the first, but not the last, reason to call this matrix a Lyapunov matrix of system (2.1).

Definition 2.4. Matrix (2.11) is called a Lyapunov matrix of system (2.1) associated with matrix W.

In the following sections the Lyapunov matrices of system (2.1) will be studied in detail. Here we prove only that they are continuous functions of τ.

Lemma 2.2. *Lyapunov matrix (2.11) depends continuously on $\tau \geq 0$.*

Proof. The statement is a direct consequence of Lemma 2.1 and the fact that $K(t)$ is continuous for $t \geq 0$. $\square$

2.6 Lyapunov Matrices: Basic Properties

As was mentioned at the end of the previous section, matrix (2.11) depends continuously on τ. In this section we launch a more detailed analysis of the basic properties of the matrix. Some of the properties will allow us to provide a new definition of the matrix.

Lemma 2.3. *The Lyapunov matrix $U(\tau)$ satisfies the dynamic property*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) = U(\tau)A_0 + U(\tau - h)A_1, \quad \tau \geq 0. \tag{2.14}$$

Proof. Let $t \geq 0$ and $\tau > 0$; then

$$\frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] = K^T(t)W\left[K(t+\tau)A_0 + K(t+\tau-h)A_1\right].$$

The exponential stability of system (2.1) implies that

$$\begin{aligned}
\left\| \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] \right\| &\leq \|K(t)\| \, \|W\| \, \|K(t+\tau)\| \, \|A_0\| \\
&\quad + \|K(t)\| \, \|W\| \, \|K(t+\tau-h)\| \, \|A_1\| \\
&\leq \gamma^2 \mathrm{e}^{-\sigma(2t+\tau)} \|W\| \left(\|A_0\| + \mathrm{e}^{\sigma h} \|A_1\| \right) \\
&\leq \gamma^2 \mathrm{e}^{\sigma h} \mathrm{e}^{-2\sigma t} \|W\| \left(\|A_0\| + \|A_1\| \right).
\end{aligned}$$

On the one hand, since the integral

$$\int_0^\infty \gamma^2 \mathrm{e}^{\sigma h} \mathrm{e}^{-2\sigma t} \|W\| \left(\|A_0\| + \|A_1\| \right) \mathrm{d}t$$

converges, the integral

$$\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] \mathrm{d}t$$

converges absolutely and uniformly with respect to $\tau \geq 0$, which in turn implies the equality

$$\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] \mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\int_0^\infty K^T(t)WK(t+\tau)\mathrm{d}t\right) = \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}.$$

On the other hand,

$$\begin{aligned}\int_0^\infty \frac{\partial}{\partial \tau}\left[K^T(t)WK(t+\tau)\right] \mathrm{d}t &= \left(\int_0^\infty K^T(t)WK(t+\tau)\mathrm{d}t\right)A_0 \\ &+ \left(\int_0^\infty K^T(t)WK(t+\tau-h)\mathrm{d}t\right)A_1 = U(\tau)A_0 + U(\tau-h)A_1,\end{aligned}$$

and we arrive at the dynamic property (2.14). □

Lemma 2.4. *A Lyapunov matrix satisfies the symmetry property*

$$U(-\tau) = U^T(\tau), \quad \tau \geq 0. \tag{2.15}$$

Proof. The matrix

$$\begin{aligned}U(-\tau) &= \int_0^\infty K^T(t)WK(t-\tau)\mathrm{d}t = \int_{-\tau}^\infty K^T(\xi+\tau)WK(\xi)\mathrm{d}\xi \\ &= \int_{-\tau}^0 K^T(\xi+\tau)WK(\xi)\mathrm{d}\xi + U^T(\tau).\end{aligned}$$

Since the matrix $K(\xi) = 0_{n\times n}$, $\xi \in [-\tau, 0)$, the integral term on the right-hand side of the preceding line of equalities disappears, and we arrive at (2.15). □

Corollary 2.1. *It follows from (2.15) that matrix $U(0)$ is symmetric, $U^T(0) = U(0)$.*

Corollary 2.2. *Lyapunov matrix (2.11) is infinitely many times differentiable for $\tau \in (0, h)$.*

Indeed, the symmetry property (2.15) and Lemma 2.2 imply that the initial condition for matrix $U(\tau)$ as a solution of Eq. (2.14) is continuous, so the Lyapunov matrix is continuously differentiable. This means, in turn, that the initial condition is continuously differentiable, so the Lyapunov matrix is two times continuously differentiable. It is evident that this process may be continued up to infinity.

Lemma 2.5. *The Lyapunov matrix satisfies the algebraic property*

$$U(0)A_0 + U(-h)A_1 + A_0^T U(0) + A_1^T U(h) = -W. \tag{2.16}$$

Proof. We first differentiate the product

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[K^T(t)WK(t)\right] &= \left[K(t)A_0 + K(t-h)A_1\right]^T WK(t)\\ &\quad + K^T(t)W\left[K(t)A_0 + K(t-h)A_1\right], \quad t \geq 0,\end{aligned}$$

and then, integrating the preceding equality from $t=0$ to $t=\infty$, we get

$$\begin{aligned}-W &= \int_0^\infty \left[K(t)A_0 + K(t-h)A_1\right]^T WK(t)\mathrm{d}t\\ &\quad + \int_0^\infty K^T(t)W\left[K(t)A_0 + K(t-h)A_1\right]\mathrm{d}t\\ &= A_0^T U(0) + A_1^T U(h) + U(0)A_0 + U(-h)A_1.\end{aligned}$$

□

The symmetry property indicates that the first derivative of the Lyapunov matrix suffers discontinuity at the point $\tau = 0$.

Lemma 2.6. *Algebraic property (2.16) can be written in the form*

$$U'(+0) - U'(-0) = -W.$$

Here $U'(+0)$ and $U'(-0)$ stand respectively for the right-hand-side and the left-hand-side derivatives of the matrix $U(\tau)$ at $\tau = 0$.

Proof. Observe first that

$$U'(+0) = \lim_{\tau \to +0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = U(0)A_0 + U(-h)A_1.$$

Differentiating the symmetry property we first get

$$\frac{\mathrm{d}U(-\tau)}{\mathrm{d}\tau} = \left[\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right]^T, \quad \tau > 0$$

and then

$$\lim_{\tau \to -0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} = -\left[U(0)A_0 + U(-h)A_1\right]^T.$$

Now the statement of the lemma is a simple consequence of the equality

$$\begin{aligned}\lim_{\tau \to +0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} - \lim_{\tau \to -0} \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} &= U(0)A_0 + U(-h)A_1 \\ &\quad + \left[U(0)A_0 + U(-h)A_1\right]^T\end{aligned}$$

and algebraic property (2.16). □

Despite the fact that functional (2.13) was computed from Eq. (2.5), it seems to be instructive to demonstrate directly that the functional satisfies the equation.

Theorem 2.4. *Functional (2.13), with $U(\tau)$ given by (2.11), satisfies Eq. (2.5).*

Proof. Let $x(t)$ be a solution of system (2.1); then

$$\begin{aligned} v_0(x_t) &= \underbrace{x^T(t)U(0)x(t)}_{R_1(t)} + \underbrace{2x^T(t)\int_{-h}^{0} U(-h-\theta)A_1 x(t+\theta)\mathrm{d}\theta}_{R_2(t)} \\ &\quad + \underbrace{\int_{-h}^{0} x^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1 x(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1}_{R_3(t)}. \end{aligned}$$

We will differentiate the summands on the right-hand side of the preceding equality one by one.

It is easy to see that for the first term

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}R_1(t) &= 2x^T(t)U(0)\left[A_0x(t) + A_1x(t-h)\right] \\ &= x^T(t)\left[U(0)A_0 + A_0^T U(0)\right]x(t) + 2x^T(t)U(0)A_1x(t-h).\end{aligned}$$

By a simple change of the integration variable we present the second term in a form more suitable for subsequent differentiation:

$$R_2(t) = 2x^T(t)\int_{t-h}^{t} U(-h-\xi+t)A_1 x(\xi)\mathrm{d}\xi.$$

Then,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_2(t) &= 2\left[A_0x(t)+A_1x(t-h)\right]^T \int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t)U(-h)A_1x(t) - 2x^T(t)U(0)A_1x(t-h) \\
&\quad +2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}U(-h-\xi+t)\right)A_1x(\xi)\mathrm{d}\xi \\
&= 2x^T(t)A_0^T\int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t-h)A_1^T\int_{t-h}^{t} U(-h-\xi+t)A_1x(\xi)\mathrm{d}\xi \\
&\quad +2x^T(t)\int_{t-h}^{t}\left(\frac{\partial}{\partial t}U(-h-\xi+t)\right)A_1x(\xi)\mathrm{d}\xi \\
&\quad +x^T(t)\left[U(-h)A_1+A_1^TU(h)\right]x(t) \\
&\quad -2x^T(t)U(0)A_1x(t-h).
\end{aligned}
$$

Applying to the last term a similar change of the integration variables we have

$$
R_3(t) = \int_{t-h}^{t} x^T(\xi_1)A_1^T\left[\int_{t-h}^{t} U(\xi_1-\xi_2)A_1x(\xi_2)\mathrm{d}\xi_2\right]\mathrm{d}\xi_1.
$$

The time derivative of the term is

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) &= x^T(t)A_1^T\int_{t-h}^{t} U(t-\xi)A_1x(\xi)\mathrm{d}\xi \\
&\quad -x^T(t-h)A_1^T\int_{t-h}^{t} U(t-h-\xi)A_1x(\xi)\mathrm{d}\xi
\end{aligned}
$$

$$+\left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t)\mathrm{d}\xi\right) A_1 x(t)$$
$$-\left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t + h)\mathrm{d}\xi\right) A_1 x(t-h).$$

Since the term

$$J_1(t) = \left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t)\mathrm{d}\xi\right) A_1 x(t)$$
$$= x^T(t)A_1^T \int_{t-h}^{t} U(t-\xi)A_1 x(\xi)\mathrm{d}\xi$$

and the term

$$J_2(t) = \left(\int_{t-h}^{t} x^T(\xi)A_1^T U(\xi - t + h)\mathrm{d}\xi\right) A_1 x(t-h)$$
$$= x^T(t-h)A_1^T \int_{t-h}^{t} U(t-h-\xi)A_1 x(\xi)\mathrm{d}\xi,$$

we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}R_3(t) = 2x^T(t)A_1^T \int_{t-h}^{t} U(t-\xi)A_1 x(\xi)\mathrm{d}\xi$$
$$-2x^T(t-h)A_1^T \int_{t-h}^{t} U(t-h-\xi)A_1 x(\xi)\mathrm{d}\xi.$$

In the computed time derivatives we first collect all terms that include an integral factor. The sum of such terms is

$$S_1(t) = 2x^T(t)\int_{t-h}^{t} \left[A_0^T U(-h-\xi+t) + A_1^T U(t-\xi) + \frac{\partial}{\partial t}U(-h-\xi+t)\right] A_1 x(\xi)\mathrm{d}\xi.$$

Applying symmetry property (2.15) we find that

$$\frac{\partial}{\partial t}U(-h-\xi+t)=\left[\frac{\partial}{\partial t}U(h+\xi-t)\right]^T=-\left[\left(\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau)\right)\bigg|_{\tau=h+\xi-t}\right]^T.$$

As $\xi\in[t-h,t]$, the variable $\tau=h+\xi-t\geq 0$ and properties (2.14) and (2.15) imply that

$$\frac{\partial}{\partial t}U(-h-\xi+t)=-A_0^TU(-h-\xi+t)-A_1^TU(-\xi+t).$$

The preceding equality means that the sum of the terms with an integral factor disappears.

Now we collect in the computed time derivatives the algebraic terms. The sum of the terms is

$$S_2(t)=x^T(t)\left[U(0)A_0+A_0^TU(0)+U(-h)A_1+A_1^TU(h)\right]x(t).$$

Property (2.16) implies that the preceding quadratic form coincides with $-w(x(t))$; therefore

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0(x_t)=-x^T(t)Wx(t),\quad t\geq 0.$$

□

2.7 Lyapunov Matrices: Limit Case

It is interesting to see what happens with Lyapunov matrices of system (2.1) when the delay term disappears. It may occur in two cases.

In the first case matrix A_1 tends to $0_{n\times n}$, and the limit system is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=A_0x(t).$$

Assume that matrix A_1 tends to $0_{n\times n}$ in such a way that the system remains exponentially stable. It follows from (2.13) that in this case the functional $v_0(x_t)$ tends to the quadratic form $x^T(t)U(0)x(t)$. The symmetry property (2.15) implies that matrix $U(0)$ is symmetric, and the algebraic property (2.16) turns into the classical Lyapunov matrix equation for $U(0)$:

$$U(0)A_0+A_0^TU(0)=-W.$$

In the second case, the delay tends to zero, $h\to+0$, and the new limit system is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}=(A_0+A_1)x(t).$$

Once again, assume that the system remains exponentially stable when $h \to +0$; then functional (2.13) tends to the quadratic form $x^T(t)U(0)x(t)$. Symmetry property (2.15) implies that matrix $U(0)$ is symmetric, and algebraic property (2.16) takes the form of the classical Lyapunov matrix equation

$$U(0)[A_0 + A_1] + [A_0 + A_1]^T U(0) = -W$$

for the new limit system.

This brief analysis provides an additional justification for calling matrix $U(\tau)$ a Lyapunov matrix of system (2.1).

2.8 Lyapunov Matrices: New Definition

Two serious limitations are associated with the definition of Lyapunov matrices by means of improper integral (2.11). The first one is that this definition is applicable to exponentially stable systems only. The second one is that the definition is of little help from a computational point of view. Indeed, it demands a preliminary computation of the fundamental matrix $K(t)$ for $t \in [0, \infty)$, which by itself is a difficult task, and the consequent evaluation of integral (2.11) for different values of τ.

In this section we remove the assumption that system (2.1) is exponentially stable and propose a new definition of the Lyapunov matrices, which will serve for unstable systems as well. But first we prove the following result.

Theorem 2.5. *Let $\widetilde{U}(\tau)$ be a solution of Eq. (2.14) that satisfies properties (2.15) and (2.16). If we define the functional $\widetilde{v}_0(\varphi)$, $\varphi \in PC([-h,0], R^n)$, by formula (2.13), where the matrix $U(\tau)$ is replaced by the matrix $\widetilde{U}(\tau)$, then the functional is such that along the solutions of system (2.1) the following equality holds:*

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}_0(x_t) = -x^T(t)Wx(t), \quad t \geq 0.$$

Proof. A direct inspection shows that the only properties of the matrix $U(\tau)$ that were used in the proof of Theorem 2.4 are those given in (2.14)–(2.16). Since the matrix $\widetilde{U}(\tau)$ satisfies the properties, the functional $\widetilde{v}_0(\varphi)$ satisfies Eq. (2.5) as well. $\square$

Theorem 2.5 justifies the following definition.

Definition 2.5. We say that the matrix $U(\tau)$ is a Lyapunov matrix of system (2.1) associated with a symmetric matrix W if it satisfies properties (2.14)–(2.16).

On the one hand, the new definition makes it possible to overcome the first limitation of the original definition of the Lyapunov matrices – the exponential stability assumption. On the other hand, it poses a new question: Does Definition 2.5

define for the case of exponentially stable system (2.1) the same Lyapunov matrix as that defined by improper integral (2.11)? The following statement provides an affirmative answer to this question.

Theorem 2.6. *Let system (2.1) be exponentially stable. Then matrix (2.11) is the unique solution of Eq. (2.14) that satisfies properties (2.15) and (2.16).*

Proof. Indeed, matrix (2.11) satisfies Eq. (2.14) and properties (2.15) and (2.16) (Lemmas 2.3–2.5). Assume that there are two matrices, $U_j(\tau)$, $j=1,2$, that satisfy these three properties. We define two functionals of the form (2.13). The first one, $v_0^{(1)}(\varphi)$, with matrix $U(\tau)=U_1(\tau)$, and the other one, $v_0^{(2)}(\varphi)$, with matrix $U(\tau)=U_2(\tau)$. Then, by Theorem 2.5,

$$\frac{\mathrm{d}}{\mathrm{d}t}v_0^{(j)}(x_t)=-x^T(t)Wx(t),\quad j=1,2.$$

Since the difference $\Delta v(x_t)=v_0^{(2)}(x_t)-v_0^{(1)}(x_t)$ satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\Delta v(x_t)=0,\quad t\geq 0,$$

we conclude that the identity

$$\Delta v(x_t(\varphi))=\Delta v(\varphi),\quad t\geq 0,$$

holds along any solution of system (2.1). The exponential stability of the system implies that $x_t(\varphi)\to 0_h$ as t tends to ∞. This means that $\Delta v(x_t(\varphi))\to 0$ as t tend to ∞, and we arrive at the conclusion that for any initial function $\varphi\in PC([-h,0],R^n)$ the following equality holds:

$$\begin{aligned}0=\Delta v(\varphi)\,&=\,\varphi^T(0)\Delta U(0)\varphi(0)+2\varphi^T(0)\int_{-h}^{0}\Delta U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta\\&+\int_{-h}^{0}\varphi^T(\theta_1)A_1^T\left[\int_{-h}^{0}\Delta U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1.\end{aligned}\qquad(2.17)$$

Here matrix $\Delta U(\tau)=U_2(\tau)-U_1(\tau)$.

Let us select a vector $\gamma\in R^n$ and define the initial function

$$\varphi^{(1)}(\theta)=\begin{cases}0, & \theta\in[-h,0)\\ \gamma, & \theta=0\end{cases}.$$

For the initial function equality (2.17) takes the form

$$0=\Delta v(\varphi^{(1)})=\gamma^T\Delta U(0)\gamma.$$

The preceding equality holds for an arbitrary vector γ. Because matrix $\Delta U(0)$ is symmetric, we obtain that

$$\Delta U(0) = 0_{n\times n}. \tag{2.18}$$

Given two vectors $\gamma,\mu \in R^n$ and $\tau \in (0,h]$, let us select $\varepsilon > 0$ such that $-\tau+\varepsilon < 0$ and define the new initial function

$$\varphi^{(2)}(\theta) = \begin{cases} \mu, & \theta \in [-\tau, -\tau+\varepsilon), \\ \gamma, & \theta = 0, \\ 0, & \text{for all other points of segment } [-h,0]. \end{cases}$$

It is a matter of direct calculations to demonstrate that

$$\begin{aligned} \Delta v(\varphi^{(2)}) = 2 \int_{-\tau}^{-\tau+\varepsilon} \gamma^T \Delta U(-h-\theta) A_1 \mu \mathrm{d}\theta \\ + \int_{-\tau}^{-\tau+\varepsilon} \left[\int_{-\tau}^{-\tau+\varepsilon} \mu^T A_1^T \Delta U(\theta_1-\theta_2) A_1 \mu \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1. \end{aligned}$$

Let ε be sufficiently small; then

$$2 \int_{-\tau}^{-\tau+\varepsilon} \gamma^T \Delta U(-h-\theta) A_1 \mu \mathrm{d}\theta = 2\varepsilon\gamma^T \Delta U(\tau-h) A_1 \mu + o(\varepsilon)$$

and

$$\int_{-\tau}^{-\tau+\varepsilon} \left[\int_{-\tau}^{-\tau+\varepsilon} \mu^T A_1^T \Delta U(\theta_1-\theta_2) A_1 \mu \mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 = o(\varepsilon).$$

Here the notation $o(\varepsilon)$ stands for a quantity that satisfies the condition

$$\lim_{\varepsilon\to+0} \frac{o(\varepsilon)}{\varepsilon} = 0.$$

Since the equality $\Delta v(\varphi^{(2)}) = 0$ holds for any sufficiently small $\varepsilon > 0$, we conclude that

$$2\gamma^T \Delta U(\tau-h) A_1 \mu = 0.$$

Because the preceding equality holds for any choice of vectors $\gamma,\mu \in R^n$, we obtain that

$$\Delta U(\tau-h) A_1 = 0_{n\times n}. \tag{2.19}$$

This is true for any $\tau \in (0,h]$. By continuity arguments we obtain that equality (2.19) remains true on the closed segment $[0,h]$.

The matrices $U_j(\tau)$, $j = 1,2$, satisfy Eq. (2.14), so the matrix $\Delta U(\tau)$ does the same:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0 + \Delta U(\tau - h)A_1, \quad \tau \geq 0.$$

Condition (2.19) implies that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Delta U(\tau) = \Delta U(\tau)A_0, \quad \tau \in [0,h].$$

Because matrix $\Delta U(\tau)$ satisfies (2.18), we immediately obtain the identity

$$\Delta U(\tau) = 0_{n\times n}, \quad \tau \in [0,h],$$

which means that $U_2(\tau) = U_1(\tau)$. This ends the proof. □

2.9 Lyapunov Matrices: Existence and Uniqueness Issues

Definition 2.5 raises the question of when a Lyapunov matrix exists. In other words, we are interested in conditions under which Eq. (2.14) admits a solution that satisfies properties (2.15) and (2.16). Theorem 2.6 provides a partial answer to the question. Here we give a detailed account of the existence and uniqueness of Lyapunov matrices.

First we prove that a Lyapunov matrix $U(\tau)$ provides a solution of a special boundary value problem for an auxiliary system of delay-free linear matrix differential equations. To this end we introduce two auxiliary matrices:

$$Y(\tau) = U(\tau), \quad Z(\tau) = U(\tau - h), \quad \tau \in [0,h]. \tag{2.20}$$

Lemma 2.7. *Let $U(\tau)$ be a Lyapunov matrix associated with a symmetric matrix W; then auxiliary matrices (2.20) satisfy the delay-free system of matrix equations*

$$\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) = Y(\tau)A_0 + Z(\tau)A_1, \quad \frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) = -A_1^T Y(\tau) - A_0^T Z(\tau) \tag{2.21}$$

and the boundary value conditions

$$Y(0) = Z(h), \qquad A_0^T Y(0) + Y(0)A_0 + A_1^T Y(h) + Z(0)A_1 = -W. \tag{2.22}$$

Proof. The first equation in (2.21) is a direct consequence of Eq. (2.14). To derive the second equation, we observe that $Z(\tau) = U^T(h-\tau)$, $\tau \in [0,h]$, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) &= \left[\frac{\mathrm{d}}{\mathrm{d}\tau}U(h-\tau)\right]^T = -\left[U(h-\tau)A_0 + U(-\tau)A_1\right]^T \\ &= -A_1^T U(\tau) - A_0^T U(\tau - h) = -A_1^T Y(\tau) - A_0^T Z(\tau).\end{aligned}$$

The first boundary value condition follows immediately from (2.20), whereas the second one is the algebraic property (2.16) written in the terms of the auxiliary matrices. □

Now we show that, conversely, any solution of the boundary value problem (2.21) and (2.22) generates a Lyapunov matrix associated with W.

Theorem 2.7. *If a pair $(Y(\tau),Z(\tau))$ satisfies (2.21) and (2.22), then the matrix*

$$U(\tau)=\frac{1}{2}\left[Y(\tau)+Z^T(h-\tau)\right], \tau\in[0,h], \tag{2.23}$$

is a Lyapunov matrix associated with W if we extend it to $[-h,0)$ by setting $U(-\tau)=U^T(\tau)$ for $\tau\in(0,h]$.

Proof. We check that matrix (2.23) satisfies the conditions of Definition 2.5.

Since we define this matrix on $[-h,0)$ by setting $U(-\tau)=U^T(\tau)$, to verify the symmetry property we only need to check that the matrix

$$U(0)=\frac{1}{2}\left[Y(0)+Z^T(h)\right]$$

is symmetric. The first boundary value condition, $Y(0)=Z(h)$, implies that

$$U(0)=\frac{1}{2}\left[Y(0)+Y^T(0)\right],$$

which proves the desired symmetry property.

Now we address the algebraic property. First we observe that the following matrix equalities hold:

$$\begin{aligned}U(0)A_0+A_0^TU(0)&=\frac{1}{2}\left[Y(0)+Y^T(0)\right]A_0+\frac{1}{2}A_0^T\left[Y(0)+Y^T(0)\right]\\&=\frac{1}{2}\left[Y(0)A_0+A_0^TY(0)\right]+\frac{1}{2}\left[Y(0)A_0+A_0^TY(0)\right]^T\end{aligned}$$

and

$$\begin{aligned}U(-h)A_1+A_1^TU(h)&=\frac{1}{2}\left[Y(h)+Z^T(0)\right]^TA_1+\frac{1}{2}A_1^T\left[Y(h)+Z^T(0)\right]\\&=\frac{1}{2}\left[Z(0)A_1+A_1^TY(h)\right]+\frac{1}{2}\left[Z(0)A_1+A_1^TY(h)\right]^T.\end{aligned}$$

Therefore,

$$R=U(0)A_0+A_0^TU(0)+U(-h)A_1+A_1^TU(h)$$

$$
\begin{aligned}
&= \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0) + Z(0)A_1 + A_1^T Y(h)\right] \\
&\quad + \frac{1}{2}\left[Y(0)A_0 + A_0^T Y(0) + Z(0)A_1 + A_1^T Y(h)\right]^T.
\end{aligned}
$$

The second boundary value condition in (2.22) implies that

$$
R = -\frac{1}{2}W - \frac{1}{2}W^T = -W.
$$

Finally, we check the dynamic property. The matrix $U(\tau)$ satisfies the equation

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}U(\tau) &= \frac{1}{2}\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} + \frac{1}{2}\frac{\mathrm{d}Z^T(h-\tau)}{\mathrm{d}\tau} \\
&= \frac{1}{2}\left[Y(\tau)A_0 + Z(\tau)A_1\right] - \frac{1}{2}\left[-A_1^T Y(h-\tau) - A_0^T Z(h-\tau)\right]^T \\
&= \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right]A_0 + \frac{1}{2}\left[Y(h-\tau) + Z^T(\tau)\right]^T A_1 \\
&= U(\tau)A_0 + U(\tau-h)A_1, \quad \tau \in [0,h].
\end{aligned}
$$

Thus, by Definition 2.5, matrix (2.23) is a Lyapunov matrix associated with W. □

Corollary 2.3. *If the boundary value problem (2.21) and (2.22) admits a unique solution $(Y(\tau), Z(\tau))$, then the matrix*

$$
U(\tau) = Y(\tau), \quad \tau \in [0,h],
$$

is a unique Lyapunov matrix associated with W.

Proof. First we show that if a pair $(Y(\tau), Z(\tau))$ satisfies (2.21) and (2.22), then the pair

$$
\left(\widetilde{Y}(\tau), \widetilde{Z}(\tau)\right) = \left(Z^T(h-\tau), Y^T(h-\tau)\right) \tag{2.24}
$$

also satisfies the conditions. It follows directly from (2.24) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Y}(\tau) &= -\left[-A_1^T Y(h-\tau) - A_0^T Z(h-\tau)\right]^T = \widetilde{Y}(\tau)A_0 + \widetilde{Z}(\tau)A_1, \\
\frac{\mathrm{d}}{\mathrm{d}\tau}\widetilde{Z}(\tau) &= -\left[Y(h-\tau)A_0 + Z(h-\tau)A_1\right]^T = -A_1^T\widetilde{Y}(\tau) - A_0^T\widetilde{Z}(\tau).
\end{aligned}
$$

Now, we check that matrices (2.24) satisfy the first boundary value condition in (2.22):

$$
\widetilde{Y}(0) - \widetilde{Z}(h) = \left[Z(h) - Y(0)\right]^T = 0_{n\times n}.
$$

And, finally, let us check the second boundary value condition in (2.22):

$$\begin{aligned}\widetilde{R} &= \widetilde{Y}(0)A_0 + A_0^T\widetilde{Y}(0) + A_1^T\widetilde{Y}(h) + \widetilde{Z}(0)A_1\\ &= Z^T(h)A_0 + A_0^T Z^T(h) + A_1^T Z^T(0) + Y^T(h)A_1\\ &= \left[A_0^T Y(0) + Y(0)A_0 + Z(0)A_1 + A_1^T Y(h)\right]^T\\ &= -W.\end{aligned}$$

Because the boundary value problem (2.21) and (2.22) admits a unique solution, we conclude that

$$Y(\tau) = Z^T(h-\tau), \quad \tau \in [0,h],$$

and therefore

$$U(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right] = Y(\tau), \quad \tau \in [0,h],$$

is a Lyapunov matrix associated with W. □

We present now an important condition for system (2.1). For the delay-free case this condition is well known and guarantees that the classical Lyapunov matrix Eq. (2.11) admits a unique solution for any matrix W.

Definition 2.6. We say that system (2.1) satisfies the Lyapunov condition if the spectrum of the system,

$$\Lambda = \left\{ s \mid \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right) = 0 \right\},$$

does not contain a point s_0 such that $-s_0$ also belongs to the spectrum, or, put another way, there are no eigenvalues of the system arranged symmetrically with respect to the origin of the complex plane.

Remark 2.3. If system (2.1) satisfies the Lyapunov condition, then it has no eigenvalues on the imaginary axis of the complex plane.

The following statement will play an important role in the proof of Theorem 2.8.

Lemma 2.8. *If system (2.21) admits a solution $(Y(\tau), Z(\tau))$ of the boundary value problem (2.22) with $W = 0_{n\times n}$, then*

$$Y(\tau) = Z(h+\tau), \quad \tau \in R. \tag{2.25}$$

Proof. We verify first that the matrices $Y(\tau)$ and $Z(\tau)$ satisfy the second-order matrix differential equation

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tau^2} = \frac{\mathrm{d}X}{\mathrm{d}\tau}A_0 - A_0^T\frac{\mathrm{d}X}{\mathrm{d}\tau} + A_0^T X A_0 - A_1^T X A_1. \tag{2.26}$$

To this end, we differentiate the first equation of system (2.21):

$$\frac{\mathrm{d}^2Y(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}A_0 + \frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}A_1.$$

The last term on the right-hand side of the preceding equality can be expressed by means of the second equation of (2.21) as follows:

$$\frac{\mathrm{d}Z(\tau)}{\mathrm{d}\tau}A_1 = -A_1^T Y(\tau)A_1 - A_0^T Z(\tau)A_1.$$

Then

$$\frac{\mathrm{d}^2Y(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau}A_0 - A_1^T Y(\tau)A_1 - A_0^T Z(\tau)A_1.$$

The first equation of (2.21) allows us to present the last term on the right-hand side of the preceding equality in the form

$$-A_0^T Z(\tau)A_1 = -A_0^T\left[\frac{\mathrm{d}Y(\tau)}{\mathrm{d}\tau} - Y(\tau)A_0\right].$$

And we arrive at the conclusion that $Y(\tau)$ satisfies Eq. (2.26). Similar manipulations prove that the matrix $Z(\tau)$ is a solution of the equation as well.

Any solution of (2.26) is uniquely determined by the initial conditions, $X(\tau_0)$, $X'(\tau_0)$. For $W = 0_{n\times n}$ the second condition in (2.22) can be transformed as follows:

$$\begin{aligned} 0_{n\times n} &= Y(0)A_0 + Z(0)A_1 + A_0^T Z(h) + A_1^T Y(h) \\ &= Y'(0) - Z'(h). \end{aligned}$$

If we add to the preceding equality the first condition from (2.22), $Y(0) = Z(h)$, then the identity (2.25) becomes evident. □

Now everything is ready to present the main result of the section.

Theorem 2.8. *System (2.1) admits a unique Lyapunov matrix associated with a given symmetric matrix W if and only if the system satisfies the Lyapunov condition.*

Proof. Sufficiency: Given a symmetric matrix W, according to Theorem 2.7, we can compute a Lyapunov matrix associated with W if there exists a solution of the boundary value problem (2.21) and (2.22). In what follows, we demonstrate that under the Lyapunov condition the boundary value problem admits a unique solution.

Let system (2.1) satisfy the Lyapunov condition. System (2.21) is linear and time invariant. To define a particular solution of the system, one must know the initial matrices $Y_0 = Y(0)$, $Z_0 = Z(0)$. This means that, in total, the initial matrices have $2n^2$ unknown components. Conditions (2.22) provide a system of $2n^2$ scalar linear algebraic equations in $2n^2$ unknown components of the initial matrices. The algebraic system admits a unique solution if and only if the unique solution of the system with $W = 0_{n\times n}$ is the trivial one. Assume by contradiction that there exists a nontrivial solution, (Y_0, Z_0), of the algebraic system with $W = 0_{n\times n}$. The initial matrices generate a nontrivial solution, $(Y(\tau), Z(\tau))$, $\tau \in [0,h]$, of the boundary value problem (2.21) and (2.22) with $W = 0_{n\times n}$. The nontrivial solution can be presented as a sum of eigenmotions of system (2.21):

$$Y(\tau) = \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} P_\nu(\tau), \quad Z(\tau) = \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} Q_\nu(\tau).$$

Here s_ν, $\nu = 0,1,\ldots,N$, are distinct eigenvalues of system (2.21) and $P_\nu(\tau)$ and $Q_\nu(\tau)$ are polynomials with matrix coefficients. The solution $(Y(\tau), Z(\tau))$ is nontrivial, so at least one of the polynomials $P_\nu(\tau)$, say $P_0(\tau)$, is nontrivial, because otherwise $Y(\tau) \equiv 0_{n\times n}$, and identity (2.25) implies that $Z(\tau) \equiv 0_{n\times n}$. Let polynomial $P_0(\tau)$ be of degree ℓ,

$$P_0(\tau) = \sum_{j=0}^{\ell} \tau^j B_j,$$

where B_j, $j = 0,1,\ldots \ell$, are constant $n\times n$ matrices, and $B_\ell \neq 0_{n\times n}$. It follows from Lemma 2.8 that $Y(\tau) = Z(h+\tau)$, and therefore

$$P_0(\tau) = \mathrm{e}^{s_0 h} Q_0(\tau + h).$$

Hence $Q_0(\tau)$ is also a nontrivial polynomial of degree ℓ,

$$Q_0(\tau) = \sum_{j=0}^{\ell} \tau^j C_j,$$

where $C_\ell = \mathrm{e}^{-s_0 h} B_\ell$.

Taking into account (2.25), we present the first matrix equation in (2.21) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Y(\tau) = Y(\tau) A_0 + Y(\tau - h) A_1.$$

And we obtain that

$$\begin{aligned} 0_{n\times n} = & \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} \right] \\ & - \sum_{\nu=0}^{N} \mathrm{e}^{s_\nu \tau} \left[P_\nu(\tau) A_0 + \mathrm{e}^{-s_\nu h} P_\nu(\tau - h) A_1 \right]. \end{aligned}$$

Because all eigenvalues s_ν, $\nu = 0,1,\ldots,N$, are distinct, the preceding equality implies that for each ν the polynomial identity

$$0_{n\times n} = s_\nu P_\nu(\tau) + \frac{\mathrm{d}P_\nu(\tau)}{\mathrm{d}\tau} - P_\nu(\tau) A_0 - \mathrm{e}^{-s_\nu h} P_\nu(\tau - h) A_1$$

holds. In the polynomial identity for $\nu = 0$ we collect the terms of the highest degree ℓ. The sum of these terms is equal to a zero matrix, so we arrive at the matrix equality

$$B_\ell \left(s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right) = 0_{n\times n}.$$

Because $B_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det \left(s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right) = 0,$$

and we conclude that s_0 is an eigenvalue of the original system (2.1).

The second equation of system (2.21) and the identity $Y(\tau) = Z(\tau + h)$ imply that

$$\frac{\mathrm{d}}{\mathrm{d}\tau} Z(\tau) = -A_1^T Z(\tau + h) - A_0^T Z(\tau).$$

The preceding identity generates the new set of polynomial identities

$$0_{n\times n} = s_\nu Q_\nu(\tau) + \frac{\mathrm{d}Q_\nu(\tau)}{\mathrm{d}\tau} + A_1^T Q_\nu(\tau + h) + A_0^T Q_\nu(\tau), \quad \nu = 0, 1, \ldots, N.$$

If in the identity for $\nu = 0$ we collect the terms of the highest degree ℓ, then

$$\left[s_0 I + A_0 + d^{s_0 h} A_1 \right]^T C_\ell = 0_{n\times n}.$$

As $C_\ell \neq 0_{n\times n}$, the preceding equality holds only if

$$\det \left[(-s_0) I - A_0 - d^{-(-s_0)h} A_1 \right] = 0.$$

And we conclude that $-s_0$ is an eigenvalue of system (2.1). This means that system (2.1) does not satisfy the Lyapunov condition. But this contradicts the theorem condition. The contradiction proves that the only solution of the boundary value problem (2.21), (2.22), with $W = 0_{n\times n}$, is the trivial one. Therefore, the boundary value problem (2.21), (2.22) admits a unique solution for any symmetric W, and this solution generates a Lyapunov matrix associated with W (Theorem 2.7).

Necessity: Now let us assume that system (2.1) does not satisfy the Lyapunov condition, i.e., the spectrum of the system contains a point s_0 such that $-s_0$ also belongs to the spectrum. Then there exist nonzero vectors $\gamma, \mu \in C^n$ such that

$$\mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0, \quad \left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0)h} A_1 \right]^T \gamma = 0.$$

We show that in this case there exists a nontrivial solution $(Y(\tau), Z(\tau))$, $\tau \in [0, h]$, of the boundary value problem (2.21), (2.22) with $W = 0_{n\times n}$. To check this, we set $Y(\tau) = \mathrm{e}^{s_0 \tau} \gamma \mu^T$ and $Z(\tau) = \mathrm{e}^{s_0(\tau - h)} \gamma \mu^T$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau) &= s_0 \mathrm{e}^{s_0\tau}\gamma\mu^T = \mathrm{e}^{s_0\tau}\gamma\mu^T(A_0 + \mathrm{e}^{-s_0h}A_1)\\ &= Y(\tau)A_0 + Z(\tau)A_1,\\ \frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau) &= s_0 \mathrm{e}^{s_0(\tau-h)}\gamma\mu^T = \mathrm{e}^{s_0(\tau-h)}(-A_0^T - \mathrm{e}^{s_0h}A_1^T)\gamma\mu^T\\ &= -A_1^T Y(\tau) - A_0^T Z(\tau).\end{aligned}$$

It is evident that $Y(\tau) = Z(\tau + h)$, so

$$Y(0) = Z(h), \text{ and } \left.\frac{\mathrm{d}}{\mathrm{d}\tau}Y(\tau)\right|_{\tau=0} = \left.\frac{\mathrm{d}}{\mathrm{d}\tau}Z(\tau)\right|_{\tau=h}.$$

By Theorem 2.7, the nontrivial solution generates the following nontrivial Lyapunov matrix associated with $W = 0_{n\times n}$:

$$U_0(\tau) = \frac{1}{2}\left[Y(\tau) + Z^T(h-\tau)\right] = \frac{1}{2}\left[\mathrm{e}^{s_0\tau}\gamma\mu^T + \mathrm{e}^{-s_0\tau}\mu\gamma^T\right].$$

Assume now that for a given symmetric matrix W there exists a Lyapunov matrix $U(\tau)$. It is evident that the matrix $U(\tau)+U_0(\tau)$ is also a Lyapunov matrix associated with W. This contradicts the theorem condition. The contradiction shows that our assumption that (2.1) does not satisfy the Lyapunov condition is wrong. This ends the proof of the necessity part. □

Corollary 2.4. *The Lyapunov matrix $U(\tau)$ associated with W satisfies the second-order delay-free matrix equation*

$$\frac{\mathrm{d}^2U(\tau)}{\mathrm{d}\tau^2} = \frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}A_0 - A_0^T\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau} + A_0^T U(\tau)A_0 - A_1^T U(\tau)A_0, \qquad \tau \geq 0,$$

and the following boundary value conditions:

$$\begin{aligned}\left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} &= U(0)A_0 + U^T(h)A_1,\\ \left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0} + \left(\left.\frac{\mathrm{d}U(\tau)}{\mathrm{d}\tau}\right|_{\tau=+0}\right)^T &= -W.\end{aligned}$$

Let us now see what happens when system (2.1) does not satisfy the Lyapunov condition. But first we prove a needed technical result.

Lemma 2.9. *Given two nontrivial vectors $\gamma, \mu \in C^n$, there exists a real symmetric matrix W_0 such that $\gamma^T W_0 \mu \neq 0$.*

Proof. If there exists an index j such that $\gamma_j\mu_j \neq 0$, then the matrix $W_0 = e_je_j^T$, where e_j denotes the jth unit column vector, satisfies the desired condition $\gamma^T W_0\mu = \gamma_j\mu_j \neq 0$. If $\gamma_j\mu_j = 0$ for all j, then there exist indices j and k, $j \neq k$, such that $\gamma_j \neq 0$ and $\gamma_k = 0$, while $\mu_j = 0$ and $\mu_k \neq 0$. Hence, setting $W_0 = e_je_k^T + e_ke_j^T$ we obtain $\gamma^T W_0\mu = \gamma_j\mu_k \neq 0$. □

Theorem 2.9. *If system (2.1) does not satisfy the Lyapunov condition, then there is a symmetric matrix W for which a Lyapunov matrix associated with W does not exist.*

Proof. Assume by contradiction that for any symmetric matrix W there exists a Lyapunov matrix associated with W. Since system (2.1) does not satisfy the Lyapunov condition, then there exists an eigenvalue s_0 such that $-s_0$ is also an eigenvalue of the system. Let γ and μ be eigenvectors corresponding to these eigenvalues. System (2.1) admits two solutions of the form

$$x^{(1)}(t) = \mathrm{e}^{s_0t}\gamma, \qquad x^{(2)}(t) = \mathrm{e}^{-s_0t}\mu.$$

By Lemma 2.9, there exists a symmetric matrix W_0 such that $\gamma^T W_0\mu \neq 0$. According to our assumption, there is a Lyapunov matrix $U(\tau)$ associated with W_0. Let us define the bilinear functional

$$\begin{aligned} z(\varphi,\psi) &= \varphi^T(0)U(0)\psi(0) + \varphi^T(0)\int_{-h}^{0} U(-h-\theta)A_1\psi(\theta)\mathrm{d}\theta \\ &\quad + \left(\int_{-h}^{0} \varphi^T(\theta)A_1^T U(h+\theta)\mathrm{d}\theta\right)\psi(0) \\ &\quad + \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\psi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1. \end{aligned}$$

Here it is assumed that $\varphi,\psi \in PC([-h,0],R^n)$. Given two solutions of (2.1), $x(t,\varphi)$ and $x(t,\psi)$, one can verify by direct calculation that

$$\frac{\mathrm{d}z(x_t(\varphi),x_t(\psi))}{\mathrm{d}t} = -x^T(t,\varphi)W_0x(t,\psi), \quad t \geq 0.$$

On the one hand, this means that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}z(x_t^{(1)},x_t^{(2)}) = -\left[x^{(1)}(t)\right]^T W_0x^{(2)}(t) = -\gamma^T W_0\mu \neq 0. \tag{2.27}$$

On the other hand, the direct substitution of the solutions into the bilinear functional yields

$$z(x_t^{(1)},x_t^{(2)}) = \gamma^T \left[U(0) + \int_{-h}^{0} U(-h-\theta)A_1 \mathrm{e}^{s_0\theta}\mathrm{d}\theta + \int_{-h}^{0} A_1^T U(h+\theta)\mathrm{e}^{-s_0\theta}\mathrm{d}\theta \right.$$
$$\left. + \int_{-h}^{0} \mathrm{e}^{s_0\theta_1} A_1^T \left(\int_{-h}^{0} \mathrm{e}^{-s_0\theta_2} U(\theta_1-\theta_2)\mathrm{d}\theta_2 \right) A_1 \mathrm{d}\theta_1 \right] \mu.$$

Observe that the matrix in the square brackets on the right-hand side of the preceding equality does not depend on t and

$$\frac{\mathrm{d}}{\mathrm{d}t} z(x_t^{(1)},x_t^{(2)}) = 0, \quad t \geq 0.$$

But this contradicts inequality (2.27). Hence, our assumption is not true, and for symmetric matrix W_0 the associated Lyapunov matrix does not exist. □

2.10 Lyapunov Matrices: Computational Issue

It is evident that the availability of constructive procedures for the computation of the Lyapunov matrices is crucial for a successful application of the quadratic functionals to the analysis of time-delay systems. It was shown in the previous section that the computation of Lyapunov matrices can be reduced to the construction of a solution of a special boundary value problem for a system of delay-free linear matrix differential equations (see Theorem 2.7). In Theorem 2.8 it was shown that the Lyapunov condition guarantees that the boundary value problem admits a unique solution.

With the help of the Kronecker product of matrices [13, 15, 25] the matrix boundary value problem (2.21), (2.22) can be written in vector form, which simplifies the computation of the solution of the problem. To this end, we define the vectorization operation

$$\mathrm{vec}\,(Q) = q,$$

where $q \in R^{n^2}$ is obtained from $Q \in R^{n\times n}$ by stacking up its columns. The operation satisfies the equality

$$\mathrm{vec}\,(AQB) = (A \otimes B)q,$$

where the matrix

$$A \otimes B = \begin{pmatrix} b_{11}A & b_{21}A & \cdots & b_{n1}A \\ b_{12}A & b_{22}A & \cdots & b_{n2}A \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n}A & b_{21}A & & b_{nn}A \end{pmatrix}$$

is known as the Kronecker product of matrices A and B.

System (2.28) takes the vector form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = L\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix}, \qquad L = \begin{pmatrix} I\otimes A_0 & I\otimes A_1 \\ -A_1^T\otimes I & -A_0^T\otimes I \end{pmatrix}. \tag{2.28}$$

Here $y(\tau) = \mathrm{vec}(Y(\tau))$ and $z(\tau) = \mathrm{vec}(Z(\tau))$. Boundary value conditions (2.22) take the form

$$M\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} + N\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = -\begin{pmatrix} 0 \\ w \end{pmatrix}, \tag{2.29}$$

where $w = \mathrm{vec}\,(W)$ and

$$M = \begin{pmatrix} I\otimes I & 0_{n\times n}\otimes 0_{n\times n} \\ A_0^T\otimes I + I\otimes A_0 & I\otimes A_1 \end{pmatrix},\ N = \begin{pmatrix} 0_{n\times n}\otimes 0_{n\times n} & -I\otimes I \\ A_1^T\otimes I & 0_{n\times n}\otimes 0_{n\times n} \end{pmatrix}. \tag{2.30}$$

It follows from system (2.28) that

$$\begin{pmatrix} y(h) \\ z(h) \end{pmatrix} = \mathrm{e}^{Lh}\begin{pmatrix} y(0) \\ z(0) \end{pmatrix}.$$

Substituting the preceding equality into boundary value condition (2.29) we obtain the algebraic system for the initial vectors

$$\left[M + N\mathrm{e}^{Lh}\right]\begin{pmatrix} y(0) \\ z(0) \end{pmatrix} = -\begin{pmatrix} 0 \\ w \end{pmatrix}. \tag{2.31}$$

Assume that the preceding system admits a solution; then this solution generates the corresponding solution of system (2.28),

$$\begin{pmatrix} y(\tau) \\ z(\tau) \end{pmatrix} = \mathrm{e}^{L\tau}\begin{pmatrix} y(0) \\ z(0) \end{pmatrix},$$

and a solution $(Y(\tau), Z(\tau))$ of the boundary value problem (2.21), (2.22). By Theorem 2.7 we obtain a Lyapunov matrix, associated with W; see (2.23).

We conclude this section with a criterion that system (2.1) satisfies the Lyapunov condition.

Theorem 2.10. *System (2.1) satisfies the Lyapunov condition if and only if the following condition holds:*

$$\det\left(M + N\mathrm{e}^{Lh}\right) \neq 0.$$

Proof. Necessity: System (2.1) satisfies the Lyapunov condition. It has been shown in the proof of the necessity part of Theorem 2.8 that under this condition the only solution of boundary value problem (2.21), (2.22), with $W = 0_{n\times n}$, is the trivial one.

Therefore, the only solution of system (2.31), with $w = 0$, is the trivial one. This implies that the matrix $\left(M + N\mathrm{e}^{Lh}\right)$ is nonsingular.

Sufficiency: Because the matrix $\left(M + N\mathrm{e}^{Lh}\right)$ is nonsingular, for any given w system (2.31) admits a unique solution. Therefore, by Corollary 2.3, for any symmetric matrix W system (2.1) admits a unique Lyapunov matrix $U(\tau)$. According to Theorem 2.8, this implies that system (2.1) satisfies the Lyapunov condition. □

2.11 Complete Type Functionals

One of the conditions of Theorem 2.3 states that the functional $v(\varphi)$ should admit a quadratic lower bound of the form

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n),$$

where $\alpha_1 > 0$. Surprisingly enough, no such bound has been found for functional (2.13). An attempt, undertaken in [26], resulted only in a local cubic lower bound for the functional. The following example confirms that no such quadratic lower bound for the functional exists.

Example 2.1 (A.P. Zhabko). The scalar equation

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = -x(t-1), \quad t \geq 0,$$

is exponentially stable. This means that there exist $\gamma \geq 1$ and $\sigma > 0$ such that the inequality

$$|x(t,\varphi)| \leq \gamma \mathrm{e}^{-\sigma t} \|\varphi\|_1, \quad t \geq 0,$$

holds along any solution of the equation.

For a given $\varepsilon \in (0,1)$ we define the initial function

$$\widetilde{\varphi}(\theta) = \begin{cases} \varepsilon, & \theta \in [-1,-1+\varepsilon) \\ 0, & \theta \in [-1+\varepsilon,0) \\ \varepsilon^2, & \theta = 0. \end{cases}$$

It is clear that $\|\widetilde{\varphi}\|_1 = \sup_{\theta \in [-1,0]} |\widetilde{\varphi}(\theta)| = \varepsilon$. The corresponding solution, $x(t,\widetilde{\varphi})$, evaluated by the step-by-step method is of the following form:

For $t \in [0,1]$

$$x(t,\widetilde{\varphi}) = \begin{cases} \varepsilon(\varepsilon - t), & t \in [0,\varepsilon], \\ 0, & t \in (\varepsilon,1]; \end{cases}$$

For $t \in [1,2]$

$$x(t,\widetilde{\varphi}) = \begin{cases} -\varepsilon^2(t-1) + \frac{1}{2}\varepsilon(t-1)^2, & t \in [1, 1+\varepsilon], \\ -\frac{1}{2}\varepsilon^3, & t \in (1+\varepsilon, 2]. \end{cases}$$

Since $|x(t,\widetilde{\varphi})| \leq \frac{1}{2}\varepsilon^3$ for $t \in [1,2]$, the exponential stability of the equation implies that the inequality

$$|x(t,\widetilde{\varphi})| \leq \gamma \frac{1}{2}\varepsilon^3 \mathrm{e}^{-\sigma(t-2)}$$

holds for $t \geq 2$. We now estimate the value $v_0(\widetilde{\varphi})$:

$$\begin{aligned} v_0(\widetilde{\varphi}) &= \int_0^\infty x^2(t,\widetilde{\varphi})\mathrm{d}t = \int_0^1 x^2(t,\widetilde{\varphi})\mathrm{d}t + \int_1^2 x^2(t,\widetilde{\varphi})\mathrm{d}t + \int_2^\infty x^2(t,\widetilde{\varphi})\mathrm{d}t \\ &\leq \frac{1}{3}\varepsilon^5 + \frac{1}{4}\left(1 + \frac{\gamma^2}{2\sigma}\right)\varepsilon^6. \end{aligned}$$

This demonstrates that the functional $v_0(\varphi)$ does not allow a quadratic lower bound of the form $\alpha_1 |\varphi(0)|^2 \leq v_0(\varphi)$ with $\alpha_1 > 0$; otherwise the inequality

$$\alpha_1 \varepsilon^2 \leq \frac{1}{3}\varepsilon^5 + \frac{1}{4}\left(1 + \frac{\gamma^2}{2\sigma}\right)\varepsilon^6$$

should hold for any $\varepsilon \in (0,1)$.

The preceding example shows that to obtain a functional satisfying the conditions of Theorem 2.3, we need a certain modification of functional (2.13). We are now ready to present this modification.

Theorem 2.11. *Given three symmetric matrices W_j, $j = 0,1,2$, let us define the functional*

$$w(\varphi) = \varphi^T(0)W_0\varphi(0) + \varphi^T(-h)W_1\varphi(-h) + \int_{-h}^0 \varphi^T(\theta)W_2\varphi(\theta)\mathrm{d}\theta. \tag{2.32}$$

If there exists a Lyapunov matrix $U(\tau)$ associated with the matrix $W = W_0 + W_1 + hW_2$ and $v_0(\varphi)$ is functional (2.13) with this Lyapunov matrix, then the time derivative of the modified functional

$$v(\varphi) = v_0(\varphi) + \int_{-h}^0 \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta \tag{2.33}$$

along the solutions of system (2.1) is such that the following equality holds:

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -w(x_t), \quad t \geq 0.$$

Proof. Indeed, the time derivative of the first term, $v_0(x_t)$, is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} v_0(x_t) = -x^T(t)\left[W_0 + W_1 + hW_2\right]x(t).$$

The time derivative of the second term

$$\begin{aligned} R(t) &= \int_{-h}^{0} x^T(t+\theta)\left[W_1 + (h+\theta)W_2\right]x(t+\theta)\mathrm{d}\theta \\ &= \int_{t-h}^{t} x^T(s)\left[W_1 + (h+s-t)W_2\right]x(s)\mathrm{d}s \end{aligned}$$

is equal to

$$\frac{\mathrm{d}}{\mathrm{d}t} R(t) = x^T(t)\left[W_1 + hW_2\right]x(t) - x^T(t-h)W_1 x(t-h) - \int_{t-h}^{t} x^T(s)W_2 x(s)\mathrm{d}s.$$

The sum of the time derivatives coincides with $-w(x_t)$. □

Definition 2.7. We say that functional (2.33) is of the complete type if matrices W_j, $j = 0, 1, 2$, are positive definite.

Lemma 2.10. *Let system (2.1) be exponentially stable. Given positive-definite matrices W_j, $j = 0, 1, 2$, there exists $\alpha_1 > 0$ such that the complete type functional (2.33) admits the following quadratic lower bound:*

$$\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi), \quad \varphi \in PC([-h, 0], R^n). \tag{2.34}$$

Proof. Consider the modified functional

$$\widetilde{v}(\varphi) = v(\varphi) - \alpha \|\varphi(0)\|^2.$$

Here α is a real parameter. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} \widetilde{v}(x_t) = -\widetilde{w}(x_t),$$

where

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\alpha x^T(t)\left[A_0 x(t) + A_1 x(t-h)\right] \\ &\geq (x^T(t), x^T(t-h))W(\alpha)\begin{pmatrix} x(t) \\ x(t-h)\end{pmatrix}.\end{aligned}$$

The matrix

$$W(\alpha) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \alpha\begin{pmatrix} A_0 + A_0^T & A_1 \\ A_1^T & 0_{n\times n}\end{pmatrix}.$$

Since the block diagonal matrix on the right-hand side of the preceding equality is positive definite, there exists $\alpha = \alpha_1 > 0$ such that the matrix $W(\alpha_1)$ is positive definite, too. This means that for $\alpha = \alpha_1$ the functional $\widetilde{w}(x_t) \geq 0$. The exponential stability of system (2.1) makes it possible to present the modified functional as follows:

$$\widetilde{v}(\varphi) = \int_0^\infty \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0.$$

The last inequality proves that (2.34) holds for $\alpha_1 > 0$. □

Lemma 2.11. *Let system (2.1) satisfy the Lyapunov condition (Definition 2.6). Given symmetric matrices W_j, $j = 0,1,2$, for some positive α_2 functional (2.33) satisfies the inequality*

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

Proof. To prove the inequality, we introduce the following notations:

$$\nu = \max_{\theta\in[0,h]} \|U(\theta)\|, \quad a = \|A_1\|.$$

Now we estimate the terms of functional (2.33). It is evident that

$$R_1 = \varphi^T(0)U(0)\varphi(0) \leq \nu\|\varphi(0)\|^2 \leq \nu\|\varphi\|_h^2$$

and

$$R_2 = 2\varphi^T(0)\int_{-h}^0 U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \leq 2\nu a h\|\varphi\|_h^2.$$

For the next term we have

$$\begin{aligned}R_3 &= \int_{-h}^0 \varphi^T(\theta_1)A_1^T\left[\int_{-h}^0 U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\ &\leq \nu a^2 h^2\|\varphi\|_h^2.\end{aligned}$$

Finally, we estimate the additional term as follows:

$$R_4 = \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta$$
$$\leq h\left(\|W_1\| + h\|W_2\|\right)\|\varphi\|_h^2 .$$

Collecting the estimations we conclude that

$$v(\varphi) \leq \alpha_2 \|\varphi\|_h^2 ,$$

where

$$\alpha_2 = \nu\,(1+ah)^2 + h\left(\|W_1\| + h\|W_2\|\right).$$

□

We return now to Theorem 2.3 and show that its conditions are necessary for the exponential stability of system (2.1).

Theorem 2.12. *System (2.1) is exponentially stable if and only if there exists a functional* $v : PC([-h,0],R^n) \to R$ *such that the following conditions are satisfied.*

1. $\alpha_1 \|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \|\varphi\|_h^2$*, for some positive* α_1*,* α_2*.*
2. *For some* $\beta > 0$ *the inequality*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) \leq -\beta \|x(t)\|^2 , \quad t \geq 0,$$

holds along the solutions of the system.

Proof. Sufficiency follows from Theorem 2.3.

Necessity is a direct consequence of Lemmas 2.10 and 2.11. □

2.12 Applications

In this section we present some applications of Lyapunov matrices and quadratic functionals.

2.12.1 Quadratic Performance Index

We consider a control system of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t).$$

Given a control law

$$\widetilde{u}(t) = Mx(t-h), \quad t \geq 0, \tag{2.35}$$

a closed-loop system is of the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = A_0 x(t) + A_1 x(t-h), \quad t \geq 0, \tag{2.36}$$

where $A_0 = A$ and $A_1 = BM$.

Assume that the closed-loop system is exponentially stable, and define the value of the quadratic performance index

$$J(\widetilde{u}) = \int_0^\infty \left[y^T(t)Py(t) + u^T(t)Qu(t) \right] \mathrm{d}t. \tag{2.37}$$

Here P and Q are given symmetric matrices of the appropriate dimensions. The value of the index can be presented in the form

$$J(\widetilde{u}) = \int_0^\infty \left[x^T(t,\varphi)W_0 x(t,\varphi) + x^T(t-h,\varphi)W_1 x(t-h,\varphi) \right] \mathrm{d}t,$$

where $\varphi \in PC([-h,0],R^n)$ is an initial function of the solution $x(t,\varphi)$ of the closed-loop system (2.36) and the matrices $W_0 = C^T PC$ and $W_1 = M^T QM$.

Theorem 2.13. *The value of the performance index (2.37) for the stabilizing control law (2.35) is equal to*

$$J(\widetilde{u}) = v_0(\varphi) + \int_{-h}^0 \varphi^T(\theta)W_1\varphi(\theta)\mathrm{d}\theta,$$

where $v_0(\varphi)$ is functional (2.13) computed for the Lyapunov matrix $U(\tau)$ associated with the matrix $W = W_0 + W_1 = C^T PC + M^T QM$.

2.12.2 Exponential Estimates

In this section we apply the complete type functionals, defined in the previous section, to derive an exponential estimate for the solutions of system (2.1). We begin with the following statement.

Lemma 2.12. *Let system (2.1) be exponentially stable. Given positive-definite matrices W_j, $j = 0,1,2$, for the complete type functional (2.33), there exist positive constants β_ℓ, $\ell = 1,2$, such that*

$$\beta_1 \|\varphi(0)\|^2 + \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta \leq v(\varphi), \quad \varphi \in PC([-h,0],R^n). \tag{2.38}$$

Proof. To prove inequality (2.38), we consider the functional

$$\widetilde{v}(\varphi) = v(\varphi) - \beta_1 \|\varphi(0)\|^2 - \beta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where β_1 and β_2 are real parameters. Along the solutions of system (2.1) the functional is such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{v}(x_t) = -\widetilde{w}(x_t), \quad t \geq 0.$$

Here

$$\begin{aligned}\widetilde{w}(x_t) &= w(x_t) + 2\beta_1 x^T(t)\left[A_0 x(t) + A_1 x(t-h)\right] + \beta_2 \left[\|x(t)\|^2 - \|x(t-h)\|^2\right] \\ &\geq \left[x^T(t), x^T(t-h)\right] Q(\beta_1,\beta_2) \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}.\end{aligned}$$

The matrix

$$Q(\beta_1,\beta_2) = \begin{pmatrix} W_0 & 0_{n\times n} \\ 0_{n\times n} & W_1 \end{pmatrix} + \beta_1 \begin{pmatrix} A_0 + A_0^T & A_1 \\ A_1^T & 0_{n\times n} \end{pmatrix} + \beta_2 \begin{pmatrix} I & 0_{n\times n} \\ 0_{n\times n} & -I \end{pmatrix}.$$

Since the matrices W_0 and W_1 are positive definite, there exist positive constants β_1, β_2 for which the matrix $Q(\beta_1,\beta_2)$ is positive definite. For such a choice of the parameters the following inequality holds along the solutions of system (2.1):

$$\widetilde{w}(x_t) \geq 0, \quad t \geq 0.$$

The preceding inequality implies that

$$\widetilde{v}(\varphi) = \int_{0}^{\infty} \widetilde{w}(x_t(\varphi))\mathrm{d}t \geq 0,$$

whence (2.38) follows immediately. □

Lemma 2.13. *Let system (2.1) satisfy the Lyapunov condition (Definition 2.6). Given symmetric matrices W_j, $j = 0,1,2$, for functional (2.33), there exist positive* *constants* δ_ℓ, $\ell = 1,2$, *such that*

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n). \tag{2.39}$$

Proof. We will use the notations introduced in the proof of Lemma 2.11. It is evident that the first two terms of the complete type functional (2.33) admit the upper bounds

$$R_1 = \varphi^T(0)U(0)\varphi(0) \leq \nu \|\varphi(0)\|^2$$

and

$$\begin{aligned} R_2 &= 2\varphi^T(0) \int_{-h}^{0} U(-h-\theta)A_1\varphi(\theta)\mathrm{d}\theta \\ &\leq \nu a h \|\varphi(0)\|^2 + \nu a \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta. \end{aligned}$$

For the next term we have

$$\begin{aligned} R_3 &= \int_{-h}^{0} \varphi^T(\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1\varphi(\theta_2)\mathrm{d}\theta_2 \right] \mathrm{d}\theta_1 \\ &\leq \nu a^2 \left[\int_{-h}^{0} \|\varphi(\theta)\| \mathrm{d}\theta \right]^2 \leq \nu a^2 h \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta. \end{aligned}$$

Finally, we estimate the additional term

$$\begin{aligned} R_4 &= \int_{-h}^{0} \varphi^T(\theta)\left[W_1 + (h+\theta)W_2\right]\varphi(\theta)\mathrm{d}\theta \\ &\leq (\|W_1\| + h\|W_2\|) \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta. \end{aligned}$$

Collecting the estimations we conclude that

$$v(\varphi) \leq \delta_1 \|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \mathrm{d}\theta,$$

where

$$\delta_1 = \nu\,(1+ah), \quad \delta_2 = a\delta_1 + (\|W_1\| + h\,\|W_2\|)\,. \qquad \square$$

We show how an exponential estimate for the solutions of system (2.1) can be derived with the use of complete type functionals.

Theorem 2.14. *System (2.1) is exponentially stable if and only if it admits a complete type functional $v(\varphi)$ such that for some $\alpha_1 > 0$ and $\alpha_2 > 0$ the following condition is satisfied:*

$$\alpha_1 \,\|\varphi(0)\|^2 \leq v(\varphi) \leq \alpha_2 \,\|\varphi\|_h^2, \quad \varphi \in PC([-h,0],R^n).$$

Proof. Sufficiency: Let $v(\varphi)$ be a complete type functional that satisfies the theorem conditions. There exist positive-definite matrices $W_j,\ j = 0,1,2$ such that the functional satisfies the equality

$$\frac{\mathrm{d}}{\mathrm{d}t} v(x_t) = -w(x_t), \quad t \geq 0,$$

where

$$w(x_t) = x^T(t)W_0x(t) + x^T(t-h)W_1x(t-h) + \int_{-h}^{0} x^T(t+\theta)W_2x(t+\theta)\mathrm{d}\theta.$$

First we show that there exists $\sigma > 0$ for which the inequality

$$\frac{\mathrm{d}v(x_t)}{\mathrm{d}t} + 2\sigma v(x_t) \leq 0, \quad t \geq 0, \tag{2.40}$$

holds. Indeed, on the one hand, according to Lemma 2.13, we can find positive $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$v(\varphi) \leq \delta_1 \,\|\varphi(0)\|^2 + \delta_2 \int_{-h}^{0} \|\varphi(\theta)\|^2 \,\mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n).$$

On the other hand, it is evident that

$$w(\varphi) \geq \lambda_{\min}(W_0)\,\|\varphi(0)\|^2 + \lambda_{\min}(W_2) \int_{-h}^{0} \|\varphi(\theta)\|^2 \,\mathrm{d}\theta, \quad \varphi \in PC([-h,0],R^n),$$

where $\lambda_{\min}(W)$ stands for the minimal eigenvalue of a symmetric matrix W. We take $\sigma > 0$, which satisfies the inequalities

$$2\sigma\delta_1 \leq \lambda_{\min}(W_0), \quad 2\sigma\delta_2 \leq \lambda_{\min}(W_2).$$

It is evident that such σ satisfies (2.40).

Now, inequality (2.40) implies that

$$v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

Then the theorem condition makes it possible to derive the inequalities

$$\alpha_1 \|x(t,\varphi)\|^2 \leq v(x_t(\varphi)) \leq v(\varphi)\mathrm{e}^{-2\sigma t} \leq \alpha_2 \|\varphi\|_h^2 \mathrm{e}^{-2\sigma t}, \quad t \geq 0.$$

And we arrive at the desired exponential estimate

$$\|x(t,\varphi)\| \leq \gamma \|\varphi\|_h \mathrm{e}^{-\sigma t}, \quad t \geq 0,$$

where

$$\gamma = \sqrt{\frac{\alpha_2}{\alpha_1}}.$$

Necessity: This part of the proof follows from Theorem 2.11 and Lemmas 2.10 and 2.11. □

Remark 2.4. It is worth mentioning that the exponential estimate obtained by Theorem 2.14 depends on the choice of positive-definite matrices W_j, $j = 0, 1, 2$. These matrices may serve as free parameters for optimization of the estimate. A special choice of matrices W_j, $j = 0, 1, 2$, may result in a tighter exponential estimate for the solutions of system (2.1). We do not try here to optimize the estimate.

We can obtain an exponential estimate for the solutions of a time-delay system even if it is not exponentially stable. Indeed, assume that the spectrum of system (2.1),

$$\Lambda = \left\{ s \;\middle|\; \det\left(sI - A_0 - \mathrm{e}^{-sh}A_1\right) = 0 \right\},$$

lies in the half-plane

$$\Gamma = \{ s \; | Re(s) < \Delta \}.$$

Then the spectrum of the modified system,

$$\frac{\mathrm{d}}{\mathrm{d}t} y(t) = (A_0 - \Delta I) y(t) + \mathrm{e}^{-\Delta h} A_1 y(t-h), \quad t \geq 0,$$

is as follows:

$$\begin{aligned} \widetilde{\Lambda} &= \left\{ s \;\middle|\; \det\left[(s+\Delta)I - A_0 - \mathrm{e}^{-(s+\Delta)h} A_1\right] = 0 \right\} \\ &= \{ s - \Delta \mid s \in \Lambda \}. \end{aligned}$$

This implies that the modified system is exponentially stable. Observe that the solutions of these systems satisfy the identity

$$y(t,\widetilde{\varphi}) = \mathrm{e}^{-\Delta t} x(t,\varphi), \quad t \geq -h, \tag{2.41}$$

where $\widetilde{\varphi}(\theta) = \mathrm{e}^{-\Delta\theta}\varphi(\theta)$, $\theta \in [-h,0]$. Since the modified system is exponentially stable, we can apply Theorem 2.14 to the system and compute $\widetilde{\gamma} \geq 1$ and $\widetilde{\sigma} > 0$ such that

$$\|y(t,\widetilde{\varphi})\| \leq \widetilde{\gamma}\|\widetilde{\varphi}\|_h \mathrm{e}^{-\widetilde{\sigma}t}, \quad t \geq 0.$$

It follows from identity (2.41) that the solutions of system (2.1) satisfy the inequality

$$\|x(t,\varphi)\| \leq \gamma\|\varphi\|_h \mathrm{e}^{\sigma t}, \quad t \geq 0,$$

where $\gamma = \mathrm{e}^{|\Delta|h}\widetilde{\gamma}$, and $\sigma = \widetilde{\sigma} + \Delta$. We may apply this procedure for $\Delta = \Delta_0 + \varepsilon$, where

$$\Delta_0 = \max_{s\in\Lambda} Re(s)$$

and ε is a given positive number.

2.12.3 Critical Values of Delay

In this section an interesting connection between the spectrum of the original system (2.1) and that of the auxiliary system (2.21) will be established. But first we prove the following theorem.

Theorem 2.15. *Let λ_0 be an eigenvalue of system (2.21). Then $-\lambda_0$ is also an eigenvalue of the system.*

Proof. Because λ_0 is an eigenvalue of system (2.21), there exists a nontrivial pair (Y_0,Z_0) of $n\times n$ matrices such that

$$\lambda_0 Y_0 = Y_0A_0 + Z_0A_1, \quad \lambda_0 Z_0 = -A_1^T Y_0 - A_0^T Z_0.$$

Applying the transposition to the preceding matrix equalities we obtain that the pair

$$(Y_1,Z_1) = \left(Z_0^T, Y_0^T\right)$$

satisfies the following matrix equalities:

$$-\lambda_0 Y_1 = Y_1A_0 + Z_1A_1, \quad -\lambda_0 Z_1 = -A_1^T Y_1 - A_0^T Z_1.$$

This means that $-\lambda_0$ is also an eigenvalue of system (2.21). □

The following theorem provides a connection between the spectrum of time-delay system (2.1) and that of delay-free system (2.21). This connection may be effectively used for the computation of critical delay values of system (2.1), i.e., the values for which system (2.1) admits an eigenvalue on the imaginary axis of the complex plane.

Theorem 2.16. *If s_0 is an eigenvalue of time-delay system (2.1) such that $-s_0$ is also an eigenvalue of the system, then s_0 and $-\,s_0$ belong to the spectrum of delay-free system (2.21).*

Proof. Since points s_0 and $-s_0$ belong to the spectrum of time-delay system (2.1), there exist two nonzero vectors $\gamma, \mu \in \mathbb{C}^n$ such that

$$\mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0,$$

$$\left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0)h} A_1 \right]^T \gamma = 0.$$

Now, premultiplying the first equality by γ and postmultiplying the second one by $\mathrm{e}^{-s_0 h}\mu^T$, we obtain

$$\gamma\mu^T \left[s_0 I - A_0 - \mathrm{e}^{-s_0 h} A_1 \right] = 0_{n\times n},$$

$$\left[(-s_0) I - A_0 - \mathrm{e}^{-(-s_0)h} A_1 \right]^T \mathrm{e}^{-s_0 h}\gamma\mu^T = 0_{n\times n}.$$

If we set $Y_0 = \gamma\mu^T$ and $Z_0 = \mathrm{e}^{-s_0 h}\gamma\mu^T$, then the preceding equalities take the form

$$s_0 Y_0 = Y_0 A_0 + Z_0 A_1, \quad \text{and} \quad s_0 Z_0 = -A_1^T Y_0 - A_0^T Z_0.$$

This means that s_0 belongs to the spectrum of delay-free system (2.21). By Theorem 2.15, $-s_0$ belongs to the spectrum as well. □

Remark 2.5. The spectrum Λ of delay system (2.1) depends on the delay value h, whereas the spectrum of delay-free system (2.21) does not depend on h. In particular, this means that system (2.1) remains exponentially stable (unstable) for all values of delay $h \geq 0$ if the spectrum of system (2.21) has no common points with the imaginary axis of the complex plane. On the other hand, the common points represent possible crossing points of the imaginary axis through which eigenvalues of system (2.1) may migrate from one half-plane to the other as the system delay h varies.

2.12.4 Robustness Bounds

It is well known that Lyapunov functions for delay-free systems are effectively used for estimating the robustness bounds for perturbed systems. The main contribution of this section consists in the demonstration that complete type functionals may also provide reasonable robustness bounds for time-delay systems.

Consider a perturbed system of the form

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = (A_0 + \Delta_0)\, y(t) + (A_1 + \Delta_1)\, y(t-h), \quad t \geq 0. \tag{2.42}$$

Here matrices Δ_0 and Δ_1 are unknown but such that

$$\|\Delta_k\| \leq \rho_k, \; k = 0, 1. \tag{2.43}$$

Let system (2.1) be exponentially stable. We would like to find some conditions on ρ_0 and ρ_1 under which system (2.42) remains stable for all Δ_0 and Δ_1 satisfying (2.43). To this end, we will use a complete type functional $v(\varphi)$ defined by formula (2.33).

We compute the time derivative of the functional along the solutions of perturbed system (2.42).

Lemma 2.14. *The time derivative of functional (2.33) along the solutions of perturbed system (2.42) is of the form*

$$\frac{\mathrm{d}}{\mathrm{d}t} v(y_t) = -w(y_t) + 2\left[\Delta_0 y(t) + \Delta_1 y(t-h)\right]^T l(y_t), \quad t \geq 0,$$

where

$$l(y_t) = U(0)y(t) + \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta.$$

Proof. Recall that $v(y_t)$ is written as follows:

$$\begin{aligned} v_0(y_t) = {} & y^T(t)U(0)y(t) + 2y^T(t)\int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta \\ & + \int_{-h}^{0} y^T(t+\theta_1)A_1^T \left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1 y(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1 \\ & + \int_{-h}^{0} y^T(t+\theta)\left[W_1 + (h+\theta)W_2\right] y(t+\theta)\mathrm{d}\theta. \end{aligned}$$

The time derivative of the first term,

$$R_1(t) = y^T(t)U(0)y(t),$$

is

$$\frac{\mathrm{d}R_1(t)}{\mathrm{d}t} = 2y^T(t)U(0)\left[(A_0+\Delta_0)y(t)+(A_1+\Delta_1)y(t-h)\right].$$

For the next term,

$$R_2(t) = 2y^T(t)\int_{-h}^{0} U(-h-\theta)A_1y(t+\theta)\mathrm{d}\theta,$$

we have

$$\begin{aligned}\frac{\mathrm{d}R_2(t)}{\mathrm{d}t} &= 2\left[(A_0+\Delta_0)y(t)+(A_1+\Delta_1)y(t-h)\right]^T\int_{-h}^{0} U(-h-\theta)A_1y(t+\theta)\mathrm{d}\theta\\ &\quad +2y^T(t)U(-h)A_1y(t)-2y^T(t)U(0)A_1y(t-h)\\ &\quad -2y^T(t)\int_{-h}^{0}\left[U'(h+\theta)\right]^T A_1y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

The time derivative of the double integral

$$R_3(t) = \int_{-h}^{0} y^T(t+\theta_1)A_1^T\left[\int_{-h}^{0} U(\theta_1-\theta_2)A_1y(t+\theta_2)\mathrm{d}\theta_2\right]\mathrm{d}\theta_1$$

is of the form

$$\begin{aligned}\frac{\mathrm{d}R_3(t)}{\mathrm{d}t} &= 2y^T(t)\int_{-h}^{0}\left[U(\theta)A_1\right]^T A_1y(t+\theta)\mathrm{d}\theta\\ &\quad -2y^T(t-h)A_1^T\int_{-h}^{0} U(-h-\theta)A_1y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

And, finally, the time derivative of the last term,

$$R_4 = \int_{-h}^{0} y^T(t+\theta)\left[W_1+(h+\theta)W_2\right]y(t+\theta)\mathrm{d}\theta,$$

is

$$\begin{aligned}\frac{\mathrm{d}R_4(t)}{\mathrm{d}t} &= y^T(t)\left[W_1+hW_2\right]y(t)-y^T(t-h)W_1y(t-h)\\ &\quad -\int_{-h}^{0} y^T(t+\theta)W_2y(t+\theta)\mathrm{d}\theta.\end{aligned}$$

Now, repeating the arguments applied in the proof of Theorem 2.4 we arrive at the desired expression for the derivative of $v(y_t)$. □

Let

$$\nu = \max_{\theta\in[0,h]} \|U(\theta)\|, \quad a = \|A_1\|.$$

Then the following estimates hold:

$$J_1(t) = 2y^T(t)\Delta_0^T U(0)y(t) \le 2\nu\rho_0 \|y(t)\|^2,$$

$$J_2(t) = 2y^T(t-h)\Delta_1^T U(0)y(t) \le \nu\rho_1 \left[\|y(t)\|^2 + \|y(t-h)\|^2\right],$$

$$J_3(t) = 2y^T(t)\Delta_0^T \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta$$

$$\le h\nu\rho_0 a \|y(t)\|^2 + \nu\rho_0 a \int_{-h}^{0} \|y(t+\theta)\|^2 \mathrm{d}\theta,$$

$$J_4(t) = 2y^T(t-h)\Delta_1^T \int_{-h}^{0} U(-h-\theta)A_1 y(t+\theta)\mathrm{d}\theta$$

$$\le h\nu\rho_1 a \|y(t-h)\|^2 + \nu\rho_1 a \int_{-h}^{0} \|y(t+\theta)\|^2 \mathrm{d}\theta.$$

From the preceding inequalities we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}v(y_t) \le -w(y_t) + \nu\left[2\rho_0 + h\rho_0 a + \rho_1\right]\|y(t)\|^2 + \nu\rho_1\left[1 + ha\right]\|y(t-h)\|^2$$

$$+\nu\left[\rho_0 + \rho_1\right] a \int_{-h}^{0} \|y(t+\theta)\|^2 \mathrm{d}\theta,$$

and we arrive at the following statement.

Theorem 2.17. *Let system (2.1) be exponentially stable. Given positive-definite matrices W_0, W_1, W_2, system (2.42) remains exponentially stable for all Δ_0 and Δ_1 satisfying (2.43) if the following inequalities hold:*

1. $\lambda_{\min}(W_0) \ge \nu\left[2\rho_0 + h\rho_0 a + \rho_1\right]$,
2. $\lambda_{\min}(W_1) \ge \nu\rho_1\left[1 + ha\right]$,
3. $\lambda_{\min}(W_2) \ge \nu\left[\rho_0 + \rho_1\right] a$.

Remark 2.6. Theorem 2.17 remains true if we assume that the uncertain matrices Δ_0 and Δ_1 are continuous functions of t and x_t.

2.13 Notes and References

The first work dedicated to the construction of quadratic Lyapunov functionals with a given time derivative was that by Repin [63]. In this seminal contribution a quadratic functional of a general form was suggested. The time derivative of the functional was computed, and then, equating the derivative to the prescribed one, a system of equations for the matrices that define the functional was derived. The system includes a linear matrix partial differential equation, ordinary matrix differential equations, and algebraic relations between the matrices. Under some simplifying assumptions the system was reduced to a system of two matrix differential equations similar to that given by (2.21). Many essential elements needed for the computation of Lyapunov matrices, some in explicit form and some in implicit form, can be found there. Without a doubt this three-page contribution has had a profound impact on research in the area.

In the paper by Datko, [7], the main object was a presentation of an infinite-dimensional version of the Lyapunov–Krasovskii approach to the stability analysis of linear time-delay systems. In particular, the paper provides an interpretation from the operator point of view of the results given in [63].

The paper by Castelan and Infante [4] is dedicated to the following initial value problem:

$$\frac{\mathrm{d}Q(\tau)}{\mathrm{d}\tau} = AQ(\tau) + BQ^T(h-\tau), \quad Q\left(\frac{h}{2}\right) = K, \tag{2.44}$$

where K is a given $n \times n$ matrix. It is worth mentioning that the dynamic property in [63] was written in this form. In that paper, it was shown that for any given K the initial value problem admits a unique solution. An exhaustive analysis of the solution space of Eq. (2.44) is presented in the paper as well. The reader may find there interesting observations on the spectrum of an auxiliary system, similar to that presented in Sect. 2.12.3.

More interesting for us is the second paper by the same authors, [28]. There, for the first time, the three basic properties of Lyapunov matrices are explicitly indicated. Once again, following the tradition established in [63], the dynamic property was written in the form of Eq. (2.44). The symmetry property did not receive its due attention but was simply mentioned as a property of a matrix $\widetilde{Q}(\tau)$ similar to that of the improper integral (2.11). The algebraic property was introduced as a bridge connecting the matrices $Q(\tau)$ and $\widetilde{Q}(\tau)$. What is very important for us is that the paper discusses functionals similar to that of the complete type. For these functionals quadratic lower and upper bounds of the form given in Lemmas 2.12 and 2.13 were provided. The main goal of the paper was to demonstrate that functionals may be effectively applied to the computation of upper exponential estimates of the solutions of time-delay systems. Unfortunately, at one of the intermediate steps, namely, in the computation of an upper bound for a functional, the desired exponential estimate was explicitly used. The paper also contains a remark that for the case of exponentially stable systems functionals of the form (2.13) do not admit quadratic lower bounds.

The next serious breakthrough in this direction was made in the paper by Huang [26], where the existence of lower bounds for functionals of the form (2.13) is studied. The paper demonstrates, for the case of exponentially stable systems, that functionals admit local cubic lower bounds of the form

$$\alpha \|\varphi(0)\|^3 \leq v_0(\varphi), \quad \varphi \in C([-h,0],R^n), \text{ and } \|\varphi\|_h \leq H,$$

where α and H are two positive constants. It is important to note here that this result, as well as others in the present contribution, have been proven for a very general class of linear time-delay systems. This was probably the first paper to state explicitly that Lyapunov matrices are completely defined by three basic properties: dynamic, symmetry, and algebraic properties. The most important result presented in that paper is the existence theorem. The theorem states that if a time-delay system satisfies the Lyapunov condition, then for any symmetric matrix W there exists a corresponding Lyapunov matrix. Additionally, an explicit frequency domain expression for the Lyapunov matrix is given as well.

Theorem 2.8 was proven in [40]. The complete type functionals were introduced in [42], where some robustness bounds are derived as well. Complete type functionals are applied to the computation of exponential estimates of the solutions of system (2.1) in [38]. A brief account of the theory of Lyapunov matrices and functionals appears in [36].

A detailed account of the application of functionals of the form (2.13) to the computation of various quadratic performance indices can be found in an interesting book [55]; see also [8, 9, 14, 22, 27, 51, 52].

In the paper by Louisell [53] a relation between the spectrum of a time-delay system and that of an auxiliary delay-free system of matrix equations is established. That is, it is shown that any pure imaginary eigenvalue of a time-delay system is also an eigenvalue of an auxiliary system. The statement was obtained for the case of neutral type systems with one delay. In some sense the statement of Theorem 2.16 is a generalization of this important result.

The following open problem is one of the most important problems related to Lyapunov matrices and deserves to be mentioned here: Find the conditions of the exponential stability of system (2.1) expressed in the terms of a Lyapunov matrix $U(\tau)$, associated with a positive-definite matrix W. The first result in this direction was obtained by Mondie [56], where some necessary and sufficient conditions are derived for the case of scalar equations.

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