

CHAPTER 1

Introductory Survey

*Excuse me if I alone introduce myself:
I am the Prologue.*

— R. LEONCAVALLO, Pagliacci

Order and chaos, invariant tori, KAM theory, resonances, Arnold web, diffusion ..., these are “keywords” in the theory of dynamical systems. But for mathematicians who are not directly involved in this area they may sound a bit vague. To grasp what they are about, consider the following question: Given a conservative mechanical system, thus without dissipative forces, what is its ultimate fate? Without further information the answer probably cannot be given in general. However, if we restrict ourselves to the special, but very important, case of a *slightly perturbed integrable system*, we can claim that crucial progresses have been achieved. An example is our solar system, in which the perturbations are the interactions between planets. In this book we will suggest how to guess an answer, and on the way all these keywords will come into play.

The motion of an integrable system is totally ordered, in a sense that will be specified below. The central question is: Does a very small perturbation destroy this order completely? Before the fundamental work of Kolmogorov (1954), two completely opposite answers were given. For the astronomers, interested in the computation of perturbed orbits, the answer was (more or less tacitly) negative: for them the affirmative answer seemed to be a disaster, making meaningless their series expansions. On the contrary, for the

statistical physicists, interested in the possibility of applying the ergodic theorem, the answer was decisively affirmative. That of Kolmogorov was, in some sense, a Solomon's verdict: the ultimate fate of a slightly perturbed orbit depends on the initial conditions, so that for the same perturbed system there coexist, in general, orbits that stay *forever* in the neighborhood of an unperturbed one, and other orbits that depart indefinitely.

What is the distinguishing feature that makes an orbit ordered or chaotic? The answer is given by the celebrated Kolmogorov–Arnold–Moser (KAM) theorem stated first in Kolmogorov (1954). Its proof is rather complicated and also the statement itself cannot be given without anticipating some concepts of symplectic geometry and analytical mechanics.

1.1 Configuration Space and Lagrangian Dynamics

Consider a mechanical system with n degrees of freedom, let q^1, q^2, \dots, q^n be the configuration coordinates, and denote with a dot their time derivative. The n Lagrange equations

$$\frac{d}{dt} \frac{\partial L(q, \dot{q})}{\partial \dot{q}} - \frac{\partial L(q, \dot{q})}{\partial q} = 0, \quad q = (q^1, q^2, \dots, q^n)$$

are derived from the variational principle

$$\delta \int_{t_1}^{t_2} L(q, \dot{q}) dt = 0,$$

where the variations keep fixed the initial and final points. For natural systems, the Lagrangian L is usually given by the difference between the kinetic and potential energy.

The Lagrange equations admit a geometrical interpretation, which is the reason for their practical usefulness. In fact, in the particular case of a point constrained on a smooth surface, they are the projection of the fundamental law of dynamics onto the tangent plane, thus avoiding the introduction of reaction forces. For a mechanical system the surface is replaced by its natural generalization, an n -dimensional manifold Q , namely the *configuration space*. The kinetic energy is positive definite, homogeneous, and bilinear in the components \dot{q}^k of the generalized velocity, thus defining a double symmetric tensor that equips Q with a metric (or Riemannian) structure and establishing an isomorphism between the tangent space and its dual, the cotangent space. This is referred to as the ability to raise or lower the indices. The manifold Q is *flat* or *Euclidean* if and only if a certain differential condition is satisfied (i.e., when the Riemann tensor, constructed with the first and second derivatives of the metric tensor, vanishes identically).

In this case it is possible to choose Cartesian coordinates, reducing the representation of the metric tensor to the identity matrix. However, in general Q is *not* flat.

In this context, the Lagrange equations represent a real cornerstone, like all the great intellectual achievements. They bring the Newtonian perspective to a natural conclusion and, having forces and accelerations as basic ingredients, convey the information in n equations of second order. At the same time, they are the starting point of a far-reaching path: the apparently unpretentious wish to express them as a set of $2n$ equations of the first order, also reveals that the tangent bundle TQ (i.e., the union of Q and its tangent spaces) has a sort of metric, namely the *symplectic structure*. The next section is a brief digression to state some definitions.

1.2 Symplectic Manifolds

A $2n$ -dimensional manifold is said to be *symplectic* if it is equipped with a field of closed and regular 2-forms (i.e., double antisymmetric tensors). As in the Riemannian case, with the 2-form one can evaluate in any point the “scalar product” of two tangent vectors; moreover, the regularity property yields the isomorphism between the tangent and cotangent space and in turn the ability to raise or lower indices. The closure property is, in some sense, similar to the vanishing of the Riemann tensor: the Darboux theorem states that in an open neighborhood of a symplectic manifold the 2-form Ω takes a *canonical* expression in a suitable *canonical* coordinate system, its representation matrix being

$$\Omega = \begin{pmatrix} 0_n & -1_n \\ 1_n & 0_n \end{pmatrix}.$$

The canonical coordinates are therefore similar to the Cartesian coordinates of a Euclidean space. Usually, the first n canonical coordinates are named q^k , the remaining p_k , so that the canonical 2-form takes the expression $\Omega = \sum_{k=1}^n dp_k \wedge dq^k$; the wedge symbol \wedge means *exterior product*, i.e., the antisymmetric part of the tensor product. Due to the closure property, a potential 1-form $\Theta = \sum_{k=1}^n p_k dq^k$ does exist at least locally and $\Omega = d\Theta$ is its exterior derivative.

The analogy between Cartesian and canonical coordinates can be taken further. The rigid transformations of a Euclidean space are rotations and translations and have the property to leave the metric invariant. Their counterparts, which leave the symplectic structure invariant, are the *symplectic transformations* that, in the canonical case, send the old q, p into the new $Q(q, p), P(q, p)$ through the relations

$$p_k = \frac{\partial W(q, Q)}{\partial q^k}, \quad P_k = -\frac{\partial W(q, Q)}{\partial Q^k},$$

where $W(q, Q)$ is a *generating function* satisfying the two relations

$$\sum_{k=1}^n p_k dq^k - P_k dQ^k = dW(q, Q), \quad \det \left(\frac{\partial^2 W}{\partial q^h \partial Q^k} \right) \neq 0.$$

Indeed, by exterior differentiating the first relation, one may check that the old canonical 2-form is still transformed into a canonical one, since $dd \equiv 0$; the latter relation simply ensures that the transformation is invertible.

A continuous one-parameter group of symplectic transformations can be generated as follows. Define the *Hamiltonian vector fields* as those obtained, first by differentiating a scalar function $f(q, p)$, called the *Hamiltonian*, and then by raising the indices with the symplectic structure; sometimes they are called the *symplectic gradient* of the Hamiltonian $f(q, p)$. In the class of generic vector fields defined on a symplectic manifold, the Hamiltonian ones occupy a privileged position, since a vector field generates a symplectic flow (i.e., leaving the symplectic structure invariant) if and only if it is Hamiltonian.

Lastly, let us define the *Poisson bracket* $\{f, g\}$ between two functions on a symplectic manifold as the scalar product of their gradients. Equivalently, it can be seen as the Lie (or directional) derivative of one of the two functions with respect to the vector field generated by the symplectic gradient of the other. In the canonical case we recover the well-known definition:

$$\{f, g\} = \sum_{k=1}^n \frac{\partial f}{\partial q^k} \frac{\partial g}{\partial p_k} - \frac{\partial g}{\partial q^k} \frac{\partial f}{\partial p_k}.$$

It is easy to see that the flows generated by the symplectic gradients of the two functions commute if and only if $\{f, g\} = 0$.

1.3 Phase Space and Hamiltonian Dynamics

In order to pass from the n equations of second order of the Lagrangian dynamics to $2n$ equivalent equations of first order, the most natural choice is to promote the components of the generalized velocity to independent variables by setting $\dot{q}^k = v^k$. Now we make a discovery. On TQ define the 1-form $\Theta_L = \sum_{k=1}^n \frac{\partial L}{\partial v^k} dq^k$ (this means that the coefficients of the dv^k terms are all null), so that the 2-form $\Omega_L = d\Theta_L$ is closed and, due to the regularity of the Riemannian structure, also regular: therefore the tangent bundle TQ becomes a symplectic manifold. Moreover, define the *Hamiltonian* function $H(q, v) = \sum_{k=1}^n \frac{\partial L}{\partial v^k} v^k - L(q, v)$. It is a simple matter of calculations to show that the symplectic flow generated by the symplectic gradient of the Hamiltonian $H(q, v)$ is equivalent to the Lagrange equations.

The definition of Θ_L and Ω_L shows that the symplectic structure of TQ is *not* in canonical form. In general, finding the canonical coordinates is not

an easy task, but, fortunately, in the present case it is readily seen that the *Legendre transformation*

$$(q, v) \rightarrow (q, p) \quad \text{where } p_k(q, v) = \frac{\partial L(q, v)}{\partial v^k}$$

achieves our aim, sending Θ_L into the canonical $\Theta = \sum_{k=1}^n p_k dq^k$. In other words, by the Legendre transformation we pass from the tangent bundle TQ to the *cotangent bundle* T^*Q (the union of Q and all the dual spaces of the tangent spaces); this, in order to use the natural, or canonical, coordinates of the symplectic structure. The p_k terms are named (*canonical momenta*), and the regularity of the Riemannian metric ensures that the relation $p_k(q, v)$ can be inverted, thus giving the velocities as functions of coordinates and momenta. T^*Q is usually referred as the *phase space* of the system.

With a little abuse of notation, we write $H(q, p) = H(q, v(q, p))$. Recalling that the Lagrange equations are expressed in terms of the symplectic flow generated by the symplectic gradient of the Hamiltonian, and that now the symplectic structure is the canonical one, we have that the celebrated *Hamilton equations*

$$\dot{q}^k = \frac{\partial H(q, p)}{\partial p_k}, \quad \dot{p}_k = -\frac{\partial H(q, p)}{\partial q_k}$$

are equivalent to the Lagrange equations. Hereafter we will work in canonical coordinates.

1.4 The Liouville and Arnold Theorems

Given a Hamiltonian, finding explicit solutions of the related, usually non-linear, equations is in general a hopeless task. The very few cases in which this is possible share the property that the problem can be reduced to the quadrature, i.e., to invert functions and perform integrations. This is the case of *complete integrability*.

THEOREM 1.1 (Liouville) *A sufficient condition for the complete integrability of an n -dimensional Hamiltonian system is that there exist n first integrals $\Phi_k(q, p)$ that are independent and in involution, that is $\{\Phi_i, H\} = 0$ and $\{\Phi_i, \Phi_k\} = 0, \forall i, k$.*

For the proof, one basically seeks a canonical transformation sending the first integrals into the new momenta, so that the equations of the transformed Hamiltonian, which will depend only on the momenta, are trivially integrable. Clearly, the transformation exists if and only if the first integrals are in involution, since this holds true for any n -tuple of canonical

momenta. For every n -tuple $\alpha_1, \dots, \alpha_n$ of constants, the generating function appears as the potential of a vector field in Q that comes from inverting the relations $\Phi_k(q, p) = \alpha_k$ with respect to the momenta. This vector field is therefore known and finding its potential leads to performing n integrations.

Varying the constants, the n relations $\Phi_k(q, p) = \alpha_k$ determine a foliation of the $2n$ -dimensional phase space T^*Q in n -dimensional level hypersurfaces. What is the topology of these hypersurfaces? At first glance one can say nothing on this topology, which depends on the analytical expression of the first integrals. But here the involutivity condition, which in turn is a direct consequence of the canonical structure, plays a key role.

THEOREM 1.2 (Arnold) *Given a completely integrable n -dimensional Hamiltonian system, the compact and connected components of the level surfaces of the first integrals are diffeomorphic to an n -dimensional torus. Moreover, there exist (locally) canonical coordinates called action-angle coordinates, such that the action variables parametrize the set of the tori whereas the angles parametrize the points on a torus. The Hamiltonian, expressed as a function of these coordinates, depends only on the actions, so that the dynamical evolution is a uniform rotation on an invariant torus.*

The key point in the proof consists of viewing the functions $\Phi_k(q, p)$ as Hamiltonians generating flows that, by involutivity, respect the foliation and commute with one another. It is natural to think (though this is the central point of the proof) that the sole n -dimensional compact hypersurface carrying n independent and commuting flows is the product of n circles, i.e., the torus \mathbb{T}^n . To find the action-angle variables I_j, φ^k , $j, k = 1, \dots, n$, let γ_i be the cycles on the torus generated by I_i and φ^i the corresponding angles. In order that I_j, φ^k are canonical coordinates, we require that the two 1-forms $\sum_{k=1}^n p_k dq^k$ and $\sum_{k=1}^n I_k d\varphi^k$ differ by an exact 1-form, whose integral along a cycle is consequently zero. Hence

$$\sum_{k=1}^n \oint_{\gamma_i} I_k d\varphi^k = \sum_{k=1}^n \oint_{\gamma_i} p_k dq^k.$$

On the left-hand side $d\varphi^k = 0$, $\forall k \neq i$, and I_i is constant along the cycle since, by definition, it is just the Hamiltonian generating γ_i . Therefore we define

$$I_i = \frac{1}{2\pi} \sum_{k=1}^n \oint_{\gamma_i} p_k(q, \alpha) dq^k.$$

The action variables are therefore invertible functions of the first integrals Φ 's only, thus arranged in a system of n independent first integrals in involution. We have therefore found a canonical transformation that sends the

old Hamiltonian $H(p, q)$ into a new Hamiltonian $K(I)$, which now depends only on the actions. The Hamilton equations are

$$\dot{I}_k = -\frac{\partial K(I)}{\partial \varphi^k} = 0, \quad \dot{\varphi}^k = \frac{\partial K(I)}{\partial I^k},$$

i.e., the actions are first integrals (as already known) and the angles evolve linearly in time.

Notice that all the completely integrable systems with the same dimensions are *locally* isomorphic, being described by a foliation in tori, but differ for the singularity distribution. As an elementary example, let us consider a pendulum: the phase space is a cylinder and exhibits two equilibrium positions, the first is stable, the second unstable; this latter is a homoclinic point and is joined to itself by two separatrices, which are dynamically covered in infinite time (see Figure 3.3 on page 117). Cutting out these singularities, we are left with three disconnected components, each of them diffeomorphic to the product of a circle with an open interval of the real line: the inside of the two separatrices is the oscillatory or libration¹ zone, whereas the other two are the circulation zones. Comparing the harmonic oscillator with the pendulum, one sees that now the phase space is the plane minus the origin, and the system is isomorphic only to the libration zone of the pendulum.

1.5 Quasi-Integrable Hamiltonian Systems and KAM Theorem

Let us consider a completely integrable system to which we add a “small” nonintegrable perturbation or, in brief, a *quasi-integrable* system. The Hamiltonian will be of the type

$$H(I, \varphi) = H_0(I) + \varepsilon H_p(I, \varphi), \quad \varepsilon \ll 1. \quad (1.5.1)$$

As said previously, the central question is: Does this very small perturbation destroy the foliation in tori completely? Before proceeding, we consider a numerical experiment encompassing the essence of the problem, as will be clear later in the course of the book. Let us consider the *standard map*, introduced by Chirikov (1979) and regarding a symplectic transformation $S: x \rightarrow x'$ of the plane into itself:

$$x'_1 = x_1 + \varepsilon \sin x_2, \quad x'_2 = x_2 + x'_1.$$

For $\varepsilon = 0$ the evolution of the two variables is very simple: x_1 , which is of the action type, stays unchanged while x_2 , an angular variable, grows

¹ From the Latin *libra*, i.e., balance.

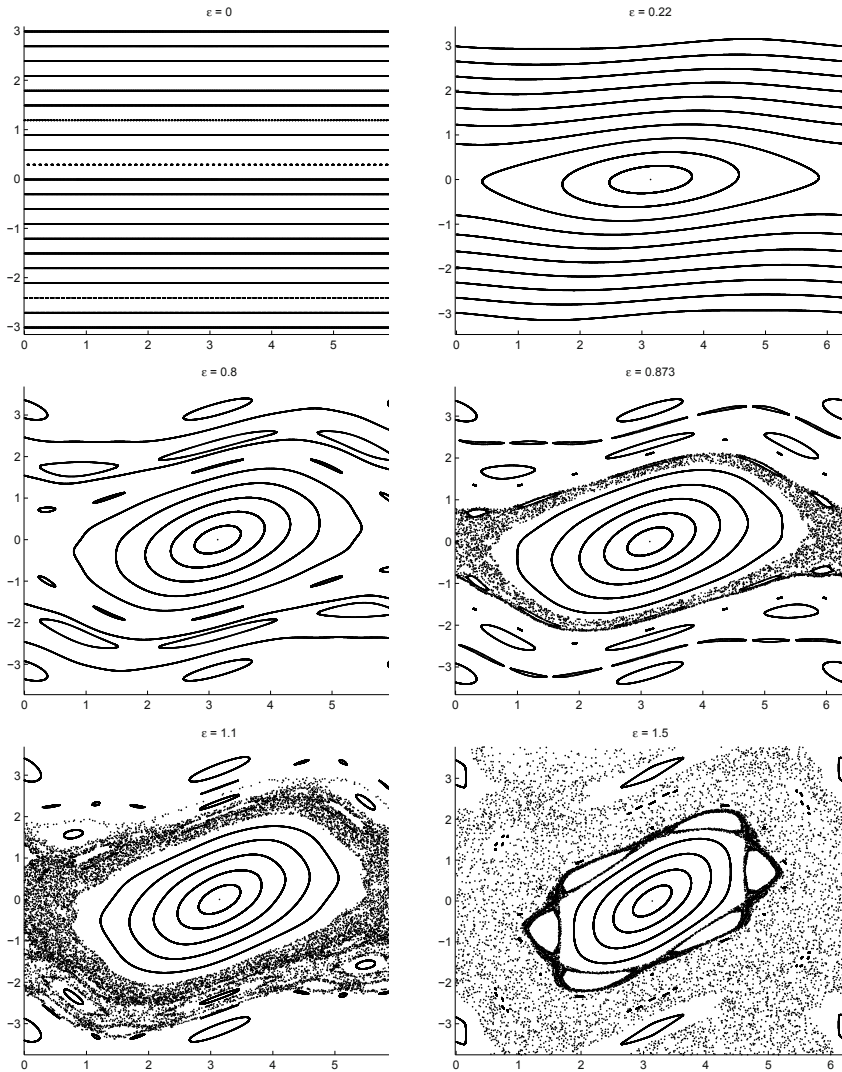


Figure 1.1: Rise of chaos in the standard map.

linearly: see Figure 1.1. Turning on the perturbation with $\varepsilon = 0.22$, the topology of the foliation changes abruptly, strongly resembling that of the pendulum, and what are called the *resonant* tori appear. Increasing the perturbation to $\varepsilon = 0.80$, some tori of the circulation zone are destroyed and replaced by a chain of adjacent resonant tori, even though the overall regularity of the motion seems preserved. However, increasing further to

$\varepsilon = 0.873, 1.1$, and 1.5 one sees that the foliation is progressively replaced by more and more wide zones of chaotic evolution, with the survival of some small islands of order. Notice, moreover, that also some tori of the libration zone break down and are replaced by a chain of resonant tori of second level, which in turn generate their own chaos.

Going back to the generic perturbed Hamiltonian (1.5.1), we see that the problem is no longer solvable through quadratures. Then, we proceed looking for a canonical transformation $I, \varphi \rightarrow I', \varphi'$, that differs from the identity by a quantity of order ε , such that the transformed Hamiltonian is integrable up to the second order terms. One may iterate this procedure, pushing the perturbation to the third order, and so on. If the process converges (but this is the key point), by increasing the order one obtains better and better approximations.

Let the canonical transformation $\varphi, I \mapsto \varphi', I'$ be generated by $W = \varphi \cdot I' + \varepsilon S(\varphi, I')$, that is

$$I = I' + \varepsilon \frac{\partial S}{\partial \varphi}, \quad \varphi' = \varphi + \varepsilon \frac{\partial S}{\partial I'},$$

where S is, for the moment, unknown. Define the *averaged* perturbation Hamiltonian

$$\overline{H_p}(I) = \frac{1}{(2\pi)^n} \int_0^{2\pi} \cdots \int_0^{2\pi} H_p(\varphi, I) d\varphi^1 \cdots d\varphi^n$$

and the *frequency vector*

$$\omega = \omega(I') = \left(\frac{\partial H_0}{\partial I} \right)_{I=I'}.$$

As one easily verifies, if we are able to find a function S that solves the *homological equation*

$$\omega \cdot \frac{\partial S}{\partial \varphi} + H_p(\varphi, I') - \overline{H_p}(I') = 0,$$

we succeed in pushing the perturbation to the second order. Iterating this procedure, we hope to end up with a canonical transformation $\text{Can}^\infty : I, \varphi \rightarrow I^\infty, \varphi^\infty$ and a completely integrable Hamiltonian $H_0^\infty(I^\infty)$. Therefore (but, we stress again, provided the procedure converges) the phase space of the perturbed Hamiltonian system would be foliated by n -dimensional hypersurfaces diffeomorphic to tori. The perturbation would simply cause a deformation of the original tori, i.e., those related to the unperturbed Hamiltonian, without destroying the well-ordered pattern.

The critical points in pursuing the outlined program are the following two: the solution of the homological equation and the convergence of the sequence of canonical transformations leading to $\text{Can}^\infty : I, \varphi \mapsto I^\infty, \varphi^\infty$.

Let us consider the first point. In order to solve the homological equation, we resort to Fourier series

$$H_p(\varphi, I') - \overline{H_p(I')} = \sum_{k \neq 0} H_k(I') e^{ik \cdot \varphi}, \quad S(\varphi, I') = \sum_{k \neq 0} S_k(I') e^{ik \cdot \varphi},$$

where $k = k_1, \dots, k_n$ is a vector with integer components. We drop the term $k = 0, \dots, 0$ since the mean value of S would be annihilated by the differentiation operator; this also imposes that the remaining part in the homological equation have null average. The *formal* solution of the homological equation is

$$S_k(I') = \frac{iH_k(I')}{\omega \cdot k},$$

which, however, shows that we are facing a serious convergence problem: clearly, there exist frequency vectors such that $\omega \cdot k = 0$ for some k , and this makes the formal solution meaningless. Such a frequency vector is called *resonant*, and it is characterized by the reciprocal rationality of its components. We must thus exclude such resonant terms and, moreover, those terms for which $\omega \cdot k$ is much smaller than the corresponding H_k in the numerator. This is the celebrated problem of the “small divisors” or “small denominators.”

The situation may appear hopeless, but, fortunately, a classical result in Diophantine theory guarantees that $\omega \cdot k$ can be bounded from below, without yielding an empty set. More precisely, the inequality

$$|\omega \cdot k| \geq \frac{\gamma}{|k|^n} \quad \forall k \in \mathbb{Z}^n - \{0\}, \quad |k| \stackrel{\text{def}}{=} \sum_j |k_j|,$$

for some positive γ is satisfied by a set of real vectors ω of large relative measure, the complement of this set having Lebesgue measure $\mathcal{O}(\gamma)$. This inequality is referred to as the *Diophantine condition*. This is a key point. In fact, if H_p is analytic, it is easy to prove that its Fourier coefficients H_k decay exponentially with $|k|$, while $\frac{1}{|\omega \cdot k|}$ grows at most as a power, thanks to the Diophantine condition. This enables us to prove the convergence of the formal expansion.

The first point is thus overcome but at a price: the frequency vectors that do not satisfy the Diophantine condition, hence the corresponding tori, must be excluded, and the foliation of the phase space by invariant tori is lost: conserved and destroyed tori are mixed together, the first ones forming a complicated Cantor set.

The second point is technically more difficult. Basically, one fixes a torus to which there corresponds a frequency vector satisfying the Diophantine condition, then proves that, if the perturbation parameter is sufficiently small, the procedure converges to Can^∞ . The proof, however, requires a further condition: in order to keep the frequency vector fixed when higher

order terms of the perturbation come into play (a necessary condition, since in its, no matter how small, neighborhood there are other vectors *not* satisfying the Diophantine condition), one must slightly change the torus; hence, it is necessary that the *frequency map* $I \rightarrow \omega(I)$ be at least locally invertible, thus $\det \left(\frac{\partial \omega_i}{\partial I_k} \right) = \det \left(\frac{\partial^2 H_0}{\partial I_i \partial I_k} \right) \neq 0$.

The above discussion is summarized by the famous KAM theorem.

THEOREM 1.3 (KAM) *Given the perturbed Hamiltonian*

$$H(\varphi, I) = H_0(I) + \varepsilon H_p(\varphi, I),$$

with $\det \left(\frac{\partial^2 H_0}{\partial I_i \partial I_j} \right) \neq 0$, for every set I^ of the actions such that the unperturbed frequencies $\omega(I^*) = \frac{\partial H_0}{\partial I}(I^*)$ satisfy the Diophantine condition, the tori $I^* = \text{constant}$ survive, though slightly deformed, with respect to sufficiently small perturbations.*

Notice that the destroyed tori are not *completely* replaced by chaotic orbits. Indeed, inside a resonance we can find an adapted Hamiltonian (called *normal resonant*), which turns out to be the one corresponding to a slightly perturbed pendulum. Again, from the KAM theorem one expects the existence of regular resonant tori, which are obtained by deforming those of the unperturbed, thus integrable, pendulum; these, in turn, can develop secondary resonances with their related normal resonant Hamiltonians, then the pattern repeats itself endlessly. The chaos is restricted to the orbits starting in the thin stochastic layer surrounding the separatrices of the pendulum created by the resonances. These orbits “hesitate” among libration and clockwise or counterclockwise circulation, giving rise to chaotic dynamics through the mechanism of the *homoclinic tangle*.

1.6 Geography of the Phase Space

At this point, we are able to sketch the overall structure of the phase space of a quasi-integrable system, taking into account that the KAM theorem changes our point of view: from now on, our attention will be focused not on the single orbits but on the tori, since all the orbits on the same torus share the same destiny. Among other things, this drastically simplifies the work, reducing from $2n$ to n the number of the classifying parameters, for which we may use indifferently the actions or the relative frequency vector.

When the perturbative parameter grows, the nature of the phase space changes, covering, in ascending order, three different situations.

- (i) *KAM*: essentially all points are regular, almost all unperturbed tori are conserved, and the dynamics is basically controlled by the KAM the-

orem. The system is in practice indistinguishable from a completely integrable one.

- (ii) *Nekhoroshev*: the measure of the destroyed tori is small but not negligible. They form an *Arnold web*, which in the frequency space is given by frequencies satisfying the resonance relations $\omega \cdot k = 0$, along with a neighborhood decreasing exponentially with the order $\sum_{j=1}^n |k_j|$ of the resonance itself. The Arnold web is therefore the union of the neighborhoods of all the hyperplanes of codimension one through the origin and with rational slope. Assume for simplicity $n = 3$. In [Figure 3.2](#) on page 116 a section with the plane $\omega_3 = 1$ in the 3-dimensional frequency space is shown, thus with equation $k_1 \omega_1 + k_2 \omega_2 + k_3 = 0$: the “skeleton” is formed by the lines whose slope and intersection with the axes take rational values, “fleshed out” by the resonance strips. The Arnold web is *connected, open, and dense* in the action space with, however, a relative small measure vanishing with the square root of the perturbative parameter. On a 2-dimensional energy surface of the action space an image of [figure 3.2](#) appears, distorted under the diffeomorphism given by the local inverse of the frequency map $I \rightarrow \omega(I)$. The dynamics is still controlled almost everywhere by the KAM theorem except for the Arnold web, where it is controlled by the Nekhoroshev theorem. A point of a stochastic layer orbit (i.e., exactly on the border of a resonance) can in principle travel along the whole Arnold web, reaching the neighborhood of every point in action space but in a very long time, which grows exponentially with the inverse of the perturbative parameter. This phenomenon, whose existence is not in general proven, is known as *Arnold diffusion*.
- (iii) *Chirikov*: the global measure of the resonances does not leave any place for invariant tori, and the dynamics is no longer controlled by the KAM and Nekhoroshev theorems but by the Chirikov overlapping criterion. When the resonances overlap, the motion can jump from one resonance to another, giving rise to large-scale diffusion with a time scale much shorter with respect to the Arnold diffusion. The system is fully chaotic.

1.7 Numerical Tools

To better understand the KAM theorem, it is useful to proceed with some numerical examples regarding case (ii), which is surely the most interesting. Several tools can be used.

The Poincaré section is a long-standing method, very effective for systems with two degrees of freedom, thus with a 4-dimensional phase space

and a 3-dimensional hypersurface of constant energy. Sectioning with a plane and recording the points where it is crossed by an orbit, one can visualize the trace of the torus, if any, around which the orbit winds; a non-structured dust will denote instead a chaotic orbit. Two examples are given in [figure 3.5](#) on page 121 and in [Figure 5.1](#) on page 149.

As already pointed out, the usefulness of the method is clearly restricted to systems with just two degrees of freedom; moreover, if the perturbation is very small, the resonances are extremely thin and may escape from the visualization. In fact, the method is not in the spirit of the KAM theorem, since it focuses attention on the orbits, instead of on the tori. In contrast, the following methods are based on frequency analysis, and as such are tori-oriented.

The Fast Fourier Transform (FFT) is the implementation of the elementary Fourier transform and is applicable to the output of a numerical integration. If all the computed frequencies are a linear combination with integer coefficients of some n fundamental ones, the spectrum is regular and the motion winds around a KAM torus.

The Frequency Modified Fourier Transform (FMFT) allows one to find the spectrum of a “signal” $Z(t)$, but seeking numerically the maximum of the function

$$\phi(\omega) = \frac{1}{2\pi} \int_{-T}^T Z(t) e^{-i\omega t} dt.$$

The output is decisively more accurate, but nothing is perfect, and trying to resolve two very close frequencies yields a slightly imprecise result.

The Frequency Modulation Indicator (FMI) exploits just this imprecision to detect the resonances. It associates to each n -tuple of action values, hence to each torus, a number that measures how much the fundamental frequencies are frequency modulated. Indeed, for a KAM torus the n fundamental frequencies (i.e., those coming from the first component in the Fourier analysis) are time-constant; on the contrary, inside a resonance the superimposed pendulum causes a frequency modulation. Without going into detail on how this frequency modulation is numerically detected, we can reach the conclusion that a picture plotting the FMI values as a function of the actions will be able to represent the distribution of the resonances.

In [Figure 8.11](#) on page 249 an example is given for a system with three degrees of freedom. The Arnold web in the action space shows up clearly, and distortions of the pattern in [Figure 3.2](#) appear. The dark blue indicates negligible values of the FMI corresponding to KAM tori, whereas light blue, yellow, and red indicate intermediate and high values, i.e., resonances and chaos. Zooming into a resonance shows that the structure repeats over and over. The outcome of these numerical experiments is the concrete possibility of detecting more and more resonances, as long as we can afford to pay the price of computational complexity and time costs.

With regard to the Arnold diffusion, its existence has been proven to be possible, while its practical relevance is an open question; in particular it is unknown if the phenomenon is generic for every quasi-integrable Hamiltonian system. With the same software used for the computation of the FMI, one can numerically measure the possible drift of the values of the fundamental frequencies in a long-time orbit, thus recognizing a transition through different tori. Some preliminary results (*very* expensive in CPU time terms) seem to suggest that actually the phenomenon is generic: as expected, the points starting on the edge of a resonance, thus in the thin stochastic layer surrounding the separatrices, travel but very slowly along the resonance strips.

Equipped with such analytical and numerical tools, we can tackle some concrete examples regarding the perturbed Kepler (i.e., two-body) problem and the multi-body gravitational problem.

1.8 The Perturbed Kepler Problem

The starting point is the isomorphism between the regularized Kepler problem and the geodesic flow on the sphere. To get a geometrical insight, consider for simplicity the 2-dimensional Kepler problem and a geodesic circle on a 2-dimensional sphere which can be safely rotated into the position

$$X_1 = \sin s, \quad X_2 = -\cos \beta \cos s, \quad X_3 = \sin \beta \cos s.$$

The vector (X_1, X_2, X_3) is orthogonal to the vector $(0, \sin \beta, \cos \beta)$; β measures the angle between the equator $X_3 = 0$ and the circle in question, whereas s is the angle along the circle itself. The definition $Y_k = \frac{dX_k}{ds}$ yields

$$Y_1 = \cos s, \quad Y_2 = \cos \beta \sin s, \quad Y_3 = -\sin \beta \sin s.$$

Since the explicit form of the extended stereographic mapping (see [figure 2.1](#) on page 19) is

$$x_k = \frac{X_k}{1 - X_{n+1}}, \quad y_k = Y_k (1 - X_{n+1}) + X_k Y_{n+1},$$

the image of the circle and its tangent vector under the extended stereographic projection is

$$\begin{aligned} x_1 &= \frac{\sin s}{1 - \sin \beta \cos s}, & x_2 &= -\frac{\cos \beta \cos s}{1 - \sin \beta \cos s}, \\ y_1 &= \cos s - \sin \beta, & y_2 &= \cos \beta \sin s. \end{aligned}$$

With $q_k = y_k$ and $E = \sin \beta$, this takes the form

$$q_1 = \cos s - E, \quad q_2 = \sqrt{1 - E^2} \sin s,$$

which is the representation of an ellipse of eccentricity $E = \sin \beta$ in terms of the eccentric anomaly s . Moreover, with $p_k = -x_k$ we get

$$p_1^2 + (p_2 - \tan \beta)^2 = 1 + \tan^2 \beta,$$

which is the representation of a circle in the Cartesian plane $p_1 p_2$, which is the *hodograph curve*. Therefore, the trajectory of the moving point of the Kepler problem is the direct projection, followed by a translation, of a geodesic circle onto the equatorial plane, while the trajectory of the velocity is the stereographic projection.

Generalizing to the 3-dimensional case, some facts appear relevant. First, the group $SO(4)$, which acts isometrically on the 3-dimensional sphere, is the symmetry group of the Kepler problem; then, the $SO(2)$ group generates the motion on the geodesic circle; lastly, the dynamical evolution of position and velocity can be parametrized with two orthogonal vectors spanning the circle itself. Roughly speaking, the two groups and the couple of vectors fit together to form the dynamical group $SO(2,4)$.

Taking the two orthogonal vectors as dynamical variables also turns out to be suited for studying the perturbed case, for example the hydrogen atom in electric and magnetic fields. The cotangent bundle to the 3-dimensional sphere, i.e., the phase space of the regularized Kepler problem, is twofold reduced. Thus we arrive to a 2-dimensional spheroid, on which the intersections of the level surfaces of the perturbation Hamiltonian describe globally the essence of the dynamics, obviously up to fast oscillations. See, e.g., [Figures 8.1–8.4](#) on pages 240 and 241.

1.9 The Multi-Body Gravitational Problem

Deducing the motion of bodies interacting gravitationally is probably the most important mechanical problem but also the most difficult. Already the three-body problem is not integrable, even if the masses are very small but of comparable size, and this fact generally prevents the use of perturbative methods.

Some important exceptions are: the planar three-body problem, which admits a global treatment in its two limit, i.e., lunar and planetary cases; then the classical 3-dimensional planetary problem. By the *planetary problem* one means the mechanical system consisting of a body of large mass, the “Sun,” and other bodies much smaller, the “planets,” interacting through gravitational forces. By the *lunar problem* one means the system consisting of a small body, the “Moon,” rotating around the “Earth,” with a third body, the “Sun,” much more distant.

Let us consider the planar case. The planar system is first reduced to four degrees of freedom thanks to its translational invariance; then, averaging along the unperturbed motion, it is further reduced to two degrees of

freedom. The averaged Hamiltonian inherits, from the original one, the rotational plane invariance, and this symmetry results in a further reduction to a system with one degree of freedom, hence integrable.

In the 3-dimensional case, let us consider the three-body planetary problem, the extension to the generic case being straightforward. The system is easily reduced to two uncoupled Kepler problems plus a perturbative term proportional to the inverse of the distance Δ between the two planets. The secular Hamiltonian is obtained by averaging Δ^{-1} along the unperturbed motion, i.e., along the Keplerian ellipses. Unfortunately, this is a nontrivial task, which cannot be carried out in a closed form and requires two preliminary steps. With the first step the expression of the two position vectors is put in a suitable form, i.e., as a function of an evolutionary parameter, closely related to the time, and of five constant parameters characterizing the ellipse. The second step consists in a series expansion of Δ^{-1} with respect to eccentricity and inclination.

To accomplish the first step, one could use the Keplerian elements of the orbit, but they suffer from the drawback of being singular for orbits that are circular and/or lying on the reference (ecliptic) plane. In contrast, the Poincaré variables are regular for orbits with small eccentricities and inclinations, and are thus well suited for studying the planetary problem.

The Keplerian elements of the orbit have a clear geometrical interpretation: semimajor axis and numerical eccentricity fix size and shape of the ellipse, while inclination, longitude of the ascending node, and argument of the pericenter are the three Euler angles fixing the spatial orientation of the ellipse. In contrast, the Poincaré variables are usually defined in a purely algebraic manner and lack a geometrical interpretation. This makes finding the expansion of the two position vectors somewhat involved and awkward, which surely does not simplify the subsequent series development and averaging process. It can be shown that exploiting the geometry of the $SO(3)$ group allows one to write the expression of the Keplerian motion in a very suitable form. Then the development is performed in such a way that it is immediate, by direct inspection, to detect the terms that vanish under the averaging process. This produces a drastic simplification and allows us to smartly group the surviving terms in a reasonable and adequate manner, the final result being an even, real-valued polynomial in the Poincaré canonical variables. Taking into account only the first quadratic terms, one gets the classical Lagrange–Laplace planetary theory, whose dynamics is compared with the true one: see [Figures 9.10](#) on page 298, [9.11](#) on page 299, [9.13](#) on page 301, and [9.14](#) on page 302.

Lastly, two numerical examples of the distribution of the resonances in our solar system are computed with the FMI method: see [Figure 9.15](#) on page 303.

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