# Preface

*Knowledge without theory is blind*
*and without practice is void.*

— I. Kant

The Newtonian program, well known by every student, is conceptually simple and attractive: given a mass distribution and the forces acting on it, write the differential equations arising from the fundamental law of dynamics and solve them in order to obtain the motion. Unfortunately, things are not so simple, and in the course of the program one encounters at least two essential and unavoidable obstacles.

First, we are not able in general to solve (technically, to integrate) a system of differential equations. Yes, every young student has learned how to tackle the harmonic oscillator, the two-body problem, or the free rigid body. But it is discouraging that these systems, along with a few others discovered mainly in the second half of the past century, exhaust the small list of integrable systems.

But even if one possessed magically an analytical formula giving exactly the time evolution, it would still be scarcely useful for various reasons. For example, the motion is in general very complicated, and following the solution in its wandering does not give valuable information about the nature of the phenomenon. What is more, a possible regularity in the motion is difficult to detect by simply inspecting the dynamical evolution of the physical coordinates. Another frequent difficulty is the extreme sensitivity to the initial conditions ("butterfly effect"), which in practice makes the concept of solution itself meaningless. But this should not come as a surprise: after all, everybody has felt a sense of frustration looking at the numerical solution of some three-dimensional systems, being unable to extract a meaning from the entangled trajectory appearing on the monitor.

The aim of this book is to show how to overcome these difficulties and grasp the essence of the dynamics in the particular but very important and significant case of quasi-integrable systems, i.e., integrable systems slightly perturbed by other forces. A paradigmatic case is the solar system, where the perturbations are the interactions among the planets. Besides their practical importance, these systems are also extremely interesting from a mathematical point of view, exhibiting an intricate and fascinating structure known as the "Arnold web."

In the book these systems will be studied both from the analytical and the numerical point of view. With regard to the first point, I think that it is impossible to overestimate the importance of the role played by the symplectic structure of the phase space or, in more traditional language, by the Hamiltonian form of the equations of motion. This structure is the natural one of the phase space, exactly as the Euclidean structure is the fundamental one of our physical space. It is the symplectic structure that forces the solutions of the integrable systems to evolve linearly on tori (products of circles) with some fundamental frequencies, providing the framework without which the two main theorems of perturbation theory, i.e., KAM and Nekhoroshev, could not even be enunciated. It is thus not surprising that, as already devised by the great founders of the analytical mechanics in the nineteenth century, one should constantly utilize the symplectic (canonical) coordinates adapted to the foliation in tori, the action-angle variables, which deeply reveal the hidden properties of the perturbed motions. Exploiting the advanced techniques of perturbation theory, many examples of reduction to normal form will be given, i.e., to an integrable, hence approximate form that however reproduces the true dynamics well.

In order to compare the approximate with the true dynamics one needs numerical methods. In the book I present some tools recently introduced: the Frequency Modified Fourier Transform (a refinement of the Discrete Fourier Transform), the Wavelets (which allow one to find the instantaneous frequency) and the Frequency Modulation Indicator (which detects the distribution of the resonances among the fundamental frequencies). The reader may also find many figures that well illustrate the effectiveness of the methods and, above all, the relative software. This is surely the main feature of the book: the reader himself can and is encouraged to reproduce the various figures of the book and experiment with other situations, exploring the details of various quasi-integrable systems. I am convinced that the union of theory and practice is the main route to try to master an argument that is considered difficult.

But a more profound motivation in resorting to numerical computations arises from some lack of reliability that to a certain extent every mathematician experiences when facing a theorem proof that is particularly lengthy and intricate, and that looks more like a rhetoric speech to persuade the reader than the granitic statement of an unquestionable truth. The numerical experiments become so an essential completion of the traditional proof, reversing Truesdell's thesis of "the computer: ruin of science and threat to mankind."

I'd like to make it clear that no knowledge on computer programming is needed in order to use the software: you only have to access directly to a MATLAB installation or, subordinately, to install a free reader. Indeed, the programs support a graphical user interface and require one only to click on buttons and menu having a hopefully clear meaning: see the final appendices to the book. The supplied programs in the accompanying CD can be downloaded as an `iso` image from the publisher's website by entering the book's ISBN (978-0-8176-8369-6) into http://extras.springer.com/ and are the following.

(i) POINCARE program analyzes symplectic maps with the aid of the Frequency Modulation Indicator.

(ii) HAMILTON program analyzes Hamiltonian systems with the Frequency Modulation Indicator.

(iii) LAGRANGE program regards the Lagrange points in the three-body problem.

(iv) KEPLER program studies the perturbations of the Kepler problem.

(v) LAPLACE program concerns the dynamics of a solar system.

I used part of the material presented here in some courses on Celestial Mechanics, Hamiltonian Systems, and Perturbation Theory, addressed to advanced undergraduate students. I think that the book may serve as an introduction to specialistic literature and to a serious study of perturbation theory, with particular emphasis on the KAM and Nekhoroshev theorems. The two theorems are proved in the book skipping some details, like the technical proof of bounding inequalities, which in a first approach (and also in a second) are more distracting than illuminating, and trying instead to stress the conceptual points. But I hope that professional researchers may also find this book useful, thanks to its enclosed software.

Briefly, the plan of the work is the following. In Chapter 1 a somewhat detailed account of the whole book is given, which should also help the reader to not lose the thread of the argument. Chapter 2 contains the basic concepts of differential geometry, Lie groups, and analytical mechanics, which Chapter 3 applies to perturbation theory. Chapter 4 deals with numerical integration of ordinary differential equations and Chapter 5 with some tools useful to numerically detect order and chaos. The final four chapters are devoted to the applications, i.e., to the perturbations of the Kepler problem, as the hydrogen atom in an electric and magnetic field, and to the planetary problem. These concrete applications are not only physically interesting but are also significant examples of how to investigate in general quasi-integrable Hamiltonian systems, combining the techniques of the reduction to normal form with the numerical analysis of how order, chaos, and resonances are distributed in phase space.

It is always useful to listen to several different voices on the same argument. Three books in particular are highly recommendable: Celletti (2010), Morbidelli (2002), and Ferraz-Mello (2007). More or less they cover the same topics of the present book, with a major emphasis on the applications but without including any software. A good introduction to this book is Tabor (1989).

I thank very much the two anonymous referees for all the useful comments and suggestions, which I've (almost) fully included.

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