Preface

Frame theory is nowadays a fundamental research area in mathematics, computer science, and engineering with many exciting applications in a variety of different fields. Introduced in 1952 by Duffin and Schaeffer, its significance for signal processing has been revealed in the pioneering work by Daubechies, Grossman, and Meyer in 1986. Since then frame theory has quickly become the key approach whenever redundant, yet stable, representations of data are required. Frames in finite-dimensional spaces, i.e., finite frames, are a very important class of frames due to their significant relevance in applications. This book is the first comprehensive introduction to both the theory and applications of finite frames, with various chapters outlining diverse directions of this intriguing research area.

Today, frame theory provides an extensive framework for the analysis and decomposition of signals in a stable and redundant way, accompanied by various reconstruction procedures. Its main methodological ingredients are the representation systems which form a frame. In fact, a frame can be regarded as the most natural generalization of the notion of an orthonormal basis. To be more specific, let $(\varphi_i)_{i=1}^M$ be a family of vectors in $\mathbb{R}^N$ or $\mathbb{C}^N$. Then these vectors form a frame if there exist constants $0 < A \leq B < \infty$ such that $A\|x\|_2 \leq \|(\langle x, \varphi_i\rangle)_{i=1}^M\|_{\ell_2} \leq B\|x\|_2$ holds for all x in the underlying space. The constants A and B determine the condition of a frame, which is optimal for $A = B = 1$, leading to the class of Parseval frames. It is evident that the notion of a frame allows the inclusion of redundant systems in the sense of overcomplete systems. This is key to the resilience of frames to disturbances (such as, e.g., noise, erasures, and quantization) of the frame coefficients $(\langle x, \varphi_i\rangle)_{i=1}^M$ associated with a signal x. These frame coefficients can be utilized, for instance, for edge detection in an image, for the transmission of a speech signal, or for recovery of missing data. Although the analysis operator $x \mapsto (\langle x, \varphi_i\rangle)_{i=1}^M$ maps a signal into a higher-dimensional space, frame theory also provides efficient methods for reconstructing the signal.

New theoretical insights and novel applications are continually arising, because the underlying principles of frame theory are basic ideas which are fundamental to a wide canon of areas of research. In this sense, frame theory might be regarded as partly belonging to applied harmonic analysis, functional analysis, and operator theory, as well as numerical linear algebra and matrix theory. Some of its countless applications are in biology, geophysics, imaging sciences, quantum computing, speech recognition, and wireless communication, to name a few.

In this book we depict the current state of the research in finite frame theory and cover the progress which has been made in the area over the last twenty years. It is suitable for both a researcher who is interested in the latest developments in finite frame theory, and also for a graduate student who seeks an introduction into this exciting research area.

This book comprises (in addition to the introduction) twelve chapters, which cover a variety of topics in the theory and applications of finite frames written by well-known leading experts in the subject. The necessary background for the subsequent chapters is provided in the comprehensive introductory chapter on finite frame theory. The twelve chapters can be divided into four topical groups: Frame properties (Chaps. 2–4), special classes of frames (Chaps. 5 and 6), applications of frames (Chaps. 7–11), and extensions of the concept of a frame (Chaps. 12 and 13). Every chapter contains the current state of its respective field and can be read independently of the others. We now provide a brief summary of the content of each chapter.

Chapter 1 provides a comprehensive introduction to the basics of finite frame theory. After answering the question *why frames?*, background material from Hilbert space theory and operator theory is presented. The authors then introduce the basics of finite frame theory and the operators connected with a frame. After this preparation the reader is equipped with an overview of well-known results on the reconstruction of signals, the construction of special frames, frame properties, and applications.

Chapter 2 deals with constructing frames with prescribed lengths of the frame vectors and a prescribed spectrum of the frame operator. Several years of research have now led to a complete solution of this problem, which is presented in this chapter. The authors show in great detail how methods stemming from the Spectral Tetris algorithm can be utilized to achieve an algorithmic solution to the problem.

Chapter 3 is devoted to the problem of partitioning a frame into a minimal number of linearly independent or a maximal number of spanning subsets. A direct application of the Rado-Horn theorem would solve the first problem, but it is much too inefficient and does not make use of frame properties. However, the authors improve the Rado-Horn theorem and derive various results solving the problem in special cases using frame properties.

Chapter 4 accommodates the fact that (besides analytic and algebraic properties) frames can also be analyzed from a geometric standpoint. Accompanied by several examples, it is shown how methods from algebraic geometry can be successfully exploited to obtain local coordinate systems on the algebraic variety of frames with prescribed frame operator and frame vector lengths. After that, angles and metrics on the Grassmannian variety are defined. They are then used to prove that the generic Parseval frames are dense in the class of Parseval frames. The chapter ends with a survey of results on signal reconstruction without phase from an algebraic geometry viewpoint.

Chapter 5 establishes a connection between finite group theory and finite frame theory. The frames of investigation are called group frames. These are frames which are induced by unitary group actions on the underlying Hilbert space; harmonic frames are a special class of group frames. One of the highlights of the chapter is the utilization of group frames to construct equiangular frames, which are most desirable in applications due to their resilience to erasures.

Chapter 6 provides a basic self-contained introduction to Gabor frames on finite Abelian groups. In the first half of the chapter the main ideas of Gabor analysis in signal processing are illuminated, and fundamental results for Gabor frames are proved. The second half deals with geometric properties such as linear independence, coherence, and the restricted isometry property for Gabor synthesis matrices, which then gives rise to the utilization of Gabor frames in compressed sensing.

Chapter 7 studies the suitability of frames for signal recovery from encoded, noisy, or erased data with controllable accuracy. After providing a survey of results on the resilience of frames with respect to noisy measurements, the author analyzes the effects of erasures and error correction. One main result states that equiangular and random Parseval frames are optimally robust against such disturbances.

Chapter 8 considers frame quantization, which is essential for the process of digitizing analog signals. The authors introduce the reader to the ideas and principles of memoryless scalar quantization as well as to first order and higher order sigma-delta quantization algorithms, and discuss their performance in terms of the reconstruction error. In particular, it is shown that an appropriate choice of quantization scheme and encoding operator leads to an error decaying exponentially with the oversampling rate.

Chapter 9 surveys recent work on sparse signal processing which has become a novel paradigm in the last year. The authors address problems such as exact or lossy recovery, estimation, regression, and support detection of sparse signals in both the deterministic and probabilistic regimes. The significance of frames for this methodological approach is, for instance, shown by revealing the special role of equal norm tight frames for detecting the presence of a sparse signal in noise.

Chapter 10 considers the connection of finite frames and filter banks. After the introduction of basic related operations, such as convolution, downsampling, the discrete Fourier transform, and the Z-transform, the polyphase representation of filter banks is proved to hold, and its properties and advantages are discussed. Thereafter, the authors show how various desiderata for the frame connected with a filter bank can be realized.

Chapter 11 is split into two parts. The first part presents a variety of conjectures stemming from diverse areas of research in pure and applied mathematics as well as engineering. Intriguingly, all these conjectures are equivalent to the famous 1959 Kadison-Singer problem. The second part is devoted to the Paulsen problem, which is formulated in pure frame theoretical terms and is also still unsolved.

Chapter 12 presents one generalization of frames, called probabilistic frames. The collection of these frames is a set of probability measures which contains the usual finite frames as point measures. The authors present the basic properties of probabilistic frames and survey a range of areas such as directional statistics, in which this concept implicitly appears.

Chapter 13 introduces fusion frames, which are a generalization of frames designed for and perfectly suited to model distributed processing. They analyze signals by projecting them onto multidimensional subspaces, in contrast to frames which consider only one-dimensional projections. Various results are reviewed, including fusion frame constructions, sparse recovery from fusion frame measurements, and specific applications of fusion frames.

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