

Preface

This book provides an introduction to geometric algebra and its application to diverse areas of mathematics. It maintains the spirit of its predecessor, *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*, and as such it has many unique features not seen in any other undergraduate textbook. It provides many innovative ways of looking at geometrical ideas and topics for student research and thesis projects.

The material has been developed over the many years that the author has taught undergraduate courses at the Universidad de Las Américas-Puebla, Mexico, in linear algebra, vector calculus, differential geometry, numerical analysis, modern algebra, and number theory. Whereas this book cannot be considered a textbook for all of these different subjects, there is a common theme they all share: they can all be efficiently formulated using the unified geometric number system advocated here. Geometric algebra, which has undergone extensive development in the second half of the twentieth Century, has its origins in the seminal works of Grassmann, Hamilton, and Clifford in the nineteenth century.

The book begins with the introduction of the spectral basis in modular number systems and in modular polynomials. This often overlooked concept provides insight into and greatly simplifies the proofs of many basic theorems in number theory and the corresponding closely related structure theorems of a linear operator. Since geometric numbers obey exactly the same algebraic rules as square matrices of real numbers, the languages are completely compatible and structure theorems that are valid for one are equally valid for the other.

The concept of a matrix as an array of numbers with an unintuitive multiplication rule hardly provides a geometric way of looking at things. Nevertheless, matrices have proven to be an extremely effective computational tool and have played a major role in the development of diverse areas of mathematics. Geometric algebra rectifies this defect by providing a geometric perspective, and many new algebraic tools. Combining both of these powerful systems by simply considering matrices whose elements are geometric numbers adds much needed geometric content and flexibility to both languages. The author hopes that this book captures both the idea and the

spirit of the powerful geometric number system that has kept him going since he learned about the subject as a graduate student at Arizona State University many years ago.

We assume readers to have had undergraduate differential and integral calculus, a first course in modern algebra, and the mathematical maturity that an upper-level mathematics or physics undergraduate student might be expected to have. The many topics covered in the book should also appeal to first-year graduate students in mathematics, physics, engineering and computer science. Any unfamiliarity that a reader might have regarding mathematical terminology can be quickly overcome by a quick reference to the unlimited resources on the internet. We also recommend that the reader has knowledge of and access to symbolic mathematical software such as Mathematica or Maple. Such software considerably lightens the computational work required and makes for easy verification of results. A simple Mathematica package is provided for calculating the spectral basis for a modular polynomial.

There are three main groupings of interrelated core chapters:

- Chapters 1–5 introduce the fundamental concepts of a spectral basis of modular numbers and modular polynomials with applications in number theory, numerical analysis, and linear algebra. The hyperbolic numbers, introduced alongside the well-known complex numbers, are used to solve the cubic equation and provide a mathematical foundation for the theory of special relativity. The geometric extension of the real numbers is achieved by introducing new *anticommuting* square roots of plus or minus one which represent orthogonal directions in successively higher dimensions.
- Chapters 7–10 lay down the ideas of linear and multilinear algebra. Matrices of geometric numbers are considered throughout. New proofs of the Cayley–Hamilton Theorem, Gram–Schmidt orthogonalization, and the spectral decomposition of a linear operator are given in geometric algebra, as well as a comprehensive geometric interpretation of complex eigenvalues and eigenvectors in an Hermitian (definite or indefinite) inner product space.
- Chapters 13–16 develop the basic ideas of vector calculus and differential geometry in the context of geometric algebra. The classical integration theorems are derived from a single fundamental theorem of calculus. Manifolds are embedded in Euclidean or pseudo-Euclidean spaces and consequently have both intrinsic and extrinsic curvature, characterized by the projection and shape operators. Highlighted is a special treatment of conformal mappings and the conformal Weyl tensor, which have applications in physics and engineering.

Chapter 6 covers some of the more traditional topics in linear algebra which are not otherwise used in this book. Chapters 11, 12, 17, and 18 provide additional breadth and scope by treating the symmetric group, by giving a novel look at the concept of space-time in special relativity, by laying down the basic ideas of projective geometry, and by giving an introduction to Lie algebras and Lie groups, topics which are not usually covered in an undergraduate course. In the Table of Contents, a “*” is used to indicate those sections which are considerably more technical and may be omitted on first reading.

The author is indebted to many students and colleagues from around the world who have contributed much during the many stages of the development of these ideas. Foremost he is indebted to David Hestenes, who first introduced him to geometric algebra many years ago as a graduate student at Arizona State University during the years 1965–1971. The author is indebted to Roman Duda of the Polish Academy of Sciences, Bernard Jancewicz, Jan Łopuszański, and Zbigniew Oziewicz of the Institute of Theoretical Physics in Wrocław, Poland, and Stony Brook University, for encouragement and support during these difficult early years. He wants to thank Rafal Ablamowicz (USA), Timothy Havel (USA), William Baylis (Canada), and Pertti Lounesto (Finland), who contributed in different ways to the writing of this book. In addition, he wants to thank Jaime Keller for inviting him to Mexico, Luis Verde-Star (Mexico), Waldyr Rodrigues (Brazil), Josep Parra (Spain), and Eduardo Bayro-Corrochano (Mexico). Among former students and now sometimes collaborators, he wants to thank José María Pozo (Spain), Marco Antonio Rodríguez (Mexico), Omar Leon Sanchez (Mexico), and Alejandra C. Vicente (Mexico). Graphics design artist Ana Sánchez Stone was a great help with all of the figures in Chap. 11 and in particular Fig. 11.6. The author is greatly indebted to Universidad de Las Américas-Puebla and Sistemas Nacionales de Investigadores de México, for many years of support. This book could not have been written without the constant support and encouragement over the years by my wife, Wanda, my mother and father and is a present for my mother's 100th birthday.

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New Foundations in Mathematics

The Geometric Concept of Number

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2013, XIV, 370 p. 55 illus., 32 illus. in color., Hardcover

ISBN: 978-0-8176-8384-9

A product of Birkhäuser Basel