Chapter 2
Graph-Coloring Problems

A very important graph parameter is the chromatic number. For a given graph, it is the smallest number of colors for which a coloring of the vertices exists such that adjacent vertices receive different colors. The search for a proof of the four color theorem—stating that every planar map can be colored with four colors such that adjacent countries receive different colors (Fig. 2.1)—has certainly been one of the driving sources [Ore67, Saa72, Tho98] of graph theory for a long time. Presently, graph coloring plays an important role in several real-world applications and still engages exciting research.

Fig. 2.1 A proper four-coloring of a map

In this chapter we will present an important side story, the story of a conjecture formulated by Martin Kneser in 1955 that remained unsolved until 1977 (published in 1978 [Lov78]). The revolutionary method with which László Lovász settled the notorious conjecture can be seen as the origin of the field with which this book deals. Guided by some deep insight, Lovász associated a simplicial complex to a

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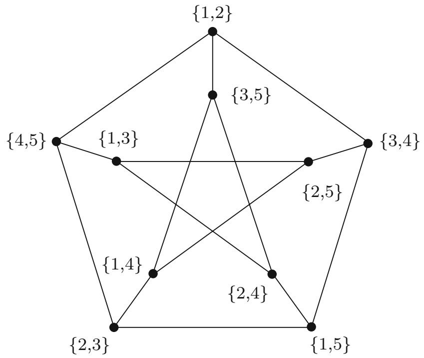

Fig. 2.2 The Petersen graph in the guise of the Kneser graph $KG_{5,2}$

graph in such a way that the topology of the complex provides some information about the chromatic number of the graph, thereby transforming a discrete problem into a topological one. The main tool he employed was the Borsuk–Ulam theorem. His proof, and the efforts to understand it, have triggered a considerable amount of research. By now, Lovász's original proof has gone through many transformations and inspired alternative proofs even until very recently. We will touch upon most of the ideas involved in the several proofs that emerged over the last decades.

2.1 The Kneser Conjecture

Kneser's original conjecture was published in 1955 as an exercise in *Jahresberichte der DMV* [Kne55], the yearly account of the German Mathematical Society. Apparently, at this time Kneser has been thinking about quadratic forms. But the connection to the conjecture seems to have been forgotten. Kneser stated his conjecture originally as a problem about sets, but the translation into a graph-theoretic problem is straightforward. Denote the set of k-subsets of $[n]$ by $\binom{[n]}{k}$.

Definition 2.1. The *Kneser graph* $KG_{n,k}$ for $n \geq 2$, $k \geq 1$, has vertex set $\binom{[n]}{k}$, and any two vertices $u, v \in \binom{[n]}{k}$ are adjacent if and only if they are disjoint, i.e., $u \cap v = \emptyset$.

To get a feeling for these graphs, let's consider first some cases with easy parameters. For $k = 1$, we obtain the complete graph on n vertices, i.e., any two vertices are adjacent. For $n = 2k$, each k-set is adjacent to and only to its complement. In other words, we obtain a complete matching, i.e., a set of disjoint edges covering all vertices. For $2k > n$, we obtain a set of vertices without any edges. For this reason, we will restrict ourselves to the cases $2k \leq n$ in the sequel. The first interesting case appears already for $n = 5$ and $k = 2$. Figure 2.2 shows the Kneser graph $KG_{5,2}$ which is the famous Petersen graph [Wes05, Die06].

As was already observed by Kneser, $KG_{n,k}$ admits a proper coloring with $n - 2k + 2$ colors simply as follows:

$$c : \binom{[n]}{k} \longrightarrow [n - 2k + 2],$$

$$u \longmapsto \min\{\min\{x : x \in u\}, n - 2k + 2\}.$$

We have to check that vertices receiving the same color are not adjacent. Consider two vertices $u, v \in \binom{[n]}{k}$ with $c(u) = c(v) = c$. If $c < n - 2k + 2$, then $c \in u \cap v$; otherwise, $u, v \subseteq \{n - 2k + 2, \ldots, n\}$. But $\{n - 2k + 2, \ldots, n\}$ contains $2k - 1$ elements, and hence u and v cannot be disjoint. The coloring witnesses the upper bound $\chi(KG_{n,k}) \leq n - 2k + 2$ for the chromatic number of the Kneser graph. Kneser conjectured that this bound was sharp, in other words, it is not smaller than $n - 2k + 2$.

Theorem 2.2 (Lovász [Lov78]). *The chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$.*

We will discuss Lovász's proof in more detail in the next section. After Imre Bárány had learned about Lovász' proof in 1978, he came up with a fairly short proof of Kneser's conjecture. Both proofs have different strengths. While Lovász's proof involves a theorem of deep insight that yields a lower bound for the chromatic number of any graph, and then specializes to the family of Kneser graphs, Bárány's proof is a fairly direct and elegant application of the Borsuk–Ulam theorem, but does not shed as much light on general graph-coloring problems.

The first proof we will discuss is the most recent proof by Greene [Gre02]. It is a tricky simplification of Bárány's proof.

Proof (topological). Assume that for some n and k the chromatic number of $KG_{n,k}$ is less than $n - 2k + 2$, and let $c : \binom{[n]}{k} \to \{1, \ldots, n - 2k + 1\}$ be a proper coloring. Set $d = n - 2k + 1$ and choose a set X of n vectors on the d-dimensional sphere $\mathbb{S}^d$ such that any $d + 1$ of them constitute a linearly independent set. Identify these n vectors with the ground set $[n]$. In other words, each vertex of $KG_{n,k}$ corresponds to a set of k vectors on the sphere. In order to apply the Borsuk–Ulam theorem, we will construct d open sets $U_1, \ldots, U_d$ and one closed set A covering $\mathbb{S}^d$. Let

$$U_i = \left\{x \in \mathbb{S}^d : \text{there exists a } k\text{-set } S \subset X, c(S) = i, S \subset H(x)\right\},$$

where $H(x) = \{y \in \mathbb{S}^d : \langle x, y\rangle > 0\}$ is the open hemisphere with pole x.

Now let $A = \mathbb{S}^d \setminus (U_1 \cup \cdots \cup U_d)$ be the complement. We show that none of the sets contains a pair of antipodal points, hence obtaining a contradiction to the Borsuk–Ulam theorem, Theorem 1.6(4).

Consider $x \in U_i$, i.e., $H(x)$ contains a k-subset of X colored with color i. Since $H(x)$ and $H(-x)$ are disjoint and c is a proper coloring, $H(-x)$ cannot contain a k-subset of X colored with i as well, and hence $-x \notin U_i$.

Now assume $\pm x \in A$. By definition of A, neither $H(x)$ nor $H(-x)$ contains a k-subset of X. Hence there must be at least $n - 2(k-1) = n - 2k + 2 = d + 1$ points of X lying on the equator $\{y \in \mathbb{S}^d : \langle x, y \rangle = 0\}$, which is contained in a subspace of dimension d. This contradicts the condition that any $d+1$ vectors of X are linearly independent. $\square$

The second proof we discuss is quite recent as well and due to Jiří Matoušek [Mat04]. It is considered to be the first combinatorial proof of Kneser's conjecture. The topological chore is an application of Tucker's lemma in the very special case that the vertices of the first barycentric subdivision of ∂Q^{n+1} are labeled. Exercise 12 on page 33 is concerned with a simple combinatorial proof for Tucker's lemma in this case.

Proof (combinatorial). Assume that there is a proper coloring $c : \binom{[n]}{k} \to \{2k-1, 2k, \dots, n-1\}$ of $KG_{n,k}$ with $n - 2k + 1$ colors. This will eventually yield a contradiction to Tucker's lemma. For this we need a little notation. Let $K = \mathrm{sd}^1\, \partial Q^n$ be the first barycentric subdivision of the boundary of the cross polytope Q^n. As explained in Exercise 12 on page 33, we can identify $\mathrm{vert}(K)$ with the set $\mathcal{Q}_n$ of nonempty subsets $v \subset \{\pm 1, \dots, \pm n\}$ such that $v \cap -v = \emptyset$. For any $S \in \mathcal{Q}_n$, let $S_+ = \{i : i > 0, i \in S\}$, resp. $S_- = \{i : i > 0, -i \in S\}$. Let $\succeq$ be an arbitrary linear order of the subsets of $[n]$ such that whenever $|A| > |B|$, then $A \succ B$. The existence of such an extension is the subject of Exercise 1 in the appendix on page 206. Now for any $A \subseteq [n]$ with $|A| \geq k$, set $\bar{c}(A) = c(A')$, where $A' \subseteq A$ is the set of the k smallest numbers in A.

We now define a labeling $\lambda : \mathcal{Q}_n \to \{\pm 1, \dots, \pm(n-1)\}$ by

$$\lambda(S) = \begin{cases} |S|, & \text{if } |S| < 2k-1 \text{ and } S_+ \succeq S_-, \\ -|S|, & \text{if } |S| < 2k-1 \text{ and } S_- \succeq S_+, \\ \bar{c}(S_+), & \text{if } |S| \geq 2k-1 \text{ and } S_+ \succeq S_-, \\ -\bar{c}(S_-), & \text{if } |S| \geq 2k-1 \text{ and } S_- \succeq S_+. \end{cases}$$

This labeling is antipodal and does not yield complementary edges and therefore yields the desired contradiction. The antipodality is obvious, and in order to see that there are no complementary edges, assume to the contrary that there are $S, T \in \mathcal{Q}_n$ forming an edge in K with $\lambda(S) = -\lambda(T) > 0$. By definition of barycentric subdivision, S is contained in T or vice versa. Since the other case works analogously, we consider only the case $S \subset T$. Hence $|S| < |T|$, and by definition of λ, we must have $\bar{c}(S_+) = \bar{c}(T_-)$. But $S_+ \cap T_- \subseteq T_+ \cap T_- = \emptyset$, which yields a contradiction to the fact that c was a proper coloring. $\square$

2.2 Lovász's Complexes

In this section we will associate several simplicial complexes to graphs. All of these constructions are due to László Lovász [Lov78, BK07]. These complexes will be used to give lower bounds on the chromatic number.

While the conditions on a proper coloring of a graph—no monochromatic edge—is a local condition, the chromatic number captures a global phenomenon. A good example is an odd cycle, which has chromatic number 3. It can be colored vertex for vertex along the cycle with two colors until the last vertex, where the true value of the chromatic number is revealed. In order to obtain bounds for the chromatic number, it is therefore necessary to capture the global behavior of the graph in some way. There are quite a few global invariants for topological spaces. In that respect, it seems natural to try to assign a topological space to a graph in such a way that the global topological properties of the space reflect some global property of the graph.

The Neighborhood Complex

We will now describe Lovász's neighborhood complex, the first construction of a simplicial complex that we associate with a graph.

Let $G = (V, E)$ be a finite simple graph. Let the *neighborhood complex* $\mathcal{N}(G)$ be the simplicial complex with vertex set V and simplices given by subsets $A \subseteq V$ such that all vertices in A have a common neighbor. As a first example consider Fig. 2.3.

Note that the neighborhood complex of a graph without edges is empty, and as soon as the graph has an edge it is nonempty.

The neighborhood complex of an odd cycle is an odd cycle of the same length. In fact, if the odd cycle has the vertex set $\{0, 1, \ldots, 2k\}$ in such a way that two vertices

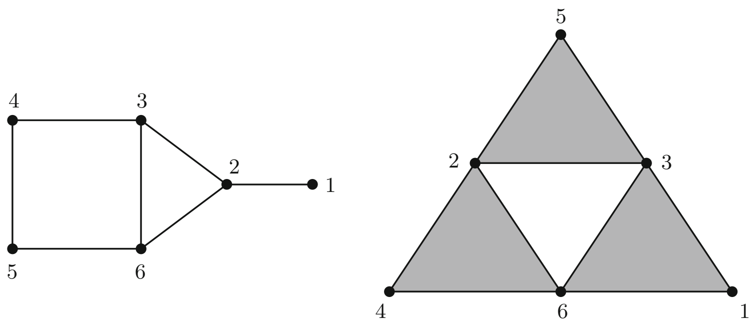

Fig. 2.3 A graph G along with its neighborhood complex

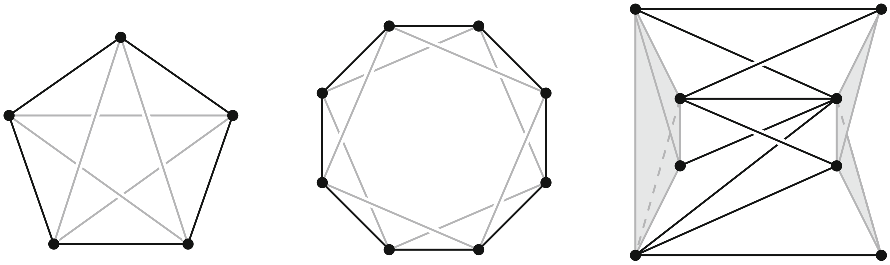

Fig. 2.4 Neighborhood complex of an odd cycle, of an even cycle, and of a bipartite graph

are adjacent if and only if they differ by one modulo $2k + 1$, then the neighborhood complex is a 1-dimensional complex with edge set

$$\big\{\{0,2\},\{2,4\},\ldots,\{2k-2,2k\},\{2k,1\},\{1,3\},\ldots,\{2k-1,0\}\big\}.$$

In the same manner, the neighborhood complex of an even cycle (of length greater than or equal to 6) consists of two cycles, each half the length of the original cycle. And more generally, if $G = (V \dot{\cup} V', E)$ is a bipartite graph with independent sets V and V', then each simplex of $\mathcal{N}(G)$ is contained in either V or V', and hence the complex is not connected. Figure 2.4 illustrates all of these cases, where in each case the graph is sketched in black and the neighborhood complex in gray. In the case of the third bipartite graph, the facets of the neighborhood complex consist of a 3-dimensional simplex and two 2-dimensional simplices sharing an edge.

As a last class of examples, we consider the neighborhood complexes $\mathcal{N}(K_n)$ of the complete graph K_n. Let us denote the vertex set of K_n by $[n]$. Then each nonempty proper subset $A \subset [n]$ has a common neighbor and therefore is a simplex of $\mathcal{N}(K_n)$. Thus, $\mathcal{N}(K_n)$ is the boundary complex of the simplex on the vertex set $[n]$ and hence a sphere of dimension $n - 2$.

We have already found some indication of the phenomenon that global properties of the neighborhood complex capture information about the chromatic number. If the neighborhood complex of a graph is nonempty, then the graph has at least one edge and therefore has chromatic number at least two. If we encounter a nonempty connected neighborhood complex of a graph G, then we already know that it cannot be bipartite and hence has chromatic number at least three.

The emerging pattern is perpetuating, as the following theorem says. We will provide an easy proof on page 50. Before we state Lovász's theorem, we should briefly remind ourselves of the topological notion of k-connectedness as defined on page 170. For more on this, and the subsequently used concepts of order topology, we refer to Appendices B and C.

Theorem 2.3 (Lovász [Lov78]). *Let $G = (V, E)$ be a finite simple graph. If the neighborhood complex $\mathcal{N}(G)$ of G is k-connected, then the graph has chromatic number at least $k + 3$. In other words,*

$$\chi(G) \geq \text{conn}(|\mathcal{N}(G)|) + 3.$$

Observe the general applicability of the theorem. Whenever we are interested in the chromatic number of a graph G, we may determine its neighborhood complex, and every lower bound we obtain for the connectivity of $\mathcal{N}(G)$ yields a lower bound for the chromatic number of G. In particular, Lovász obtained the first proof of the Kneser conjecture by showing that $\operatorname{conn}(|\mathcal{N}(KG_{n,k})|) = n - 2k - 1$. We start by giving a very beautiful and short proof of this fact.

Proposition 2.4. *The neighborhood complex $\mathcal{N}(KG_{n,k})$ of the Kneser graph is homotopy equivalent to a wedge of spheres of dimension $n - 2k$. In particular,* $\operatorname{conn}(|\mathcal{N}(KG_{n,k})|) = n - 2k - 1$.

Proof. We prove that the face poset $\mathcal{F}(\mathcal{N}(KG_{n,k}))$ of $\mathcal{N}(KG_{n,k})$, i.e., the set of nonempty faces of $\mathcal{N}(KG_{n,k})$ partially ordered by inclusion, is homotopy equivalent to the following part, $B_{n,k}$, of the Boolean lattice

$$B_{n,k} = \{S \subseteq [n] : k \leq |S| \leq n - k\}$$

By Corollary C.10 on page 206, the poset $B_{n,k}$ is lexicographically shellable. In particular, it has the homotopy type of a wedge of spheres of dimension $n - 2k$.

Define the order-preserving maps

$$\begin{aligned} f : \mathcal{F}(\mathcal{N}(KG_{n,k})) &\longrightarrow B_{n,k}, \\ F &\longmapsto \bigcup F = \{x \in [n] : \text{exists } v \in F \text{ such that } x \in v\}, \end{aligned}$$

and

$$\begin{aligned} g : B_{n,k} &\longrightarrow \mathcal{F}(\mathcal{N}(KG_{n,k})), \\ A &\longmapsto \binom{A}{k} = \{S \subset A : |S| = k\}. \end{aligned}$$

Then obviously $f \circ g = \mathrm{id}_{B_{n,k}}$, and for each $F \in \mathcal{F}(\mathcal{N}(KG_{n,k}))$, we have $\mathrm{id}_{\mathcal{F}(\mathcal{N}(KG_{n,k}))}(F) = F \subseteq (g \circ f)(F)$. By the order homotopy lemma, Lemma C.3, the maps $\mathrm{id}_{\mathcal{F}(\mathcal{N}(KG_{n,k}))}$ and $g \circ f$ are homotopic, and hence f is a homotopy equivalence. □

The Neighbor Set Function ν

Returning to the situation of a general graph G, we define ν to be the *neighbor set function*, i.e., for a subset $A \subseteq V$ of vertices of G, define $\nu(A)$ to be the set of all vertices in V that are adjacent to all vertices in A, i.e.,

$$\nu(A) = \{v \in V : v \text{ is adjacent to } a \text{ for all } a \in A\}.$$

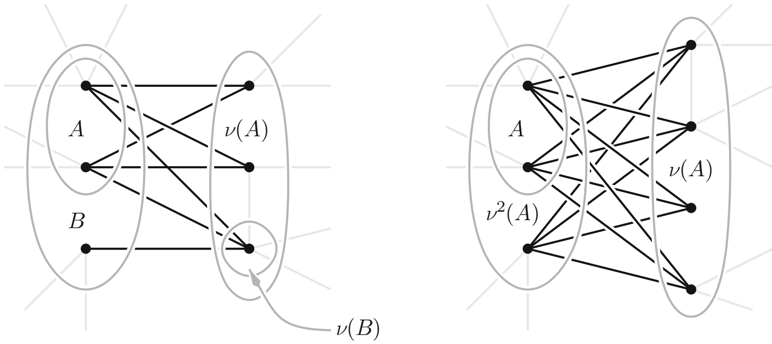

Fig. 2.5 Proof of statements 1 and 2 in Proposition 2.5

Note that for each $A \subseteq V$, the two sets A and $\nu(A)$ are the vertex sets of a complete bipartite subgraph of G, i.e., each vertex of A is adjacent to each vertex of $\nu(A)$.

With this notation, the simplices of $\mathcal{N}(G)$ are precisely given by subsets $A \subseteq V$ such that $\nu(A) \neq \emptyset$.

Proposition 2.5. *Let $G = (V, E)$ be a finite simple graph. The neighbor set function $\nu : \mathcal{P}(V) \to \mathcal{P}(V)$ of the power set of V to itself satisfies the following:*

1. *If $A \subseteq B \subseteq V$, then $\nu(B) \subseteq \nu(A)$.*
2. *$A \subseteq \nu^2(A)$ for any $A \subseteq V$.*
3. *$\nu(A) = \nu^3(A)$ for any $A \subseteq V$.*

Proof. For the first two assertions we confine ourselves to a "proof by picture" [Pól56] as given in Fig. 2.5.

The third statement is an obvious application of the first two. □

Note that the function $g \circ f$ in the proof of Proposition 2.4 is the function ν^2 for the Kneser graph. In detail,

$$\begin{aligned}(g \circ f)(F) &= g\left(\bigcup F\right) = \binom{\bigcup F}{k} \\ &= \left\{A \subseteq [n] : |A| = k, A \cap v = \emptyset \text{ for all } v \in \binom{[n] \setminus \bigcup F}{k}\right\} \\ &= \{A \subseteq [n] : |A| = k, A \cap v = \emptyset \text{ for all } v \in \nu(F)\} \\ &= \nu^2(F).\end{aligned}$$

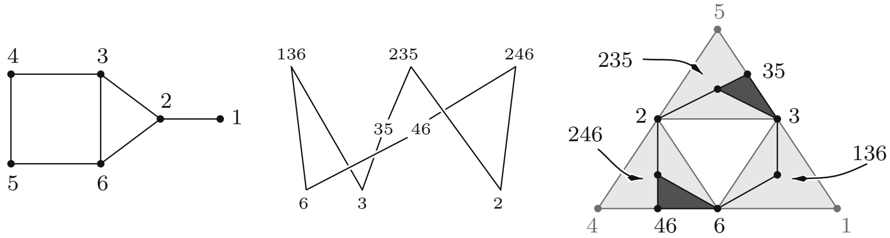

Fig. 2.6 A graph, its poset of closed sets, and the Lovász complex

The Lovász Complex

We will now concentrate on the image of the neighbor set function ν in general and use it to define a new simplicial complex that we will call the Lovász complex, $\mathcal{L}(G)$, of a graph G. It is very closely related to the neighborhood complex, $\mathcal{N}(G)$. In fact, $\mathcal{L}(G)$ will be a strong deformation retract of $\mathcal{N}(G)$ that has more structural properties. The richer structure will allow more topological tools to be used, namely the Borsuk–Ulam theorem.

A set $A \subseteq V$ in the image of ν has the property that $\nu^2(A) = A$. To see this, suppose that $A = \nu(B)$. Then $\nu^2(A) = \nu^3(B) = \nu(B) = A$, by property 3 in Proposition 2.5. We will call sets A with this property *closed*, since ν has the properties of a closure operator. Denote by $C(G)$ the set of all nonempty proper subsets of V that are closed, i.e.,

$$C(G) = \left\{A \subset V : A \neq \emptyset,\ A \neq V,\ \nu^2(A) = A\right\}.$$

For example, in the case of a Kneser graph $KG_{n,k}$, a set $F \subseteq \binom{[n]}{k}$ of vertices is closed if and only if $\nu^2(F) = \binom{\bigcup F}{k} = F$.

Let $G = (V, E)$ be a finite simple graph. Let the *Lovász complex* $\mathcal{L}(G)$ be the order complex of the partially ordered set $(C(G), \subseteq)$, i.e., the order complex of all nonempty proper closed subsets of V ordered by inclusion. In particular, note that each element of $C(G)$ is a face of the neighborhood complex $\mathcal{N}(G)$, and hence $\mathcal{L}(G)$ will be a subcomplex of the first barycentric subdivision of the neighborhood complex. For the example graph in Fig. 2.3, the graph itself, the poset of closed sets, and the Lovász complex as a subcomplex of $\operatorname{sd}\mathcal{N}(G)$ are shown in Fig. 2.6. Note how the neighbor set function ν acts on this complex.

In particularly nice situations, every face of $\mathcal{N}(G)$ is closed. One such case is that of the complete graph K_n on the vertex set $[n]$. In this case $\nu(A) = [n] \setminus A$. As we have seen already, $\mathcal{N}(K_n)$ is the boundary complex of an $(n-1)$-dimensional simplex. Hence, $\mathcal{L}(G)$ is the order complex of its face poset, and therefore the barycentric subdivision of the simplex boundary. Figure 2.7 shows the neighborhood

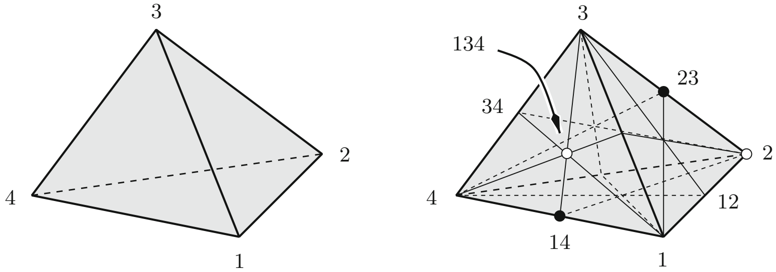

Fig. 2.7 The neighborhood and Lovász complexes of K_4

and Lovász complexes for the complete graph K_4. It also illustrates that the neighbor set function ν is topologically the antipodal map on $\mathcal{L}(K_n)$, i.e., for example $\nu(\{2\}) = \{134\}$ and $\nu(\{1,2\}) = \{3,4\}$.

Proposition 2.6. *There is a homeomorphism $\varphi : |\mathcal{L}(K_n)| \to \mathbb{S}^{n-2}$ that is $\mathbb{Z}_2$-equivariant, i.e., $\varphi(\nu(x)) = -\varphi(x)$.*

Proof. We may geometrically realize $\mathcal{L}(K_n)$ as the boundary of the standard $(n-1)$-dimensional simplex whose points are given by convex combinations $\sum_{i=1}^{n} t_i e_i$, with $t_i \geq 0$, $\sum_{i=1}^{n} t_i = 1$, and $t_i = 0$ for at least one i. A vertex $A \subset [n]$ of $\mathcal{L}(K_n)$ then corresponds to the point $e_A = \frac{1}{|A|} \sum_{i \in A} e_i$. Note that by definition, the induced action on these vertices is given by $\nu(e_A) = e_{\nu(A)}$. Moreover, observe that we may identify the sphere $\mathbb{S}^{n-2}$, together with its antipodal action, with the subspace $S = \mathbb{S}^{n-1} \cap \{x : \sum_{i=1}^{n} x_i = 0\} \subseteq \mathbb{R}^n$. We will construct an equivariant homeomorphism $\varphi : |\mathcal{L}(K_n)| \to S$ in two steps. We will first define φ on the points corresponding to vertices of $\mathcal{L}(K_n)$, and then extend the map predetermined by the $\mathbb{Z}_2$-equivariance.

Let's denote the center of the $(n-1)$-dimensional standard simplex by $c = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$. Then define φ for any $A \subset [n]$ by

$$e_A \longmapsto \frac{e_A - c}{\|e_A - c\|}.$$

We claim that φ is $\mathbb{Z}_2$-equivariant on the set of points e_A, $A \subset [n]$. In order to show this, it suffices to show that c lies on the line segment between e_A and $\nu(e_A)$. But clearly

$$c = \frac{|A|}{n} e_A + \left(1 - \frac{|A|}{n}\right) \nu(e_A),$$

since $(1 - \frac{|A|}{n}) = \frac{|\nu(A)|}{n}$. We may now extend the map to all of $|\mathcal{L}(K_n)|$. An arbitrary point of $|\mathcal{L}(K_n)|$ is given by $\sum_{i=1}^{k} t_i e_{A_i}$ for some chain $A_1 \subset \cdots \subset A_k \subset [n]$ and $t_i \geq 0$, $\sum_{i=1}^{k} t_i = 1$. Extending φ by

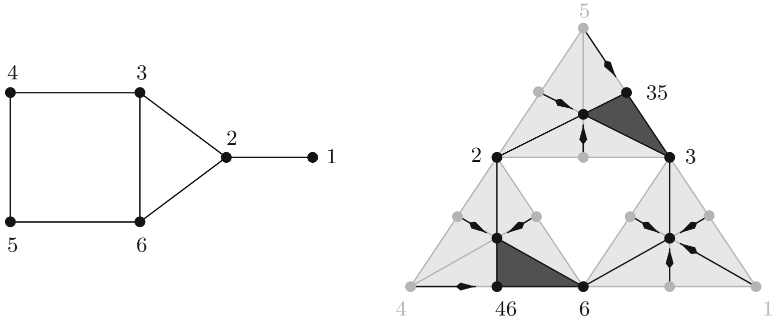

Fig. 2.8 The retraction given by ν^2 in the proof of Proposition 2.7

$$\varphi\left(\sum_{i=1}^{k} t_i e_{A_i}\right) = \frac{\sum_{i=1}^{k} t_i \varphi(e_{A_i})}{\|\sum_{i=1}^{k} t_i \varphi(e_{A_i})\|}$$

yields a continuous map, which is equivariant by definition. It is left to the reader to show the bijectivity of the resulting map. □

For an alternative proof of the previous proposition we refer to Exercise 8 on page 142.

Proposition 2.7. *The Lovász complex $\mathcal{L}(G)$ is a strong deformation retract of the neighborhood complex $\mathcal{N}(G)$. In particular, the two complexes are homotopy equivalent.*

Proof. Consider the map $\nu^2 : \mathcal{F}(\mathcal{N}(G)) \to \mathcal{F}(\mathcal{N}(G))$. As remarked before, since $\nu^3 = \nu$, the image of this map is $C(G)$. Let $i : C(G) \to \mathcal{F}(\mathcal{N}(G))$ be the inclusion map. Then $\nu^2 \circ i = \mathrm{id}_{C(G)}$, and $(i \circ \nu^2)(A) \supseteq A = \mathrm{id}_{\mathcal{F}(\mathcal{N}(G))}(A)$ for all $A \in \mathcal{F}(\mathcal{N}(G))$. Hence, by the order homotopy lemma, Lemma C.3, the order complex $\Delta(C(G))$ is a strong deformation retract of $\Delta(\mathcal{F}(\mathcal{N}(G)))$, which in turn is the first barycentric subdivision of $\mathcal{N}(G)$. The retraction map ν^2 for our example graph is illustrated in Fig. 2.8. □

A $\mathbb{Z}_2$-Action on $\mathcal{L}(G)$

We will now turn our attention to the richer structural properties of the Lovász complex $\mathcal{L}(G)$. This was first investigated by James Walker [Wal83].

By statement 3 of Proposition 2.5, the map ν induces a bijective simplicial map from $\mathcal{L}(G)$ to itself. First of all, it is a self-inverse bijection of the vertices, and furthermore it is order-reversing. Hence, ν maps inclusion chains to inclusion chains, i.e., simplices to simplices.

Since $\nu^2 = \mathrm{id}$ on $\mathcal{L}(G)$ we may identify the pair $\{\mathrm{id}, \nu\}$ with the 2-element group $\mathbb{Z}_2$. Note that ν leaves no $A \subseteq V$ fixed, and therefore ν provides a free $\mathbb{Z}_2$-action on the Lovász complex.

We know already that in the case of the Kneser graphs we have homotopy equivalences $|\mathcal{L}(KG_{n,k})| \simeq |\mathcal{N}(KG_{n,k})| \simeq |\Delta(B_{n,k})|$. Now observe that $B_{n,k}$ is also equipped with a fixed-point-free, order-reversing involution μ given by taking complements, i.e., $\mu(A) = [n] \setminus A$ for $A \in B_{n,k}$.

The content of the next proposition is that these two $\mathbb{Z}_2$-actions are compatible. We will not need it in the sequel, but we discuss it in order to get more accustomed to the Lovász complex and the neighbor set function ν.

Proposition 2.8. *There is a $\mathbb{Z}_2$-equivariant homeomorphism from $\mathcal{L}(KG_{n,k})$ to the order complex $\Delta(B_{n,k})$.*

Proof. We are essentially proving a stronger form of Proposition 2.4 along the same lines. Let $f' : C(G) \to B_{n,k}$ be defined by $F \mapsto \bigcup F$ and let $g' : B_{n,k} \to C(G)$ be given by $A \mapsto \binom{A}{k}$. Then $f' \circ g' = \mathrm{id}_{B_{n,k}}$ is clear, and $(g' \circ f')(F) = \nu^2(F) = F = \mathrm{id}_{C(G)}(F)$ for all $F \in C(G)$. Both maps, f' and g', are order-preserving, and hence the order complexes of both posets are homeomorphic.

Now compute

$$f'(\nu(F)) = f'(\nu(\binom{\bigcup F}{k})) = f'(\binom{[n] \setminus \bigcup F}{k}) = [n] \setminus \bigcup F = \mu(f'(F)),$$

which yields the $\mathbb{Z}_2$-equivariance. □

Note that the map f in the proof of Proposition 2.4 actually factors through $C(G)$, as $f = f' \circ \nu^2$.

Graph Homomorphisms and Induced Maps

We now turn our attention to the proof of Lovász's theorem, Theorem 2.3. An important property of the constructions of the neighborhood and Lovász complexes is the property that every graph homomorphism yields a simplicial map of the associated complexes.

Let $G = (V, E)$ and $H = (V', E')$ be two finite simple graphs with neighbor set functions ν and ν'. Let $f : G \to H$ be a graph homomorphism, i.e., a map $f : V \to V'$ with the property that for every edge $uv \in E$, the image $f(u)f(v)$ is an edge of H.

We will abuse notation and write $f(A)$ for the image of a subset $A \subseteq V$ under f. As Fig. 2.9 demonstrates, the inclusion $f(\nu(A)) \subseteq \nu'(f(A))$ holds for all $A \subseteq V$.

This implies, in particular, that if $\nu(A) \neq \emptyset$, then $\nu'(f(A)) \supseteq f(\nu(A)) \neq \emptyset$. Hence, f induces a simplicial map $\mathcal{N}(f) : \mathcal{N}(G) \to \mathcal{N}(H)$ by $A \mapsto f(A)$.

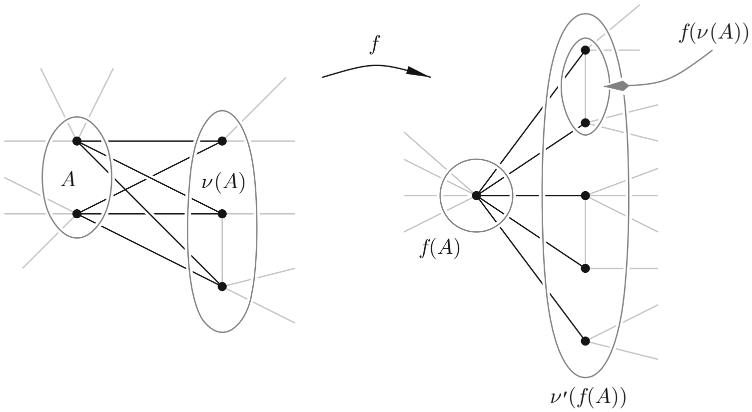

Fig. 2.9 The interplay between the neighbor set functions and graph homomorphisms

As Fig. 2.9 shows, in general the inclusion $f(\nu(A)) \subseteq \nu'(f(A))$ can be proper. Hence, in order to obtain an induced map on the Lovász complex, we have to take the closure of the image under f. The induced map $\mathcal{L}(f) : \mathcal{L}(G) \to \mathcal{L}(H)$ is given on the vertices of $\mathcal{L}(G)$ by $A \mapsto (\nu')^2(f(A))$. It yields a simplicial map by the basic observation that $(\nu')^2(f(A)) \subseteq (\nu')^2(f(B))$ for any $A \subseteq B \subseteq V$. The map $\mathcal{L}(f)$ respects the neighbor set functions, i.e., $\mathcal{L}(f)$ is equivariant with respect to the $\mathbb{Z}_2$-actions that ν and ν' induce. In order to see this we need the following lemma.

Lemma 2.9. *For any graph homomorphism $f : G \to H$, the following relation holds for any closed set $A \in C(G)$:*

$$(\nu')^2(f(\nu(A)) = \nu'(f(A)).$$

Proof. We know already that $f(\nu(A)) \subseteq \nu'(f(A))$ for all A. We therefore have

$$(\nu')^2(f(\nu(A))) \subseteq (\nu')^3(f(A)) = \nu'(f(A))$$

and

$$\nu'(f(A)) = \nu'(f(\nu^2(A))) = \nu'(f(\nu(\nu(A)))) \subseteq (\nu')^2(f(\nu(A))). \qquad \square$$

Now consider an inclusion chain $\{A_0 \subset \cdots \subset A_k\}$ of closed sets of G, i.e., a k-simplex of the Lovász complex $\mathcal{L}(G)$. Then

$$\begin{aligned}\mathcal{L}(f)(\nu(\{A_0 \subset \cdots \subset A_k\})) &= \mathcal{L}(f)(\{\nu(A_k) \subset \cdots \subset \nu(A_0)\}) \\ &= \{(\nu')^2(f(\nu(A_k))) \subset \cdots \subset (\nu')^2(f(\nu(A_0)))\} \\ &= \{\nu'(f(A_k)) \subset \cdots \subset \nu'(f(A_0))\} \\ &= \nu'(\{f(A_0) \subset \cdots \subset f(A_k)\}) \\ &= \nu'(\mathcal{L}(f)(\{A_0 \subset \cdots \subset A_k\})).\end{aligned}$$

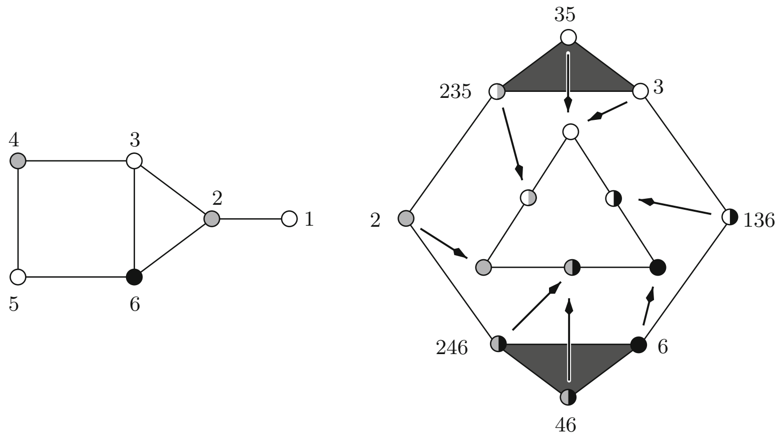

Fig. 2.10 A three-coloring of G and the induced map of the Lovász complexes

We summarize the previous insights.

Proposition 2.10. *Any graph homomorphism $f : G \to H$ induces a $\mathbb{Z}_2$-equivariant simplicial map $\mathcal{L}(f) : \mathcal{L}(G) \to \mathcal{L}(H)$.* □

Before we apply this to prove Theorem 2.3, let's consider an enlightening example. The graph homomorphisms we are mostly interested in are induced by a coloring of the graph. In fact, if $c : V(G) \to [m]$ is a proper m-coloring of the graph, then c induces a graph homomorphism $G \to K_m$. And conversely, every graph homomorphism $G \to K_m$ yields an m-coloring of the graph. In other words, the chromatic number of a graph G equals

$$\chi(G) = \min\{m \geq 0 : \text{there exists a graph homomorphism } G \to K_m\}.$$

As an example, we consider a three-coloring of our example graph. We color the vertices with colors white, gray, and black. Figure 2.10 shows the coloring along with the induced map $\mathcal{L}(G) \to \mathcal{L}(K_3)$.

Proof (of Theorem 2.3). Assume that G possesses a proper m-coloring, i.e., there exists a graph homomorphism $f : G \to K_m$. Since $\mathcal{N}(G)$ is k-connected by assumption, so is $\mathcal{L}(G)$ by Proposition 2.7. Hence, there exists a $\mathbb{Z}_2$-equivariant map $\psi : \mathbb{S}^{k+1} \to |\mathcal{L}(G)|$, where $\mathbb{Z}_2$ acts on $\mathbb{S}^{k+1}$ via the antipodal map. Such a map can easily be constructed inductively using a $\mathbb{Z}_2$-invariant triangulation of the sphere such as, for example, that given by the boundary complex of the cross polytope. The details are given in the proof of Proposition D.13 on page 216. Together with Proposition 2.6, we obtain the following composition of $\mathbb{Z}_2$-equivariant maps:

$$\mathbb{S}^{k+1} \xrightarrow{\psi} |\mathcal{L}(G)| \xrightarrow{|\mathcal{L}(f)|} |\mathcal{L}(K_m)| \xrightarrow{\varphi} \mathbb{S}^{m-2}.$$

By the Borsuk–Ulam theorem we have $m - 2 \geq k + 1$, and hence $m \geq k + 3$. □

Finally, we obtain a proof of Kneser's conjecture along the lines of Lovász's original proof.

Corollary 2.11. *For the family of Kneser graphs $KG_{n,k}$ we obtain $\chi(KG_{n,k}) \geq n - 2k + 2$.*

Proof. By Theorem 2.3 and Proposition 2.4 we obtain

$$\chi(KG_{n,k}) \geq \text{conn}(|\mathcal{N}(KG_{n,k})|) + 3 = n - 2k - 1 + 3 = n - 2k + 2. \qquad \square$$

2.3 A Conjecture by Lovász

This section is devoted to a more recent development. It is about a general approach to endowing the category of graphs with topological structure, and in fact can be seen as a generalization of the concepts we discussed in the previous sections of this chapter. The concept was introduced by László Lovász, and the story line develops along a conjecture by him claiming a somewhat analogous statement to Theorem 2.3. The conjecture was proved by Eric Babson and Dmitry Kozlov in 2005 [BK07, Koz07]. A shorter and very elegant proof was later found by Carsten Schultz [Schu06]. We will present his argument and follow in many respects his original article.

By the definition of the neighbor set function ν, which assigns the common neighbors to a set of vertices in a graph, pairs $A, \nu(A)$ are the shores of complete bipartite subgraphs. What does it mean for two sets $A, B \subseteq V$ to be the two shores of a complete bipartite subgraph of G? A fancy way to say it is that every choice of vertices $u \in A$ and $v \in B$ induces a graph homomorphism $\varphi : K_2 \to G$ defined by $\varphi(0) = u$ and $\varphi(1) = v$. Compare Fig. 2.11. In terms of the neighbor set function ν, this amounts to requiring that $A \subseteq \nu(B)$. Note that this implies $B \subseteq \nu^2(B) \subseteq \nu(A)$.

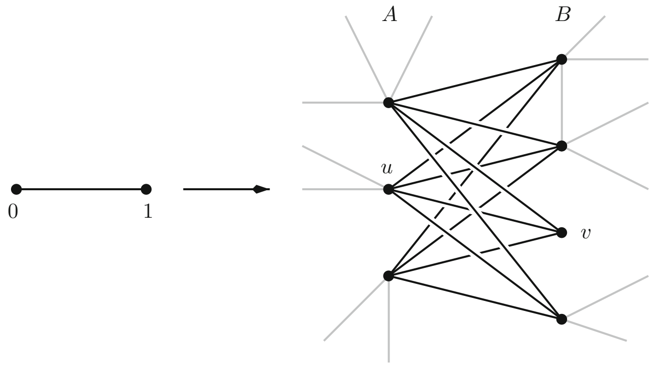

Fig. 2.11 Graph homomorphisms $K_2 \to G$ and shores of bipartite subgraphs

Hom *Complexes*

The interpretation above leads to the following generalization of graph homomorphisms. Let $T = (V', E')$ and $G = (V, E)$ be graphs. A *multihomomorphism* from T to G is a map $\varphi : V' \to \mathcal{P}(V) \setminus \{\emptyset\}$ associating a nonempty subset of V to every vertex of T such that every function $f : V' \to V$, with $f(v) \in \varphi(v)$ for all $v \in V$, is a graph homomorphism from T to G.

Note that indeed, a multihomomorphism has the property that for any pair $u, v \in V'$ of adjacent vertices, the sets $\varphi(u)$ and $\varphi(v)$ are the vertex sets of complete bipartite subgraphs of G. In particular, any multihomomorphism from K_2 to a graph G is given by nonempty sets $A, B \subset V$ that are vertex sets of a complete bipartite subgraph of G.

Each multihomomorphism φ from T to G can be identified with a product of geometric simplices contained in

$$\prod_{u \in V(T)} \Delta_{|V(G)|-1}$$

as follows. We clearly may identify the subsets of $V(G)$ with the faces of $\Delta_{|V(G)|-1}$. Then, for each $u \in V(T)$, the subset $\varphi(u) \subseteq V(G)$ defines a face $F_u^{\varphi} \subseteq \Delta_{|V(G)|-1}$. With the multihomomorphism φ, we now associate the product

$$\prod_{u \in V(T)} F_u^{\varphi} \subseteq \prod_{u \in V(T)} \Delta_{|V(G)|-1}.$$

We denote the set of multihomomorphisms from T to G by $\mathrm{Hom}(T, G)$ and denote its *geometric realization* by

$$|\,\mathrm{Hom}(T, G)\,| = \bigcup_{\substack{\varphi : V(T) \to \mathcal{P}(V(G)) \setminus \{\emptyset\} \\ \text{multihom.}}} \left(\prod_{u \in V(T)} F_u^{\varphi} \right) \subseteq \prod_{u \in V(T)} \Delta_{|V(G)|-1}.$$

First Examples

As a first example, consider $\mathrm{Hom}(K_2, C_3)$, where $C_3 =$  is a cycle of length 3. The multihomomorphisms in this case are given by all ordered pairs (A, B) describing two shores of a complete bipartite subgraph. If we denote the vertices of C_3 by $0, 1, 2$, these pairs are

$$(0, 1), (0, 2), (1, 0), (1, 2), (2, 0), (2, 1),$$

$$(01, 2), (02, 1), (12, 0), (0, 12), (1, 02), (2, 01),$$

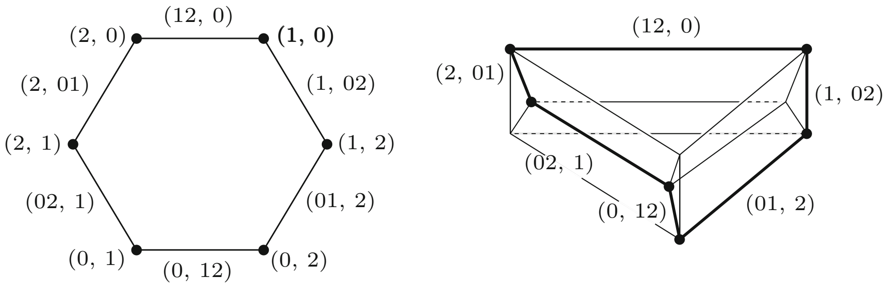

Fig. 2.12 $|\operatorname{Hom}(K_2, C_3)|$ by itself and as a subcomplex of $\Delta_2 \times \Delta_2$

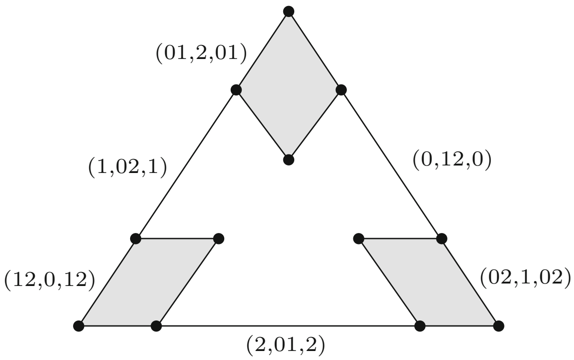

Fig. 2.13 The space $|\operatorname{Hom}(P_2, C_3)|$

where we have abused notation by writing $(0, 1)$ instead of $(\{0\}, \{1\})$ and $(01, 2)$ instead of $(\{0, 1\}, \{2\})$, etc. The geometric realization of $\operatorname{Hom}(K_2, C_3)$ consists of six edges forming a circle. Figure 2.12 shows $|\operatorname{Hom}(K_2, C_3)|$ by itself and as a subcomplex of the product

$$\prod_{u \in V(K_2)} \Delta_{|V(C_3)|-1} = \Delta_2 \times \Delta_2.$$

As with the Lovász complex $\mathcal{L}(C_{2r+1})$ of an odd cycle, $|\operatorname{Hom}(K_2, C_{2r+1})|$ is always homeomorphic to a circle, i.e., a 1-dimensional sphere, and $|\operatorname{Hom}(K_2, C_{2r})|$ is homeomorphic to two disjoint circles.

The next example, $\operatorname{Hom}(P_2, C_3)$, for $P_2 = \bullet\!\!-\!\!\bullet\!\!-\!\!\bullet$ a path of length 2, involves higher-dimensional cells. Each multihomomorphism is now given by a triple (A, B, C) such that each of the pairs (A, B) and (B, C) are the shores of complete bipartite subgraphs of C_3. Typical examples of dimension one and two are $(0, 12, 0)$ and $(12, 0, 12)$. Figure 2.13 shows the space $|\operatorname{Hom}(P_2, C_3)|$ with the 1- and 2-dimensional cells labeled.

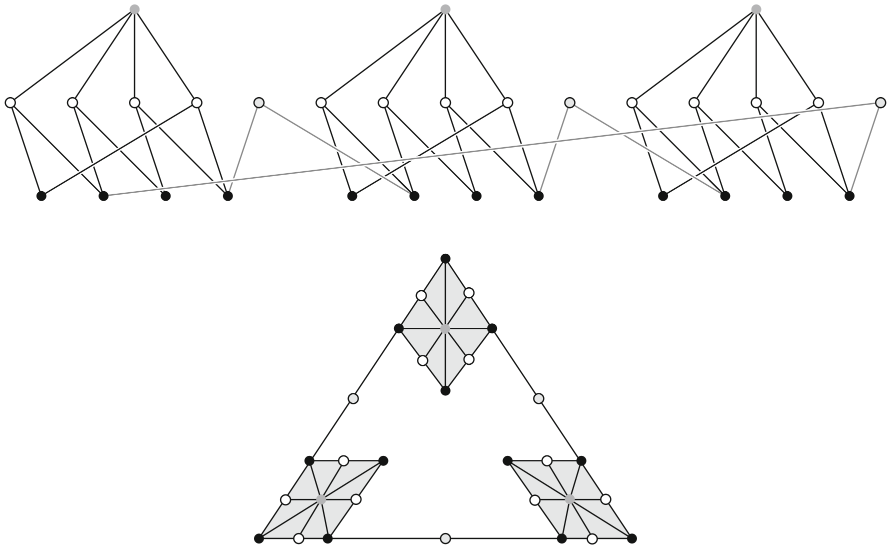

Fig. 2.14 The poset $(\mathrm{Hom}(P_2, C_3), \subseteq)$ and the order complex $\Delta(\mathrm{Hom}(P_2, C_3))$

The Partially Ordered Set Structure

The set $\mathrm{Hom}(T, G)$ of multihomomorphisms from T to G is partially ordered by inclusion, i.e., $\varphi \subseteq \psi$ holds for $\varphi, \psi \in \mathrm{Hom}(T, G)$ if and only if $\varphi(u) \subseteq \psi(u)$ for all $u \in V(T)$. The order corresponds to the partial inclusion order of cells

$$\prod_{u \in V(T)} F_u^{\varphi} \subseteq |\,\mathrm{Hom}(T, G)\,|, \ \varphi \in \mathrm{Hom}(T, G).$$

We will always refer to this partial order when considering $\mathrm{Hom}(T, G)$ as a partially ordered set.

In particular, we can assign the order complex to this partial order and obtain a simplicial complex $\Delta(\mathrm{Hom}(T, G))$ whose geometric realization $|\Delta(\mathrm{Hom}(T, G))|$ is homeomorphic to $|\,\mathrm{Hom}(T, G)\,|$. This is the content of Exercise 11 on page 68. Figure 2.14 shows the Hasse diagram of the poset $(\mathrm{Hom}(P_2, C_3), \subseteq)$ and the order complex $\Delta(\mathrm{Hom}(P_2, C_3))$.

Just as we can compose graph homomorphisms, we can do so with graph multihomomorphisms. We suggestively use the following notation:

$$\begin{aligned} \mathrm{Hom}(T, G) \times \mathrm{Hom}(G, H) &\longrightarrow \mathrm{Hom}(T, H), \\ (\varphi, \psi) &\longmapsto \varphi * \psi, \end{aligned}$$

where the latter is defined by

$$(\varphi * \psi)(u) = \{w \in V(H) : \text{there exists } v \in \varphi(u) \text{ such that } w \in \psi(v)\}.$$

It is an easy exercise to see that $\varphi * \psi$ is a graph multihomomorphism from T to H that respects the inclusion order, i.e., $\varphi \subseteq \varphi'$ and $\psi \subseteq \psi'$ implies $\varphi * \psi \subseteq \varphi' * \psi'$. This implies that $*$ induces a simplicial map on the order complex. Compare Exercise 12.

Proposition 2.12. *For any three graphs T, G, H there is a continuous map*

$$* : |\operatorname{Hom}(T, G)| \times |\operatorname{Hom}(G, H)| \longrightarrow |\operatorname{Hom}(T, H)|,$$

which, restricted to graph homomorphisms, is identical to ordinary composition. Moreover, $*$ satisfies the associativity law.

Proof. The result follows from Lemma C.2, i.e., the fact that there is a homeomorphism $|\Delta(P \times Q)| \cong |\Delta(P)| \times |\Delta(Q)|$. The details are left to Exercise 13. □

$\mathbb{Z}_2$-*Structure*

As in the previous section, we need more structure (such as a free $\mathbb{Z}_2$-action) on the spaces we are considering. Note that $T = K_2$, as well as any cycle $T = C_n$, admits a self-inverse automorphism flipping an edge, i.e., a graph isomorphism $\gamma : T \to T$ with $\gamma^2 = \mathrm{id}_T$ and such that there exists an edge $uv \in E(T)$ with $\gamma(u) = v$ (and hence $\gamma(v) = u$). Let K_2 and the odd cycle C_{2r+1} have vertex sets $\{0, 1\}$ and $\{0, 1, \dots, 2r\}$, respectively. Denote by $\alpha : K_2 \to K_2$ the automorphism given by $\alpha(i) = 1 - i$, and by $\beta : C_{2r+1} \to C_{2r+1}$ the automorphism given by $\beta(i) = 2r - i$. See Fig. 2.15.

Proposition 2.13. *Any self-inverse graph automorphism $\gamma : T \to T$ flipping an edge induces a free $\mathbb{Z}_2$-action on $|\operatorname{Hom}(T, G)|$ for any graph G via*

$$|\operatorname{Hom}(T, G)| \longrightarrow |\operatorname{Hom}(T, G)|,$$
$$x \longmapsto \gamma * x.$$

Proof. It is clear that the map is self-inverse, since $\gamma * (\gamma * x) = (\gamma * \gamma) * x = x$. We need to check that it is fixed-point-free. Assume that the edge flipped by γ has vertices u and v. If $x \in \prod_{w \in V(T)} F_w^{\varphi} \subseteq |\operatorname{Hom}(T, G)|$, then clearly $\gamma * x \in \prod_{w \in V(T)} F_w^{\gamma * \varphi}$. Now

$$F_u^{\varphi} \cap F_u^{\gamma * \varphi} = F_u^{\varphi} \cap F_v^{\varphi} = \emptyset,$$

and hence $\gamma * x \neq x$. □

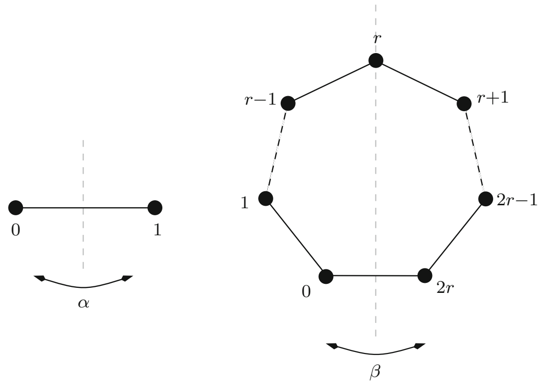

Fig. 2.15 The edge-flipping automorphisms α of K_2 and β of C_{2r+1}

Corollary 2.14. *For any graph G, the map α induces a free $\mathbb{Z}_2$-action on $|\operatorname{Hom}(K_2, G)|$ and β does so on $|\operatorname{Hom}(C_{2r+1}, G)|$.* □

Lovász and Hom *Complexes*

As we already discussed, the partially ordered set $\operatorname{Hom}(K_2, G)$ may be identified with the set of all pairs (A, B), with $A, B \subseteq V(G)$ nonempty sets that are the two shores of a complete bipartite graph ordered by componentwise inclusion. The correspondence is given by $\varphi(0) = A$ and $\varphi(1) = B$. Now $(\alpha * \varphi)(0) = \varphi(\alpha(0)) = \varphi(1) = B$, and similarly, $(\alpha * \varphi)(1) = A$. Hence, on the set of pairs the $\mathbb{Z}_2$-action is given by $(A, B) \mapsto (B, A)$. In particular, if we consider a pair $(A, \nu(A))$ for a closed set $A \subseteq V(G)$, we obtain $(A, \nu(A)) \mapsto (\nu(A), A)$. Hence, in the first coordinate we are left with the $\mathbb{Z}_2$-action of ν on the set $C(G)$ of closed subsets.

After these considerations we might expect a close relationship between the Lovász complex $\mathcal{L}(G) = \Delta C(G)$ and $\operatorname{Hom}(K_2, G)$.

The poset $\operatorname{Hom}(K_2, G)$ has been introduced before. Its order complex is a version of the *box complex* and was thoroughly investigated by Matoušek and Ziegler [MZ04]. One of their results is the following.

Proposition 2.15. *There exist simplicial $\mathbb{Z}_2$-maps*

$$f : \operatorname{sd} \mathcal{L}(G) \to \Delta \operatorname{Hom}(K_2, G)$$

and $g : \operatorname{sd}(\Delta \operatorname{Hom}(K_2, G)) \to \operatorname{sd} \mathcal{L}(G)$.

Proof. A vertex of $\operatorname{sd} \mathcal{L}(G)$ is given by an inclusion chain $\{A_0 \subset A_1 \subset \cdots \subset A_k\}$ of nonempty closed sets of vertices of the graph G. The $\mathbb{Z}_2$-action on these chains is induced by the $\mathbb{Z}_2$-action on $\mathcal{L}(G)$, and is given by

$$\nu(\{A_0 \subset A_1 \subset \cdots \subset A_k\}) = \{\nu(A_k) \subset \cdots \subset \nu(A_1) \subset \nu(A_0)\}.$$

Define $f : \operatorname{sd}\mathcal{L}(G) \to \Delta\operatorname{Hom}(K_2, G)$ on the vertices by

$$f(\{A_0 \subset A_1 \subset \cdots \subset A_k\}) = (A_0, \nu(A_k)).$$

Then f is well defined, since $A_0 \subseteq \nu(\nu(A_k)) = A_k$ and $\nu(A_k) \subseteq \nu(A_0)$. It is $\mathbb{Z}_2$-equivariant, since

$$f(\nu(\{A_0 \subset A_1 \subset \cdots \subset A_k\})) = \big(\nu(A_k), \nu(\nu(A_0))\big) = (\nu(A_k), A_0).$$

In order to verify that f is simplicial, we consider two chains

$$\{A_0 \subset A_1 \subset \cdots \subset A_k\} \subseteq \{A'_0 \subset A'_1 \subset \cdots \subset A'_l\}.$$

In this case, $A'_0 \subseteq A_0 \subset A_k \subseteq A'_l$ must hold, and therefore $A'_0 \subseteq A_0$ and $\nu(A'_l) \subseteq \nu(A_k)$.

Let us now construct the map $g : \operatorname{sd}(\Delta\operatorname{Hom}(K_2, G)) \to \operatorname{sd}\mathcal{L}(G)$. A vertex c of $\operatorname{sd}(\Delta\operatorname{Hom}(K_2, G))$ is given by an inclusion chain of pairs $c = \{(A_0, B_0) \subset (A_1, B_1) \subset \cdots \subset (A_k, B_k)\}$ with the property $A_i \subseteq \nu(B_i)$ and $B_i \subseteq \nu(A_i)$, for $i = 0, \ldots, k$. Consider the chain of inclusions

$$\nu^2(A_0) \subseteq \nu^2(A_1) \subseteq \cdots \subseteq \nu^2(A_k) \subseteq \nu^3(B_k) = \nu(B_k) \subseteq \nu(B_{k-1}) \subseteq \cdots \subseteq \nu(B_0)$$

of nonempty closed sets. Define $g(c)$ to be the inclusion chain that one obtains by eliminating repeated sets in this chain. This map is easily seen to be simplicial and $\mathbb{Z}_2$-equivariant. □

In fact, it is not hard to show that $|\mathcal{L}(G)|$ and $|\operatorname{Hom}(K_2, G)|$ are $\mathbb{Z}_2$-homotopy equivalent. We will discuss this at the end of the section.

By applying Proposition 2.6 we obtain the following corollary.

Corollary 2.16. *There are $\mathbb{Z}_2$-equivariant maps $\mathbb{S}^{n-2} \to |\operatorname{Hom}(K_2, K_n)| \to \mathbb{S}^{n-2}$.* □

Along the lines of the proof of Lovász's theorem, Theorem 2.3, we obtain a new version of Lovász's theorem.

Corollary 2.17. *For any graph G, there is the following bound on the chromatic number:*

$$\chi(G) \geq \operatorname{conn}|\operatorname{Hom}(K_2, G)| + 3.$$

Proof. Assume that $|\operatorname{Hom}(K_2, G)|$ is k-connected and that there exists an m-coloring of G, i.e., a graph homomorphism $f : G \to K_m$. Then there exists the following sequence of $\mathbb{Z}_2$-maps:

$$\mathbb{S}^{k+1} \longrightarrow |\operatorname{Hom}(K_2, G)| \longrightarrow |\operatorname{Hom}(K_2, K_m)| \longrightarrow \mathbb{S}^{m-2}.$$

By the Borsuk–Ulam theorem we obtain $m - 2 \geq k + 1$. □

We now turn our attention to the $\mathbb{Z}_2$-homotopy equivalence of $|\mathcal{L}(G)|$ and $|\operatorname{Hom}(K_2, G)|$. The following proof is due to Schultz [Schu10].

In the first step we will replace $\mathcal{L}(G)$ by a $\mathbb{Z}_2$-homeomorphic copy. To this end, recall the construction of the interval order of a partially ordered set as introduced on page 203. We will apply it to the partially ordered set $(C(G), \subseteq)$. In this case,

$$\operatorname{Int}(C(G)) = \{(A, B) : A, B \in C(G), A \subseteq B\}$$

ordered by $(A, B) \preceq (A', B')$ if and only if $A \subseteq A'$ and $B' \subseteq B$.

By Proposition C.5, the geometric realizations of $\mathcal{L}(G) = \Delta(C(G))$ and $\Delta \operatorname{Int}(C(G))$ are homeomorphic. It is an easy exercise to see that under this homeomorphism the $\mathbb{Z}_2$-action on $\operatorname{Int}(C(G))$ defined by $(A, B) \mapsto (\nu(B), \nu(A))$ corresponds to the $\mathbb{Z}_2$-action on $\mathcal{L}(G)$.

Hence, it suffices to prove the following proposition.

Proposition 2.18. *The partial orders* $\operatorname{Int}(C(G))$ *and* $\operatorname{Hom}(K_2, G)$ *are* $\mathbb{Z}_2$-*homotopy equivalent.*

Proof. Define the maps

$$\begin{aligned} f : \operatorname{Hom}(K_2, G) &\longrightarrow \operatorname{Int}(C(G)), \\ (A, B) &\longmapsto (\nu^2(A), \nu(B)), \end{aligned}$$

and

$$\begin{aligned} g : \operatorname{Int}(C(G)) &\longrightarrow \operatorname{Hom}(K_2, G), \\ (A, B) &\longmapsto (A, \nu(B)). \end{aligned}$$

These maps are easily seen to be order-preserving and $\mathbb{Z}_2$-equivariant. Now

$$(f \circ g)(A, B) = f(A, \nu(B)) = (\nu^2(A), \nu^2(B)) = (A, B)$$

and

$$(g \circ f)(A, B) = g(\nu^2(A), \nu(B)) = (\nu^2(A), \nu^2(B)) \supseteq (A, B).$$

In other words, $f \circ g = \mathrm{id}_{\operatorname{Int}(C(G))}$ and $g \circ f \geq \mathrm{id}_{\operatorname{Hom}(K_2,G)}$ with respect to the order on $\operatorname{Hom}(K_2, G)$. By the order homotopy lemma, Lemma C.3 and the following Remark C.4, the composition $g \circ f$ is $\mathbb{Z}_2$-homotopic to the identity. □

Note that in fact, the previous proof yields that the map g is injective and that the geometric realization of its image is a strong deformation retract of $|\operatorname{Hom}(K_2, G)|$.

Lovász's Conjecture

Contemplating Corollary 2.17, it seems natural to replace K_2 by some other graph in order to obtain new, and maybe stronger, lower bounds for the chromatic number. The existence of a graph homomorphism from K_2 to G witnesses a lower bound of two for the chromatic number, and the existence of a homomorphism of an odd cycle C_{2r+1} to G a lower bound of three. This raises the question whether $\operatorname{Hom}(C_{2r+1}, G)$ can also be used to obtain a general lower bound for the chromatic number. Lovász conjectured that

$$\chi(G) \geq \operatorname{conn}|\operatorname{Hom}(C_{2r+1}, G)| + 4.$$

We will show something slightly stronger. In order to do so we will need a measure for the topological complexity of a space with $\mathbb{Z}_2$-action resembling that of connectivity, but taking the $\mathbb{Z}_2$-action into account.

Definition 2.19. Let X be a topological space with a $\mathbb{Z}_2$-action. The *$\mathbb{Z}_2$-index of X* is defined to be

$$\operatorname{ind}(X) = \min\{k \geq 0 : \text{there exists a continuous } \mathbb{Z}_2\text{-map } X \to \mathbb{S}^k\},$$

i.e., the smallest dimension k such that X can be mapped equivariantly to the k-dimensional sphere endowed with the antipodal action.

Our main example of a space with $\mathbb{Z}_2$-action is the sphere $\mathbb{S}^n$ with the antipodal action. It has $\operatorname{ind}(\mathbb{S}^n) = n$ by the Borsuk–Ulam theorem, Theorem 1.6.

Definition 2.20. Let X be a topological space with a $\mathbb{Z}_2$-action. The *$\mathbb{Z}_2$-coindex of X* is defined to be

$$\text{co-ind}(X) = \max\{k \geq 0 : \text{there exists a continuous } \mathbb{Z}_2\text{-map } \mathbb{S}^k \to X\},$$

i.e., the largest dimension k such that the k-dimensional sphere endowed with the antipodal action can be mapped equivariantly to X.

Again for our main example we have, by the Borsuk–Ulam theorem, Theorem 1.6, that $\text{co-ind}(\mathbb{S}^n) = n$.

Lemma 2.21. *Let X be a topological space with a free $\mathbb{Z}_2$-action. Then the following inequalities hold:*

$$\operatorname{ind}(X) \geq \text{co-ind}(X) \geq \operatorname{conn}(X) + 1.$$

Proof. Let $k = \text{co-ind}(X)$ and $l = \text{ind}(X)$. Then there exists the following composition of $\mathbb{Z}_2$-equivariant maps:

$$\mathbb{S}^k \longrightarrow X \longrightarrow S^l.$$

And hence, by the Borsuk–Ulam theorem, $\text{ind}(X) = l \geq k = \text{co-ind}(X)$.

Now let $k = \text{conn}(X)$. Then, by Proposition D.13, there exists a $\mathbb{Z}_2$-equivariant map $\mathbb{S}^{k+1} \to X$, and hence $\text{co-ind}(X) \geq k + 1 = \text{conn}(X) + 1$. □

We will now prove a stronger version of Lovász's theorem, Theorem 2.3, from page 42.

Proposition 2.22. *For any finite simple graph G, the following lower bound holds for the chromatic number:*

$$\chi(G) \geq \text{ind}(|\,\text{Hom}(K_2, G)\,|) + 2.$$

Proof. Let $m = \chi(G)$ be the chromatic number of G. Hence, there exists a graph homomorphism $\varphi : G \to K_m$. This yields a map

$$\text{Hom}(K_2, G) \longrightarrow \text{Hom}(K_2, K_m),$$

$$\psi \longmapsto \psi * \varphi,$$

which is $\mathbb{Z}_2$-equivariant. Applying Corollary 2.16, we obtain a $\mathbb{Z}_2$-equivariant map $|\,\text{Hom}(K_2, G)\,| \to \mathbb{S}^{m-2}$, and hence

$$\text{ind}(|\,\text{Hom}(K_2, G)\,|) + 2 \leq m = \chi(G).$$ □

$\mathbf{Hom}(K_2, C_{2r+1})$

We now turn our attention toward a proof of Lovász's conjecture concerning the relation between $\chi(G)$ and $\text{Hom}(C_{2r+1}, G)$. Since we understand the relation between $\chi(G)$ and $\text{Hom}(K_2, G)$, and we have the map

$$|\,\text{Hom}(K_2, C_{2r+1})\,| \times |\,\text{Hom}(C_{2r+1}, G)\,| \longrightarrow |\,\text{Hom}(K_2, G)\,|,$$

we will first investigate $|\,\text{Hom}(K_2, C_{2r+1})\,|$. Figure 2.16 depicts the order complex $\Delta(\text{Hom}(K_2, C_{2r+1}))$. The gray bullets (● and ●) correspond to graph multihomomorphisms $K_2 \to C_{2r+1}$ that actually are graph homomorphisms such as $\varphi(0) = \{i\}$ and $\varphi(1) = \{i + 1\}$. For each edge e of C_{2r+1}, there are exactly

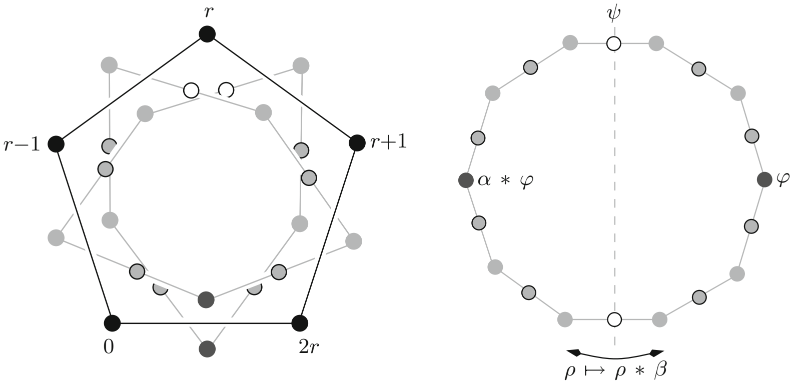

Fig. 2.16 The complex $\Delta(\mathrm{Hom}(K_2, C_{2r+1}))$ for $r = 2$

two homomorphisms mapping K_2 to e. The bullets with a black rim (◎ and ○) correspond to graph multihomomorphisms $\varphi : K_2 \to C_{2r+1}$ such as $\varphi(0) = \{i\}$ and $\varphi(1) = \{i-1, i+1\}$.

We want to consider two particular elements $\varphi, \psi \in \mathrm{Hom}(K_2, C_{2r+1})$. They are defined by

$$\varphi(i) = \begin{cases} \{0\}, & \text{if } i = 0, \\ \{2r\}, & \text{if } i = 1, \end{cases} \qquad \psi(i) = \begin{cases} \{r\}, & \text{if } i = 0, \\ \{r-1, r+1\}, & \text{if } i = 1, \end{cases}$$

and they are shown in Fig. 2.16 as dark gray bullets. These elements induce continuous maps $f, g : |\mathrm{Hom}(C_{2r+1}, G)| \to |\mathrm{Hom}(K_2, G)|$ via $f(x) = \varphi * x$ and $g(x) = \psi * x$. Let's see how these maps behave with respect to the free $\mathbb{Z}_2$-actions given by β and α, as in Corollary 2.14:

$$f(\beta * x) = \varphi * (\beta * x) = (\varphi * \beta) * x = (\alpha * \varphi) * x = \alpha * (\varphi * x) = \alpha * f(x)$$
$$g(\beta * x) = \psi * (\beta * x) = (\psi * \beta) * x = \psi * x = g(x).$$

Here we used the fact that $\varphi * \beta = \alpha * \varphi$ and $\psi * \beta = \beta$, as is easily checked. In fact, the map $\rho \mapsto \rho * \beta$ corresponds to reflection along the dotted gray line in Fig. 2.16. To summarize, f is equivariant with respect to the $\mathbb{Z}_2$-actions, whereas g is constant on each orbit, i.e., constant on each pair $x, \beta * x$. Moreover, f and g are homotopic, since φ and ψ are connected by a path, i.e., a continuous map $h : [0,1] \to |\mathrm{Hom}(K_2, C_{2r+1})|$ with $h(0) = \varphi$ and $h(1) = \psi$; see Fig. 2.16. The homotopy from f to g is now given by

$$\begin{aligned} H : |\mathrm{Hom}(C_{2r+1}, G)| \times [0,1] &\longrightarrow |\mathrm{Hom}(K_2, G)|, \\ (x,t) &\longmapsto h(t) * x. \end{aligned}$$

Let us call $\mathbb{Z}_2$-equivariant maps *odd* and maps that are constant on each orbit *even*.

Lemma 2.23. *Let X, Y be free $\mathbb{Z}_2$-spaces with $Y \neq \emptyset$, and $f, g : X \to Y$ homotopic continuous maps, such that f is odd and g even. Then*

$$\operatorname{ind}(Y) \geq \text{co-ind}(X) + 1.$$

Before we prove the lemma, we will use it to prove Lovász's conjecture. We obtain the following inequality immediately.

Corollary 2.24. *The previously considered maps f and g imply*

$$\operatorname{ind}(|\operatorname{Hom}(K_2, G)|) \geq \text{co-ind}(|\operatorname{Hom}(C_{2r+1}, G)|) + 1. \qquad \square$$

The stronger form of Lovász's 1978 theorem, Theorem 2.3, as shown in Proposition 2.22, yields

$$\begin{aligned}\chi(G) &\geq \operatorname{ind}(|\operatorname{Hom}(K_2, G)|) + 2 \geq \text{co-ind}\,|\operatorname{Hom}(C_{2r+1}, G)| + 3\\ &\geq \operatorname{conn}|\operatorname{Hom}(C_{2r+1}, G)| + 4,\end{aligned}$$

and we have therefore obtained a proof of Lovász's conjecture that we will state as a theorem in slightly stronger form than conjectured.

Theorem 2.25 (Babson, Kozlov [BK07]). *For any graph G, the inequality $\chi(G) \geq \text{co-ind}\,|\operatorname{Hom}(C_{2r+1}, G)| + 3$ holds.* $\square$

The proof of Lemma 2.23 relies on several concepts from the previous chapter. In particular, the strong Ky Fan theorem plays an essential role.

Proof (of Lemma 2.23). Assume to the contrary that $\operatorname{ind}(Y) \leq \text{co-ind}(X)$ and let $k = \text{co-ind}(X)$. Then there exist the compositions of continuous maps

$$\bar{f}, \bar{g} : \mathbb{S}^k \longrightarrow X \xrightarrow{f,g} Y \longrightarrow \mathbb{S}^k,$$

so that by the $\mathbb{Z}_2$-equivariance of the first and last maps, the map $\bar{f}$ is odd, and $\bar{g}$ is even. Since f and g are homotopic, so are $\bar{f}$ and $\bar{g}$. Let $H : \mathbb{S}^k \times [0, 1] \to \mathbb{S}^k$ be the homotopy from $\bar{f}$ to $\bar{g}$.

We need a simplicial version of H with the property that on the boundary the two simplicial maps maintain their parity. This can be done easily. We start with a simplicial version of $\mathbb{S}^k$ by considering the boundary $\partial Q^{k+1} = |\Gamma^k|$ of the $(k+1)$-dimensional cross polytope. Denote by $\mathcal{F}$ the face poset of the corresponding geometric complex Γ^k, and let $S = \Delta(\mathcal{F} \times \mathcal{I})$, where $\mathcal{I}$ is the poset $(\{0, 1\}, <)$. See Fig. 2.17 for an illustration.

Then $|S| \cong \mathbb{S}^k \times [0, 1]$, and hence we can assume that H is a map from $|S|$ to $|\Gamma^k|$. Now let r be large enough that there exists a simplicial approximation $\mathcal{H} : \operatorname{sd}^r S \to \Gamma^k$, where $\mathcal{H}$ restricted to the two boundary components is odd and even,

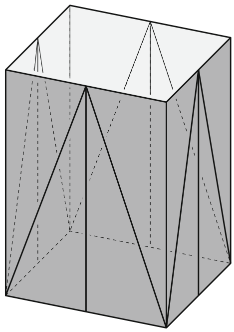

Fig. 2.17 The complex $S = \Delta(\mathcal{F} \times \mathcal{I})$ for $k = 1$

respectively. This can be done easily by making choices consistent with the $\mathbb{Z}_2$-action on the boundary components, since $\bar{f}$ and $\bar{g}$ were odd and even, respectively. Compare the construction of simplicial approximation on page 217 and Exercise 16 on page 68.

By identifying $\pm e_i$ with $\pm i$, the map $\mathcal{H}$ can be considered a labeling function $\mathrm{vert}(\mathrm{sd}^r\, S) \to \{\pm 1, \ldots, \pm(k+1)\}$. Since there are no edges $\mathrm{conv}(\{+e_i, -e_i\})$ in Γ^k, this labeling does not admit complementary edges. Moreover, note that there are no $(k+1)$-dimensional alternating simplices in $\mathrm{sd}^r\, S$, since there are too few labels. And therefore, by the theorem of Ky Fan for pseudomanifolds as discussed in Exercise 13 on page 33, the number of $+$-alternating k-dimensional simplices on the boundary is even.

On the boundary component, where the labeling is odd, the number of $+$-alternating k-simplices is odd by the weak version of Ky Fan's theorem, Theorem 1.8. But on the other component, which has an even labeling, obviously the number of $+$-alternating k-simplices is even. Since the sum of an odd and an even number cannot be even, we have reached a contradiction! □

Theorem 2.25 can be phrased in a slightly stronger way using the concept of *cohomological index*, as shown by Schultz [Schu06]. This concept might prove useful because it can actually be computed, in contrast to the difficult determination of indices, coindices, and connectivity.

2.4 Classes with Good Topological Lower Bounds for the Chromatic Number

The family of Kneser graphs is one example of a graph class for which the bound on the chromatic number obtained from the index of the associated spaces $|\,\mathrm{Hom}(K_2, G)\,|$ is sharp. It is certainly interesting to see a general scheme to create families of graphs in which this phenomenon occurs. One such scheme is the generalized Mycielski construction.

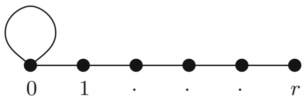

Fig. 2.18 The graph P_r^0: a path with a loop at 0

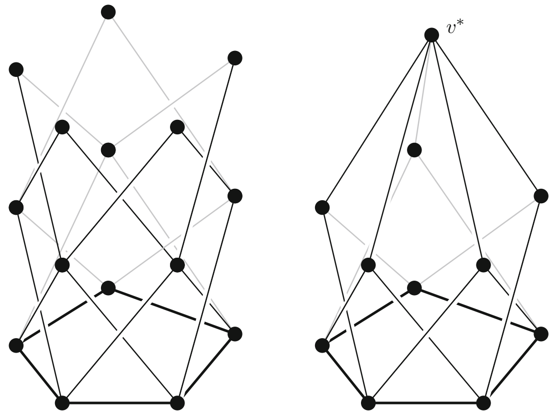

Fig. 2.19 The construction of the Mycielski graph M_2C_5

The Generalized Mycielski Construction

Let G be a finite simple graph and $r \geq 2$ a natural number. We are going to define the *generalized Mycielski graph* M_rG *of* G *with parameter* r. To do so, consider the graph P_r^0, a path of length r with vertex set $\{0, 1, \ldots, r\}$ with an additional loop at 0. See Fig. 2.18.

Consider the graph $G \times P_r^0$, which amounts to a copy of G attached to the product of G with a path P_r of length r, i.e.,

$$G \times P_r^0 = G \times \{0\} \cup G \times P_r.$$

The set $V(G) \times \{r\}$ is an independent set in this graph. We will identify all of these vertices in order to obtain M_rG, i.e.,

$$M_rG = G \times P_r^0 / \, (V(G) \times \{r\}).$$

Let us denote the single vertex that is obtained by the identification of $V(G) \times \{r\}$ by v^*. Figure 2.19 shows the two steps of the construction for the 5-cycle $G = C_5$ and $r = 2$, which in this case results in the Grötzsch graph.

This construction increases the chromatic number by at most one, i.e.,

$$\chi(G) + 1 \geq \chi(M_rG).$$

Assume that $c : V(G) \to [k]$ is a proper coloring of G. Then $\bar{c} : V(M_r G) = (V(G) \times \{0, 1, \ldots, r-1\}) \cup \{v^*\} \to [k+1]$ defined by $\bar{c}(v,t) = c(v)$ and $\bar{c}(v^*) = k+1$ is a proper coloring of $M_r G$, as is easily checked. Now the following lemma shows that the topological lower bound on the chromatic number also increases by exactly one.

Lemma 2.26. *Given any finite simple graph G, we have*

$$\text{co-ind}(|\operatorname{Hom}(K_2, M_r G)|) \geq \text{co-ind}(|\operatorname{Hom}(K_2, G)|) + 1.$$

Proof. Let $k = \text{co-ind}(|\operatorname{Hom}(K_2, G)|)$, and let $f : \mathbb{S}^k \to |\operatorname{Hom}(K_2, G)|$ be a $\mathbb{Z}_2$-equivariant map. We have to construct a $\mathbb{Z}_2$-equivariant map $g : \mathbb{S}^{k+1} \to |\operatorname{Hom}(K_2, M_r G)|$. In order to do so, we will work with the following homeomorphic model of $\mathbb{S}^{k+1}$. We know from Appendix B that

$$\mathbb{S}^{k+1} \cong \mathbb{S}^k * \mathbb{S}^0,$$

which is easily seen to be homeomorphic to

$$\mathbb{S}^k \times [-1, +1]/\sim,$$

where $\sim$ is defined by $(x,t) \sim (x',t')$ if either $(x,t) = (x',t')$ or $t = t' \in \{\pm 1\}$. In other words, we obtain $\mathbb{S}^{k+1}$ by taking the cylinder $\mathbb{S}^k \times [-1,+1]$ and collapsing its ends, $\mathbb{S}^k \times \{-1\}$ and $\mathbb{S}^k \times \{+1\}$, each to a point. Obviously, the antipodal action on this model of the sphere is induced by the map $\nu(x,t) = (-x,-t)$ on the cylinder. It therefore suffices to construct a $\mathbb{Z}_2$-equivariant map

$$\mathbb{S}^k \times [-1,+1] \to |\operatorname{Hom}(K_2, M_r G)|$$

that is constant on each end $\mathbb{S}^k \times \{-1\}$ and $\mathbb{S}^k \times \{+1\}$.

Now consider the sequence of poset maps

$$\operatorname{Hom}(K_2, G) \times \operatorname{Hom}(K_2, P_r^0) \xrightarrow{\times} \operatorname{Hom}(K_2, G \times P_r^0) \xrightarrow{\pi} \operatorname{Hom}(K_2, M_r G),$$

where π is induced from the projection

$$G \times P_r^0 \to M_r G.$$

Note that this map is, by construction, equivariant with respect to the $\mathbb{Z}_2$-action given by α. Since we already have the map $f : \mathbb{S}^k \to |\operatorname{Hom}(K_2, G)|$, we are now going to investigate $\operatorname{Hom}(K_2, P_r^0)$. It turns out that $|\operatorname{Hom}(K_2, P_r^0)|$ is an interval with the multihomomorphism $i \mapsto \{0\}$ in its center. Figure 2.20 shows the partially ordered set $\operatorname{Hom}(K_2, P_3^0)$ and (in this 1-dimensional case also) the simplicial complex $\Delta(\operatorname{Hom}(K_2, P_r^0))$.

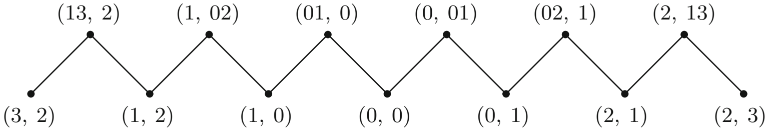

Fig. 2.20 $\mathrm{Hom}(K_2, P_3^0)$

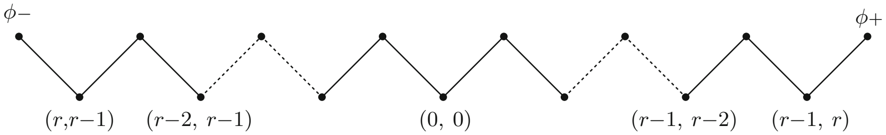

Fig. 2.21 The extension Q of $\mathrm{Hom}(K_2, P_r^0)$

Note that the $\mathbb{Z}_2$-action α on $|\,\mathrm{Hom}(K_2, P_r^0)\,|$ is given by switching both ends of the interval. We now extend the poset $\mathrm{Hom}(K_2, P_r^0)$ to a poset Q by adding two new elements ϕ_+ and ϕ_- and extend the above poset map $\pi \circ \times$ to a map

$$\mathrm{Hom}(K_2, G) \times Q \xrightarrow{\varphi} \mathrm{Hom}(K_2, M_r G).$$

The extended order is defined by the cover relations $\phi_+ > (r-1, r)$ and $\phi_- > (r, r-1)$. In general, we obtain a picture as shown in Fig. 2.21.

We extend the action of α to Q by defining $\alpha * \phi_\pm = \phi_\mp$, and hence there exists an equivariant homeomorphism $g : [-1, +1] \to |Q|$ with respect to the $\mathbb{Z}_2$-action $t \mapsto -t$ on the interval.

Now the extension of the poset map $\pi \circ \times$ is defined by $\varphi((A, B), \phi_+) = (V(G) \times \{r-1\}, \{v^*\})$ and $\varphi((A, B), \phi_-) = \varphi(\alpha * ((B, A), \phi_+)) = \alpha * \varphi((B, A), \phi_+) = (\{v^*\}, V(G) \times \{r-1\})$. This obviously yields an equivariant poset map that is constant when restricted to either $\mathrm{Hom}(K_2, G) \times \{\phi_+\}$ or $\mathrm{Hom}(K_2, G) \times \{\phi_-\}$. Altogether, we obtain the following equivariant continuous map

$$\mathbb{S}^k \times [-1, 1] \xrightarrow{f \times g} |\,\mathrm{Hom}(K_2, G)\,| \times |Q| \xrightarrow{|\varphi|} |\,\mathrm{Hom}(K_2, M_r G)\,|,$$

which is constant on each end $\mathbb{S}^k \times \{-1\}$ and $\mathbb{S}^k \times \{+1\}$ as desired. □

This lemma, together with Proposition 2.22, Lemma 2.21, and the previous observation, yields the following inequalities:

$$\begin{aligned}\chi(G) + 1 \geq \chi(M_r G) &\geq \text{co-ind}(|\,\mathrm{Hom}(K_2, M_r G)\,|) + 2 \\ &\geq \text{co-ind}(|\,\mathrm{Hom}(K_2, G)\,|) + 3.\end{aligned}$$

In the case that the topological lower bound on the chromatic number of G is tight, i.e., $\chi(G) = \text{co-ind}(|\,\mathrm{Hom}(K_2, G)\,|) + 2$, we obtain

$$\chi(G) + 1 \geq \chi(M_r G) \geq \text{co-ind}(|\operatorname{Hom}(K_2, M_r G)|) + 2 \geq \chi(G) + 1.$$

In other words, the chromatic number increases by exactly one and the topological lower bound remains tight. By iterating this procedure we obtain the following.

Proposition 2.27. *Let G be a graph with $\chi(G) = \text{co-ind}(|\operatorname{Hom}(K_2, G)|) + 2$ and $r_1, \dots, r_s \geq 2$. Then, for the iterated Mycielski construction $H = M_{r_1}(M_{r_2}(\cdots M_{r_s}(G)\cdots))$, we obtain*

$$\chi(H) = \text{co-ind}(|\operatorname{Hom}(K_2, H)|) + 2 = \chi(G) + s.$$ □

The immediate examples of graphs with tight topological lower bound are cycles and the family of Kneser graphs. Indeed, we have for even cycles

$$\chi(C_{2r}) = 2 = \text{co-ind}(|\operatorname{Hom}(K_2, C_{2r})|) + 2,$$

for odd cycles

$$\chi(C_{2r+1}) = 3 = \text{co-ind}(|\operatorname{Hom}(K_2, C_{2r+1})|) + 2,$$

and for the Kneser graphs

$$\chi(KG_{n,k}) = n - 2k + 2 = \text{co-ind}(|\operatorname{Hom}(K_2, KG_{n,k})|) + 2.$$

Exercises

1. Show that for any $n > d \geq 1$, there exists a set X of n vectors on the d-dimensional sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$ such that any subset $S \subseteq X$ with $|S| = d + 1$ elements is linearly independent.
2. Show that the sets U_i defined in the proof of the Kneser conjecture by Greene on page 39 are open.
3. Prove the following lemma, known as Gale's lemma [Gal56]. For every $d \geq 0$ and $k \geq 1$, there exists a subset $X \subseteq \mathbb{S}^d$ of $2k + d$ points such that every open hemisphere contains at least k points of X, i.e., for all $x \in \mathbb{S}^d$, the intersection $X \cap \{y \in \mathbb{S}^d : \langle x, y\rangle > 0\}$ contains at least k elements. Hint: Consider the set of points $\{(-1)^i(1, i, i^2, \dots, i^d) : i = 1, \dots, 2k + d\}$.
4. Give a proof of Lovász's theorem along the lines of Greene's proof using only open sets. You will probably find a proof that was originally given by Bárány [Bár78]. Hint: Use Gale's lemma from the previous exercise.
5. Consider the following induced subgraph $SG_{n,k}$ of the Kneser graph $KG_{n,k}$ defined by Schrijver [Sch78]. The vertices are given by all k-subsets S of $[n]$ such that S does not contain a pair of consecutive numbers modulo n, i.e., none of the pairs $\{1, 2\}, \{2, 3\}, \dots, \{n - 1, n\}, \{n, 1\}$ is contained is S. We call these sets *stable*. Since $SG_{n,k}$ is an induced subgraph of $KG_{n,k}$, any two stable sets are adjacent if and only if they are disjoint. Prove that $\chi(SG_{n,k}) = \chi(KG_{n,k}) = n - 2k + 2$. Hint: Use the ideas of the previous two exercises and the following interesting observation on polynomials.

Observation: Let $p(x)$ be a polynomial of degree at most d, with real coefficients. Then there exists a stable k-subset S of $[2k+d]$ such that $(-1)^i p(i) > 0$ whenever $i \in S$.

6. Show that the graphs $SG_{n,k}$ defined in the previous exercise are vertex critical with respect to the chromatic number, i.e., after removing an arbitrary vertex, the chromatic number drops by at least one. Remark: This is a hard exercise. At some point you might want to get some inspiration from Schrijver's article [Sch78].
7. Describe the neighborhood and Lovász complexes of $SG_{2n+1,n}$ and, furthermore, the action of ν on the Lovász complex.
8. Let G be a finite simple graph and assume that $\text{ind}(\mathcal{L}(G)) \geq l + m + 2$. Show that G has a complete bipartite $K_{l,m}$ as a subgraph. This result is due to Csorba, Lange, Schurr, and Wassmer [CLSW04].
9. Show that there exists a homeomorphism from $\mathbb{S}^{n-2}$ to the subspace $S = \mathbb{S}^{n-1} \cap \{x : \sum_{i=1}^{n} x_i = 0\} \subseteq \mathbb{R}^n$ that is equivariant with respect to the antipodal actions as needed in the proof of Proposition 2.6 on page 46.
10. Show that the map φ defined in the proof of Proposition 2.6 on page 46 is bijective.
11. Show that there is a natural homeomorphism between $|\,\text{Hom}(T, G)\,|$ and $|\Delta(\text{Hom}(T, G))|$.
12. Show that the definition of composition $*$ of graph multihomomorphisms as given on page 55 is well defined, associative, and respects the inclusion order.
13. Fill in the details of the proof of Proposition 2.12.
14. Show that the $\mathbb{Z}_2$-action on $\text{Int}(C(G))$ as defined on page 58 corresponds to the $\mathbb{Z}_2$-action on $\mathcal{L}(G)$ under the homeomorphism of the geometric realizations.
15. Show that in fact, $|\,\text{Hom}(K_2, K_n)\,|$ is $\mathbb{Z}_2$-equivariant *homeomorphic* to the sphere $\mathbb{S}^{n-2}$ endowed with the antipodal action.
16. Show that, as needed in the proof of Lemma 2.23, there exists a simplicial approximation $\mathcal{H}$ of H such that $\mathcal{H}$ is odd on one of the boundary k-spheres and even on the other.
17. Let G be a finite graph with $\chi(G) = \text{ind}(|\,\text{Hom}(K_2, G)\,|) + 2$ and let $c : V(G) \to C$ be a proper coloring with $|C| = \chi(G)$. Show that for every partition $C = A \dot{\cup} B$ of the color set with $A, B \neq \emptyset$, there exists a complete bipartite subgraph $K_{|A|,|B|}$ of G such that one of the shores is colored with all of A, and the other with all of B. This result is due to Simonyi and Tardos [ST07].
18. Show by elementary means that the Mycielski graph M_2G has chromatic number $\chi(M_2G) = \chi(G) + 1$.

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