Chapter 2
Green's Dyad for Plane-Layered Media

The Green's dyad, which is the electric-field response to a delta-function vector current source, plays a principal role in volume-integral equations, as we shall see later. In this chapter we develop the theory of the Green's dyad for plane-parallel layered media. In Chap. 9 we extend the development to multilayered media with cylindrical geometry.

2.1 Eigenmodes of Anisotropic Media

We will consider plane-parallel bodies of infinite extent in the (x,y) plane, which are made up of layers of homogeneous, anisotropic material. To be specific, we consider magnetic host materials that are characterized by the following biaxial generalized electrical permittivity matrix:

$$\varepsilon_h = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}, \tag{2.1}$$

where the entries are generalized permittivities $\varepsilon + \sigma/j\omega$.

Maxwell's equations for an electrically anisotropic body are

$$\begin{aligned} \nabla \times \mathbf{E} &= -j\omega\mu_h\mathbf{H} - j\omega(\mu(\mathbf{r}) - \mu_h)\mathbf{H} \\ &= -j\omega\mu_h\mathbf{H} + \mathbf{J}_m \\ \nabla \times \mathbf{H} &= j\omega\varepsilon_h \cdot \mathbf{E} + j\omega(\varepsilon(\mathbf{r}) - \varepsilon_h) \cdot \mathbf{E} \\ &= j\omega\varepsilon_h \cdot \mathbf{E} + \mathbf{J}_e, \end{aligned} \tag{2.2}$$

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where $\mathbf{J}_m$ and $\mathbf{J}_e$ are anomalous magnetic and electric currents that account for the presence of flaws, or anomalies, in the otherwise-uniform host material. From here on we drop the subscript h on the generalized host permittivity and permeability.

Because of the material anisotropy, it is convenient to work with a matrix formulation of these equations that has been useful in crystal optics, plasmas, and microwave devices [13–21]. If the body is homogeneous with respect to (x,y), then Maxwell's equations can be Fourier transformed with respect to (x,y) and written as the following four-vector matrix differential equation in the spectral domain:

$$\partial_z \tilde{\mathbf{e}} = \mathbf{S}\cdot\tilde{\mathbf{e}} + \mathbf{U}\cdot\tilde{\mathbf{J}}, \tag{2.3}$$

$$\tilde{E}_z = \frac{k_y}{\varepsilon_z\omega}\tilde{H}_x - \frac{k_x}{\varepsilon_z\omega}\tilde{H}_y + \frac{j}{\varepsilon_z\omega}\tilde{J}_{ez}, \tag{2.4}$$

$$\tilde{H}_z = \frac{-k_y}{\mu\omega}\tilde{E}_x + \frac{k_x}{\mu\omega}\tilde{E}_y - \frac{j}{\mu\omega}\tilde{J}_{mz}, \tag{2.5}$$

where the tilde denotes a function defined in the transform domain (k_x,k_y), and

$$\tilde{\mathbf{e}} = \begin{bmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{H}_x \\ \tilde{H}_y \end{bmatrix}; \tilde{\mathbf{J}} = \begin{bmatrix} \tilde{J}_{ex} \\ \tilde{J}_{ey} \\ \tilde{J}_{ez} \\ \tilde{J}_{mx} \\ \tilde{J}_{my} \\ \tilde{J}_{mz} \end{bmatrix}. \tag{2.6}$$

The subscript e denotes an electric current and m denotes a magnetic current. The matrices in (2.3) are given by

$$\mathbf{S} = -\begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & d \\ \alpha & \beta & 0 & 0 \\ \gamma & \delta & 0 & 0 \end{bmatrix}; \mathbf{U} = \begin{bmatrix} 0 & 0 & k_x/\omega\varepsilon_z & 0 & 1 & 0 \\ 0 & 0 & k_y/\omega\varepsilon_z & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -k_x/\omega\mu \\ -1 & 0 & 0 & 0 & 0 & -k_y/\omega\mu \end{bmatrix}. \tag{2.7}$$

The entries of $\mathbf{S}$ are given in terms of the entries of (2.1) by

$$\begin{aligned} a &= \frac{j}{\omega\varepsilon_z}k_xk_y; \alpha = \frac{j}{\omega\mu}(-\mu\varepsilon_{yx}\omega^2 - k_xk_y) \\ b &= \frac{j}{\omega\varepsilon_z}(\mu\varepsilon_z\omega^2 - k_x^2); \beta = \frac{j}{\omega\mu}(-\mu\varepsilon_y\omega^2 + k_x^2) \\ c &= \frac{j}{\omega\varepsilon_z}(-\mu\varepsilon_z\omega^2 + k_y^2); \gamma = \frac{j}{\omega\mu}(\mu\varepsilon_x\omega^2 - k_y^2) \\ d &= -\frac{j}{\omega\varepsilon_z}k_xk_y; \delta = \frac{j}{\omega\mu}(\mu\varepsilon_{xy}\omega^2 + k_xk_y). \end{aligned} \tag{2.8}$$

When $\tilde{\mathbf{J}}$ is a surface current confined to $z = z'$, i.e., $\tilde{\mathbf{J}} = \tilde{\mathbf{J}}_s \delta(z - z')$, then integration of (2.3) produces

$$\tilde{\mathbf{e}}^{(+)} - \tilde{\mathbf{e}}^{(-)} = \mathbf{U} \cdot \tilde{\mathbf{J}}_s, \tag{2.9}$$

which is called the equation of discontinuity. The superscript $(+)$ denotes the limit z approaches z' from above and the superscript $(-)$ denotes the limit from below. Equation (2.9) will be used in the next section to compute the Green's dyad for a layered workpiece.

Starting with these equations, Roberts [24] has developed a fairly complete theory of normal modes of biaxial anisotropic media. This work is based on, and extends, earlier work performed at Sabbagh Associates [22, 23, 25–29]. From here on we specialize the theory developed in [24] to the case to be considered here, in which the media involved are transversely isotropic to the z-coordinate. The dielectric permittivity tensor, in its principal-axis coordinate system, then takes the form

$$\varepsilon = \begin{bmatrix} \varepsilon_t & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}. \tag{2.10}$$

The entries in $\mathbf{S}$ now become

$$\begin{aligned}
a &= \frac{j}{\omega\varepsilon_z} k_x k_y; \alpha = \frac{j}{\omega\mu}(-k_x k_y) \\
b &= \frac{j}{\omega\varepsilon_z}(\mu\varepsilon_z\omega^2 - k_x^2); \beta = \frac{j}{\omega\mu}(-\mu\varepsilon_t\omega^2 + k_x^2) \\
c &= \frac{j}{\omega\varepsilon_z}(-\mu\varepsilon_z\omega^2 + k_y^2); \gamma = \frac{j}{\omega\mu}(\mu\varepsilon_t\omega^2 - k_y^2) \\
d &= -\frac{j}{\omega\varepsilon_z} k_x k_y; \delta = \frac{j}{\omega\mu}(k_x k_y).
\end{aligned} \tag{2.11}$$

Let's introduce some notation: $k_x^2 + k_y^2 = k_t^2$, $\omega^2\mu\varepsilon_t = \Omega_t^2$, $\omega^2\mu\varepsilon_z = \Omega_z^2$, $\varepsilon = \varepsilon_t/\varepsilon_z$. Then the eigenvalues of $\mathbf{S}$ are

$$\lambda_1 = \sqrt{k_t^2 - \Omega_t^2} \;\; \lambda_2 = -\lambda_1 \;\; \lambda_3 = \sqrt{\varepsilon}\sqrt{k_t^2 - \Omega_z^2} \;\; \lambda_4 = -\lambda_3 \;. \tag{2.12}$$

The linearly-independent eigenvectors that correspond to these eigenvalues are:

$$\mathbf{v}_1 = \begin{bmatrix} -j\omega\mu_0 k_y \\ j\omega\mu_0 k_x \\ \lambda_1 k_x \\ \lambda_1 k_y \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} -j\omega\mu_0 k_y \\ j\omega\mu_0 k_x \\ -\lambda_1 k_x \\ -\lambda_1 k_y \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} \lambda_3 k_x \\ \lambda_3 k_y \\ j\omega\varepsilon_t k_y \\ -j\omega\varepsilon_t k_x \end{bmatrix} \mathbf{v}_4 = \begin{bmatrix} \lambda_3 k_x \\ \lambda_3 k_y \\ -j\omega\varepsilon_t k_y \\ j\omega\varepsilon_t k_x \end{bmatrix}. \tag{2.13}$$

When $k_x = k_y = 0$, the following are linearly-independent eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\sqrt{\dfrac{\varepsilon_t}{\mu_0}} \end{bmatrix} \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \sqrt{\dfrac{\varepsilon_t}{\mu_0}} \end{bmatrix} \mathbf{v}_3 = \begin{bmatrix} 0 \\ \sqrt{\dfrac{\mu_0}{\varepsilon_t}} \\ 1 \\ 0 \end{bmatrix} \mathbf{v}_4 = \begin{bmatrix} 0 \\ -\sqrt{\dfrac{\mu_0}{\varepsilon_t}} \\ 1 \\ 0 \end{bmatrix}. \tag{2.14}$$

When we substitute $\mathbf{v}_1,\ \mathbf{v}_2$ of (2.13) into (2.4), with the source currents set to zero, we find that $\tilde{E}_z = 0$; hence, $\mathbf{v}_1,\ \mathbf{v}_2$ are transverse electric (TE) modes, with respect to z. Similarly, $\mathbf{v}_3,\ \mathbf{v}_4$ are transverse magnetic (TM) modes. Note that the TE modes are orthogonal to the TM modes. This will facilitate the computation of the Green's dyadic. $\mathbf{v}_1$ and $\mathbf{v}_3$ are downward-traveling waves in the z-direction; i.e., they represent waves that travel in the negative z-direction. $\mathbf{v}_2,\ \mathbf{v}_4$ are upward-traveling waves (in the positive z-direction). We see from (2.4) and (2.5) that all modes are TEM (transverse electric and magnetic) with respect to z for $k_x = k_y = 0$.

2.2 Green Dyad for Plane-Parallel Layered Media

Though the theory of eigenmodes that was developed in the preceding section is applicable to the class of transverse-isotropic media, we will apply it only to isotropic media. Hence, $\varepsilon_t = \varepsilon_z = \varepsilon$ in (2.10), which means that $\lambda_1 = \lambda_3$ in (2.13).

The flaw, or other anomaly, in a workpiece produces an anomalous current which is to be determined as the solution of a volume-integral equation. The kernel of this equation is a Green dyad for a plane-parallel layered workpiece, and we turn our attention to determining this dyad.

Consider the system shown in Fig. 2.1. The point-source of electric or magnetic current is in region 0, and we want to compute the fields in this region, or in any other region. This will give us the Green dyad. If the source is an electric current, and the field electric, then the resulting dyad is called electric–electric. If the source is a magnetic current, and the field electric, then the dyad is called electric–magnetic, and so-on.

We write for the system of four-vectors in the ith region

$$\mathbf{v}_{1i} = \begin{bmatrix} -j\omega\mu_i k_y \\ j\omega\mu_i k_x \\ \lambda_i k_x \\ \lambda_i k_y \end{bmatrix} \mathbf{v}_{2i} = \begin{bmatrix} -j\omega\mu_i k_y \\ j\omega\mu_i k_x \\ -\lambda_i k_x \\ -\lambda_i k_y \end{bmatrix} \mathbf{v}_{3i} = \begin{bmatrix} \lambda_i k_x \\ \lambda_i k_y \\ j\omega\hat{\varepsilon}_i k_y \\ -j\omega\hat{\varepsilon}_i k_x \end{bmatrix} \mathbf{v}_{4i} = \begin{bmatrix} \lambda_i k_x \\ \lambda_i k_y \\ -j\omega\hat{\varepsilon}_i k_y \\ j\omega\hat{\varepsilon}_i k_x \end{bmatrix}. \tag{2.15}$$

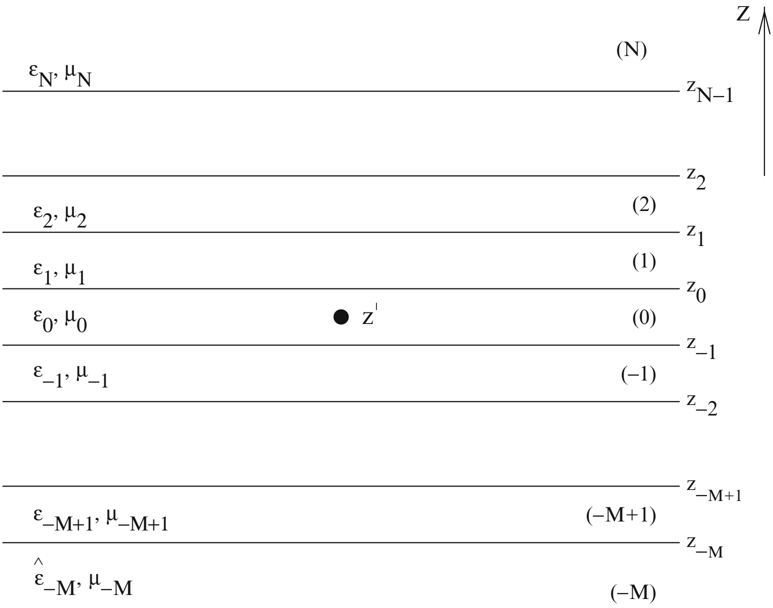

Fig. 2.1 Plane-parallel layered workpiece. The source is in region 0

2.2.1 Infinite-Space Green Dyad

First of all, we will compute the infinite-space Green dyad, which is the dyad associated with an infinite, uniform, unlayered medium. This corresponds to the situation wherein region 0 of Fig. 2.1 extends to $z = \pm\infty$; i.e., $z_0 = \infty$, and $z_{-1} = -\infty$.

Figure 2.2 shows a source located at $z = z'$ in region 0, together with the appropriate field eigenvectors on each side of z'. This choice of the eigenvectors is consistent with the fact that $\mathbf{v}_1$ and $\mathbf{v}_3$ travel in the negative z-direction, and $\mathbf{v}_2$ and $\mathbf{v}_4$ are positively traveling waves. The superscript, (inc), denotes "incident" fields due to the source.

If the source is an electric current vector, then the right-hand side of (2.9) consists of the three electric excitation vectors:

$$\mathbf{f}_x^{(e)} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\tilde{J}_{ex} \end{bmatrix} \mathbf{f}_y^{(e)} = \begin{bmatrix} 0 \\ 0 \\ \tilde{J}_{ey} \\ 0 \end{bmatrix} \mathbf{f}_z^{(e)} = \begin{bmatrix} k_x\tilde{J}_{ez}/\omega\hat{\varepsilon}_0 \\ k_y\tilde{J}_{ez}/\omega\hat{\varepsilon}_0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.16}$$

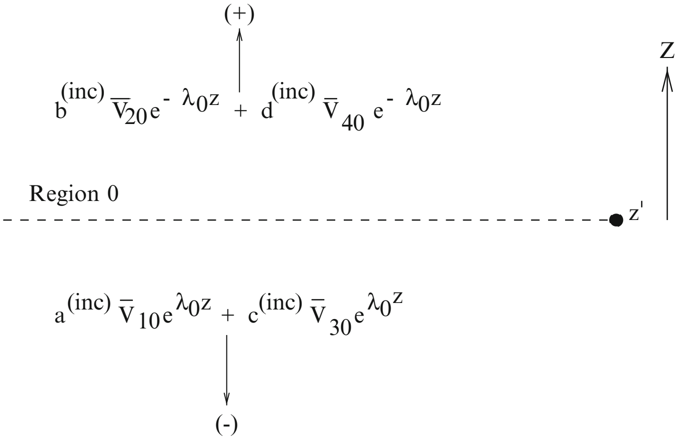

Fig. 2.2 Showing the eigenvectors used to calculate the infinite-space Green dyad. The source is located at $z = z'$

and if the source is a magnetic current, then the right-hand side consists of the magnetic excitation vectors:

$$\mathbf{f}_x^{(m)} = \begin{bmatrix} 0 \\ -\tilde{J}_{mx} \\ 0 \\ 0 \end{bmatrix} \mathbf{f}_y^{(m)} = \begin{bmatrix} \tilde{J}_{my} \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{f}_z^{(m)} = \begin{bmatrix} 0 \\ 0 \\ -k_x\tilde{J}_{mz}/\omega\mu_0 \\ -k_y\tilde{J}_{mz}/\omega\mu_0 \end{bmatrix} . \tag{2.17}$$

Equation (2.9) becomes therefore

$$-a^{(\mathrm{inc})}\mathbf{v}_{10}e^{\lambda_0 z'} + b^{(\mathrm{inc})}\mathbf{v}_{20}e^{-\lambda_0 z'} - c^{(\mathrm{inc})}\mathbf{v}_{30}e^{\lambda_0 z'} + d^{(\mathrm{inc})}\mathbf{v}_{40}e^{-\lambda_0 z'} = \mathbf{f}\,, \tag{2.18}$$

where $\mathbf{f}$ denotes one of the six excitation vectors in (2.16) or (2.17). We will return to this shortly.

Take the dot product of (2.18) with respect to the TE-mode vectors, $\mathbf{v}_{10}$, $\mathbf{v}_{20}$, and get

$$\begin{aligned} -A^{(i)}\alpha_{11} + B^{(i)}\alpha_{12} &= F_1 \\ -A^{(i)}\alpha_{12} + B^{(i)}\alpha_{22} &= F_2\,, \end{aligned} \tag{2.19}$$

where we have used the orthogonality of the TE and TM modal vectors to eliminate $c^{(\mathrm{inc})}$ and $d^{(\mathrm{inc})}$. Here

$$\begin{aligned} A^{(i)} &= a^{(\mathrm{inc})}e^{\lambda_0 z'} \\ B^{(i)} &= b^{(\mathrm{inc})}e^{-\lambda_0 z'} \\ F_1 &= \mathbf{v}_{10}\cdot\mathbf{f} \end{aligned}$$

$$\begin{aligned}
F_2 &= \mathbf{v}_{20} \cdot \mathbf{f} \\
\alpha_{11} &= \mathbf{v}_{10} \cdot \mathbf{v}_{10} = k_t^2(\lambda_0^2 - \omega^2 \mu_0^2) \\
\alpha_{12} &= \mathbf{v}_{10} \cdot \mathbf{v}_{20} = -k_t^2(\lambda_0^2 + \omega^2 \mu_0^2) \\
\alpha_{22} &= \mathbf{v}_{20} \cdot \mathbf{v}_{20} = k_t^2(\lambda_0^2 - \omega^2 \mu_0^2).
\end{aligned} \tag{2.20}$$

Upon solving (2.19) for $A^{(i)}$, $B^{(i)}$, and then using (2.20) we get

$$\begin{aligned}
a^{(\mathrm{inc})} &= \frac{F_1\alpha_{22} - F_2\alpha_{12}}{-\alpha_{11}\alpha_{22} + \alpha_{12}^2} e^{-\lambda_0 z'} \\
b^{(\mathrm{inc})} &= \frac{-F_2\alpha_{11} + F_1\alpha_{12}}{-\alpha_{11}\alpha_{22} + \alpha_{12}^2} e^{\lambda_0 z'} \ .
\end{aligned} \tag{2.21}$$

Hence, the contribution of the TE-modes to the infinite-space Green dyad is

$$\begin{aligned}
&\frac{-F_2\alpha_{11} + F_1\alpha_{12}}{-\alpha_{11}\alpha_{22} + \alpha_{12}^2} \mathbf{v}_{20} e^{-\lambda_0(z-z')} \ , \ z \geq z' \\
&\frac{F_1\alpha_{22} - F_2\alpha_{12}}{-\alpha_{11}\alpha_{22} + \alpha_{12}^2} \mathbf{v}_{10} e^{\lambda_0(z-z')} \ , \ z \leq z' \ .
\end{aligned} \tag{2.22}$$

To compute the TM-mode contribution, take the dot product of (2.18) with respect to $\mathbf{v}_{30}$, $\mathbf{v}_{40}$. Proceeding as before, we get for the TM-mode contribution

$$\begin{aligned}
&\frac{F_4\beta_{33} - F_3\beta_{34}}{\beta_{33}\beta_{44} - \beta_{34}^2} \mathbf{v}_{40} e^{-\lambda_0(z-z')} \ , \ z \geq z' \\
-&\frac{F_3\beta_{44} - F_4\beta_{34}}{\beta_{33}\beta_{44} - \beta_{34}^2} \mathbf{v}_{30} e^{\lambda_0(z-z')} \ , \ z \leq z' \ ,
\end{aligned} \tag{2.23}$$

where

$$\begin{aligned}
F_3 &= \mathbf{v}_{30} \cdot \mathbf{f} \\
F_4 &= \mathbf{v}_{40} \cdot \mathbf{f} \\
\beta_{33} &= \mathbf{v}_{30} \cdot \mathbf{v}_{30} = k_t^2(\lambda_0^2 - \omega^2 \hat{\varepsilon}_0^2) \\
\beta_{34} &= \mathbf{v}_{30} \cdot \mathbf{v}_{40} = k_t^2(\lambda_0^2 + \omega^2 \hat{\varepsilon}_0^2) \\
\beta_{44} &= \mathbf{v}_{40} \cdot \mathbf{v}_{40} = k_t^2(\lambda_0^2 - \omega^2 \hat{\varepsilon}_0^2) \ .
\end{aligned} \tag{2.24}$$

Therefore, upon combining (2.22) and (2.23) we get for the composite infinite-field Green dyad:

$$\left[\frac{-F_2\alpha_{11}+F_1\alpha_{12}}{-\alpha_{11}\alpha_{22}+\alpha_{12}^2}\mathbf{v}_{20}+\frac{F_4\beta_{33}-F_3\beta_{34}}{\beta_{33}\beta_{44}-\beta_{34}^2}\mathbf{v}_{40}\right]e^{-\lambda_0(z-z')} \quad , \ z\geq z'$$

$$\left[\frac{F_1\alpha_{22}-F_2\alpha_{12}}{-\alpha_{11}\alpha_{22}+\alpha_{12}^2}\mathbf{v}_{10}-\frac{F_3\beta_{44}-F_4\beta_{34}}{\beta_{33}\beta_{44}-\beta_{34}^2}\mathbf{v}_{30}\right]e^{\lambda_0(z-z')} \quad , \ z\leq z' \ . \tag{2.25}$$

The Fs that appear in (2.25) are defined in (2.20) and (2.24). They are computed by taking the dot product of the eigenvectors of (2.15) (with $i=0$) with the excitation vectors in (2.16) and (2.17). For example, the value of F_1 that is associated with electric excitation in the x-direction is given by $F_1^{(ex)} = \mathbf{v}_{10}\cdot\mathbf{f}_x^{(e)} = -\lambda_0 k_y\tilde{J}_{ex}$. Continuing in this manner, we find:

$$\begin{aligned}
F_1^{(ex)} &= -\lambda_0 k_y \tilde{J}_{ex}, F_1^{(mx)} = -j\omega\mu_0 k_x \tilde{J}_{mx}\\
F_2^{(ex)} &= \lambda_0 k_y \tilde{J}_{ex}, F_2^{(mx)} = -j\omega\mu_0 k_x \tilde{J}_{mx}\\
F_3^{(ex)} &= j\omega\hat{\varepsilon}_0 k_x \tilde{J}_{ex}, F_3^{(mx)} = -\lambda_0 k_y \tilde{J}_{mx}\\
F_4^{(ex)} &= -j\omega\hat{\varepsilon}_0 k_x \tilde{J}_{ex}, F_4^{(mx)} = -\lambda_0 k_y \tilde{J}_{mx}\\
F_1^{(ey)} &= \lambda_0 k_x \tilde{J}_{ey}, F_1^{(my)} = -j\omega\mu_0 k_y \tilde{J}_{my}\\
F_2^{(ey)} &= -\lambda_0 k_x \tilde{J}_{ey}, F_2^{(my)} = -j\omega\mu_0 k_y \tilde{J}_{my}\\
F_3^{(ey)} &= j\omega\hat{\varepsilon}_0 k_y \tilde{J}_{ey}, F_3^{(my)} = \lambda_0 k_x \tilde{J}_{my}\\
F_4^{(ey)} &= -j\omega\hat{\varepsilon}_0 k_y \tilde{J}_{ey}, F_4^{(my)} = \lambda_0 k_x \tilde{J}_{my}\\
F_1^{(ez)} &= 0, F_1^{(mz)} = -\frac{\lambda_0 k_t^2}{\omega\mu_0}\tilde{J}_{mz}\\
F_2^{(ez)} &= 0, F_2^{(mz)} = \frac{\lambda_0 k_t^2}{\omega\mu_0}\tilde{J}_{mz}\\
F_3^{(ez)} &= \frac{\lambda_0 k_t^2}{\omega\hat{\varepsilon}_0}\tilde{J}_{ez}, F_3^{(mz)} = 0\\
F_4^{(ez)} &= \frac{\lambda_0 k_t^2}{\omega\hat{\varepsilon}_0}\tilde{J}_{ez}, F_4^{(mz)} = 0 \ .
\end{aligned} \tag{2.26}$$

When these are combined with the α's and β's of (2.20) and (2.24), we get for the infinite-space Green dyad

$$\begin{aligned}&[J_2\mathbf{v}_{20}+J_4\mathbf{v}_{40}]\,e^{-\lambda_0(z-z')},z\geq z'\\&[J_1\mathbf{v}_{10}+J_3\mathbf{v}_{30}]\,e^{\lambda_0(z-z')},z\leq z'\ ,\end{aligned}\tag{2.27}$$

where

$$\begin{array}{lcccc}
 & J_1 & J_2 & J_3 & J_4\\[2mm]
(ex): & \dfrac{k_y\tilde{J}_{ex}}{2\lambda_0k_t^2} & \dfrac{k_y\tilde{J}_{ex}}{2\lambda_0k_t^2} & \dfrac{jk_x\tilde{J}_{ex}}{2\omega\hat{\varepsilon}_0k_t^2} & \dfrac{jk_x\tilde{J}_{ex}}{2\omega\hat{\varepsilon}_0k_t^2}\\[4mm]
(ey): & -\dfrac{k_x\tilde{J}_{ey}}{2\lambda_0k_t^2} & -\dfrac{k_x\tilde{J}_{ey}}{2\lambda_0k_t^2} & \dfrac{jk_y\tilde{J}_{ey}}{2\omega\hat{\varepsilon}_0k_t^2} & \dfrac{jk_y\tilde{J}_{ey}}{2\omega\hat{\varepsilon}_0k_t^2}\\[4mm]
(ez): & 0 & 0 & -\dfrac{\tilde{J}_{ez}}{2\lambda_0\omega\hat{\varepsilon}_0} & \dfrac{\tilde{J}_{ez}}{2\lambda_0\omega\hat{\varepsilon}_0}\\[4mm]
(mx): & -\dfrac{jk_x\tilde{J}_{mx}}{2k_t^2\omega\mu_0} & \dfrac{jk_x\tilde{J}_{mx}}{2k_t^2\omega\mu_0} & \dfrac{k_y\tilde{J}_{mx}}{2\lambda_0k_t^2} & -\dfrac{k_y\tilde{J}_{mx}}{2\lambda_0k_t^2}\\[4mm]
(my): & -\dfrac{jk_y\tilde{J}_{my}}{2k_t^2\omega\mu_0} & \dfrac{jk_y\tilde{J}_{my}}{2k_t^2\omega\mu_0} & -\dfrac{k_x\tilde{J}_{my}}{2\lambda_0k_t^2} & \dfrac{k_x\tilde{J}_{my}}{2\lambda_0k_t^2}\\[4mm]
(mz): & \dfrac{\tilde{J}_{mz}}{2\lambda_0\omega\mu_0} & \dfrac{\tilde{J}_{mz}}{2\lambda_0\omega\mu_0} & 0 & 0.
\end{array}\tag{2.28}$$

When we introduce the vectors of (2.15) into (2.27), and then use (2.28), we obtain an expression for the x- and y-components of the field in terms of the source components. The z-component of the fields are obtained from (2.4) and (2.5). For instance, we will compute the x-component of the electric field, due to the x-component of electric current:

$$\begin{aligned}&\left\{\begin{array}{ll}\left[\dfrac{-j\omega\mu_0k_y^2\tilde{J}_{ex}}{2\lambda_0k_t^2}+\dfrac{jk_x^2\lambda_0\tilde{J}_{ex}}{2\omega\hat{\varepsilon}_0k_t^2}\right]e^{-\lambda_0(z-z')}, & z\geq z'\\[4mm]\left[\dfrac{-j\omega\mu_0k_y^2\tilde{J}_{ex}}{2\lambda_0k_t^2}+\dfrac{jk_x^2\lambda_0\tilde{J}_{ex}}{2\omega\hat{\varepsilon}_0k_t^2}\right]e^{\lambda_0(z-z')}, & z\leq z'\end{array}\right.\\[4mm]&=-j\omega\mu_0\left(1-\frac{k_x^2}{k_0^2}\right)\frac{e^{-\lambda_0|z-z'|}}{2\lambda_0}\tilde{J}_{ex}\ .\end{aligned}\tag{2.29}$$

The coefficient of $\tilde{J}_{ex}$ in (2.29) is the xx-component of the Fourier-domain, infinite-space, electric–electric Green dyad $\tilde{G}^{(ee)}_{(0)}$. The complete expression for the dyad is

$$\tilde{\mathbf{G}}^{(ee)}_{(0)}(k_x,k_y;z,z') = -j\omega\mu_0 \begin{bmatrix} 1-k_x^2/k_0^2 & -k_xk_y/k_0^2 & \pm jk_x\lambda_0/k_0^2 \\ -k_xk_y/k_0^2 & 1-k_y^2/k_0^2 & \pm jk_y\lambda_0/k_0^2 \\ \pm jk_x\lambda_0/k_0^2 & \pm jk_y\lambda_0/k_0^2 & 1+\lambda_0^2/k_0^2 \end{bmatrix} \frac{e^{-\lambda_0|z-z'|}}{2\lambda_0}$$

$$+j\frac{\delta(z-z')}{\omega\hat{\varepsilon}_0}\mathbf{a}_z\mathbf{a}_z \ , \tag{2.30}$$

where $(+)$ sign goes with $z > z'$ and $(-)$ with $z < z'$. The term $j\dfrac{\delta(z-z')}{\omega\hat{\varepsilon}_0}\mathbf{a}_z\mathbf{a}_z$ is called the "depolarizing" term [34] and follows from the last term in (2.4). A similar analysis holds for the magnetic–magnetic Green dyad and produces the same expression as (2.30), except that μ_0 is everywhere replaced by $-\hat{\varepsilon}_0$.

The spectral-domain, infinite-space, magnetic–electric Green dyad is given by

$$\tilde{\mathbf{G}}^{(em)}_{(0)}(k_x,k_y;z,z') = \begin{bmatrix} 0 & \pm\lambda_0 & -jk_y \\ \mp\lambda_0 & 0 & jk_x \\ jk_y & -jk_x & 0 \end{bmatrix} \frac{e^{-\lambda_0|z-z'|}}{2\lambda_0} \ . \tag{2.31}$$

It is not difficult to show that

$$\tilde{\mathbf{G}}^{(me)}_{(0)} = \tilde{\mathbf{G}}^{(em)}_{(0)} \ . \tag{2.32}$$

2.2.2 Layered-Space Green Dyad

The infinite-space dyad that we computed in the previous section serves as the "incident" field in the layered medium of Fig. 2.1. We are interested in computing the fields produced when the incident dyad is scattered from the layers.

As before, we focus our attention on region 0, but this time assume that it has a finite upper-boundary, z_0, and lower-boundary, z_{-1}, as shown in Fig. 2.3. Region 1, which lies above region 0, is a half-space, as is region -1, which lies below region 0.

The infinite-field Green dyad is shown in Fig. 2.3. Because of the singularity of this dyad, we split it into that part, $[J_2\mathbf{v}_{20}+J_4\mathbf{v}_{40}]e^{-\lambda_0(z-z')}$, which is valid for $z \geq z'$, and $[J_1\mathbf{v}_{10}+J_3\mathbf{v}_{30}]e^{\lambda_0(z-z')}$, which is valid for $z \leq z'$. The scattered field, $(a_0\mathbf{v}_{10}+c_0\mathbf{v}_{30})e^{\lambda_0 z}+(b_0\mathbf{v}_{20}+d_0\mathbf{v}_{40})e^{-\lambda_0 z}$, is continuous throughout region 0.

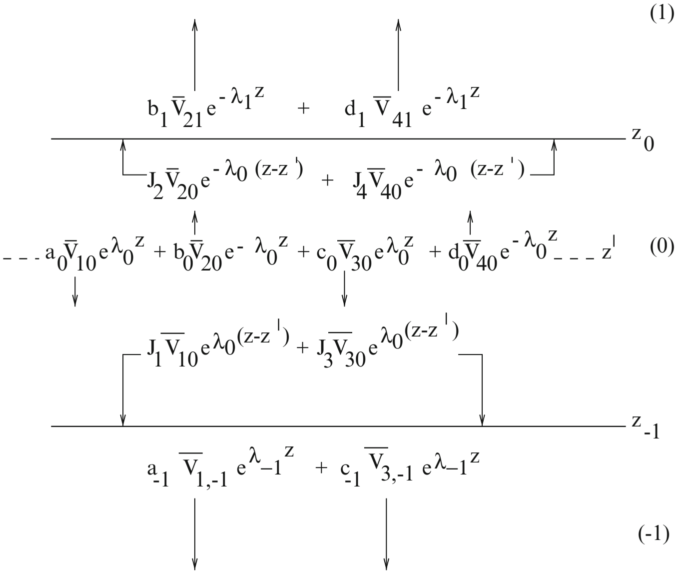

Fig. 2.3 Showing the source, at z', in region 0, together with the incident and scattered fields. Regions 1 and -1 are half-spaces

Continuity of the fields at $z = z_0$ requires that

$$b'_1\mathbf{v}_{21} + d'_1\mathbf{v}_{41} - a'_0\mathbf{v}_{10} - c'_0\mathbf{v}_{30} = b'_0\mathbf{v}_{20} + d'_0\mathbf{v}_{40} + J'_2e^{\lambda_0 z'}\mathbf{v}_{20} + J'_4e^{\lambda_0 z'}\mathbf{v}_{40}\ , \quad (2.33)$$

where

$$\begin{array}{c} b'_1 = b_1e^{-\lambda_1 z_0},\ d'_1 = d_1e^{-\lambda_1 z_0},\ a'_0 = a_0e^{\lambda_0 z_0},\ b'_0 = b_0e^{-\lambda_0 z_0} \\ c'_0 = c_0e^{\lambda_0 z_0},\ d'_0 = d_0e^{-\lambda_0 z_0},\ J'_2 = J_2e^{-\lambda_0 z_0},\ J'_4 = J_4e^{-\lambda_0 z_0} \end{array} . \quad (2.34)$$

b'_1, d'_1, a'_0, c'_0 are scattered fields at z_0 and b'_0, d'_0, J'_2, and J'_4 are incident upon z_0.

Continuity of the fields at the lower boundary, $z = z_{-1}$, requires that

$$a'_{-1}\mathbf{v}_{1,-1} + c'_{-1}\mathbf{v}_{3,-1} - b'_0\mathbf{v}_{20} - d'_0\mathbf{v}_{40} = a'_0\mathbf{v}_{10} + c'_0\mathbf{v}_{30} + J'_1e^{-\lambda_0 z'}\mathbf{v}_{10} + J'_3e^{-\lambda_0 z'}\mathbf{v}_{30}, \quad (2.35)$$

where now

$$\begin{array}{c} a'_{-1} = a_{-1}e^{\lambda_{-1}z_{-1}},\ c'_{-1} = c_{-1}e^{\lambda_{-1}z_{-1}},\ a'_0 = a_0e^{\lambda_0 z_{-1}},\ b'_0 = b_0e^{-\lambda_0 z_{-1}} \\ c'_0 = c_0e^{\lambda_0 z_{-1}},\quad d'_0 = d_0e^{-\lambda_0 z_{-1}},\quad J'_1 = J_1e^{\lambda_0 z_{-1}},\quad J'_3 = J_3e^{\lambda_0 z_{-1}} \end{array} . \quad (2.36)$$

In this case, a'_{-1}, c'_{-1}, b'_0, d'_0 are scattered fields at z_{-1} and c'_0, a'_0, J'_1, and J'_3 are incident upon z_{-1}.

Equations (2.33) and (2.35) include TE- and TM-modes. They are easier to solve if we separate these modes. Take the dot product of (2.33) with respect to the TE-modes, $\mathbf{v}_{21}$ and $\mathbf{v}_{10}$ and get

$$\begin{aligned} b_1' \mathbf{v}_{21} \cdot \mathbf{v}_{21} - a_0' \mathbf{v}_{10} \cdot \mathbf{v}_{21} &= b_0' \mathbf{v}_{20} \cdot \mathbf{v}_{21} + J_2' e^{\lambda_0 z'} \mathbf{v}_{20} \cdot \mathbf{v}_{21} \\ b_1' \mathbf{v}_{21} \cdot \mathbf{v}_{10} - a_0' \mathbf{v}_{10} \cdot \mathbf{v}_{10} &= b_0' \mathbf{v}_{20} \cdot \mathbf{v}_{10} + J_2' e^{\lambda_0 z'} \mathbf{v}_{20} \cdot \mathbf{v}_{10}, \end{aligned} \tag{2.37}$$

where

$$\begin{aligned} \mathbf{v}_{21} \cdot \mathbf{v}_{21} &= k_t^2 (\lambda_1^2 - \omega^2 \mu_1^2) \\ \mathbf{v}_{10} \cdot \mathbf{v}_{21} &= -k_t^2 (\omega^2 \mu_0 \mu_1 + \lambda_0 \lambda_1) \\ \mathbf{v}_{10} \cdot \mathbf{v}_{10} &= k_t^2 (\lambda_0^2 - \omega^2 \mu_0^2) \\ \mathbf{v}_{20} \cdot \mathbf{v}_{21} &= k_t^2 (\lambda_0 \lambda_1 - \omega^2 \mu_0 \mu_1) \\ \mathbf{v}_{20} \cdot \mathbf{v}_{10} &= -k_t^2 (\omega^2 \mu_0^2 + \lambda_0^2) \ . \end{aligned} \tag{2.38}$$

The solution of (2.37) is

$$\begin{aligned} b_1 &= \frac{2\mu_0 \lambda_0 e^{(\lambda_1 - \lambda_0) z_0}}{\mu_0 \lambda_1 + \mu_1 \lambda_0} (b_0 + e^{\lambda_0 z'} J_2) \\ a_0 &= \frac{(\mu_1 \lambda_0 - \mu_0 \lambda_1) e^{-2\lambda_0 z_0}}{\mu_1 \lambda_0 + \mu_0 \lambda_1} (b_0 + e^{\lambda_0 z'} J_2) \ . \end{aligned} \tag{2.39}$$

Next, compute the TM-modes by taking the dot product of (2.33) with $\mathbf{v}_{41}$ and $\mathbf{v}_{30}$:

$$\begin{aligned} d_1' \mathbf{v}_{41} \cdot \mathbf{v}_{41} - c_0' \mathbf{v}_{30} \cdot \mathbf{v}_{41} &= d_0' \mathbf{v}_{40} \cdot \mathbf{v}_{41} + J_4' e^{\lambda_0 z'} \mathbf{v}_{40} \cdot \mathbf{v}_{41} \\ d_1' \mathbf{v}_{41} \cdot \mathbf{v}_{30} - c_0' \mathbf{v}_{30} \cdot \mathbf{v}_{30} &= d_0' \mathbf{v}_{40} \cdot \mathbf{v}_{30} + J_4' e^{\lambda_0 z'} \mathbf{v}_{40} \cdot \mathbf{v}_{30} \ , \end{aligned} \tag{2.40}$$

where

$$\begin{aligned} \mathbf{v}_{41} \cdot \mathbf{v}_{41} &= k_t^2 (\lambda_1^2 - \omega^2 \hat{\varepsilon}_1^2) \\ \mathbf{v}_{30} \cdot \mathbf{v}_{41} &= k_t^2 (\omega^2 \hat{\varepsilon}_0 \hat{\varepsilon}_1 + \lambda_0 \lambda_1) \\ \mathbf{v}_{30} \cdot \mathbf{v}_{30} &= k_t^2 (\lambda_0^2 - \omega^2 \hat{\varepsilon}_0^2) \\ \mathbf{v}_{40} \cdot \mathbf{v}_{41} &= k_t^2 (\lambda_0 \lambda_1 - \omega^2 \hat{\varepsilon}_0 \hat{\varepsilon}_1) \\ \mathbf{v}_{40} \cdot \mathbf{v}_{30} &= k_t^2 (\omega^2 \hat{\varepsilon}_0^2 + \lambda_0^2) \ . \end{aligned} \tag{2.41}$$

The solution of (2.40) is

$$
\begin{aligned}
d_1 &= \frac{2\hat{\varepsilon}_0\lambda_0 e^{(\lambda_1-\lambda_0)z_0}}{\hat{\varepsilon}_0\lambda_1+\hat{\varepsilon}_1\lambda_0}(d_0+J_4e^{\lambda_0 z'}) \\
c_0 &= \frac{(\lambda_1\hat{\varepsilon}_0-\lambda_0\hat{\varepsilon}_1)e^{-2\lambda_0 z_0}}{\lambda_1\hat{\varepsilon}_0+\lambda_0\hat{\varepsilon}_1}(d_0+J_4e^{\lambda_0 z'}) \;. \qquad (2.42)
\end{aligned}
$$

We repeat the analysis for (2.35). First, for the TE-modes:

$$
\begin{aligned}
a'_{-1}\mathbf{v}_{1,-1}\cdot\mathbf{v}_{1,-1} - b'_0\mathbf{v}_{1,-1}\cdot\mathbf{v}_{20} &= (a'_0+J'_1e^{-\lambda_0 z'})\mathbf{v}_{10}\cdot\mathbf{v}_{1,-1} \\
a'_{-1}\mathbf{v}_{1,-1}\cdot\mathbf{v}_{20} - b'_0\mathbf{v}_{20}\cdot\mathbf{v}_{20} &= (a'_0+J'_1e^{-\lambda_0 z'})\mathbf{v}_{10}\cdot\mathbf{v}_{20} \;, \qquad (2.43)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{v}_{1,-1}\cdot\mathbf{v}_{1,-1} &= k_t^2(\lambda_{-1}^2-\omega^2\mu_{-1}^2) \\
\mathbf{v}_{20}\cdot\mathbf{v}_{1,-1} &= -k_t^2(\omega^2\mu_0\mu_{-1}+\lambda_0\lambda_{-1}) \\
\mathbf{v}_{20}\cdot\mathbf{v}_{20} &= k_t^2(\lambda_0^2-\omega^2\mu_0^2) \\
\mathbf{v}_{10}\cdot\mathbf{v}_{1,-1} &= k_t^2(\lambda_0\lambda_{-1}-\omega^2\mu_0\mu_{-1}) \\
\mathbf{v}_{20}\cdot\mathbf{v}_{10} &= -k_t^2(\omega^2\mu_0^2+\lambda_0^2) \;. \qquad (2.44)
\end{aligned}
$$

The solution of (2.43) is

$$
\begin{aligned}
a_{-1} &= \frac{2\mu_0\lambda_0 e^{(\lambda_0-\lambda_{-1})z_{-1}}}{\mu_0\lambda_{-1}+\mu_{-1}\lambda_0}(a_0+e^{-\lambda_0 z'}J_1) \\
b_0 &= \frac{(\mu_{-1}\lambda_0-\mu_0\lambda_{-1})e^{2\lambda_0 z_{-1}}}{\mu_{-1}\lambda_0+\mu_0\lambda_{-1}}(a_0+e^{-\lambda_0 z'}J_1) \;. \qquad (2.45)
\end{aligned}
$$

The TM-mode equations

$$
\begin{aligned}
c'_{-1}\mathbf{v}_{3,-1}\cdot\mathbf{v}_{3,-1} - d'_0\mathbf{v}_{40}\cdot\mathbf{v}_{3,-1} &= (c'_0+J'_3e^{-\lambda_0 z'})\mathbf{v}_{30}\cdot\mathbf{v}_{3,-1} \\
c'_{-1}\mathbf{v}_{40}\cdot\mathbf{v}_{3,-1} - d'_0\mathbf{v}_{40}\cdot\mathbf{v}_{40} &= (c'_0+J'_3e^{-\lambda_0 z'})\mathbf{v}_{30}\cdot\mathbf{v}_{40} \;, \qquad (2.46)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{v}_{3,-1}\cdot\mathbf{v}_{3,-1} &= k_t^2(\lambda_{-1}^2-\omega^2\hat{\varepsilon}_{-1}^2) \\
\mathbf{v}_{40}\cdot\mathbf{v}_{3,-1} &= k_t^2(\omega^2\hat{\varepsilon}_0\hat{\varepsilon}_{-1}+\lambda_0\lambda_{-1}) \\
\mathbf{v}_{40}\cdot\mathbf{v}_{40} &= k_t^2(\lambda_0^2-\omega^2\hat{\varepsilon}_0^2) \\
\mathbf{v}_{30}\cdot\mathbf{v}_{3,-1} &= k_t^2(\lambda_0\lambda_{-1}-\omega^2\hat{\varepsilon}_0\hat{\varepsilon}_{-1}) \\
\mathbf{v}_{40}\cdot\mathbf{v}_{30} &= k_t^2(\omega^2\hat{\varepsilon}_0^2+\lambda_0^2) \;, \qquad (2.47)
\end{aligned}
$$

have for their solutions

$$c_{-1} = \frac{2\hat{\varepsilon}_0\lambda_0 e^{(\lambda_0-\lambda_{-1})z_{-1}}}{\hat{\varepsilon}_0\lambda_{-1}+\hat{\varepsilon}_{-1}\lambda_0}(c_0+J_3e^{-\lambda_0 z'})$$
$$d_0 = \frac{(\hat{\varepsilon}_0\lambda_{-1}-\hat{\varepsilon}_{-1}\lambda_0)e^{2\lambda_0 z_{-1}}}{\hat{\varepsilon}_0\lambda_{-1}+\hat{\varepsilon}_{-1}\lambda_0}(c_0+J_3e^{-\lambda_0 z'})\ . \tag{2.48}$$

We introduce the following transmission and reflection coefficients:

$$\begin{aligned}
T_1^{(E)} &= \frac{2\mu_0\lambda_0}{\mu_0\lambda_1+\mu_1\lambda_0}; T_1^{(M)} = \frac{2\hat{\varepsilon}_0\lambda_0}{\hat{\varepsilon}_0\lambda_1+\hat{\varepsilon}_1\lambda_0}\\
R_1^{(E)} &= \frac{\mu_1\lambda_0-\mu_0\lambda_1}{\mu_1\lambda_0+\mu_0\lambda_1}; R_1^{(M)} = \frac{\hat{\varepsilon}_0\lambda_1-\hat{\varepsilon}_1\lambda_0}{\hat{\varepsilon}_1\lambda_0+\hat{\varepsilon}_0\lambda_1}\\
T_{-1}^{(E)} &= \frac{2\mu_0\lambda_0}{\mu_0\lambda_{-1}+\mu_{-1}\lambda_0}; T_{-1}^{(M)} = \frac{2\hat{\varepsilon}_0\lambda_0}{\hat{\varepsilon}_0\lambda_{-1}+\hat{\varepsilon}_{-1}\lambda_0}\\
R_{-1}^{(E)} &= \frac{\mu_{-1}\lambda_0-\mu_0\lambda_{-1}}{\mu_{-1}\lambda_0+\mu_0\lambda_{-1}}; R_{-1}^{(M)} = \frac{\hat{\varepsilon}_0\lambda_{-1}-\hat{\varepsilon}_{-1}\lambda_0}{\hat{\varepsilon}_{-1}\lambda_0+\hat{\varepsilon}_0\lambda_{-1}}
\end{aligned} \tag{2.49}$$

The final expressions for the mode coefficients are given in terms of these scattering parameters:

$$a_0 = \frac{R_1^{(E)}R_{-1}^{(E)}e^{-\lambda_0(2T+z')}J_1 + R_1^{(E)}e^{-\lambda_0(2z_0-z')}J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}$$

$$b_0 = \frac{R_{-1}^{(E)}e^{\lambda_0(2z_{-1}-z')}J_1 + R_1^{(E)}R_{-1}^{(E)}e^{-\lambda_0(2T-z')}J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}$$

$$c_0 = \frac{R_1^{(M)}R_{-1}^{(M)}e^{-\lambda_0(2T+z')}J_3 + R_1^{(M)}e^{-\lambda_0(2z_0-z')}J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$d_0 = \frac{R_{-1}^{(M)}e^{\lambda_0(2z_{-1}-z')}J_3 + R_1^{(M)}R_{-1}^{(M)}e^{-\lambda_0(2T-z')}J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$b_1 = \frac{T_1^{(E)}R_{-1}^{(E)}e^{-\lambda_0(T-z_{-1}+z')}J_1 + T_1^{(E)}e^{-\lambda_0(z_0-z')}J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}e^{\lambda_1 z_0}$$

$$d_1 = \frac{T_1^{(M)} R_{-1}^{(M)} e^{-\lambda_0 (T - z_{-1} + z')} J_3 + T_1^{(M)} e^{-\lambda_0 (z_0 - z')} J_4}{1 - R_1^{(M)} R_{-1}^{(M)} e^{-2\lambda_0 T}} e^{\lambda_1 z_0}$$

$$a_{-1} = \frac{T_{-1}^{(E)} e^{-\lambda_0 (z' - z_{-1})} J_1 + T_{-1}^{(E)} R_1^{(E)} e^{-\lambda_0 (T + z_0 - z')} J_2}{1 - R_1^{(E)} R_{-1}^{(E)} e^{-2\lambda_0 T}} e^{-\lambda_{-1} z_{-1}}$$

$$c_{-1} = \frac{T_{-1}^{(M)} e^{-\lambda_0 (z' - z_{-1})} J_3 + T_{-1}^{(M)} R_1^{(M)} e^{-\lambda_0 (T + z_0 - z')} J_4}{1 - R_1^{(M)} R_{-1}^{(M)} e^{-2\lambda_0 T}} e^{-\lambda_{-1} z_{-1}} , \qquad (2.50)$$

where $T = z_0 - z_{-1}$ is the height of the source region.

In order to determine the scattered fields within the source region, we start with the expression $a_0 \mathbf{v}_{10} e^{\lambda_0 z} + b_0 \mathbf{v}_{20} e^{-\lambda_0 z} + c_0 \mathbf{v}_{30} e^{\lambda_0 z} + d_0 \mathbf{v}_{40} e^{-\lambda_0 z}$ and use (2.15):

$$\tilde{E}_x = e^{\lambda_0 z}(-j\omega\mu_0 k_y a_0 + \lambda_0 k_x c_0) + e^{-\lambda_0 z}(-j\omega\mu_0 k_y b_0 + \lambda_0 k_x d_0)$$

$$\tilde{E}_y = e^{\lambda_0 z}(j\omega\mu_0 k_x a_0 + \lambda_0 k_y c_0) + e^{-\lambda_0 z}(j\omega\mu_0 k_x b_0 + \lambda_0 k_y d_0)$$

$$\tilde{E}_z = jk_t^2 (c_0 e^{\lambda_0 z} - d_0 e^{-\lambda_0 z}) - \frac{\delta(z - z')}{j\omega\hat{\varepsilon}_0} \tilde{J}_{ez}$$

$$\tilde{H}_x = e^{\lambda_0 z}(\lambda_0 k_x a_0 + j\omega\hat{\varepsilon}_0 k_y c_0) + e^{-\lambda_0 z}(-\lambda_0 k_x b_0 - j\omega\hat{\varepsilon}_0 k_y d_0)$$

$$\tilde{H}_y = e^{\lambda_0 z}(\lambda_0 k_y a_0 - j\omega\hat{\varepsilon}_0 k_x c_0) + e^{-\lambda_0 z}(-\lambda_0 k_y b_0 + j\omega\hat{\varepsilon}_0 k_x d_0)$$

$$\tilde{H}_z = jk_t^2 (a_0 e^{\lambda_0 z} + b_0 e^{-\lambda_0 z}) + \frac{\delta(z - z')}{j\omega\mu_0} \tilde{J}_{mz} . \qquad (2.51)$$

From here on we will drop the delta-function terms in $\tilde{E}_z$ and $\tilde{H}_z$, because they will be associated with the infinite-field Green dyad.

The coefficients $a_0,\ b_0,\ c_0,\ d_0$ that appear in (2.51) have been previously expressed in terms of $J_1,\ J_2,\ J_3,\ J_4$ in (2.50). When these results are substituted into (2.51) we get:

$$\tilde{E}_x = e^{-\lambda_0 (2T + (z' - z))} \left[\frac{-R_1^{(E)} R_{-1}^{(E)} j\omega\mu_0 k_y J_1}{1 - R_1^{(E)} R_{-1}^{(E)} e^{-2\lambda_0 T}} + \frac{R_1^{(M)} R_{-1}^{(M)} \lambda_0 k_x J_3}{1 - R_1^{(M)} R_{-1}^{(M)} e^{-2\lambda_0 T}} \right]$$

$$+ \ e^{-\lambda_0 (2T - (z' - z))} \left[\frac{-R_1^{(E)} R_{-1}^{(E)} j\omega\mu_0 k_y J_2}{1 - R_1^{(E)} R_{-1}^{(E)} e^{-2\lambda_0 T}} + \frac{R_1^{(M)} R_{-1}^{(M)} \lambda_0 k_x J_4}{1 - R_1^{(M)} R_{-1}^{(M)} e^{-2\lambda_0 T}} \right]$$

$$+ e^{-\lambda_0(2z_0-(z'+z))}\left[\frac{-R_1^{(E)}j\omega\mu_0 k_y J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}\lambda_0 k_x J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$+ e^{\lambda_0(2z_{-1}-(z'+z))}\left[\frac{-R_{-1}^{(E)}j\omega\mu_0 k_y J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_{-1}^{(M)}\lambda_0 k_x J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$\tilde{E}_y = e^{-\lambda_0(2T+(z'-z))}\left[\frac{R_1^{(E)}R_{-1}^{(E)}j\omega\mu_0 k_x J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}R_{-1}^{(M)}\lambda_0 k_y J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$+ e^{-\lambda_0(2T-(z'-z))}\left[\frac{R_1^{(E)}R_{-1}^{(E)}j\omega\mu_0 k_x J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}R_{-1}^{(M)}\lambda_0 k_y J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$+e^{-\lambda_0(2z_0-(z'+z))}\left[\frac{R_1^{(E)}j\omega\mu_0 k_x J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}\lambda_0 k_y J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$+ e^{\lambda_0(2z_{-1}-(z'+z))}\left[\frac{R_{-1}^{(E)}j\omega\mu_0 k_x J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_{-1}^{(M)}\lambda_0 k_y J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$\tilde{E}_z = e^{-\lambda_0(2T+(z'-z))}\frac{R_1^{(M)}R_{-1}^{(M)}jk_t^2 J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$+ e^{-\lambda_0(2T-(z'-z))}\frac{-R_1^{(M)}R_{-1}^{(M)}jk_t^2 J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$+ e^{-\lambda_0(2z_0-(z'+z))}\frac{R_1^{(M)}jk_t^2 J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$+ e^{\lambda_0(2z_{-1}-(z'+z))}\frac{-R_{-1}^{(M)}jk_t^2 J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}$$

$$\tilde{H}_x = e^{-\lambda_0(2T+(z'-z))}\left[\frac{R_1^{(E)}R_{-1}^{(E)}\lambda_0 k_x J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_y J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$+ e^{-\lambda_0(2T-(z'-z))}\left[\frac{-R_1^{(E)}R_{-1}^{(E)}\lambda_0 k_x J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{-R_1^{(M)}R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_y J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]$$

$$
\begin{aligned}
&+\, e^{-\lambda_0(2z_0-(z'+z))}\left[\frac{R_1^{(E)}\lambda_0 k_x J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}j\omega\hat{\varepsilon}_0 k_y J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right] \\
&+\, e^{\lambda_0(2z_{-1}-(z'+z))}\left[\frac{-R_{-1}^{(E)}\lambda_0 k_x J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{-R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_y J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}_y = {}& e^{-\lambda_0(2T+(z'-z))}\left[\frac{R_1^{(E)}R_{-1}^{(E)}\lambda_0 k_y J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{-R_1^{(M)}R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_x J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right] \\
&+\, e^{-\lambda_0(2T-(z'-z))}\left[\frac{-R_1^{(E)}R_{-1}^{(E)}\lambda_0 k_y J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_1^{(M)}R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_x J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right] \\
&+\, e^{-\lambda_0(2z_0-(z'+z))}\left[\frac{R_1^{(E)}\lambda_0 k_y J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{-R_1^{(M)}j\omega\hat{\varepsilon}_0 k_x J_4}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right] \\
&+\, e^{\lambda_0(2z_{-1}-(z'+z))}\left[\frac{-R_{-1}^{(E)}\lambda_0 k_y J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}+\frac{R_{-1}^{(M)}j\omega\hat{\varepsilon}_0 k_x J_3}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}\right]
\end{aligned}
$$

$$
\begin{aligned}
\tilde{H}_z = {}& e^{-\lambda_0(2T+(z'-z))}\frac{R_1^{(E)}R_{-1}^{(E)}jk_t^2 J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}} \\
&+\, e^{-\lambda_0(2T-(z'-z))}\frac{R_1^{(E)}R_{-1}^{(E)}jk_t^2 J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}} \\
&+\, e^{-\lambda_0(2z_0-(z'+z))}\frac{R_1^{(E)}jk_t^2 J_2}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}} \\
&+\, e^{\lambda_0(2z_{-1}-(z'+z))}\frac{R_{-1}^{(E)}jk_t^2 J_1}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}\,. \qquad (2.52)
\end{aligned}
$$

When the sources, J_1, J_2, J_3, J_4, are expressed in terms of the electric and magnetic currents, as in (2.28), then we can express the fields in (2.52) in terms of layered-space Green dyads:

$$
\begin{aligned}
\tilde{\mathbf{E}}(k_x,k_y;z,z') &= \tilde{\mathbf{G}}_{(s)}^{(ee)}(k_x,k_y;z,z')\cdot\tilde{\mathbf{J}}_e(k_x,k_y) \\
\tilde{\mathbf{E}}(k_x,k_y;z,z') &= \tilde{\mathbf{G}}_{(s)}^{(em)}(k_x,k_y;z,z')\cdot\tilde{\mathbf{J}}_m(k_x,k_y)
\end{aligned}
$$

$$\tilde{\mathbf{H}}(k_x,k_y;z,z') = \tilde{\mathbf{G}}^{(me)}_{(s)}(k_x,k_y;z,z')\cdot\tilde{\mathbf{J}}_e(k_x,k_y)$$
$$\tilde{\mathbf{H}}(k_x,k_y;z,z') = \tilde{\mathbf{G}}^{(mm)}_{(s)}(k_x,k_y;z,z')\cdot\tilde{\mathbf{J}}_m(k_x,k_y)\ . \qquad (2.53)$$

The result for the spectral-domain, electric–electric, dyadic Green function is shown here:

$$\begin{aligned}
\tilde{G}^{(ee)(s)}_{xx} &= -j\omega\mu_0\Bigg\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}\left[G^{(E)}_{1,-1}-k_x^2\left(G^{(E)}_{1,-1}\frac{1}{k_t^2}+G^{(M)}_{1,-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\\
&\quad+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}\left[G^{(E)}_{1}-k_x^2\left(G^{(E)}_{1}\frac{1}{k_t^2}+G^{(M)}_{1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\\
&\quad+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}\left[G^{(E)}_{-1}-k_x^2\left(G^{(E)}_{-1}\frac{1}{k_t^2}+G^{(M)}_{-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\tilde{G}^{(ee)(s)}_{xy} &= -j\omega\mu_0(-k_xk_y)\Bigg\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}\left[G^{(E)}_{1,-1}\frac{1}{k_t^2}+G^{(M)}_{1,-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right]\\
&\quad+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}\left[G^{(E)}_{1}\frac{1}{k_t^2}+G^{(M)}_{1}\frac{\lambda_0^2}{k_0^2k_t^2}\right]\\
&\quad+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}\left[G^{(E)}_{-1}\frac{1}{k_t^2}+G^{(M)}_{-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right]\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\tilde{G}^{(ee)(s)}_{xz} &= -j\omega\mu_0(-jk_x\lambda_0)\Bigg\{-\frac{s(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G^{(M)}_{1,-1}\frac{1}{k_0^2}-\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G^{(M)}_{1}\frac{1}{k_0^2}\\
&\quad+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G^{(M)}_{-1}\frac{1}{k_0^2}\Bigg\}\\
\tilde{G}^{(ee)(s)}_{yx} &= \tilde{G}^{(ee)(s)}_{xy}
\end{aligned}$$

$$\begin{aligned}
\tilde{G}^{(ee)(s)}_{yy} &= -j\omega\mu_0\Bigg\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}\left[G^{(E)}_{1,-1}-k_y^2\left(G^{(E)}_{1,-1}\frac{1}{k_t^2}+G^{(M)}_{1,-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\\
&\quad+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}\left[G^{(E)}_{1}-k_y^2\left(G^{(E)}_{1}\frac{1}{k_t^2}+G^{(M)}_{1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\\
&\quad+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}\left[G^{(E)}_{-1}-k_y^2\left(G^{(E)}_{-1}\frac{1}{k_t^2}+G^{(M)}_{-1}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
\tilde{G}_{yz}^{(ee)(s)} &= -j\omega\mu_0(-jk_y\lambda_0)\left\{-\frac{s(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G_{1,-1}^{(M)}\frac{1}{k_0^2}-\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G_1^{(M)}\frac{1}{k_0^2}\right.\\
&\qquad\left.+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G_{-1}^{(M)}\frac{1}{k_0^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{G}_{zx}^{(a)(ee)(s)} &= \tilde{G}_{xz}^{(a)(ee)(s)}, \quad \tilde{G}_{zx}^{(b)(ee)(s)} = -\tilde{G}_{xz}^{(b)(ee)(s)}\\
\tilde{G}_{zy}^{(a)(ee)(s)} &= \tilde{G}_{yz}^{(a)(ee)(s)}, \quad \tilde{G}_{zy}^{(b)(ee)(s)} = -\tilde{G}_{yz}^{(b)(ee)(s)}\\
\tilde{G}_{zz}^{(ee)(s)} &= -j\omega\mu_0\left\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G_{1,-1}^{(M)}\frac{k_t^2}{k_0^2}-\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G_1^{(M)}\frac{k_t^2}{k_0^2}\right.\\
&\qquad\left.-\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G_{-1}^{(M)}\frac{k_t^2}{k_0^2}\right\},
\end{aligned}
\tag{2.54}
$$

where $c(z'-z) = e^{\lambda_0(z'-z)} + e^{-\lambda_0(z'-z)}$ and $s(z'-z) = e^{\lambda_0(z'-z)} - e^{-\lambda_0(z'-z)}$. The superscript (a) denotes terms that are convolutional ("Töplitz") in z and z', i.e., depend upon $z-z'$, whereas (b) denotes terms that are correlational ("Hankel") in z and z', i.e., depend upon $z+z'$. The Gs are defined in terms of the TE and TM-mode reflection coefficients:

$$
\begin{aligned}
G_{1,-1}^{(E)} &= \frac{R_1^{(E)}R_{-1}^{(E)}}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}, \quad G_1^{(E)} = \frac{R_1^{(E)}}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}, \quad G_{-1}^{(E)} = \frac{R_{-1}^{(E)}}{1-R_1^{(E)}R_{-1}^{(E)}e^{-2\lambda_0 T}}\\
G_{1,-1}^{(M)} &= \frac{R_1^{(M)}R_{-1}^{(M)}}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}, \quad G_1^{(M)} = \frac{R_1^{(M)}}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}, \quad G_{-1}^{(M)} = \frac{R_{-1}^{(M)}}{1-R_1^{(M)}R_{-1}^{(M)}e^{-2\lambda_0 T}}.
\end{aligned}
\tag{2.55}
$$

The physical origin of the Töplitz and Hankel terms follows from the definitions in (2.55), and a graphical illustration is shown in Fig. 2.4.

The Töplitz structure, as shown in the top of Fig. 2.4, arises when the path between the source point, z', and field point, z, includes reflections from both boundaries. The total $z-$directed path length between z' and z is $z-z'+2T$ for path A and $z'-z+2T$ for path B. In each case, the length includes the difference between the z-coordinate of the source and field points.

The Hankel structure, as shown in the bottom of Fig. 2.4, arises when the path between the source and field points includes reflections from only one of the boundaries. The total path length between z' and z is $2Z_0-(z+z')$ for path A and $z+z'-2Z_{-1}$ for path B. In each case, the length includes the sum of the source and field z-coordinates.

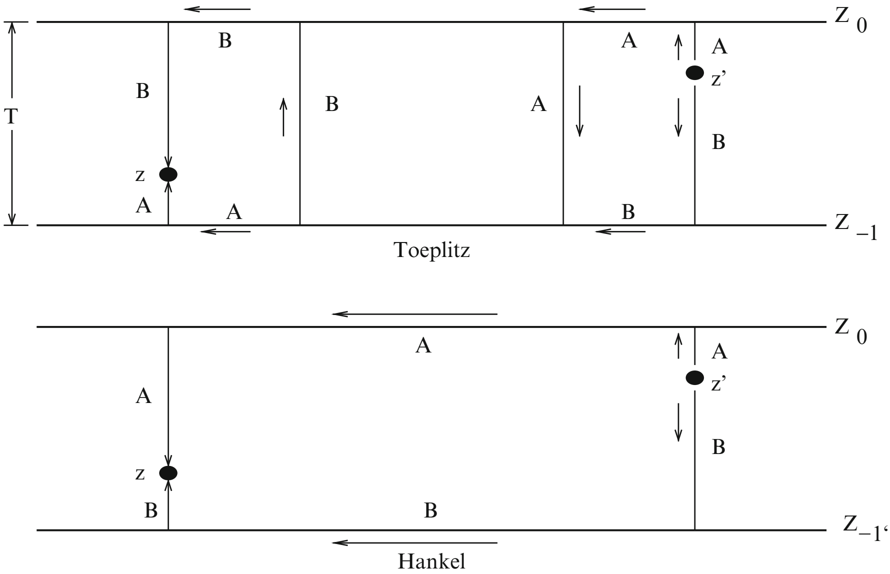

Fig. 2.4 Illustrating the difference between Töplitz (*top*) and Hankel (*bottom*) Green's functions

The spectral-domain, magnetic–magnetic, dyadic Green function is given by the dual of (2.54):

$$\begin{aligned}\tilde{G}_{xx}^{(mm)(s)} &= j\omega\hat{\varepsilon}_0 \left\{ \frac{c(z'-z)}{2\lambda_0} e^{-2\lambda_0 T} \left[G_{1,-1}^{(M)} - k_x^2 \left(G_{1,-1}^{(M)} \frac{1}{k_t^2} + G_{1,-1}^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right) \right] \right. \\ &\quad - \frac{e^{\lambda_0(z+z')}}{2\lambda_0} e^{-2\lambda_0 z_0} \left[G_1^{(M)} - k_x^2 \left(G_1^{(M)} \frac{1}{k_t^2} + G_1^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right) \right] \\ &\quad \left. - \frac{e^{-\lambda_0(z+z')}}{2\lambda_0} e^{2\lambda_0 z_{-1}} \left[G_{-1}^{(M)} - k_x^2 \left(G_{-1}^{(M)} \frac{1}{k_t^2} + G_{-1}^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right) \right] \right\} \\ \tilde{G}_{xy}^{(mm)(s)} &= j\omega\hat{\varepsilon}_0 (-k_x k_y) \left\{ \frac{c(z'-z)}{2\lambda_0} e^{-2\lambda_0 T} \left[G_{1,-1}^{(M)} \frac{1}{k_t^2} + G_{1,-1}^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right] \right. \\ &\quad - \frac{e^{\lambda_0(z+z')}}{2\lambda_0} e^{-2\lambda_0 z_0} \left[G_1^{(M)} \frac{1}{k_t^2} + G_1^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right] \\ &\quad \left. - \frac{e^{-\lambda_0(z+z')}}{2\lambda_0} e^{2\lambda_0 z_{-1}} \left[G_{-1}^{(M)} \frac{1}{k_t^2} + G_{-1}^{(E)} \frac{\lambda_0^2}{k_0^2 k_t^2} \right] \right\}\end{aligned}$$

$$\tilde{G}_{xz}^{(mm)(s)} = j\omega\hat{\varepsilon}_0(-jk_x\lambda_0)\left\{-\frac{s(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G_{1,-1}^{(E)}\frac{1}{k_0^2}+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G_1^{(E)}\frac{1}{k_0^2}\right.$$
$$\left.-\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G_{-1}^{(E)}\frac{1}{k_0^2}\right\}$$

$$\tilde{G}_{yx}^{(mm)(s)} = \tilde{G}_{xy}^{(mm)(s)}$$

$$\tilde{G}_{yy}^{(mm)(s)} = j\omega\hat{\varepsilon}_0\left\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}\left[G_{1,-1}^{(M)}-k_y^2\left(G_{1,-1}^{(M)}\frac{1}{k_t^2}+G_{1,-1}^{(E)}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\right.$$
$$-\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}\left[G_1^{(M)}-k_y^2\left(G_1^{(M)}\frac{1}{k_t^2}+G_1^{(E)}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]$$
$$\left.-\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}\left[G_{-1}^{(M)}-k_y^2\left(G_{-1}^{(M)}\frac{1}{k_t^2}+G_{-1}^{(E)}\frac{\lambda_0^2}{k_0^2k_t^2}\right)\right]\right\}$$

$$\tilde{G}_{yz}^{(mm)(s)} = j\omega\hat{\varepsilon}_0(-jk_y\lambda_0)\left\{-\frac{s(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G_{1,-1}^{(E)}\frac{1}{k_0^2}+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G_1^{(E)}\frac{1}{k_0^2}\right.$$
$$\left.-\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G_{-1}^{(E)}\frac{1}{k_0^2}\right\}$$

$$\tilde{G}_{zx}^{(a)(mm)(s)} = \tilde{G}_{xz}^{(a)(mm)(s)}, \quad \tilde{G}_{zx}^{(b)(mm)(s)} = -\tilde{G}_{xz}^{(b)(mm)(s)}$$
$$\tilde{G}_{zy}^{(a)(mm)(s)} = \tilde{G}_{yz}^{(a)(mm)(s)}, \quad \tilde{G}_{zy}^{(b)(mm)(s)} = -\tilde{G}_{yz}^{(b)(mm)(s)}$$
$$\tilde{G}_{zz}^{(mm)(s)} = j\omega\hat{\varepsilon}_0\left\{\frac{c(z'-z)}{2\lambda_0}e^{-2\lambda_0 T}G_{1,-1}^{(E)}\frac{k_t^2}{k_0^2}+\frac{e^{\lambda_0(z+z')}}{2\lambda_0}e^{-2\lambda_0 z_0}G_1^{(E)}\frac{k_t^2}{k_0^2}\right.$$
$$\left.+\frac{e^{-\lambda_0(z+z')}}{2\lambda_0}e^{2\lambda_0 z_{-1}}G_{-1}^{(E)}\frac{k_t^2}{k_0^2}\right\}. \tag{2.56}$$

Finally, we list the mixed dyadic functions, starting with the electric–magnetic:

$$\tilde{G}_{xx}^{(em)(s)} = -k_xk_y\left\{-s(z'-z)\frac{e^{-2\lambda_0 T}}{2}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{k_t{}^2}\right]-\frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_1^{(E)}-G_1^{(M)}}{k_t{}^2}\right]\right.$$
$$\left.+\frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{k_t{}^2}\right]\right\}$$

$$
\begin{aligned}
\tilde{G}_{xy}^{(em)(s)} &= e^{-2\lambda_0 T}\frac{s(z'-z)}{2}G_{1,-1}^{(M)}+e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2}G_1^{(M)}-e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2}G_{-1}^{(M)} \\
&\quad -k_y{}^2\left\{-\frac{s(z'-z)}{2}e^{-2\lambda_0 T}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{k_t{}^2}\right]-\frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_1^{(E)}-G_1^{(M)}}{k_t{}^2}\right]\right. \\
&\quad \left.+\frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{k_t{}^2}\right]\right\} \\
\tilde{G}_{xz}^{(em)(s)} &= -jk_y\left\{e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(E)}+e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_1^{(E)}+e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(E)}\right\} \\
\tilde{G}_{yx}^{(em)(s)} &= -e^{-2\lambda_0 T}\frac{s(z'-z)}{2}G_{1,-1}^{(M)}-e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2}G_1^{(M)}+e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2}G_{-1}^{(M)} \\
&\quad -k_x{}^2\left\{\frac{s(z'-z)}{2}e^{-2\lambda_0 T}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{k_t{}^2}\right]+\frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_1^{(E)}-G_1^{(M)}}{k_t{}^2}\right]\right. \\
&\quad \left.-\frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{k_t{}^2}\right]\right\} \\
\tilde{G}_{yy}^{(em)(s)} &= -\tilde{G}_{xx}^{(em)(s)} \\
\tilde{G}_{yz}^{(em)(s)} &= -jk_x\left\{-e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(E)}-e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_1^{(E)}-e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(E)}\right\} \\
\tilde{G}_{zx}^{(em)(s)} &= -jk_y\left\{-e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(M)}+e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_1^{(M)}+e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(M)}\right\} \\
\tilde{G}_{zy}^{(em)(s)} &= -jk_x\left\{e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(M)}-e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_1^{(M)}-e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(M)}\right\} \\
\tilde{G}_{zz}^{(em)(s)} &= 0\,, \qquad (2.57)
\end{aligned}
$$

and ending with the magnetic–electric:

$$
\begin{aligned}
\tilde{G}_{xx}^{(me)(s)} &= -k_x k_y \left\{ s(z'-z)\frac{e^{-2\lambda_0 T}}{2}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{{k_t}^2}\right] - \frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_{1}^{(E)}-G_{1}^{(M)}}{{k_t}^2}\right]\right. \\
&\left. \quad + \frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{{k_t}^2}\right]\right\} \\
\tilde{G}_{xy}^{(me)(s)} &= e^{-2\lambda_0 T}\frac{s(z'-z)}{2}G_{1,-1}^{(E)} - e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2}G_{1}^{(E)} + e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2}G_{-1}^{(E)} \\
&\quad - {k_y}^2\left\{\frac{s(z'-z)}{2}e^{-2\lambda_0 T}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{{k_t}^2}\right] - \frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_{1}^{(E)}-G_{1}^{(M)}}{{k_t}^2}\right]\right. \\
&\left. \quad + \frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{{k_t}^2}\right]\right\} \\
\tilde{G}_{xz}^{(me)(s)} &= -jk_y\left\{e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(M)} - e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_{1}^{(M)} - e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(M)}\right\} \\
\tilde{G}_{yx}^{(me)(s)} &= -e^{-2\lambda_0 T}\frac{s(z'-z)}{2}G_{1,-1}^{(E)} + e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2}G_{1}^{(E)} - e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2}G_{-1}^{(E)} \\
&\quad - {k_x}^2\left\{-\frac{s(z'-z)}{2}e^{-2\lambda_0 T}\left[\frac{G_{1,-1}^{(E)}-G_{1,-1}^{(M)}}{{k_t}^2}\right] + \frac{e^{\lambda_0(z'+z)}}{2}e^{-2\lambda_0 z_0}\left[\frac{G_{1}^{(E)}-G_{1}^{(M)}}{{k_t}^2}\right]\right. \\
&\left. \quad - \frac{e^{-\lambda_0(z'+z)}}{2}e^{2\lambda_0 z_{-1}}\left[\frac{G_{-1}^{(E)}-G_{-1}^{(M)}}{{k_t}^2}\right]\right\} \\
\tilde{G}_{yy}^{(me)(s)} &= -\tilde{G}_{xx}^{(me)(s)} \\
\tilde{G}_{yz}^{(me)(s)} &= -jk_x\left\{-e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(M)} + e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_{1}^{(M)} + e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(M)}\right\} \\
\tilde{G}_{zx}^{(me)(s)} &= -jk_y\left\{-e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(E)} - e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_{1}^{(E)} - e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(E)}\right\} \\
\tilde{G}_{zy}^{(me)(s)} &= -jk_x\left\{e^{-2\lambda_0 T}\frac{c(z'-z)}{2\lambda_0}G_{1,-1}^{(E)} + e^{-2\lambda_0 z_0}\frac{e^{\lambda_0(z'+z)}}{2\lambda_0}G_{1}^{(E)} + e^{2\lambda_0 z_{-1}}\frac{e^{-\lambda_0(z'+z)}}{2\lambda_0}G_{-1}^{(E)}\right\} \\
\tilde{G}_{zz}^{(me)(s)} &= 0\,. \qquad (2.58)
\end{aligned}
$$

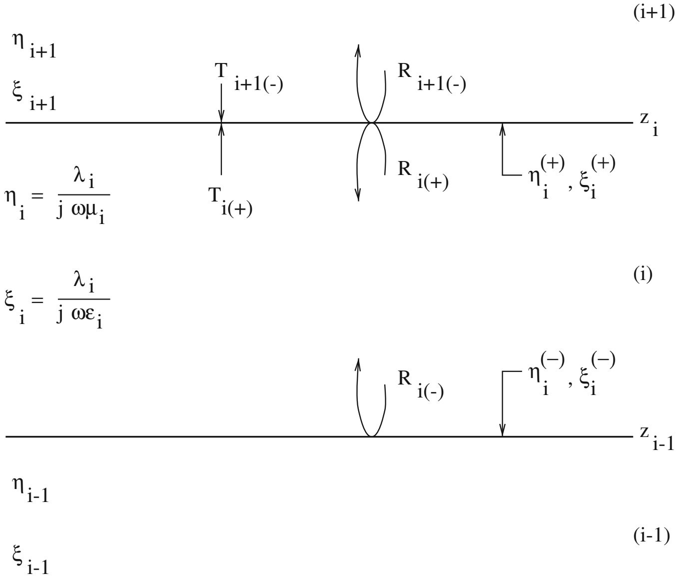

Fig. 2.5 Definition of scattering parameters and intrinsic wave-immittances in layered structures

2.2.3 A Recursion Relation for Stratified Media

The reflection and transmission coefficients of (2.49) are those of a slab surrounded by a half-space above and below it. We will derive a recursion relation that will allow the computation of the reflection coefficients, $R_{\pm 1}^{(E,M)}$, that are used in (2.55) when there are an arbitrary number of layers above and below the source slab [35]. Of course, there will ultimately be half-spaces that terminate the system at $\pm\infty$.

Let region i be bounded above by z_i and below by z_{i-1}. Region $i+1$ lies immediately above region i and region $i-1$ lies immediately below. The intrinsic wave-admittance, η_i, and wave-impedance, ξ_i, for the TE and TM modes in region i are

$$\text{(TE)}\ \eta_i = \frac{\lambda_i}{j\omega\mu_i}\ ,\ \xi_i = \frac{\lambda_i}{j\omega\hat{\varepsilon}_i}\ \text{(TM)}. \tag{2.59}$$

We define $R_{i\pm}^{(E,M)}$ to be the reflection coefficient in layer i at the interface with layer $i \pm 1$, and $T_{i\pm}^{(E,M)}$ to be the transmission coefficient, as shown in Fig. 2.5. Then, as shown in (2.49), for a slab sandwiched between two half-spaces

$$
\begin{aligned}
R_{i\pm}^{(E)} &= \frac{\mu_{i\pm1}\lambda_i - \mu_i\lambda_{i\pm1}}{\mu_i\lambda_{i\pm1} + \mu_{i\pm1}\lambda_i} = \frac{\eta_i - \eta_{i\pm1}}{\eta_i + \eta_{i\pm1}} \\
R_{i\pm}^{(M)} &= \frac{\hat{\varepsilon}_i\lambda_{i\pm1} - \hat{\varepsilon}_{i\pm1}\lambda_i}{\hat{\varepsilon}_i\lambda_{i\pm1} + \hat{\varepsilon}_{i\pm1}\lambda_i} = \frac{\xi_{i\pm1} - \xi_i}{\xi_i + \xi_{i\pm1}} \\
T_{i\pm}^{(E)} &= \frac{\mu_i}{\mu_{i\pm1}} \frac{2\eta_i}{\eta_i + \eta_{i\pm1}} \\
T_{i\pm}^{(M)} &= \frac{\hat{\varepsilon}_i}{\hat{\varepsilon}_{i\pm1}} \frac{2\xi_i}{\xi_i + \xi_{i\pm1}} \ .
\end{aligned} \tag{2.60}
$$

When we have layers above and below region i, we must replace the intrinsic wave-parameters, $\xi_{i\pm1},\ \eta_{i\pm1}$, by equivalent "load parameters," $\eta_i^{\pm},\ \xi_i^{\pm}$, which are defined to be the surface admittance and impedance in layer i at the interface with layer $i\pm1$. The surface admittance is defined to be $\tilde{H}_y/\tilde{E}_x$ at the appropriate interface, and the surface impedance is defined to be $\tilde{E}_x/\tilde{H}_y$. These ratios are the same regardless of which side of the interface they are evaluated at, because $\tilde{E}_x$ and $\tilde{H}_y$ are each continuous at the interface. Hence, we replace (2.60) by

$$
R_{i\pm}^{(E)} = \frac{\eta_i - \eta_i^{\pm}}{\eta_i + \eta_i^{\pm}} \ , \ R_{i\pm}^{(M)} = \frac{\xi_i^{\pm} - \xi_i}{\xi_i^{\pm} + \xi_i}. \tag{2.61}
$$

Let the TE-field in region i be

$$
\begin{aligned}
& a_i\mathbf{v}_{1i}e^{\lambda_i(z-z_i)} + b_i\mathbf{v}_{2i}e^{-\lambda_i(z-z_i)} \\
&= b_i\left[R_{i+}^{(E)}\mathbf{v}_{1i}e^{\lambda_i(z-z_i)} + \mathbf{v}_{2i}e^{-\lambda_i(z-z_i)}\right],
\end{aligned} \tag{2.62}
$$

then the field components at the lower-boundary, $z = z_{i-1}$, are

$$
\begin{aligned}
\tilde{E}_x &= b_i\left[R_{i+}^{(E)}(-j\omega\mu_i k_y)e^{-\lambda_i T_i} - j\omega\mu_i k_y e^{\lambda_i T_i}\right] \\
\tilde{H}_y &= b_i\left[R_{i+}^{(E)}(\lambda_i k_y)e^{-\lambda_i T_i} - \lambda_i k_y e^{\lambda_i T_i}\right].
\end{aligned} \tag{2.63}
$$

The load-admittance at $z = z_{i-1}$, therefore, is given by

$$
\begin{aligned}
\eta_i^{(-)} = \eta_{i-1}^{(+)} &= \frac{\tilde{H}_y}{\tilde{E}_x} \\
&= \frac{\lambda_i R_{i+}^{(E)} e^{-\lambda_i T_i} - \lambda_i e^{\lambda_i T_i}}{-j\omega\mu_i\left(R_{i+}^{(E)} e^{-\lambda_i T_i} + e^{\lambda_i T_i}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&= \eta_i \frac{e^{\lambda_i T_i} - R_{i+}^{(E)} e^{-\lambda_i T_i}}{e^{\lambda_i T_i} + R_{i+}^{(E)} e^{-\lambda_i T_i}} \\
&= \eta_i \frac{e^{\lambda_i T_i} - \dfrac{\eta_i - \eta_i^{(+)}}{\eta_i + \eta_i^{(+)}} e^{-\lambda_i T_i}}{e^{\lambda_i T_i} + \dfrac{\eta_i - \eta_i^{(+)}}{\eta_i + \eta_i^{(+)}} e^{-\lambda_i T_i}} \\
&= \eta_i \frac{\eta_i^{(+)} + \eta_i \tanh(\lambda_i T_i)}{\eta_i + \eta_i^{(+)} \tanh(\lambda_i T_i)} \; . \qquad (2.64)
\end{aligned}
$$

When we use $\mathbf{v}_{3i}$ and $\mathbf{v}_{4i}$, we find that the components of the TM-field at $z = z_{i-1}$ are given by

$$
\begin{aligned}
\tilde{E}_x &= d_i \left[R_{i+}^{(M)} \lambda_i k_x e^{-\lambda_i T_i} + \lambda_i k_x e^{\lambda_i T_i} \right] \\
\tilde{H}_y &= d_i \left[R_{i+}^{(M)} (-j\omega\hat{\varepsilon}_i k_x) e^{-\lambda_i T_i} + j\omega\hat{\varepsilon}_i k_x e^{\lambda_i T_i} \right]. \qquad (2.65)
\end{aligned}
$$

Hence, by an analysis similar to (2.64), we get

$$
\begin{aligned}
\xi_i^{(-)} = \xi_{i-1}^{(+)} &= \frac{\tilde{E}_x}{\tilde{H}_y} \\
&= \xi_i \frac{\xi_i^{(+)} + \xi_i \tanh(\lambda_i T_i)}{\xi_i + \xi_i^{(+)} \tanh(\lambda_i T_i)} \; . \qquad (2.66)
\end{aligned}
$$

Let the TE-field in region i below the source region be given by

$$
\begin{aligned}
& a_i \mathbf{v}_{1i} e^{\lambda_i (z - z_{i-1})} + b_i \mathbf{v}_{2i} e^{-\lambda_i (z - z_{i-1})} \\
&= a_i \left[\mathbf{v}_{1i} e^{\lambda_i (z - z_{i-1})} + R_{i-}^{(E)} \mathbf{v}_{2i} e^{-\lambda_i (z - z_{i-1})} \right], \qquad (2.67)
\end{aligned}
$$

where we treat the negatively-traveling wave, $a_i \mathbf{v}_{1i} e^{\lambda_i (z - z_{i-1})}$, as being incident on the surface $z = z_{i-1}$. Therefore, we have

$$
\begin{aligned}
\tilde{E}_x &= a_i \left[-j\omega\mu_i k_y e^{\lambda_i T_i} + R_{i-}^{(E)} (-j\omega\mu_i k_y) e^{-\lambda_i T_i} \right] \\
\tilde{H}_y &= a_i \left[\lambda_i k_y e^{\lambda_i T_i} + R_{i-}^{(E)} (-\lambda_i k_y) e^{-\lambda_i T_i} \right], \qquad (2.68)
\end{aligned}
$$

for the fields at the upper-boundary, z_i, of the ith region. Hence, the load admittance at $z = z_i$ is

$$
\begin{aligned}
\eta_i^{(+)} = \eta_{i+1}^{(-)} &= -\frac{\tilde{H}_y}{\tilde{E}_x} \\
&= \frac{\lambda_i R_{i-}^{(E)} e^{-\lambda_i T_i} - \lambda_i e^{\lambda_i T_i}}{-j\omega\mu_i\left(R_{i-}^{(E)} e^{-\lambda_i T_i} + e^{\lambda_i T_i}\right)} \\
&= \eta_i \frac{e^{\lambda_i T_i} - R_{i-}^{(E)} e^{-\lambda_i T_i}}{e^{\lambda_i T_i} + R_{i-}^{(E)} e^{-\lambda_i T_i}} \\
&= \eta_i \frac{e^{\lambda_i T_i} - \dfrac{\eta_i - \eta_i^{(-)}}{\eta_i + \eta_i^{(-)}} e^{-\lambda_i T_i}}{e^{\lambda_i T_i} + \dfrac{\eta_i - \eta_i^{(-)}}{\eta_i + \eta_i^{(-)}} e^{-\lambda_i T_i}} \\
&= \eta_i \frac{\eta_i^{(-)} + \eta_i \tanh(\lambda_i T_i)}{\eta_i + \eta_i^{(-)} \tanh(\lambda_i T_i)} \; .
\end{aligned} \tag{2.69}
$$

Similarly, for regions below the source region

$$
\xi_i^{(+)} = \xi_{i+1}^{(-)} = \xi_i \frac{\xi_i^{(-)} + \xi_i \tanh(\lambda_i T_i)}{\xi_i + \xi_i^{(-)} \tanh(\lambda_i T_i)} \; . \tag{2.70}
$$

In order to avoid large numbers in the computation of the hyperbolic tangent, we write it as

$$
\tanh(\lambda_i T_i) = \frac{e^{\lambda_i T_i} - e^{-\lambda_i T_i}}{e^{\lambda_i T_i} + e^{-\lambda_i T_i}} = \frac{1 - e^{-2\lambda_i T_i}}{1 + e^{-2\lambda_i T_i}} \; .
$$

Equations (2.64), (2.69), (2.66), and (2.70) are the iterations that produce the immittances that go into the expressions for the reflection coefficients, (2.61). The iterations are started at the interface of the last slab with an infinite half-space, for which

$$
\begin{aligned}
\eta_{N-1}^{+} = \eta_N^{+} = \eta_N, \xi_{N-1}^{+} = \xi_N^{+} = \xi_N \\
\eta_{-(M-1)}^{-} = \eta_{-M}^{-} = \eta_{-M}, \xi_{-(M-1)}^{-} = \xi_{-M}^{-} = \xi_{-M} \; .
\end{aligned} \tag{2.71}
$$

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