2
A Simple Market Model

In the simplest possible market model there are two assets (one stock and one bond), one time step and just two possible future scenarios. Many of the basic ideas of mathematical finance arise even in such a simple model. It also turns out that many more complex models can be viewed as a succession of one-period models such as this.

2.1 Model Assumptions

The aim in this chapter is to build a mathematical model of a simple financial market. To begin, we describe the mathematical objects involved and their properties. In order to keep the model tractable, it is necessary to make some simplifying assumptions. This will aid in understanding the most important basic ideas and principles of mathematical finance, while avoiding unnecessary mathematical difficulty.

The first assumption is that the model involves just two moments in time. It is often helpful to think of the first, earlier, time as "now" or "today", and of the second, later, time as "the future" or "tomorrow".

Assumption 2.1 (Two trading dates)

Trading takes place at time 0 and time 1.

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Two basic kinds of asset are traded in this model: one risk-free and one risky. These are called the *primary* or *underlying* assets. In later development there will be other types of asset or security, whose value is derived in some way from the two *primary* or *underlying* assets. These are the so-called *derivative securities*, or just *derivatives*, our main subject of study.

We normally think of the *risk-free asset* as a bond, or a unit of money in a bank or savings account, that is earning interest.

Assumption 2.2 (Risk-free asset)

The model contains a *bond* with initial price B_0. The interest rate $r \in \mathbb{R}$ is known in advance, and the value of the bond at time 1 is

$$B_1 = (1 + r)B_0.$$

Remark 2.1

The price movement of the bond is completely determined by any of the pairs (B_0, r), (B_0, B_1) and (r, B_1).

The next assumption encapsulates the idea that at the future time 1 there can be only two possible outcomes or *scenarios*.

Assumption 2.3 (Number of scenarios)

There are two possible future scenarios, denoted u and d; the set of possible scenarios is written Ω so we have $\Omega = \{\mathrm{u}, \mathrm{d}\}$. Scenario u occurs with probability $p \in (0,1)$, and scenario d occurs with probability $1 - p \in (0,1)$.

Any model with just two scenarios is called a *binary* model. The probability assignment $(p, 1-p)$ is called the *real-world probability* or the *market probability*. It turns out that this probability plays no part in what follows, which is one of the reasons why the mathematical theory of derivative pricing was described as "remarkable" in Chapter 1.

We will think of the *risky asset* as a stock or share, although in principle it could be a unit of any commodity for which the future price is unknown.

Assumption 2.4 (Risky asset)

The model contains a single *stock* whose initial price S_0 is known at time 0. The price S_1 of the stock at time 1 is a *random variable* whose value depends on the scenario. (Recall that the *stock price* or *share price* of a given stock is the market price of one share of that stock.)

Thus, at time 0 we know the initial stock price S_0 and we know that at time 1 it will be *either* $S_1(\mathrm{u})$ *or* $S_1(\mathrm{d})$. Only at time 1 will we know which it actually is. The labels of the scenarios are chosen to suggest either an upwards (u) or downwards (d) movement from the initial price S_0, and for convenience we always assume that

$$S_1(\mathrm{d}) < S_1(\mathrm{u}). \tag{2.1}$$

The final assumption on the prices of the bond and stock for the moment is that they are strictly positive.

Assumption 2.5 (Positivity of prices)

We have $B_t > 0$ (equivalently $B_0 > 0$ and $r > -1$) and $S_t > 0$ for $t = 0, 1$.

It is helpful to represent the stock prices in this model as a tree (on its side), with the branches representing the possible future scenarios, such as in Figure 2.1.

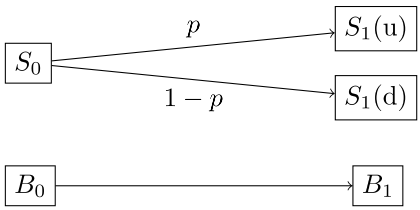

Figure 2.1 One-period single-stock binary model

A trader in this model may form a *portfolio* of assets at time 0 and hold this portfolio, unchanged, until time 1. This is formally stated as follows.

Assumption 2.6 (Divisibility, liquidity and short selling)

For any $x, y \in \mathbb{R}$ an investor may form a *portfolio* consisting of x bonds and y shares. Such a portfolio is denoted by the pair (x, y).

It is often useful to denote a portfolio by the symbol φ so we write $\varphi = (x, y)$. Assumption 2.6 deserves a few words of explanation, because it reflects some simplifications of real markets. The fact that an investor may hold a fraction of a share or bond, is referred to as *perfect divisibility* of the assets. In reality, almost perfect divisibility is achieved whenever the volume of transactions is large in comparison to the unit prices.

The fact that no bounds are imposed on the size of x and y is related to another attribute of an ideal market, namely *liquidity*. This means that assets can be bought or sold on demand, in arbitrary quantities, at the market price. In practice there are obvious restrictions on the volume of trading and the possible size of a portfolio; for example, there is only a finite amount of currency in circulation, and a finite number of shares available for purchase. However, the essence of the following mathematical development is not affected if the sizes of holdings x and y are restricted.

Investors in this model are allowed to hold negative quantities of the bond; this is equivalent to borrowing money from the bank at the declared interest rate. This money may be spent provided it is returned with interest at time 1. As with a loan from a bank, a negative holding of bonds represents an obligation to repay or return these.

In the same way, investors in this model are also permitted to hold negative quantities of shares, which is equivalent to borrowing shares from someone else, and thus having an obligation towards them. Borrowed shares may be sold for cash provided that these (or others identical to them) are returned to the original owner at time 1. This is the device known as *short selling*. In practice the lender makes a small charge for this service, which for simplicity is omitted from our simple model—see Assumption 2.7 below.

If an investor holds a positive amount of a given asset, he is said to be in a *long position* (or just *long*) in that asset; if the holding is negative, then he is said to be in a *short position* (or just *short*). So a short position in either bond or stock means that the investor has borrowed some of this asset. The act of repaying a money loan, returning a bond, or returning a borrowed share is called *settling* or *closing* the short position. Similarly a long position in an asset is *closed* by selling it for its current value.

The next section explains how we model the buying and selling of assets. The following simplifying assumption is made in order to expose the main ideas of the subsequent theory.

Assumption 2.7 (No friction)

Trading is instantaneous, borrowing and lending rates are the same, and there are no transaction costs in trading (equivalently, buying and selling prices are the same).

This is a significant simplifying assumption, because in practice the interest rate for borrowing will reflect not only the change in value of the loan but also a charge for the service provided. That is why the return on a typical savings account is less than the interest that would be charged on a loan by the same bank. Similarly, since there are costs involved in trading, in practice there are differences between buying and selling prices of shares.

Section 8.3 gives a brief introduction to models with proportional transaction costs. Sophisticated models that include other costs can be developed, but these are beyond the scope of this book.

2.2 Viability

In this section we introduce the idea of *arbitrage*, which is the key concept used to identify realistic market models and also to define the notion of a fair price for an asset. The underlying idea is that in a realistic market no-one can guarantee a riskless profit; that is, there is no possibility of *arbitrage*.

To formulate the definition of arbitrage, we must first define the *value* of a portfolio $\varphi = (x, y)$ consisting of x bonds and y shares. This is denoted by V_t^φ for $t = 0, 1$ and is defined by

$$V_t^\varphi := xB_t + yS_t. \tag{2.2}$$

The *gain* of a portfolio φ is

$$G^\varphi := V_1^\varphi - V_0^\varphi,$$

and if $V_0^\varphi \neq 0$, then the *return* is

$$R^\varphi := \frac{G^\varphi}{V_0^\varphi} = \frac{1}{V_0^\varphi}\left(V_1^\varphi - V_0^\varphi\right). \tag{2.3}$$

Observe that each of the quantities V_1^φ, G^φ and R^φ is a random variable, and therefore has two possible values depending on the scenario at time 1.

Example 2.2

There are two special cases of the return, namely when the portfolio consists of only bonds (so that $y = 0$) or only shares (so that $x = 0$). It is easy to verify that

$$R^{(x,0)} = r$$

whenever $x \neq 0$, and we refer to this by saying that the return on the bond is equal to $R^B := r$.

The return on the stock is

$$R^S := \frac{S_1}{S_0} - 1$$

since

$$R^{(0,y)} = \frac{S_1 - S_0}{S_0} = \frac{S_1}{S_0} - 1$$

whenever $y \neq 0$.

Example 2.3

Consider Model 2.2. The portfolio $\varphi = (-6, 1)$ consisting of $x = -6$ bonds and $y = 1$ share has initial value

$$V_0^\varphi = -6 \times 10 + 1 \times 75 = 15.$$

The final value, gain and return of this portfolio are given by

$$V_1^\varphi(\mathrm{u}) = -6 \times 12 + 1 \times 120 = 48, \qquad V_1^\varphi(\mathrm{d}) = -6 \times 12 + 1 \times 72 = 0,$$

$$G^\varphi(\mathrm{u}) = 48 - 15 = 33, \qquad G^\varphi(\mathrm{d}) = 0 - 15 = -15,$$

$$R^\varphi(\mathrm{u}) = \tfrac{33}{15} = 220\%, \qquad R^\varphi(\mathrm{d}) = \tfrac{-15}{15} = -100\%.$$

$$\begin{tikzcd} & \boxed{S_1(\mathrm{u}) = 120} \\ \boxed{S_0 = 75} \arrow[ur, "p = \frac{3}{4}"] \arrow[r, "1 - p = \frac{1}{4}"'] & \boxed{S_1(\mathrm{d}) = 72} \\ \boxed{B_0 = 10} \arrow[r] & \boxed{B_1 = 12} \end{tikzcd}$$

Model 2.2 One-period single-stock binary model in Example 2.3

Exercise 2.4

In Model 2.2, find the portfolio (x, y) with final value given by

$$V_1^{(x,y)}(\mathrm{u}) = 2148, \qquad V_1^{(x,y)}(\mathrm{d}) = 1236,$$

and calculate its return.

Exercise 2.5

(a) Suppose we know that the return on a portfolio (x, y) in Model 2.2 satisfies $V_1^{(x,y)}(\mathrm{u}) = 50\%$. Assuming that $y \neq 0$, find $\frac{x}{y}$, the ratio of shares to bonds in the portfolio.

(b) For a general binary model with one stock and a risk-free asset, show that if two portfolios (x, y) and (x', y') with $y \neq 0$, $y' \neq 0$, $V_0^{(x,y)} \neq 0$ and $V_0^{(x',y')} \neq 0$ satisfy $\frac{x}{y} = \frac{x'}{y'}$, then they have the same return.

Exercise 2.6

Suppose that the price of a bond in a single-period binary model is given by $B_0 = 15$ and $B_1 = 18$. A portfolio φ consisting of 30 bonds and 20 shares of a certain company has final value

$$V_1^\varphi = \begin{cases} 1140 & \text{with probability } p, \\ 1000 & \text{with probability } 1 - p. \end{cases}$$

Find the final stock price S_1 giving rise to these values.

The concept of *arbitrage* is fundamental to modern finance theory. Informally, this is a market condition that allows an investor to make a risk-free profit without initial investment, often referred to as a *free lunch*. Here is how to make it precise in our model.

Definition 2.7 (Arbitrage and viability)

(a) An *arbitrage opportunity* is a portfolio φ with $V_0^\varphi = 0$ and $V_1^\varphi \geq 0$, together with $V_1^\varphi(\omega) > 0$ for at least one $\omega \in \Omega$.

(b) A model is said to be *viable* if there are no arbitrage opportunities.

Strictly speaking we should call this an arbitrage opportunity *in bonds and stock* for reasons that we shall see later.

Exercise 2.8†

Decide whether a portfolio φ would be an arbitrage opportunity in each of the following situations:

(a) $V_0^\varphi > 0$, $V_1^\varphi(\mathrm{u}) > 0$ and $V_1^\varphi(\mathrm{d}) = 0$;

(b) $V_0^\varphi = 0$, $V_1^\varphi(\mathrm{u}) > 0$ and $V_1^\varphi(\mathrm{d}) < 0$;

(c) $V_0^\varphi = 0$, $G^\varphi(\mathrm{u}) = 0$ and $G^\varphi(\mathrm{d}) > 0$;

(d) $R^\varphi(\mathrm{u}) \geq r$, $R^\varphi(\mathrm{d}) \geq r$ and one of these inequalities is strict.

Arbitrage opportunities rarely exist in practice. If they can be found at all, they are generally beyond the reach of small investors, because the possible gain from an arbitrage opportunity is typically extremely small in comparison to the size of the transaction required to benefit from it. Moreover, in an *efficient market*, traders will move quickly to take advantage of an arbitrage opportunity. Other traders who find themselves being thus taken advantage of will move to prevent it by adjusting prices. Consequently, as well as being difficult to spot, a situation when arbitrage exists is generally short-lived because prices quickly change to eliminate arbitrage opportunities. For this reason real-world markets are effectively arbitrage-free.

The **No-Arbitrage Principle** is often referred to in the literature, but not always clearly defined. We take it to be the guiding principle that in a realistic market model there should be no arbitrage, for the reasons given above. In other words, it is only *viable* market models that are realistic. It follows that it is important to be able easily to identify viable models, and this is one major theme of this and later chapters.

The viability of a binary model is neatly characterized as follows.

Theorem 2.9

A single-period binary model with a single stock is viable if and only if

$$S_1(\mathrm{d}) < (1+r)S_0 < S_1(\mathrm{u}). \tag{2.4}$$

Remark 2.10

Equation (2.4) can equivalently be expressed as

$$\frac{S_1(\mathrm{d})}{1+r} < S_0 < \frac{S_1(\mathrm{u})}{1+r} \tag{2.5}$$

or

$$\frac{S_1(\mathrm{d})}{S_0} < 1+r < \frac{S_1(\mathrm{u})}{S_0},$$

which is in turn equivalent to

$$R^S(\mathrm{d}) < r < R^S(\mathrm{u}),$$

where R^S is the return on the stock (as computed in Example 2.2).

Proof (of Theorem 2.9)

Suppose that (2.4) holds, and let (x,y) be any portfolio with zero initial value; that is

$$0 = V_0^{(x,y)} = xB_0 + yS_0. \tag{2.6}$$

To see that this portfolio cannot be an arbitrage opportunity, examine the three possibilities for y. First, if $y=0$, it follows from (2.7) that $x=0$ and so $V_1^{(x,y)}=0$. Thus (x,y) is not an arbitrage opportunity.

Now consider the case that $y>0$. Multiplying inequality (2.4) by y and adding xB_1 leads to

$$xB_1 + yS_1(\mathrm{d}) < xB_1 + y(1+r)S_0 < xB_1 + yS_1(\mathrm{u}).$$

The middle term in this inequality is equal to

$$x(1+r)B_0 + y(1+r)S_0 = (1+r)V_0^{(x,y)} = 0,$$

while the other terms are the portfolio values in the two scenarios at time 1. Thus

$$V_1^{(x,y)}(\mathrm{d}) < 0 < V_1^{(x,y)}(\mathrm{u}),$$

and consequently (x,y) is not an arbitrage opportunity.

Finally, if $y<0$, it follows by a similar argument that

$$V_1^{(x,y)}(\mathrm{u}) < 0 < V_1^{(x,y)}(\mathrm{d})$$

so again (x,y) is not an arbitrage opportunity. We conclude that the model is viable.

Conversely, suppose that the model is viable. We will show by contradiction that (2.4) holds. Suppose to the contrary that

$$(1+r)S_0 \le S_1(\mathrm{d}) < S_1(\mathrm{u}). \tag{2.7}$$

This means that the return on the stock in both scenarios matches or exceeds the return on the riskless investment, so that we may form an arbitrage opportunity by borrowing money and investing it in the stock. In detail, consider the portfolio φ consisting of 1 share and $-S_0/B_0$ bonds. The initial value of this portfolio is

$$V_0^{\varphi} = -\frac{S_0}{B_0}B_0 + S_0 = 0.$$

Computing the final values of this portfolio and using (2.7) gives

$$V_1^{\varphi}(\mathrm{u}) = -\frac{S_0}{B_0}B_1 + S_1(\mathrm{u}) = -(1+r)S_0 + S_1(\mathrm{u}) > 0,$$

$$V_1^{\varphi}(\mathrm{d}) = -\frac{S_0}{B_0}B_1 + S_1(\mathrm{d}) = -(1+r)S_0 + S_1(\mathrm{d}) \ge 0.$$

Consequently, the portfolio $(-S_0/B_0, 1)$ is an arbitrage opportunity. This contradicts the viability of the model, and so inequality (2.7) cannot hold true.

A similar construction using the portfolio $(S_0/B_0, -1)$ gives arbitrage in the case when

$$S_1(\mathrm{d}) < S_1(\mathrm{u}) \leq (1+r)S_0. \tag{2.8}$$

Since neither (2.7) nor (2.8) is true, we conclude that (2.4) holds. □

Example 2.11

Inequality (2.4) holds for Model 2.2 because $S_0(1+r) = 75 \times 1.2 = 90$ and

$$S_1(\mathrm{d}) = 72 < 90 < 120 = S_1(\mathrm{u}).$$

Exercise 2.12

Explain why Model 2.3 is not viable, and construct an arbitrage opportunity. What would you change to make this model viable?

$$\begin{tikzcd} & \boxed{S_1(\mathrm{u}) = 170} \\ \boxed{S_0 = 150} \arrow[ur] \arrow[dr] & \\ & \boxed{S_1(\mathrm{d}) = 130} \end{tikzcd}$$

$$\boxed{B_0 = 100} \longrightarrow \boxed{B_1 = 115}$$

Model 2.3 One-period single-stock binary model in Exercise 2.12

Exercise 2.13

Construct an example of a viable model with $S_0 < S_1(\mathrm{d}) < S_1(\mathrm{u})$. What can you say about the interest rate in any such model?

2.3 Derivative Securities and the Pricing Problem

As explained in the introduction, this book is about pricing derivative securities. Before developing the theory it is useful to examine the pricing problem for a European call option, and show that an intuitive solution turns out to be wrong. This will motivate the need for the theory that ensues.

Example 2.14

Recall that the owner of a *European call option* (discussed briefly in Chapter 1) has the right (without obligation) to buy a share of the underlying stock S at time 1 at a *strike price*, say K, which is determined and written into the option contract when it is purchased at time 0. At time 1 when the actual future value of S is known, there are three possibilities. If the stock price S_1 is greater than the strike price K, we say that the option is *in the money*, and if $S_1 = K$, the model is *at the money*; otherwise, the option is *out of the money*.

At time 1, the decision of whether to exercise the option or not will depend on the stock price S_1. The owner of the option will exercise it (meaning he will exercise his right to buy a share at the fixed price K) at time 1 only if it is in the money; that is, if $S_1 > K$. In that case, the value of the option to the owner is $S_1 - K$ since it represents the right to buy a share worth S_1 for the lower price K. If, on the other hand, $S_1 \le K$, then he is able to buy a share for K or less on the open market, and there would be no sense in exercising the option. The option is therefore worthless.

We summarize this by saying that the *payoff* of the option—that is, the value to the owner—at time 1 is

$$\begin{aligned} C_1 &:= \begin{cases} S_1 - K & \text{if } S_1 > K, \\ 0 & \text{if } S_1 \le K \end{cases} \\ &= [S_1 - K]_+ \end{aligned}$$

recalling that for any $a \in \mathbb{R}$, the *positive and negative parts* of a, denoted by a_+ and a_- are defined by $a_+ := \max\{a, 0\}$ and $a_- := \max\{-a, 0\}$. (Some important properties of a_+ and a_- are given in Exercise 2.17 below.)

Notice that the payoff is never negative and could be positive, since the owner of this option will only exercise it if it is profitable to do so. Such an option is therefore a valuable asset, and a market agent should expect to pay something, let us call it π, at time 0 to purchase it. Obtaining a call option for free would amount to a free lunch or riskless profit.

The pricing problem is to determine whether there is a fair price to pay for a European call option, and if so what it is. Of course we will need to define what is meant by a fair price.

First let us consider in detail this problem for a European call option in Model 2.2. Suppose that a trader wishes to write (that is, sell) a European call option with strike $K = 96$. What price should he charge for this at time 0? As a first guess, one might take the expected payoff of the option. We have

$$C_1 = \begin{cases} [120 - 96]_+ = 24 & \text{with probability } \frac{3}{4}, \\ [72 - 96]_+ = 0 & \text{with probability } \frac{1}{4}. \end{cases}$$

The expected value of the option is

$$\tfrac{3}{4} \times 24 + \tfrac{1}{4} \times 0 = 18,$$

which would seem to be a reasonable guess for the price of this option. However, we will see that this price is *too high*—since it allows the seller to make a risk free profit as follows.

The trader should simultaneously sell a call option for the price $\pi = 18$ and create the portfolio $\varphi = (-1.95, 0.5)$ in bonds and stock. The value of this *extended portfolio* (consisting of bonds, shares *and* the option) at time 0 is

$$V_0^{\varphi} - 18 = (-1.95 \times 10) + (0.5 \times 75) - 1 \times 18 = 0$$

so there is no net outlay. An alternative way to see this is to consider the cash flow: 0.5 shares (worth 37.5) are purchased by borrowing 1.95 bonds (worth 19.5) and the proceeds 18 from selling the option.

At time 1, there are two possibilities. If the stock price moves to 120, the call option will be exercised; consequently the trader must buy a share at 120 and then sell it to the owner of the option for 96, making a loss of 24. In this case, on closing the position in bonds and shares at the same time, the trader will receive

$$V_1^{\varphi}(\mathrm{u}) - 24 = (-1.95 \times 12) + (0.5 \times 120) - 24 = 12.6.$$

If the stock price moves to 72, the option is worthless, and is not exercised. The trader has nothing to pay out. Closing his position in shares, bonds and options gives

$$V_1^{\varphi}(\mathrm{d}) - 0 = (-1.95 \times 12) + (0.5 \times 72) - 0 = 12.6.$$

This means that if option is priced at 18, a trader selling at this price is able to make a risk-free profit of 12.6 in cash—*no matter what happens at time* $t = 1$. No market agent who is aware of this would be willing to buy a call option at a price of 18, as he would certainly not wish to assist the trader in making money out of nothing. We say that 18 is an *unfair* price for the option.

Remark 2.15

The argument presented here is quite typical in the construction of arbitrage opportunities. The idea is to sell those assets that are considered to be overpriced, and/or to buy those assets that are considered (or proven) to be too cheap.

The next exercise is designed to show that if the option price is too low, a trader may create an arbitrage opportunity, giving him a risk free profit, by following a similar strategy to the one above, but buying an option, instead of selling. Of course, no other market agent would be willing to sell a call option to our trader at such a price.

Exercise 2.16

Show that if the call option with strike $K = 96$ in Model 2.2 is priced at 6, then a trader can make a risk-free profit by creating the portfolio $(3.15, -0.5)$ and buying one call option. Deduce that the price 6 is too low.

Exercise 2.17

Prove the following for any real number a:

(a) $a_- = (-a)_+$.

(b) $a = a_+ - a_-$.

(c) $|a| = a_+ + a_-$.

2.4 Fair Pricing

The theory we are about to develop centres around extending the idea of arbitrage to portfolios that include assets other than the primary ones, as in the above example. In this way we will be able to define precisely what is meant by a price being "too high" or "too low", or "fair".

Let us suppose then that we have, in addition to the primary assets, a *derivative security* D (such as the European call option in the above example) that may be traded in the markets. The derivative will have a *payoff* or value at time 1, which depends on the way the stock price S_1 behaves. The stock price S_1 itself depends on the scenario followed by the model, and so we formally define a *derivative security*, usually referred to simply as a *derivative*, as follows.

Definition 2.18 (Derivative in a single period model)

A *derivative* (or *derivative security* or *contingent claim*) is an asset whose payoff at time 1 is given by a random variable $D : \Omega \to \mathbb{R}$.

Example 2.19

A European call option as above is a derivative show payoff at time 1 is the random variable C_1 where

$$C_1(\omega) = \big[S_1(\omega) - K\big]_+$$

for $\omega \in \Omega$.

Let π denote a price at which a derivative D is traded at time 0. The pricing problem is to find those values of π that are fair. To make this precise we will need the following definitions.

Definition 2.20 (Extended portfolio, extended arbitrage opportunity)

Let D be a derivative priced at π at time 0. An *extended portfolio* involving D is a triple $\psi = (x, y, z)$ that denotes a holding of x bonds, y shares and z units of the derivative D. The *value* V_t^ψ of the extended portfolio $\psi = (x, y, z)$ at $t = 0, 1$ is given by

$$V_0^\psi := xB_0 + yS_0 + z\pi, \qquad V_1^\psi := xB_1 + yS_1 + zD. \tag{2.9}$$

An *extended arbitrage opportunity* in bonds, shares and the derivative D is an extended portfolio ψ with $V_0^\psi = 0$ and $V_1^\psi \geq 0$, and $V_1^\psi(\omega) > 0$ for at least one $\omega \in \Omega$.

A rational seller will never agree to sell D at the price π if it allows an extended arbitrage opportunity (x, y, z) with $z > 0$, because that would mean that the buyer can make a risk-free profit by buying D and creating the portfolio $(x/z, y/z)$. Likewise, an extended arbitrage opportunity with $z < 0$ would allow a seller to make a risk free profit by selling D and simultaneously creating the portfolio $(-x/z, -y/z)$.

This leads to the extension of the No-Arbitrage Principle to also include derivatives: in a rational market model the prices of derivatives will be such as to prevent extended arbitrage opportunities. In other words, in a rational market all derivatives will be traded at *fair prices*, defined precisely as follows.

Definition 2.21 (Fair price)

A *fair* or *rational* or *arbitrage-free* price for a derivative D is any price π for it that prevents extended arbitrage opportunities involving D.

Finding fair prices is clearly very important. A natural question, which the theory will answer, is whether fair prices actually exist (they do in our model, if it is viable) and whether they are unique (yes in our simple model, but not always in more general models).

2.5 Replication and Completeness

The key to finding a fair price for a derivative D is the idea of *replication*, which enables us find a unique fair price at time 0 for any derivative in our simple model.

Definition 2.22 (Replication, attainability)

A portfolio φ *replicates* a derivative D if its value at time 1 is equal to D in all possible scenarios; that is.

$$V_1^{\varphi}(\omega) = D(\omega) \quad \text{for all } \omega \in \Omega,$$

or $V_1^{\varphi} = D$ for short. If a derivative D admits a replicating portfolio, then D is called *attainable*.

The main tool for finding a fair price by means of replication is the *Law of One Price*. In its general form, it states that if two extended portfolios have the same value at time 1 for all scenarios, then in a rational market the prices at time 0 must be such as to give the same value at time 0—otherwise, there would be arbitrage. The particular case that we need is in the next theorem.

Theorem 2.23 (Law of One Price in a simple model)

If a portfolio $\varphi = (x, y)$ replicates an attainable derivative D in a viable single-period single-stock model, then there is a unique fair price D_0 for D at time 0, which is equal to the initial value of the portfolio. That is

$$D_0 = V_0^{\varphi}. \tag{2.10}$$

Proof

Let $\varphi = (x, y)$ be a replicating portfolio for D. We begin by showing that any price π other than V_0^{φ} is unfair.

Suppose first that $\pi > V_0^{\varphi}$. Then at time 0 we may sell D at the price π, and with the proceeds buy the portfolio φ which costs V_0^{φ} and still have spare cash amounting to $\pi - V_0^\varphi$, which we can invest in bonds. This amounts to constructing the extended portfolio $(x', y, -1)$, where

$$x' = x + \frac{1}{B_0}\big(\pi - V_0^\varphi\big) = \frac{1}{B_0}(\pi - yS_0).$$

The value of this portfolio at time 0 is

$$V_0^{(x',y,-1)} = (\pi - yS_0) + yS_0 - \pi = 0.$$

At time 1, the value of this portfolio in any scenario $\omega \in \Omega$ is

$$\begin{aligned} V_1^{(x',y,-1)}(\omega) &= x'B_1 + yS_1(\omega) - D(\omega) \\ &= xB_1 + (1+r)\big(\pi - V_0^\varphi\big) + yS_1(\omega) - D(\omega) \\ &= V_1^\varphi(\omega) + (1+r)\big(\pi - V_0^\varphi\big) - D(\omega). \end{aligned}$$

Since φ replicates D, we have $V_1^\varphi(\omega) = D(\omega)$, and therefore

$$V_1^{(x',y,-1)}(\omega) = (1+r)\big(\pi - V_0^\varphi\big) > 0.$$

Thus we have constructed an extended arbitrage opportunity.

If $\pi < V_0^\varphi$, then we may construct an arbitrage opportunity by buying the derivative, selling the portfolio, and investing the difference $V_0^\varphi - \pi$ in bonds. This amounts to constructing the extended portfolio $(x'', -y, 1)$ where

$$x'' = -x + \frac{1}{B_0}\big(V_0^\varphi - \pi\big) = \frac{1}{B_0}(yS_0 - \pi).$$

Arguing as in the first case, this guarantees a profit of

$$(1+r)\big(V_0^\varphi - \pi\big) > 0.$$

So any price π for D other than V_0^φ is not fair. To complete the proof, we must show that $\pi = V_0^\varphi$ is a fair price. To see this, assume that D is priced at $\pi = V_0^\varphi$, and suppose that an extended arbitrage is achieved by the extended portfolio (u, v, w) consisting of u bonds, v shares and w units of D. The value at time 0 of this portfolio is

$$\begin{aligned} V_0^{(u,v,w)} &= uB_0 + vS_0 + w\pi = uB_0 + vS_0 + wV_0^\varphi \\ &= (u + wx)B_0 + (v + wy)S_0 = V_0^{(u+wx,v+wy)}. \end{aligned}$$

At time 1, the replication of D by φ gives

$$\begin{aligned} V_1^{(u,v,w)} &= uB_1 + vS_1 + wD = uB_1 + vS_1 + wV_1^\varphi \\ &= (u + wx)B_1 + (v + wy)S_1 = V_1^{(u+wx,v+wy)}. \end{aligned}$$

Thus $(u+wx, v+wy)$ is an arbitrage opportunity in bonds and shares, which contradicts the assumption that the model is viable. Thus there can be no extended arbitrage opportunity if $\pi = V_0^{\varphi}$, so this is the unique fair price for D at time 0, and we may denote it D_0. □

This result is only applicable if the derivative D is attainable (that is, can be replicated), so we need to ask whether there is always a replicating portfolio for D, and if so how to find it. In a single-period binary model, the answer is very simple indeed: all derivative securities are attainable, and we say that the model is *complete*.

Theorem 2.24

In a single-period model with one stock, every derivative D is attainable and has a unique replicating portfolio (x, y) consisting of

$$x = \frac{1}{B_1} \frac{D(\mathrm{d})S_1(\mathrm{u}) - D(\mathrm{u})S_1(\mathrm{d})}{S_1(\mathrm{u}) - S_1(\mathrm{d})} \tag{2.11}$$

bonds and

$$y = \frac{D(\mathrm{u}) - D(\mathrm{d})}{S_1(\mathrm{u}) - S_1(\mathrm{d})} \tag{2.12}$$

shares.

Proof

At time 1, there are only two possibilities for the equation $V_1^{(x,y)} = D$, so we need to find a portfolio (x, y) satisfying the two equations

$$xB_1 + yS_1(\mathrm{u}) = D(\mathrm{u}), \qquad xB_1 + yS_1(\mathrm{d}) = D(\mathrm{d}).$$

For any D this system has a unique solution given by (2.11) and (2.12). □

Example 2.25

Consider the European call option discussed in Example 2.14. The replicating portfolio for this option is

$$x = \frac{1}{12} \frac{0 \times 120 - 24 \times 72}{120 - 72} = -3, \qquad y = \frac{24 - 0}{120 - 72} = \frac{1}{2}.$$

Consequently, the fair price of this option is

$$V_0^{(x,y)} = -3 \times 10 + \tfrac{1}{2} \times 75 = 7.5.$$

This should be contrasted with the guess of 18 in Example 2.14, which was too high and the price 6 which was shown in Exercise 2.16 to be too low.

The theory we have just developed also shows just how we arrived at the extended portfolio $(-1.95, 0.5, -1)$ to give arbitrage when the price was too high. We know that the option is replicated by the portfolio $\varphi = (-3, 0.5)$ to give the fair price of 7.5; that is, this is the price for the option that does not allow arbitrage. If D is priced at 18 then by selling at that price instead of 7.5 the difference $18 - 7.5 = 10.5$ can be invested risk free in 1.05 bonds. So, if the bond holding is now increased to $-3 + 1.05 = -1.95$ this figure contains both the bond holding needed to replicate D *and* the extra element of investment for profit. This extra bond holding gives the guaranteed profit $1.05 \times 12 = 12.6$ no matter what happens at time $t = 1$.

The extended portfolio in Exercise 2.16 that gives a risk free profit if D is priced at 6 was found in the same way.

Exercise 2.26

Let D be the derivative given by $D(\mathrm{u}) = 49$, $D(\mathrm{d}) = 21$ in Model 2.4.

(a) Show that Model 2.4 is viable.

(b) Find the replicating portfolio for D and hence the fair price D_0.

(c) Construct extended arbitrage opportunities to show that 25 is too low a price, and 40 is too high to pay for this derivative.

$$
\begin{tikzcd}
 & \boxed{S_1(\mathrm{u}) = 130} \\
\boxed{S_0 = 100} \arrow[ur] \arrow[dr] & \\
 & \boxed{S_1(\mathrm{d}) = 90}
\end{tikzcd}
$$

$$\boxed{B_0 = 100 \qquad r = 5\%}$$

Model 2.4 One-period binary model in Exercises 2.26, 2.27 and 2.37

Exercise 2.27

Suppose that the fair price at time 0 of a derivative D in Model 2.4 is $D_0 = 100$, and you are told that $D(\mathrm{u}) = 55$. What is the value of $D(\mathrm{d})$?

2.6 Risk-Neutral Probabilities

Finding the fair price of a derivative by first calculating the bond and stock holding in the replicating portfolio and then its initial value is quite laborious—

and becomes even more so when we consider models with more than one step. In this section we introduce the important concept of *risk-neutral probabilities*, which gives a much more efficient technique for finding fair prices, since it involves only the calculation of a new probability on the set Ω of possible scenarios. The fair price of any derivative is then simply its expected value under this new probability, after *discounting*.

The notion of the *discounted value* of any asset is an important one in mathematical finance. The idea is that of adjusting the future value of a random quantity to today's prices or *discounting the effect of inflation or interest rates*. This is made precise in the following definition.

Definition 2.28 (Discounted value)

Let $X : \Omega \to \mathbb{R}$ be a random variable representing a quantity (for example the payoff of a derivative) that is known at time 1. The *value of X discounted to time* 0 is the random variable

$$\bar{X} := \frac{X}{1+r}.$$

This is often referred to as simply the *discounted value* of X.

Recalling (2.5), we can say that a single-period binary model is viable if and only if the stock price at time 0 lies strictly between the two *discounted stock prices* at time 1; that is

$$\bar{S}_1(\mathrm{d}) < S_0 < \bar{S}_1(\mathrm{u}). \tag{2.13}$$

Thus, in a viable binary model, the point S_0 can be written as a weighted average (called a *convex combination*) of the two quantities $\bar{S}_1(\mathrm{u})$ and $\bar{S}_1(\mathrm{d})$. In other words, there is a number $q \in (0,1)$ such that

$$S_0 = q\bar{S}_1(\mathrm{u}) + (1-q)\bar{S}_1(\mathrm{d}), \tag{2.14}$$

where q is given explicitly by

$$q = \frac{S_0 - \bar{S}_1(\mathrm{d})}{\bar{S}_1(\mathrm{u}) - \bar{S}_1(\mathrm{d})}. \tag{2.15}$$

The key idea now is that the pair $(q, 1-q)$ can be interpreted as a new probability assignment for the scenarios u, d, and then equation (2.14) shows that S_0 is the expected value of the discounted price $\bar{S}_1$ under this new probability. That is, writing $\mathbb{Q} = (q, 1-q)$ we have

$$S_0 = \mathbb{E}_{\mathbb{Q}}(\bar{S}_1) := q\bar{S}_1(\mathrm{u}) + (1-q)\bar{S}_1(\mathrm{d}). \tag{2.16}$$

The new probability assignment $\mathbb{Q}$, called a *risk-neutral probability*, is in general *quite different from the original real-world probability* $(p, 1-p)$: it is a purely artificial mathematical probability. The key defining property of $\mathbb{Q}$ is the equality (2.16); thus we have the following formal definition.

Definition 2.29 (Risk-neutral probability)

A *risk-neutral probability* is a probability assignment $\mathbb{Q} = (q, 1-q)$ to the scenarios u, d such that (2.16) holds and $0 < q < 1$.

Notice that the original real-world probability $(p, 1-p)$ has not featured at all in the above discussion, which is summarized as follows.

Theorem 2.30

A viable one-step binary model containing one stock admits a unique risk-neutral probability $\mathbb{Q}$.

Proof

We have already shown the existence of a probability assignment $\mathbb{Q}$ on Ω satisfying equation (2.16) if the model is viable. This probability assignment is unique because, as already observed, equation (2.16) gives

$$q = \frac{S_0 - \bar{S}_1(\mathrm{d})}{\bar{S}_1(\mathrm{u}) - \bar{S}_1(\mathrm{d})}, \qquad 1 - q = \frac{\bar{S}_1(\mathrm{u}) - S_0}{\bar{S}_1(\mathrm{u}) - \bar{S}_1(\mathrm{d})}. \tag{2.17}$$

□

Example 2.31

For Model 2.2 we have

$$q = \tfrac{75 - 72/1.2}{120/1.2 - 72/1.2} = \tfrac{3}{8}$$

and we may easily verify that

$$q\bar{S}_1(\mathrm{u}) + (1-q)\bar{S}_1(\mathrm{d}) = \tfrac{3}{8}\tfrac{120}{1.2} + \tfrac{5}{8}\tfrac{72}{1.2} = 75 = S_0.$$

The defining property (2.16) may be expressed in a number of equivalent ways. First consider the *increment of the discounted price*

$$\Delta\bar{S}_1 := \bar{S}_1 - S_0. \tag{2.18}$$

We may write (2.16) as

$$\mathbb{E}_{\mathbb{Q}}(\Delta\bar{S}_1) = 0,$$

since S_0 is constant. That is, the average movement of the price after discounting is zero.

Remarks 2.32

(1) The increment $\Delta\bar{S}_1$ of the discounted price is *not* the same as the *discounted price increment*, which would be

$$\overline{\Delta S_1} := \overline{S_1 - S_0} = \frac{S_1 - S_0}{1+r}.$$

(2) In later development when there are several time steps we will again encounter this phenomenon, namely that there is on average no change in the discounted price provided the probability is chosen carefully. This is the defining property of an important concept—that of a *martingale*. In our context the discounted stock price is a one-step martingale.

A second alternative way to express equation (2.16) is

$$\mathbb{E}_{\mathbb{Q}}(S_1) = (1+r)S_0,$$

which may be rearranged to yield

$$\mathbb{E}_{\mathbb{Q}}\big(R^S\big) = r,$$

where R^S is the return of the stock (see Example 2.2). This expression explains why the probability assignment $\mathbb{Q}$ is called *risk-neutral*: under the artificial probability $\mathbb{Q}$, the expected return from the risky investment is expected to be the same as the return from an investment in the risk free asset.

The existence of such a risk-neutral probability assignment actually characterizes a viable model; we have already proved one half of the following result in Theorem 2.30.

Theorem 2.33

A one-step binary model containing one stock is viable if and only if it allows a risk-neutral probability assignment. If a risk-neutral probability exists, then it is unique.

Exercise 2.34‡

Complete the proof of Theorem 2.33 by proving that if a one-step binary model with one stock has a risk-neutral probability, then it is viable.

Hint. Use Theorem 2.9.

Risk-neutral probabilities give us a quick and convenient way of calculating the fair price of a derivative. In view of results like Theorem 2.35 below, the fair price of a derivative is sometimes called its *risk-neutral price*.

Theorem 2.35

In a single-period binary model with risk-neutral probability $\mathbb{Q}$, the fair price of any derivative D is

$$D_0 = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}(D) = \mathbb{E}_{\mathbb{Q}}(\bar{D}), \tag{2.19}$$

where $\bar{D}$ is the discounted value of D.

Proof

It follows from Theorem 2.24 that there exists a replicating portfolio (x, y) for D and we know from Theorem 2.23 that there is a unique fair price for D, namely $D_0 = V_0^{(x,y)}$. Since $D = V_1^{(x,y)} = xB_1 + yS_1$, then $\bar{D} = xB_0 + y\bar{S}_1$ and we conclude from (2.16) that

$$\mathbb{E}_{\mathbb{Q}}(\bar{D}) = \mathbb{E}_{\mathbb{Q}}(xB_0 + y\bar{S}_1) = xB_0 + y\mathbb{E}_{\mathbb{Q}}(\bar{S}_1) = xB_0 + yS_0 = V_0^{(x,y)} = D_0.$$

□

Example 2.36

In Example 2.31, we found that the risk-neutral probability assignment for Model 2.2 is $\mathbb{Q} = (\frac{3}{8}, \frac{5}{8})$. The fair price of the European call option with strike 96 of Example 2.14 is then

$$\tfrac{3}{8} \times \tfrac{24}{1.2} + \tfrac{5}{8} \times \tfrac{0}{1.2} = 7.5.$$

As Theorem 2.35 tells us, this is the same as the initial value of the replicating portfolio for this option, given in Example 2.25.

Exercise 2.37†

Compute the fair prices for a European call option C_1 with strike 110 and a European call option C'_1 with strike 120 in Model 2.4 using:

(a) Replication.

(b) Risk-neutral probabilities.

Check that these prices agree.

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