Chapter 2
Finite Element Formulations

The governing equations for problems solved by the finite element method are typically formulated by partial differential equations in their original form. These are rewritten into a weak form, such that domain integration can be utilized to satisfy the governing equations in an average sense. A functional Π is set up for the system, typically describing the energy or energy rate and implying that the solution can be found by minimization. For a generic functional, this is written as

$$\frac{\partial \Pi}{\partial u} = \frac{\partial}{\partial u}\left(\int_V f(x_i, u_i)\; dV\right) = \frac{\partial}{\partial u}\left(\sum_j f(x_i, u_i)\; \Delta V_j\right) = 0 \tag{2.1}$$

where the functional is a function of the coordinates x_i and the primary variable u_i being e.g. displacements or velocities for mechanical problems depending on the formulation. The domain integration is approximated by a summation over a finite number of elements discretizing the domain. Figure 2.1 illustrates a three-dimensional domain discretized by hexahedral elements with eight nodes. The variables are defined and solved in the nodal points, and evaluation of variables in the domain is performed by interpolation in each element. Shared nodes give rise to an assembly of elements into a global system of equations of the form

$$\mathbf{Ku} = \mathbf{f} \tag{2.2}$$

where $\mathbf{K}$ is the stiffness matrix, $\mathbf{u}$ is the primary variable and $\mathbf{f}$ is the applied load, e.g. stemming from applied tractions F on a surface S_F in Fig. 2.1. The system of equations (2.2) is furthermore subject to essential boundary conditions, e.g. prescribed displacements or velocities u along a surface S_U.

The basic aspects of available finite element formulations in terms of modeling and computation are briefly reviewed in this chapter. This will support the choice of formulation to be detailed and applied in the remaining chapters, where an electro-thermo-mechanical finite element formulation is presented

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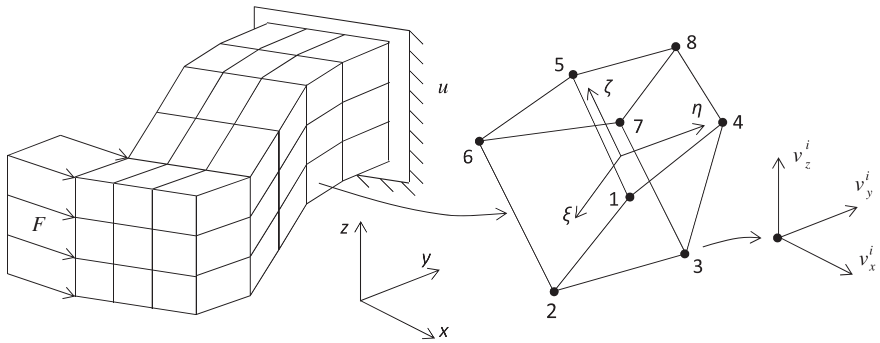

Fig. 2.1 Illustration of three-dimensional finite element model composed of isoparametric, hexahedral elements with eight nodes. Each node has three degrees of freedom for representation of vector fields and one degree of freedom for representation of scalar fields

together with a range of aspects to complete a computer program capable of modeling manufacturing processes such as metal forming and resistance welding. This chapter is focused on the mechanical formulations because they represent major differences and because the mechanical model plays a central role in the overall modeling strategy. From a process point of view the mechanical model is responsible for material flow, contact and stress distribution, and from a computational point of view is responsible for the largest amount of CPU time. In addition, the overall structure of the presented computer program is built upon the mechanical formulation with the remaining thermal and electrical modules integrated.

One fundamental difference between the finite element formulations is the governing equilibrium equation, being either quasi-static or dynamic in the modeling of manufacturing processes. Another fundamental choice to cover is the material model suited for describing the materials under consideration, bearing in mind the process to simulate and thereby the expected range of deformation and deformation rate. The available constitutive models to utilize in the material description are rigid-plastic/viscoplastic and elasto-plastic/viscoplastic.

Table 2.1, after Tekkaya and Martins [1], provides an overview of the quasi-static formulations and the dynamic formulation. The quasi-static formulations are represented by the flow formulation and the solid formulation, distinguishable by the underlying constitutive equations. The following two sections are devoted to give a brief overview of the quasi-static and dynamic formulations including their advantages and disadvantages.

Presentation of the quasi-static and dynamic formulations follows the general outline given by Tekkaya and Martins [1] and additional information can be found in major reference books by Zienkiewicz and Taylor [2], Banabic et al. [3], Wagoner and Chenot [4] and Dunne and Petrinic [5].

Table 2.1 Overview of finite element formulations and commercial computer programs applied in the metal forming industry

	Quasi-static formulations		Dynamic formulation
	Flow formulation	Solid formulation	
Equilibrium equation:	Quasi-static	Quasi-static	Dynamic
Constitutive equations:	Rigid-plastic/ viscoplastic	Elasto-plastic/ viscoplastic	Elasto-plastic/ viscoplastic
Main structure:	Stiffness matrix and force vector	Stiffness matrix and force vector	Mass and damping matrices and internal and external force vectors
Solution scheme[a]:	Implicit	Implicit	Explicit
Size of incremental step:	Large	Medium to large	Very small
CPU time per incremental step:	Medium	Medium to long	Very short
Time integration scheme[b]:	Explicit	Implicit	Explicit
Accuracy of the results (stress and strain distributions):	Medium to high	High	Medium to low
Springback and residual stresses:	No (although the basic formulation can be modified to include elastic recovery)	Yes	Yes/no
Commercial FEM computer programs related to metal forming	FORGE[c], DEFORM[c], QFORM, eesy-2-form	Abaqus (implicit), Simufact.forming, AutoForm, Marc	Abaqus (explicit), DYNA3D, PAM-STAMP

[a]Explicit/implicit if the residual force is not/is minimized at each incremental step.
[b]Explicit/implicit if the algorithm does not/does need the values of the next time step to compute the solution.
[c]Elasto-plastic options available.

2.1 Quasi-Static Formulations

The quasi-static formulations are governed by the static equilibrium equation, which in the absence of body forces takes the following form,

$$\sigma_{ij,j} = 0 \tag{2.1.1}$$

where $\sigma_{ij,j}$ denotes the partial derivatives of the Cauchy stress tensor with respect to the Cartesian coordinates x_j. This equation expresses the equilibrium in the current configuration, i.e. in the mesh following the deformation.

By employing the Galerkin method, it is possible to write an integral form of Eq. (2.1.1) that fulfills the equilibrium in an average sense over the entire domain instead of satisfying the equilibrium point-wise. This formulation allows domain integration to substitute the more tedious solution of the original differential equations. The integral over domain volume V is

$$\int_V \sigma_{ij,j} \delta u_i dV = 0 \tag{2.1.2}$$

with δu_i being an arbitrary variation in the primary unknown u_i, which is either displacement or velocity depending on the implementation. Displacement is the primary unknown in rate independent formulations and velocity is the primary unknown in rate dependent formulations.

Applying integration by parts in Eq. (2.1.2), followed by the divergence theorem and taking into account the natural and essential boundary conditions, it is possible to rewrite Eq. (2.1.2) as follows,

$$\int_V \sigma_{ij} \left(\delta u_i\right)_{,j} dV - \int_S t_i \, \delta u_i dS = 0 \tag{2.1.3}$$

where $t_i = \sigma_{ij} n_j$ denotes the tractions with direction of the unit normal vector n_j applied on the boundary surface S. Equation (2.1.3) is the "weak variational form" of Eq. (2.1.1) because the static governing equilibrium equations are now only satisfied under weaker continuity requirements.

The above listed equations together with appropriate constitutive equations enable quasi-static finite element formulations to be defined by means of the following matrix set of non-linear equations,

$$\mathbf{K}^t \mathbf{u}^t = \mathbf{F}^t \tag{2.1.4}$$

which express the equilibrium condition at the instant of time t through the stiffness matrix $\mathbf{K}$, the generalized force vector $\mathbf{F}$ resulting from the loads, pressure and friction stresses applied on the boundary. The equation system is non-linear due to the stiffness matrix's dependency of the primary unknown $\mathbf{u}$ to geometry and material properties.

The quasi-static finite element formulations utilized in the analysis of metal forming and resistance welding processes are commonly implemented in conjunction with implicit solution schemes. The main advantage of implicit schemes over alternative solutions based on explicit procedures is that equilibrium is checked at each increment of time by means of iterative procedures to minimize the residual force vector $\mathbf{R}(\mathbf{u})$, which is computed as follows in iteration number n,

$$\mathbf{R}_n^t = \mathbf{K}_{n-1}^t \mathbf{u}_n^t - \mathbf{F}^t \tag{2.1.5}$$

The non-linear set of equations (2.1.4), derived from the quasi-static implicit formulations, can be solved by different numerical techniques such as the direct iteration (also known as "successive replacement") and the Newton–Raphson methods. In the direct iteration method, the stiffness matrix is evaluated for the displacements of the previous iteration in order to reduce Eq. (2.1.4) to a linear set of equations. The method is iterative and converges linearly and unconditionally towards the solution during the earlier stages of the iteration procedure but becomes slow as the solution is approached. The standard Newton-Raphson method is an alternative iterative method based on a linear expansion of the residual $\mathbf{R}(\mathbf{u})$ near the velocity estimate at the previous iteration,

$$\mathbf{R}_n^t = \mathbf{R}_{n-1}^t + \left[\frac{\partial \mathbf{R}}{\partial \mathbf{u}}\right]_{n-1}^t \Delta \mathbf{u}_n^t = 0 \tag{2.1.6a}$$

$$\mathbf{u}_n^t = \mathbf{u}_{n-1}^t + \Delta \mathbf{u}_n^t \tag{2.1.6b}$$

This procedure is only conditionally convergent, but converges quadratically in the vicinity of the exact solution. The iterative procedures are designed in order to minimize the residual force vector $\mathbf{R}(\mathbf{u})$ to within a specified tolerance. Control and assessment is performed by means of appropriate convergence criteria.

The main advantage of the quasi-static implicit finite element formulations is that equilibrium conditions are checked at each increment of time in order to minimize the residual force vector $\mathbf{R}(\mathbf{u})$ to within a specified tolerance.
The main drawbacks in the quasi-static implicit finite element formulations are summarized as follows:

- Solution of linear systems of equations is required during each iteration;
- High computation times and high memory requirements;
- Computation time depends quadratically on the number of degrees of freedom if a direct solver is utilized, and with the Newton-Raphson method the solution is only conditionally convergent;
- The stiffness matrix is often ill-conditioned, which can turn the solution procedure unstable and deteriorate the performance of iterative solvers;
- Difficulties in dealing with complex non-linear contact and tribological boundary conditions are experienced, and that often leads to convergence problems.

2.2 Dynamic Formulation

The dynamic finite element formulation is based on the dynamic equilibrium equation in the current configuration, here written in the absence of body forces with the inertia term expressed through the mass density ρ and the acceleration $\ddot{u}_i$,

$$\sigma_{ij,j} - \rho\, \ddot{u}_i = 0 \tag{2.2.1}$$

Applying a mathematical procedure similar to that described in the previous section results in the following weak variational form of Eq. (2.2.1),

$$\int_V \rho\, \ddot{u}_i \delta u_i dV + \int_V \sigma_{ij}\, (\delta u_i)_{,j}\, dV - \int_S t_i\, \delta u_i dS = 0 \tag{2.2.2}$$

The above equation enables dynamic finite element formulations to be represented by the following matrix set of non-linear equations,

$$\mathbf{M}^t \ddot{\mathbf{u}}^t + \mathbf{F}_{\text{int}}^t = \mathbf{F}^t \tag{2.2.3}$$

which express the dynamic equilibrium condition at the instant of time t. The symbol $\mathbf{M}$ denotes the mass matrix, $\mathbf{F}_{\text{int}} = \mathbf{K}\mathbf{u}$ is the vector of internal forces resulting from the stiffness, and $\mathbf{F}$ is the generalized force vector.

The non-linear set of equations (2.2.3), derived from the dynamic formulation, is commonly solved by means of an explicit central difference time integration scheme,

$$\mathbf{M}^t \left(\frac{\dot{\mathbf{u}}^{t+1/2} - \dot{\mathbf{u}}^{t-1/2}}{\Delta\, t^{t+1/2}} \right) + \mathbf{F}^t_{\text{int}} = \mathbf{F}^t \tag{2.2.4a}$$

$$\dot{\mathbf{u}}^{t+1/2} = \left(\mathbf{M}^t\right)^{-1} \left(\mathbf{F}^t - \mathbf{F}^t_{\text{int}}\right) \Delta\, t^{t+1/2} + \dot{\mathbf{u}}^{t-1/2} \tag{2.2.4b}$$

$$\mathbf{u}^{t+1} = \mathbf{u}^t + \dot{\mathbf{u}}^{t+1/2} \Delta\, t^{t+1} \tag{2.2.4c}$$

If the mass matrix $\mathbf{M}$ in Eq. (2.2.4a, b) is diagonalized (or lumped) its inversion is trivial, and the system of differential equations decouples. Its overall solution can then be performed independently and very fast for each degree of freedom. Further reductions of the computation time per increment of time stem from utilization of reduced integration schemes that are often applied even to the deviatoric parts of the stiffness matrix, and finally numerical actions related to mass scaling and load factoring contribute. Load factoring is described ahead.

Additional computational advantages result from the fact that dynamic explicit schemes, unlike quasi-static implicit schemes, do not check equilibrium requirements at the end of each increment of time. The analogy between the dynamic equilibrium equation (2.2.1) and the ideal mass-spring vibrating system allows concluding that explicit central difference time integration schemes (frequently referred as explicit integration schemes) are conditionally stable whenever the size of the increment of time Δt satisfies

$$\Delta t \leq \frac{L_e}{\sqrt{E/\rho}} = \frac{L_e}{c_e} \tag{2.2.5}$$

where L_e is the typical size of the finite elements discretizing the domain, E is the elasticity modulus and c_e is the velocity of propagation of a longitudinal wave in the material. In case of metal forming applications, the stability condition Eq. (2.2.5) requires the utilization of very small increments of time Δt, say microseconds, and millions of increments to finish a simulation because industrial metal forming processes usually take several seconds to be accomplished. This is the reason why computer programs often make use of the following numerical actions in order to increase the increment of time Δt and, consequently, reducing the overall computation time:

- Diagonalization of the mass matrix;
- Mass scaling—by increasing the density of the material and thus artificially reducing the speed c_e of the longitudinal wave;
- Load factoring—by changing the rate of loading through an artificial increase in the velocity of the tooling as compared to the real forming velocity;
- Reduced integration of the deviatoric part of the stiffness matrix, which is usually fully integrated.

The above-mentioned numerical actions can artificially add undesirable inertia effects, and it is therefore necessary to include a damping term $\mathbf{C}^t \dot{\mathbf{u}}^t$ in (2.2.3),

$$\mathbf{M}^t \ddot{\mathbf{u}}^t + \mathbf{C}^t \dot{\mathbf{u}}^t + \mathbf{F}_{\text{int}}^t = \mathbf{F}^t \tag{2.2.6}$$

The damping term $\mathbf{C}^t \dot{\mathbf{u}}^t$ is not only necessary because of the above-mentioned numerical actions to reduce the computation time but also to ensure fast convergence of the solution towards the static solution describing the actual process.

This turns dynamic explicit formulations into close resemblance with damped mass-spring vibrating systems and justifies the reason why these formulations loose efficiency whenever the material is strain-rate sensitive or thermo-mechanical phenomena need to be taken into consideration.

The main advantages of the dynamic explicit formulations are:

- Computer programs are robust and do not present convergence problems;
- The computation time depends linearly on the number of degrees of freedom while in alternative quasi-static implicit schemes the dependency is more than linear (in case of iterative solvers) and up to quadratic (in case of direct solvers).

The main drawbacks of the dynamic explicit formulation can be summarized as follows:

- Utilization of very small time increments;
- Equilibrium after each increment of time is not checked;
- Assignment of the system damping is rather arbitrary;
- The formulation needs experienced users for adequately designing the mesh and choosing the scaling parameters for mass, velocity and damping. Otherwise it may lead to inaccurate solutions for the deformation, prediction of forming defects and distribution of the major field variables within the workpiece;
- Springback calculations are very time consuming and may lead to errors. This specific problem is frequently overtaken by combining dynamic explicit with quasi-static implicit analysis.

The last two drawbacks apply if the dynamic explicit formulations are used in the “high-speed-mode”.

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