Chapter 2
Modeling of Uncertain Systems

As discussed in Chap. 1, it is well understood that uncertainties are unavoidable in a real control system. The uncertainty can be classified into two categories: disturbance signals and dynamic perturbations. The former includes input and output disturbance (such as a gust on an aircraft), sensor noise and actuator noise, etc. The latter represents the discrepancy between the mathematical model and the actual dynamics of the system in operation. A mathematical model of any real system is always just an approximation of the true, physical reality of the system dynamics. Typical sources of the discrepancy include unmodeled (usually high-frequency) dynamics, neglected nonlinearities in the modeling, effects of deliberate reduced-order models, and system-parameter variations due to environmental changes and torn-and-worn factors. These modeling errors may adversely affect the stability and performance of a control system. In this chapter, we will discuss in detail how dynamic perturbations are usually described so that they can be accounted for in system robustness analysis and design.

2.1 Unstructured Uncertainties

Many dynamic perturbations that may occur in different parts of a system can, however, be lumped into one single perturbation block Δ, for instance, some unmodeled, high-frequency dynamics. This uncertainty representation is referred to as "unstructured" uncertainty. In the case of linear, time-invariant systems, the block Δ may be represented by an unknown transfer function matrix. The unstructured dynamics uncertainty in a control system can be described in different ways, such as is listed in the following, where $G_p(s)$ denotes the actual, perturbed system dynamics and $G_o(s)$ a nominal model description of the physical system.

1. Additive perturbation (see Fig. 2.1):

$$G_p(s) = G_o(s) + \Delta(s) \tag{2.1}$$

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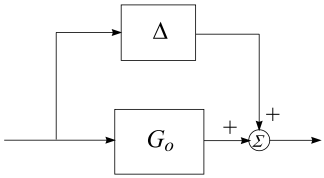

Fig. 2.1 Additive perturbation configuration

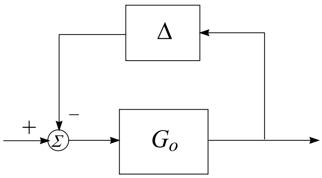

Fig. 2.2 Inverse additive perturbation configuration

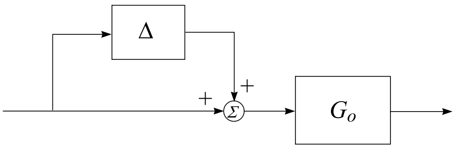

Fig. 2.3 Input multiplicative perturbation configuration

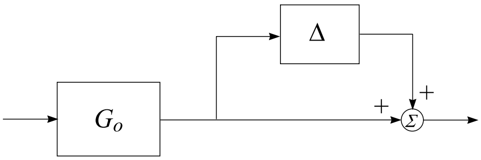

Fig. 2.4 Output multiplicative perturbation configuration

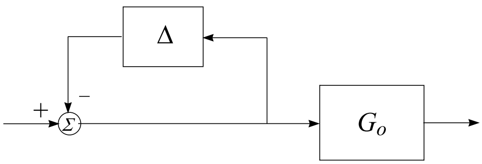

Fig. 2.5 Inverse input multiplicative perturbation configuration

2. Inverse additive perturbation (see Fig. 2.2):

$$\big(G_p(s)\big)^{-1} = \big(G_o(s)\big)^{-1} + \Delta(s) \tag{2.2}$$

3. Input multiplicative perturbation (see Fig. 2.3):

$$G_p(s) = G_o(s)\big[I + \Delta(s)\big] \tag{2.3}$$

4. Output multiplicative perturbation (see Fig. 2.4):

$$G_p(s) = \big[I + \Delta(s)\big]G_o(s) \tag{2.4}$$

5. Inverse input multiplicative perturbation (see Fig. 2.5):

$$\big(G_p(s)\big)^{-1} = \big[I + \Delta(s)\big]\big(G_o(s)\big)^{-1} \tag{2.5}$$

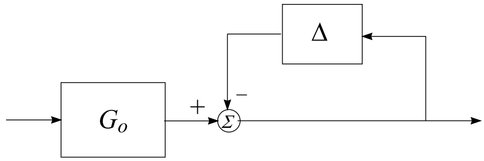

Fig. 2.6 Inverse output multiplicative perturbation configuration

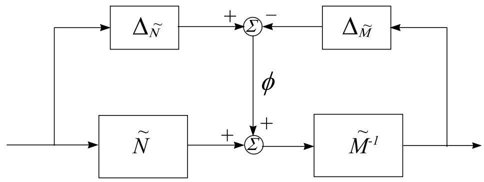

Fig. 2.7 Left coprime factor perturbations configuration

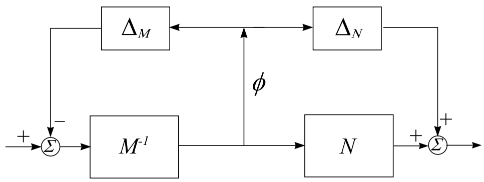

Fig. 2.8 Right coprime factor perturbations configuration

6. Inverse output multiplicative perturbation (see Fig. 2.6):

$$\big(G_p(s)\big)^{-1} = \big(G_o(s)\big)^{-1}\big[I + \Delta(s)\big] \tag{2.6}$$

7. Left coprime factor perturbations (see Fig. 2.7):

$$G_p(s) = (\tilde M + \Delta_{\tilde M})^{-1}(\tilde N + \Delta_{\tilde N}) \tag{2.7}$$

8. Right coprime factor perturbations (see Fig. 2.8):

$$G_p(s) = (N + \Delta_N)(M + \Delta_M)^{-1} \tag{2.8}$$

The additive uncertainty representations give an account of absolute error between the actual dynamics and the nominal model, while the multiplicative representations show relative errors.

In the last two representations, $(\tilde M, \tilde N)/(M, N)$ are left/right coprime factorizations of the nominal system model $G_o(s)$, respectively; and $(\Delta_{\tilde M}, \Delta_{\tilde N})/(\Delta_M, \Delta_N)$ are the perturbations on the corresponding factors [111].

The block Δ (or, $(\Delta_{\tilde M}, \Delta_{\tilde N})/(\Delta_M, \Delta_N)$ in the coprime factor perturbations cases) is uncertain, but usually is norm-bounded. It may be bounded by a known transfer function, say $\overline{\sigma}[\Delta(j\omega)] \le \delta(j\omega)$, for all frequencies ω, where δ is a known scalar function and $\overline{\sigma}[\cdot]$ denotes the largest singular value of a matrix. The uncertainty can thus be represented by a unit, norm-bounded block Δ cascaded with a scalar transfer function $\delta(s)$.

It should be noted that a successful robust control-system design would depend on, to a certain extent, an appropriate description of the perturbation considered, though theoretically most representations are interchangeable.

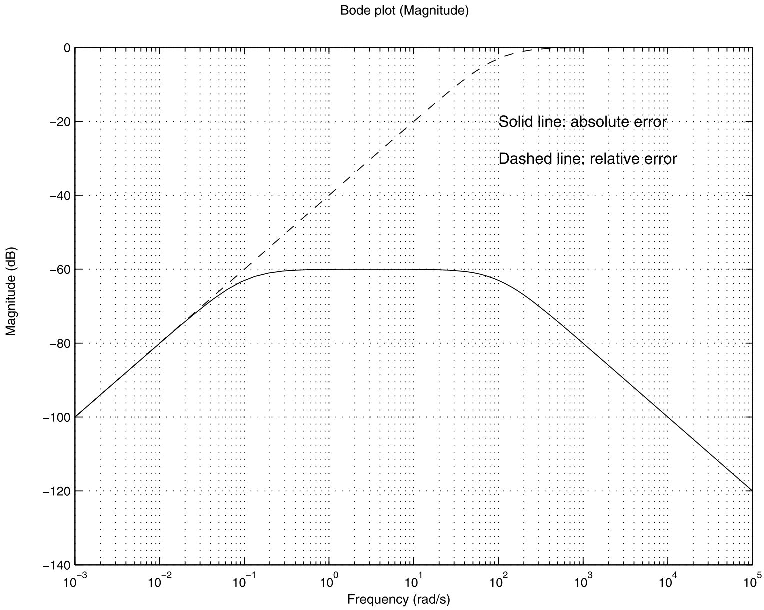

Fig. 2.9 Absolute and relative errors in Example 2.1

Example 2.1 The dynamics of many control systems may include a "slow" part and a "fast" part, for instance in a dc motor. The actual dynamics of a scalar plant may be

$$G_p(s) = g_{\text{gain}} G_{\text{slow}}(s) G_{\text{fast}}(s)$$

where g_{gain} is constant, and

$$G_{\text{slow}}(s) = \frac{1}{1+sT}; \qquad G_{\text{fast}}(s) = \frac{1}{1+\alpha sT}, \quad \alpha \ll 1$$

In the design, it may be reasonable to concentrate on the slow response part while treating the fast response dynamics as a perturbation. Let Δ_a and Δ_m denote the additive and multiplicative perturbations, respectively. It can easily be worked out that

$$\begin{aligned}\Delta_a(s) &= G_p - g_{\text{gain}} G_{\text{slow}} = g_{\text{gain}} G_{\text{slow}}(G_{\text{fast}} - 1) \\ &= g_{\text{gain}} \frac{-\alpha sT}{(1+sT)(1+\alpha sT)} \\ \Delta_m(s) &= \frac{G_p - g_{\text{gain}} G_{\text{slow}}}{g_{\text{gain}} G_{\text{slow}}} = G_{\text{fast}} - 1 = \frac{-\alpha sT}{1+\alpha sT}\end{aligned}$$

The magnitude Bode plots of Δ_a and Δ_m can be seen in Fig. 2.9, where g_{gain} is assumed to be 1. The difference between the two perturbation representations is obvious: though the magnitude of the absolute error may be small, the relative error can be large in the high-frequency range in comparison to that of the nominal plant.

2.2 Parametric Uncertainty

The unstructured uncertainty representations discussed in Sect. 2.1 are useful in describing unmodeled or neglected system dynamics. These complex uncertainties usually occur in the high-frequency range and may include unmodeled lags (time delay), parasitic coupling, hysteresis, and other nonlinearities. However, dynamic perturbations in many industrial control systems may also be caused by inaccurate description of component characteristics, torn-and-worn effects on plant components, or shifting of operating points, etc. Such perturbations may be represented by variations of certain system parameters over some possible value ranges (complex or real). They affect the low-frequency range performance and are called "parametric uncertainties".

Example 2.2 A mass–spring–damper system can be described by the following second order, ordinary differential equation:

$$m\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+c\frac{\mathrm{d}x(t)}{\mathrm{d}t}+kx(t)=f(t)$$

where m is the mass, c the damping constant, k the spring stiffness, $x(t)$ the displacement and $f(t)$ the external force. For imprecisely known parameter values, the dynamic behavior of such a system is actually described by

$$(m_o+\delta_m)\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}+(c_o+\delta_c)\frac{\mathrm{d}x(t)}{\mathrm{d}t}+(k_o+\delta_k)x(t)=f(t) \tag{2.9}$$

where m_o, c_o, and k_o denote the nominal parameter values and δ_m, δ_c and δ_k possible variations over certain ranges.

By defining the state variables x_1 and x_2 as the displacement variable and its first order derivative (velocity), the second order differential equation (2.9) may be rewritten into a standard state-space form

$$\begin{aligned}\dot{x_1}&=x_2\\ \dot{x_2}&=\frac{1}{m_o+\delta_m}\big[-(k_o+\delta_k)x_1-(c_o+\delta_c)x_2+f\big]\\ y&=x_1\end{aligned}$$

Further, the system can be represented by an analog block diagram as in Fig. 2.10.

Notice that $\frac{1}{m_o+\delta_m}$ can be rearranged as a feedback in terms of $\frac{1}{m_o}$ and δ_m. Figure 2.10 can be redrawn as in Fig. 2.11, by pulling out all the uncertain variations.

Let z_1, z_2, and z_3 be $\dot{x_2}$, x_2, and x_1, respectively, considered as another, fictitious output vector; and, d_1, d_2, and d_3 be the signals coming out from the perturbation blocks δ_m, δ_c, and δ_k, as shown in Fig. 2.11. The perturbed system can be arranged in the following state-space model and represented as in Fig. 2.12:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 1\\ -\frac{k_o}{m_o} & -\frac{c_o}{m_o}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ -1 & -1 & -1\end{bmatrix}\begin{bmatrix}d_1\\ d_2\\ d_3\end{bmatrix}+\begin{bmatrix}0\\ \frac{1}{m_o}\end{bmatrix}f$$

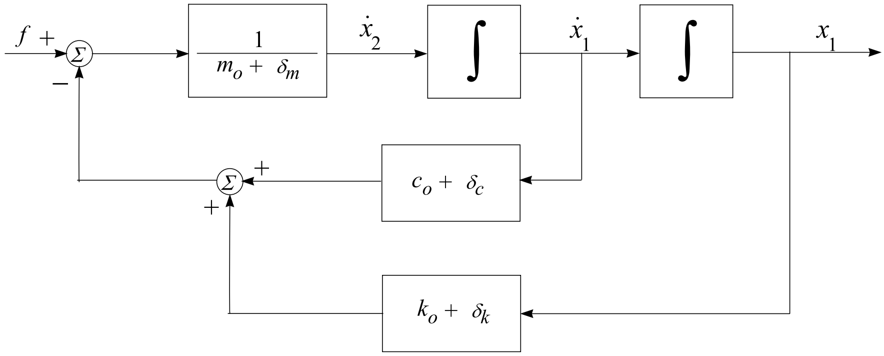

Fig. 2.10 Analog block diagram of Example 2.2

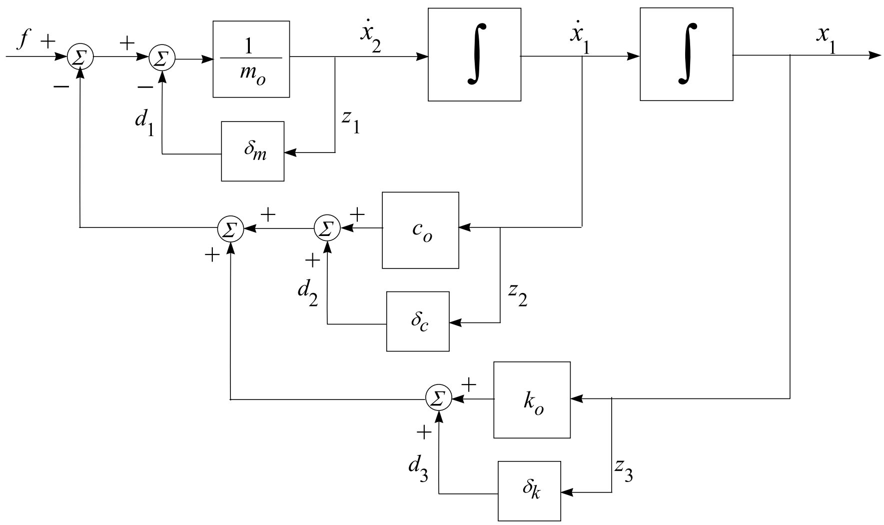

Fig. 2.11 Structured uncertainties block diagram of Example 2.2

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} -\frac{k_o}{m_o} & -\frac{c_o}{m_o} \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{m_o} \\ 0 \\ 0 \end{bmatrix} f \quad (2.10)$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The state-space model of (2.10) describes the augmented, interconnection system M of Fig. 2.12. The perturbation block Δ in Fig. 2.12 corresponds to parameter variations and is called “parametric uncertainty”. The uncertain block Δ is not a full matrix but a diagonal one. It has certain structure, hence the terminology of “structured uncertainty”. More general cases will be discussed shortly in Sect. 2.4.

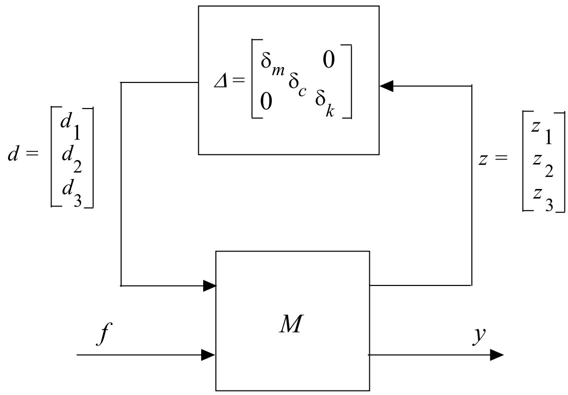

Fig. 2.12 Standard configuration of Example 2.2

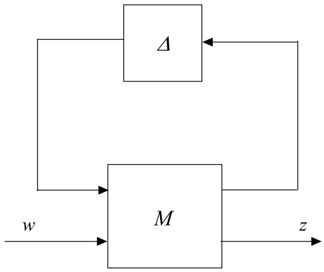

Fig. 2.13 Standard M–Δ configuration

2.3 Linear Fractional Transformations

The block diagram in Fig. 2.12 can be generalized to be a standard configuration to represent how the uncertainty affects the input/output relationship of the control system under study. This kind of representation first appeared in the circuit analysis back in the 1950s [140, 141]. It was later adopted in the robust control study [145] for uncertainty modeling. The general framework is depicted in Fig. 2.13.

The interconnection transfer function matrix M in Fig. 2.13 is partitioned as

$$M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$$

where the dimensions of M_{11} conform with those of Δ. By routine manipulations, it can be derived that

$$z = \left[M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12} \right]$$

if $(I - M_{11}\Delta)$ is invertible. When the inverse exists, we may define

$$F(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1} M_{12}$$

$F(M, \Delta)$ is called a *linear fractional transformation* (LFT) of M and Δ. Because the "upper" loop of M is closed by the block Δ, this kind of linear fractional transformation is also called an *upper linear fractional transformation* (ULFT), and denoted with a subscript u, i.e. $F_u(M, \Delta)$, to show the way of connection. Similarly,

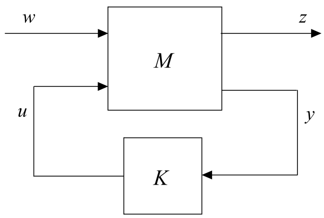

Fig. 2.14 Lower LFT configuration

there are also *lower linear fractional transformations* (LLFT) that are usually used to indicate the incorporation of a controller K into a system. Such a lower LFT can be depicted as in Fig. 2.14 and defined by

$$F_l(M, K) = M_{11} + M_{12}K(I - M_{22}K)^{-1}M_{21}$$

With the introduction of linear fractional transformations, the unstructured uncertainty representations discussed in Sect. 2.1 may be uniformly described by Fig. 2.13, with appropriately defined interconnection matrices Ms as listed below.

1. Additive perturbation:

$$M = \begin{bmatrix} 0 & I \\ I & G_o \end{bmatrix} \tag{2.11}$$

2. Inverse additive perturbation:

$$M = \begin{bmatrix} -G_o & G_o \\ -G_o & G_o \end{bmatrix} \tag{2.12}$$

3. Input multiplicative perturbation:

$$M = \begin{bmatrix} 0 & I \\ G_o & G_o \end{bmatrix} \tag{2.13}$$

4. Output multiplicative perturbation:

$$M = \begin{bmatrix} 0 & G_o \\ I & G_o \end{bmatrix} \tag{2.14}$$

5. Inverse input multiplicative perturbation:

$$M = \begin{bmatrix} -I & I \\ -G_o & G_o \end{bmatrix} \tag{2.15}$$

6. Inverse output multiplicative perturbation:

$$M = \begin{bmatrix} -I & G_o \\ -I & G_o \end{bmatrix} \tag{2.16}$$

7. Left coprime factor perturbations:

$$M = \begin{bmatrix} \begin{bmatrix} -\tilde{M}_G^{-1} \\ 0 \end{bmatrix} & \begin{bmatrix} -G_o \\ I \end{bmatrix} \\ \tilde{M}_G^{-1} & G_o \end{bmatrix} \tag{2.17}$$

where $G_o = \tilde{M}_G^{-1}\tilde{N}_G$, a left coprime factorization of the nominal plant; and, the perturbed plant is $G_p = (\tilde{M}_G + \Delta_{\tilde{M}})^{-1}(\tilde{N}_G + \Delta_{\tilde{N}})$.

8. Right coprime factor perturbations:

$$M = \begin{bmatrix} \begin{bmatrix} -M_G^{-1} & 0 \end{bmatrix} & M_G^{-1} \\ \begin{bmatrix} -G_o & I \end{bmatrix} & G_o \end{bmatrix} \tag{2.18}$$

where $G_o = N_G M_G{}^{-1}$, a right coprime factorization of the nominal plant; and, the perturbed plant is $G_p = (N_G + \Delta_N)(M_G + \Delta_M)^{-1}$.

In the above, it is assumed that $[I - M_{11}\Delta]$ is invertible. The perturbed system is thus

$$G_p(s) = F_u(M, \Delta)$$

In the coprime factor perturbation representations, (2.17) and (2.18), $\Delta = [\,\Delta_{\tilde{M}}\ \Delta_{\tilde{N}}\,]$ and $\Delta = \left[\begin{smallmatrix}\Delta_M \\ \Delta_N\end{smallmatrix}\right]$, respectively. The block Δ in (2.11)–(2.18) is supposed to be a "full" matrix, i.e. it has no specific *structure*.

2.4 Structured Uncertainties

In many robust design problems, it is more likely that the uncertainty scenario is a mixed case of those described in Sects. 2.1 and 2.2. The uncertainties under consideration would include unstructured uncertainties, such as unmodeled dynamics, as well as parameter variations. All these uncertain parts still can be taken out from the dynamics and the whole system can be rearranged in a standard configuration of (upper) linear fractional transformation $F(M, \Delta)$. The uncertain block Δ would then have the following general form:

$$\Delta = \mathrm{diag}[\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}, \Delta_1, \ldots, \Delta_f], \quad \delta_i \in \mathcal{C}, \Delta_j \in \mathcal{C}^{m_j \times m_j} \tag{2.19}$$

where $\sum_{i=1}^{s} r_i + \sum_{j=1}^{f} m_j = n$ with n is the dimension of the block Δ. We may define the set of such Δ as $\mathbf{\Delta}$. The total block Δ thus has two types of uncertain block: s repeated *scalar* blocks and f *full* blocks. The parameters δ_i of the repeated scalar blocks can be real numbers only, if further information of the uncertainties is available. However, in the case of real numbers, the analysis and design would be even harder. The full blocks in (2.19) need not be square, but by restricting them as such makes the notation much simpler.

When a perturbed system is described by an LFT with the uncertain block of (2.19), the Δ considered has a certain structure. It is thus called "structured uncertainty". Apparently, using a lumped, full block to model the uncertainty in such cases, for instance in Example 2.2, would lead to pessimistic analysis of the system behavior and produce conservative designs.

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