# Chapter 2
# Geometric Fundamentals

In this chapter we lay the geometric foundations that will serve as a basis for the topics that we shall meet later. The statements of *projective geometry*, in contrast to those of affine geometry, often allow a particularly simple formulation. The projective equivalence of polytopes and pointed polyhedra (Theorem 3.36) and Bézout's Theorem (Theorem 8.27) on the number of intersections of two algebraic curves in the plane are good examples of this. We will also introduce the notion of *convexity*, which is an irreplaceable concept in linear computational geometry.

## 2.1 Projective Spaces

The basic motivation behind the introduction of projective spaces comes from the examination of two distinct lines in an arbitrary affine plane, for example the Euclidean plane $\mathbb{R}^2$. The lines either intersect or are parallel to one another. The fundamental idea of projective geometry is to extend the affine plane so that parallel lines have an intersection point at "infinity".

For the remainder of this text, let $K$ be an arbitrary field and for any subset $A$ of a vector space $V$, let $\operatorname{lin} A$ denote the linear hull of $A$. The cases $K = \mathbb{R}$ and $K = \mathbb{C}$ are of primary interest in this book.

**Definition 2.1**

(i) Let $V$ be a finite dimensional vector space over $K$. The *projective space* $P(V)$ induced by $V$ is the set of one-dimensional subspaces of $V$. The dimension of $P(V)$ is defined as $\dim P(V) = \dim V - 1$. The function which maps a vector $v \in V \setminus \{0\}$ to the one-dimensional linear subspace $\operatorname{lin} v$ is called the *canonical projection*.

(ii) For any natural number $n$, the set $P(K^{n+1})$ is called the *$n$-dimensional projective space over $K$*. We denote it by $\mathbb{P}^n_K$ and remove the lower index $K$ if the coordinate field is clear from the context.

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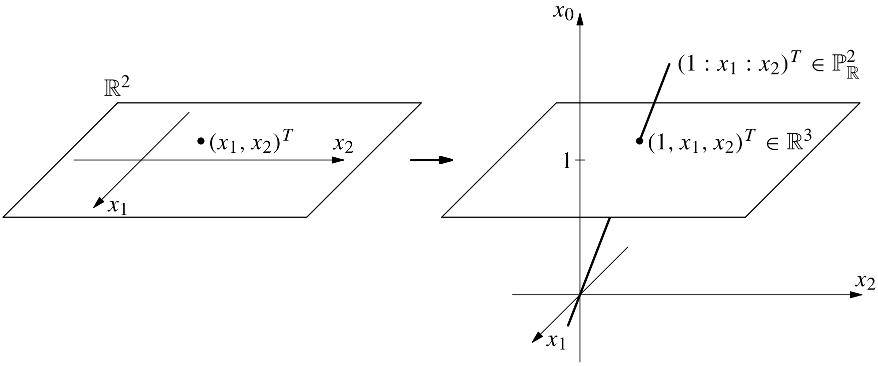


**Fig. 2.1** Embedding the Euclidean plane $\mathbb{R}^2$ into the real projective plane $\mathbb{P}^2_{\mathbb{R}}$

A one-dimensional linear subspace $U$ of $V$ is generated by an arbitrary non-zero vector $u \in U$. Thus, we can identify the projective space with the set of equivalence classes of the equivalence relation $\sim$ on $V \setminus \{0\}$, where $x \sim y$ if and only if there exists a $\lambda \in K \setminus \{0\}$ such that $x = \lambda y$.

**Definition 2.2** Let $(x_0, \dots, x_n)^T \in K^{n+1} \setminus \{0\}$ be a vector. Then $x := \operatorname{lin}\{(x_0, \dots, x_n)^T\} \in \mathbb{P}^n$. We call any element of $x \setminus \{0\}$ *homogeneous coordinates* of $x$ and write $x = (x_0 : \dots : x_n)^T$, with $(x_0 : \dots : x_n)^T = (y_0 : \dots : y_n)^T$ if and only if $(x_0, \dots, x_n)^T \sim (y_0, \dots, y_n)^T$, i.e., if there exists a $\lambda \in K \setminus \{0\}$ such that $x_i = \lambda y_i$ for $0 \le i \le n$.

We can embed the affine space $K^n$ in the projective space $\mathbb{P}^n_K$ via the injection:

$$\iota : K^n \to \mathbb{P}^n_K, \quad (x_1, \dots, x_n)^T \mapsto (1 : x_1 : \dots : x_n)^T. \tag{2.1}$$

Figure 2.1 illustrates the embedding of the Euclidean plane into the real projective plane.

The set of *ideal points* of $\mathbb{P}^n_K$ is

$$\mathbb{P}^n \setminus \iota\big(K^n\big) = \big\{(x_0 : x_1 : \dots : x_n)^T \in \mathbb{P}^n \, : \, x_0 = 0\big\}.$$

**Definition 2.3** Every subspace $U$ of a vector space $V$ defines a *projective subspace* $P(U) = \{\operatorname{lin}(u) : u \in U \setminus \{0\}\}$.

Therefore, the set of (non-empty) projective subspaces of a projective space $P(V)$ is in one-to-one correspondence with the (non-zero) linear subspaces of $V$. The set of ideal points of $\mathbb{P}^n_K$ forms a subspace of dimension $n - 1$. Also, $\operatorname{lin} \emptyset = \{0\}$ and $P(\{0\}) = \emptyset$.

Projective subspaces of dimension 0, 1 and 2 are called *points*, *lines* and *planes*, as usual. Projective subspaces of dimension $n - 1$ (i.e., codimension 1) are called *hyperplanes*. The embedding $\iota(U)$ of a $k$-dimensional subspace $U$ of $K^n$ produces a $k$-dimensional projective subspace called the *projective closure* of $U$.

*Example 2.4* Consider the projective plane $\mathbb{P}^2_K$. The projective lines of this space correspond to the two-dimensional subspaces of $K^3$. Since the intersection of any two distinct two-dimensional subspaces of $K^3$ is always one-dimensional, any two distinct lines of the projective plane have a uniquely determined intersection point.

Conversely, given any two distinct projective points there exists one unique projective line incident with both. This follows directly from the fact that the linear hull of two distinct one-dimensional subspaces of a vector space is two-dimensional.

The extension of the affine space $K^n$ to the projective space $\mathbb{P}^n_K$ simplifies many proofs by eliminating case distinctions. In the particularly interesting cases $K = \mathbb{R}$ and $K = \mathbb{C}$, the field $K$ has a locally compact (and connected) topology, inducing the product topology on $K^n$. This topology has a natural extension to the point sets $\mathbb{P}^n_{\mathbb{R}}$ and $\mathbb{P}^n_{\mathbb{C}}$ as a compactification. See Exercise 2.19.

Every hyperplane $H$ in $\mathbb{P}^n_K$ can be expressed as the kernel of a non-trivial *linear form*, that is, a $K$-linear map

$$\phi : K^{n+1} \to K, \quad x = (x_0 : \cdots : x_n)^T \mapsto u_0 x_0 + \cdots + u_n x_n \tag{2.2}$$

where the coefficients $u_0, \ldots, u_n \in K$ are not all zero. The set of all $K$-linear forms on $K^{n+1}$ yields the *dual space* $(K^{n+1})^*$. Pointwise addition and scalar multiplication turns the dual space into a vector space over $K$. The map $\phi$ defined in (2.2) is identified with the row vector $u = (u_0, \ldots, u_n)$. Clearly, every hyperplane uniquely defines the vector $u \neq 0$ up to a non-zero scalar and vice versa. In other words: hyperplanes can also be expressed in terms of homogeneous coordinates, and we simply write $H = \ker \phi = [u_0 : \cdots : u_n]$.

The following proposition shows how hyperplanes can be expressed with the help of the *inner product*

$$\langle \cdot, \cdot \rangle : K^{n+1} \times K^{n+1} \to K, \quad \langle x, y \rangle := x_0 y_0 + x_1 y_1 + \cdots + x_n y_n \tag{2.3}$$

on $K^{n+1}$. For $x \in K^{n+1}$ and $u \in (K^{n+1})^*$, we write

$$u(x) = u \cdot x = \langle x, u^T \rangle$$

where "$\cdot$" denotes standard matrix multiplication.

**Proposition 2.5** *The projective point $x = (x_0 : \cdots : x_n)^T$ lies in the projective hyperplane $u = [u_0 : \cdots : u_n]$ if and only if $\langle x, u^T \rangle = 0$.*

*Proof* Notice that the condition $\langle x, u^T \rangle = 0$ makes sense in homogeneous coordinates since it is homogeneous itself. The claim follows from the equation

$$\langle (\lambda x_0, \ldots, \lambda x_n)^T, (\mu u_0, \ldots, \mu u_n)^T \rangle = \lambda\mu (x_0 u_0 + \cdots + x_n u_n) = \lambda\mu \langle x, u^T \rangle$$

for every $\lambda, \mu \in K$. □

At the end of the book, in Theorem 12.24, we will prove a far-reaching generalization of Proposition 2.5.

*Example 2.6* As in Example 2.4, consider the affine plane $K^2$ and its projective closure, the projective plane $\mathbb{P}^2_K$. We can use the homogeneous coordinates to represent a projective line of $\mathbb{P}^2_K$. For $a, b, c \in K$ with $(b, c) \neq (0, 0)$ let

$$\ell = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in K^2 : a + bx + cy = 0 \right\}$$

be an arbitrary affine line. Then the projective line $[a : b : c]$ is the projective closure of $\ell$. It contains exactly one extra projective point that is not the image of an affine point of the embedding $\iota$. This point is the ideal point of $\ell$ and has the homogeneous coordinates $(0 : c : -b)$.

The homogeneous coordinates of every line of $K^2$ parallel to $\ell$ differ only in $a$, their first coordinate (in the projective closure). Therefore, they share the same point at infinity. All ideal points lie on the unique projective line $[1 : 0 : 0]$, which is not the projective closure of any affine line. This line is called the *ideal line*.

Ideal points in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ are often called *points at infinity* in the literature. The idea of two parallel lines “intersecting at infinity” means that the projective closures of two parallel lines in $\mathbb{R}^2$ intersect at the same ideal point of $\mathbb{P}^2_{\mathbb{R}}$.

## 2.2 Projective Transformations

A *linear transformation* is a vector space automorphism, i.e., a bijective linear map from a vector space to itself. Since projective spaces are defined in terms of vector space quotients, linear transformations induce maps between the associated projective spaces.

More precisely, let $V$ be a finite dimensional $K$-vector space and $f : V \to V$ a $K$-linear transformation. For $v \in V \setminus \{0\}$ and $\lambda \in K$ we have $f(\lambda v) = \lambda f(v)$ and therefore $f(\operatorname{lin}(v)) = \operatorname{lin}(f(v))$. As $f$ is bijective, non-zero vectors are mapped to non-zero vectors. Hence $f$ induces a *projective transformation*:

$$P(f) : P(V) \to P(V), \quad \operatorname{lin}(v) \mapsto \operatorname{lin}\big(f(v)\big).$$

For $V = K^{n+1}$, the map $f$ is usually described by a matrix $A \in \mathrm{GL}_{n+1} K$. We will therefore use the notation $[A] := P(f)$ for projective transformations. Let $P(V)$ be an $n$-dimensional projective space. A *flag* of length $k$ is a sequence of projective subspaces $(U_1, \ldots, U_k)$ with $U_1 \subsetneq U_2 \subsetneq \cdots \subsetneq U_k$. The maximal length of a flag is $n + 2$. Every *maximal flag* begins with the empty set and ends with the entire space $P(V)$.

**Theorem 2.7** *Let $P(V)$ be a finite dimensional projective space with two maximal flags $(U_0, \ldots, U_{n+1})$ and $(W_0, \ldots, W_{n+1})$. Then there exists a projective transformation $\pi : P(V) \to P(V)$ with $\pi(U_i) = W_i$.*

*Proof* Since the subspace $U_i$ is strictly larger than $U_{i-1}$, we can pick vectors $u^{(i)} \in U_i \setminus U_{i-1}$ for $i \in \{1, \dots, n+1\}$. By construction $u^{(i)}$ is linearly independent of $u^{(1)}, \dots, u^{(i-1)}$, and therefore $(u^{(1)}, \dots, u^{(n+1)})$ is a basis of $V$. Similarly we obtain a second basis $(w^{(1)}, \dots, w^{(n+1)})$ from the second maximal flag $(W_0, \dots, W_{n+1})$.

From linear algebra we know that there exists a unique invertible linear map $f : V \to V$ that maps $u^{(i)}$ to $w^{(i)}$ for all $i \in \{1, \dots, n+1\}$. Therefore $\pi := P(f)$ is a projective transformation with the properties stated in the theorem. □

An equivalent formulation of the above statement is: The group of invertible linear maps $\mathrm{GL}(V)$ operates *transitively* on the maximal flags of $P(V)$.

For a not necessarily maximal flag $\mathcal{F} = (V_1, \dots, V_k)$ we call the strictly monotone sequence of natural numbers $(\dim_K V_1, \dots, \dim_K V_k)$ the *type* of $\mathcal{F}$.

**Corollary 2.8** *Let $(U_1, \dots, U_k)$ and $(W_1, \dots, W_k)$ be two flags of $P(V)$ with the same types. Then there exists a projective transformation $\pi$ on $P(V)$ with $\pi(U_i) = W_i$.*

*Proof* Both $(U_1, \dots, U_k)$ and $(W_1, \dots, W_k)$ can be extended to maximal flags. Thus the statement follows from Theorem 2.7. □

One may think that the uniqueness of the linear transformation $f$ in the proof of Theorem 2.7 implies the uniqueness of $\pi = P(f)$. Showing that this is generally not true is the goal of the exercise below. First we clarify some terminology: A point set $M \subseteq \mathbb{P}^n$ is called *collinear* if there exists a projective line that contains all points of $M$. A quadruple $(a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)})$ of points of $\mathbb{P}^2$ is called a *quadrangle* if no subset of three points is collinear.

**Exercise 2.9** For any two quadrangles $(a^{(1)}, a^{(2)}, a^{(3)}, a^{(4)})$ and $(b^{(1)}, b^{(2)}, b^{(3)}, b^{(4)})$ there exists a projective transformation $\pi$ of $\mathbb{P}^2$ with $\pi(a^{(i)}) = b^{(i)}$ for $1 \le i \le 4$.

An *affine transformation* is a projective transformation that maps ideal points to ideal points.

**Exercise 2.10** For every affine transformation $\pi$ of $\mathbb{P}^n_K$ there exists a linear transformation $A \in \mathrm{GL}_n(K)$ and a vector $v \in K^n$ such that $\pi(\iota(x)) = \iota(Ax + v)$ for all $x \in K^n$.

## 2.3 Convexity

We begin by summarizing some notation from linear algebra to clarify the terminology and concepts that we will use. As before, let $K$ denote a field.

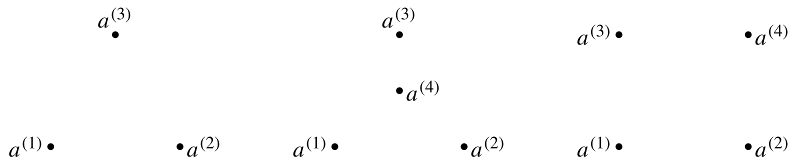


**Fig. 2.2** Affinely independent points (*left*) and affinely dependent points (*middle* and *right*) in the Euclidean plane $\mathbb{R}^2$

**Definition 2.11** Let $A \subseteq K^n$. An *affine combination* of points in $A$ is a linear combination $\sum_{i=1}^{m} \lambda^{(i)} a^{(i)}$ with $m \geq 1$, $\lambda^{(1)}, \ldots, \lambda^{(m)} \in K$, $a^{(1)}, \ldots, a^{(m)} \in A$ and $\sum_{i=1}^{m} \lambda^{(i)} = 1$. The set of all affine combinations of $A$ is called the *affine hull* of $A$ or simply aff $A$. We call the points $a^{(1)}, \ldots, a^{(m)} \in K^n$ *affinely independent* if they generate an affine subspace of dimension $m - 1$.

For example, the three points in the picture on the left hand side of Fig. 2.2 are affinely independent and each set of four or more points in the real plane (as in the middle and on the right hand side of Fig. 2.2) are affinely dependent. We set $\operatorname{aff} \emptyset = \emptyset$ and $\dim \emptyset = -1$.

The language of projective geometry allows us to describe linear algebra over an arbitrary field in geometric terms. In the case of an ordered field like the real numbers (and unlike $\mathbb{C}$) we can further exploit the geometry to obtain results. For the remaining part of this chapter, let $K$ be the field $\mathbb{R}$ of real numbers.

**Definition 2.12** Let $A \subseteq \mathbb{R}^n$. A *convex combination* of $A$ is an affine combination $\sum_{i=1}^{m} \lambda^{(i)} a^{(i)}$ which additionally satisfies $\lambda^{(1)}, \ldots, \lambda^{(m)} \geq 0$. The set conv $A$ of all convex combinations of $A$ is called the *convex hull* of $A$. A set $C \subseteq \mathbb{R}^n$ is called *convex* if it contains all convex combinations that can be obtained from it. The *dimension* of a convex set is the dimension of its affine hull.

The empty set is convex by definition. The simplest non-trivial example of a convex set is the closed interval $[a, b] \subseteq \mathbb{R}$. It is one-dimensional and is the convex hull of its end points. Analogously, for $a, b \in \mathbb{R}^n$ we define:

$$[a, b] := \left\{ \lambda a + (1 - \lambda) b : 0 \leq \lambda \leq 1 \right\} = \operatorname{conv}\{a, b\}.$$

See Fig. 2.3 for some examples.

**Exercise 2.13** A set $C \subseteq \mathbb{R}^n$ is convex if and only if for every two points $x, y \in C$, the segment $[x, y]$ is contained in $C$.

### *2.3.1 Orientation of Affine Hyperplanes*

For real numbers $a_0, a_1, \ldots, a_n$ with $(a_1, \ldots, a_n) \neq 0$ consider the affine hyperplane $H = \{x \in \mathbb{R}^n : a_0 + a_1 x_1 + \cdots + a_n x_n = 0\}$. Then $[a_0 : a_1 : \cdots : a_n]$ are the

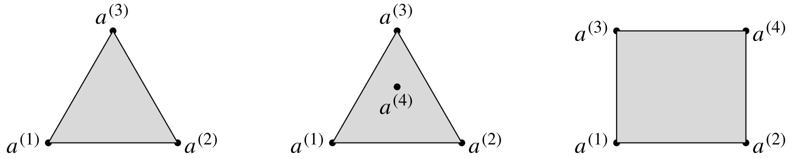


**Fig. 2.3** Convex hulls of the points from Fig. 2.2

homogeneous coordinates of its projective closure. The complement $\mathbb{R}^n \setminus H$ has two connected components,

$$H_\circ^+ := \{x \in \mathbb{R}^n : a_0 + a_1x_1 + \cdots + a_nx_n > 0\} \quad \text{and} \tag{2.4}$$

$$H_\circ^- := \{x \in \mathbb{R}^n : a_0 + a_1x_1 + \cdots + a_nx_n < 0\}. \tag{2.5}$$

These components are called the *open affine half-spaces* defined by $H$, with $H_\circ^+$ and $H_\circ^-$ attributed as *positive* and *negative*, respectively. The (*closed*) *positive half-space*

$$H^+ := \{x \in \mathbb{R}^n : a_0 + a_1x_1 + \cdots + a_nx_n \geq 0\}$$

satisfies $H^+ = H \cup H_\circ^+ = \mathbb{R}^n \setminus H_\circ^-$. The opposite half-space $H^-$ is analogously defined. The vector $(\lambda a_0, \lambda a_1, \ldots, \lambda a_n)$ defines the same affine hyperplane $H$ for any $\lambda \neq 0$, however the roles of $H^+$ and $H^-$ are reversed when $\lambda$ is negative. We will let

$$[a_0 : a_1 : \cdots : a_n]^+ := \{x \in \mathbb{R}^n : a_0 + a_1x_1 + \cdots + a_nx_n \geq 0\}$$

and analogously define $[a_0 : a_1 : \cdots : a_n]^-$. When we wish to distinguish which of the two half-spaces defined by $H$ is positive or negative, we will call $[a_0 : a_1 : \cdots : a_n]$ the *oriented homogeneous coordinates* of $H$.

We often consider a given affine hyperplane $H$ in $\mathbb{R}^n$ and use the notation $H^+$ and $H^-$ without having first fixed a coordinate representation of $H$. This is simply a notational device which enables us to differentiate between the two half-spaces; the coordinates for $H$ can always be chosen so that the notation is in accordance with the above definition.

The inner product introduced in (2.3) is the *Euclidean scalar product* on $\mathbb{R}^n$. As in Proposition 2.5 the sign of the scalar product

$$\left\langle (1, x_1, \ldots, x_n)^T, (a_0, a_1, \ldots, a_n)^T \right\rangle$$

denotes the half-space for $[a_0 : a_1 : \cdots : a_n]$ in which the point $(1, x_1, \ldots, x_n)^T$ lies.

## *2.3.2 Separation Theorems*

For $M \subseteq \mathbb{R}^n$, we let $\operatorname{int} M$ denote the *interior* of $M$. That is, the set of points $p \in M$ for which there exists an $\epsilon$-ball centered at $p$, completely contained in $M$. A set is called *open* when $\operatorname{int} M = M$ and is *closed* if it is the complement of an open set. The *closure* $\overline{M}$ of $M$ is the smallest closed set in $\mathbb{R}^n$ containing $M$. The set $\partial M := \overline{M} \setminus \operatorname{int} M$ is the *boundary* of $M$. All of these terms are defined with respect to the ambient space $\mathbb{R}^n$.

Some concepts from analysis are essential for the structure theory of convex sets. The following statements rely on two core results which are proved in Appendix B. Here, an affine hyperplane $H$ is called a *supporting hyperplane* for a convex set $C \subseteq \mathbb{R}^n$ if $H \cap C \neq \emptyset$ and $C$ is entirely contained in one of the closed affine half-spaces determined by $H$.

**Theorem 2.14** *Let $C$ be a closed and convex subset of $\mathbb{R}^n$ and $p \in \mathbb{R}^n \setminus C$ an exterior point. Then there exists an affine hyperplane $H$ with $C \subseteq H^+$ and $p \in H^-$, that meets neither $C$ nor $p$.*

The next statement is a direct consequence of Theorem 2.14.

**Corollary 2.15** *Let $C$ be a closed and convex subset of $\mathbb{R}^n$. Then every point of the boundary $\partial C$ is contained in a supporting hyperplane.*

A convex set $C \subseteq \mathbb{R}^n$ is called *full-dimensional* if $\dim C = n$. When $C$ is not full dimensional, it is often useful to use these topological concepts with respect to the affine hull. The *relative interior* $\operatorname{relint} C$ of a convex set $C$ consists of the interior points of $C$ interpreted as a subset of $\operatorname{aff} C$. Analogously, the *relative boundary* of $C$ is the boundary of $C$ as a subset of $\operatorname{aff} C$.

## 2.4 Exercises

**Exercise 2.16** Let $P(V)$ be a projective space. For every set $S \subseteq V$ the set

$$T = \big\{\operatorname{lin}\{x\} : x \in S \setminus \{0\}\big\}$$

is a subset of $P(V)$ and for the subspace $\operatorname{lin} S$ generated by $S$, $P(\operatorname{lin} S)$ is a projective subspace which we denote by $\langle T \rangle$. Prove the dimension formula

$$\dim U + \dim W = \dim\big(\langle U \cup W \rangle\big) + \dim(U \cap W)$$

for two arbitrary projective subspaces $U$ and $W$ of $P(V)$.

**Exercise 2.17** Let $K$ be any field, and let $A = (a_{ij}) \in \mathrm{GL}_{n+1} K$. Show:

(a) If $H$ is a projective hyperplane with homogeneous coordinates $(h_0 : h_1 : \cdots : h_n)$ then the image $[A]H$ under the projective transformation $[A]$ is the kernel of the linear form with coefficients $(h_0, h_1, \ldots, h_n)A^{-1}$.
(b) The projective transformation $[A]$ acting on $\mathbb{P}^n_K$ is affine if and only if $a_{12} = a_{13} = \cdots = a_{1,n+1} = 0$.

**Exercise 2.18**

(a) Every projective transformation on the real projective line $\mathbb{P}^1_{\mathbb{R}}$ (apart from the identity) has at most two fixed points.
(b) Every projective transformation on the complex projective line $\mathbb{P}^1_{\mathbb{C}}$ (apart from the identity) has at least one and at most two fixed points. (Explain why it is natural to talk about a double fixed point in the first case.)

A projective space over a topological field has a natural topology that will be discussed in the following exercise.

**Exercise 2.19** Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Show:

(a) The point set of a projective space $\mathbb{P}^n_{\mathbb{K}} = \mathbb{K}^{n+1}/\sim$ is compact with respect to the quotient topology.
(b) Every projective subspace of $\mathbb{P}^n_{\mathbb{K}}$, interpreted as a subset of the points of $\mathbb{P}^n_{\mathbb{K}}$, is compact.

**Exercise 2.20** Let $K$ be a finite field with $q$ elements.

(a) Show that the projective plane $\mathbb{P}^2_K$ has exactly $N := q^2 + q + 1$ points and equally many lines.
(b) Denote by $p^{(1)}, \ldots, p^{(N)}$ the points and by $\ell_1, \ldots, \ell_N$ the lines of $\mathbb{P}^2_K$. Furthermore, let $A \in \mathbb{R}^{N \times N}$ be the *incidence matrix* defined by

$$a_{ij} = \begin{cases} 1 & \text{if } p^{(i)} \text{ lies on } \ell_j, \\ 0 & \text{otherwise.} \end{cases}$$

Compute the absolute value of the determinant of $A$. [*Hint*: Study the matrix $A \cdot A^T$.]

**Exercise 2.21** (Carathéodory's Theorem) If $A \subseteq \mathbb{R}^n$ and $x \in \operatorname{conv} A$, then $x$ can be written as a convex combination of at most $n+1$ points in $A$. [*Hint*: Since $m \geq n+2$ points are affinely dependent, every convex combination of $m$ points in $A$ can be written as a convex combination of $m-1$ points.]

## 2.5 Remarks

For further material on projective geometry, refer to the books of Beutelspacher and Rosenbaum [13] and Richter-Gebert [88]. More detailed descriptions of convexity can be found in Grünbaum [56, §2], Webster [98] or Gruber [55]. For basic topological concepts, see the books of Crossley [30] and Hatcher [58]. Although our projective transformations are by definition always linearly induced, in other texts it is common to extend this notion to include collineations induced by field automorphisms.

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