

Preface

The famous old road from Vézelay in Burgundy to Compostela in Spain is a long one, and very few pilgrims walk the entire route. Yet every year there are those who follow some part of it. We do not expect that many readers of this book will accompany us step by step from the definition of a norm on page 3, all the way to an advanced form of the Pontryagin maximum principle in the final chapter, though we would welcome their company. In describing the itinerary, therefore, we shall make some suggestions for shorter excursions.

The book consists of four parts. The first of these is on functional analysis, the last on optimal control. It may appear that these are rather disparate topics. Yet they share the same lineage: functional analysis (Part I) was born to serve the calculus of variations (Part III), which in turn is the parent of optimal control (Part IV). Add to this observation the need for additional elements from optimization and nonsmooth analysis (Part II), and the logic of the four parts becomes clear. We proceed to comment on them in turn.

Part I: Functional analysis. The prerequisites are the standard first courses in real analysis, measure and integration, and general topology. It seems likely to us, then, that the reader's backpack already contains *some* functional analysis: Hilbert spaces, at least; perhaps more. But we must set off from somewhere, and we do not, strictly speaking, assume this. Thus, Part I serves as an *introduction* to functional analysis. It includes the essential milestones: operators, convex sets, separation, dual spaces, uniform boundedness, open mappings, weak topologies, reflexivity. . .

Our course on functional analysis leads to a destination, however, as does every worthwhile journey. For this reason, there is an emphasis on those elements which will be important later for optimization, for the calculus of variations, and for control (that is, for the rest of the book).

Thus, compactness, lower semicontinuity, and minimization are stressed. Convex functions are introduced early, together with directional derivatives, tangents, and normals. Minimization principles are emphasized. The relevance of the smoothness of the norm of a Banach space, and of subdifferentials, is explained. Integral functionals are studied in detail, as are measurable selections. Greater use of optimization is made, even in proving classical results. These topics manage to walk hand in hand quite amiably with the standard ones.

The reader to whom functional analysis is largely familiar territory will nonetheless find Part I useful as a guide to certain areas.

Part II: Optimization and nonsmooth analysis. The themes that we examine in optimization are strictly mathematical: existence, necessary conditions, sufficient conditions. The goal is certainly not to make the reader an expert in the field, in which modeling and numerical analysis have such an important place. So we are threading our way on a fairly narrow (if scenic) path. But some knowledge of the subject and its terminology, some familiarity with the multiplier rule, a good understanding of the deductive (versus the inductive) method, an appreciation of the role of convexity, are all important things to acquire for future purposes. Some students will not have this background, which is why it is supplied here.

Part II also contains a short course on nonsmooth analysis and geometry, together with closely related results on invariance of trajectories. These subjects are certainly worth a detour in their own right, and the exposition here is streamlined and innovative in some respects. But their inclusion in the text is also based on the fact that they provide essential infrastructure for later chapters.

Part III: Calculus of variations. This is meant to be a rather thorough look at the subject, from its inception to the present. In writing it, we have tried to show the reader not only the landmarks, but also the contours of this beautiful and ongoing chapter in mathematics. This is done in part by advancing in stages, along a path that reveals its history.

A notable feature in this landscape is the presence of recent advanced results on regularity and necessary conditions. In particular, we encounter a refined multiplier rule that is completely proved. We know of no textbook which has such a thing; certainly not with the improvements to be found here. Another important theme is the existence question, where we stress the need to master the direct method. This is made possible by the earlier groundwork in functional analysis. There are also substantial examples and exercises, involving such topics as viscosity solutions, nonsmooth Lagrangians, the logarithmic Sobolev inequality, and periodic trajectories.

Part IV: Optimal control. Control theory is a very active subject that regularly produces new kinds of mathematical challenges. We focus here upon optimality, a topic in which the central result is the Pontryagin maximum principle. This important theorem is viewed from several different angles, both classical and modern, so as to fully appreciate its scope. We demonstrate in particular that its extension to nonsmooth data not only unifies a variety of special cases mathematically, but is itself of intrinsic interest.

Our survey of optimal control does not neglect existence theory, without which the deductive approach cannot be applied. We also discuss Hamilton-Jacobi methods, relaxation, and regularity of optimal controls. The exercises stem in part from several fields of application: economics, finance, systems engineering, and resources. The final chapter contains general results on necessary conditions for differential inclusions. These theorems, which provide a direct route to the maximum princi-

ple and the multiplier rule, appear here for the first time in a text; they have been polished and refined for the occasion.

A full proof of a general maximum principle, or of a multiplier rule, has never been an easy thing; indeed, it has been famously hard. One may say that it has become more streamlined; it has certainly become more general, and more unified; but it has not become easy. Thus, there is a difficult stretch of road towards the end of Part IV; however, it leads to a fully self-contained text.

Intended users. The author has himself used the material of this book many times, for various courses at the first-year or second-year graduate level. Accordingly, the text has been planned with potential instructors in mind. The main question is whether to do in detail (most of) Part I, or just refer to it as needed for background material. The answer must depend on the experience and the walking speed of the audience, of course.

The author has given one-semester courses that did not stray from Part I. For some audiences, this could be viewed as a second course on functional analysis, since, as we have said, the text adopts a somewhat novel emphasis and choice of material relative to other introductions. The instructor must also decide on how much of Chapter 8 to cover (it's nothing but problems). If the circumstances of time and audience permit, one could tread much of Part I and still explore some chapters from Part II (as an introduction to optimization and nonsmooth analysis) or Part III (the calculus of variations). As an aid in doing so, Part I has been organized in such a way that certain material can be bypassed without losing one's way. We refer especially to Sections 4.3–4.4, 5.4, 6.2–6.4, and Sections 7.2–7.4 (or all of Chapter 7, if in fact the audience is familiar with Hilbert spaces).

Here is a specific example. To give a course on functional analysis and calculus of variations, one could choose to travel lightly and drop from Part I the material just mentioned. Then, Chapter 9 might be done (minus the last section, perhaps). Following this, one could skip ahead to the first three or four chapters of Part III; they constitute in themselves a viable introduction to the calculus of variations. (True, the proof of Tonelli's theorem in Chapter 16 uses unseen elements of Chapter 6, but we indicate a shortcut to a special case that circumvents this.)

For advanced students who are already competent in functional analysis, an entirely different path can be taken, in which one focuses entirely on the latter half of the book. Then Part I (and possibly Part II) can play the role of a convenient and strangely relevant appendix (one that happens to be at the front), to be consulted as needed. As regards the teaching of optimal control, we strongly recommend that it be preceded by the first three or four chapters of Part III, as well as Section 19.1 on verification functions.

In addition to whatever merits it may have as a text, we believe that the book has considerable value as a reference. This is particularly true in the calculus of variations and optimal control; its advanced results make it stand out in this respect. But its ac-

cessible presentation of certain other topics may perhaps be appreciated too (convex analysis, integral semicontinuity, measurable selections, nonsmooth analysis, and metric regularity, for example). We dare to hope that it will be of interest to both the mathematics and the control engineering communities for all of these reasons, as well as to certain related ones (such as operations research and economics).

A word about the exercises: there are hundreds. Some of them stand side by side with the text, for the reader to meet at an early stage. But additional exercises (many of them original in nature, and more difficult) lie waiting at the end of each part, in a separate chapter. Solutions (full, partial, or just hints) are given for quite a few of them, in the endnotes. A list of notation is given at the beginning of the index.

Acknowledgements.

I shall end this preface on a more personal note. My mathematical education has been founded on the excellent books and the outstanding teachers that I have met along the way. Among the latter, I have long wanted to give thanks to Thomas Higgins of the order of Christian Brothers; to Norbert Schlomiuk of l'Université de Montréal; to Basil Rattray and Michael Herschorn of McGill University; to Terry Rockafellar, Victor Klee, and Ernest Michael at the University of Washington. In the endnotes, I mention the books that have had the greatest influence on me.

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