

Chapter 2

Basic Notions and Examples

In this chapter we introduce the notion of a dynamical system, both for discrete and continuous time. We also describe many examples, including rotations and expanding maps of the circle, endomorphisms and automorphisms of the torus, and autonomous differential equations and their flows. Together with other examples introduced throughout the book, these are used to illustrate new concepts and results. We also describe some basic constructions determining new dynamical systems, including suspension flows and Poincaré maps. Finally, we consider the notion of an invariant set, both for maps and flows.

2.1 The Notion of a Dynamical System

In the case of discrete time, a dynamical system is simply a map.

Definition 2.1 Any map $f: X \rightarrow X$ is called a *dynamical system with discrete time* or simply a *dynamical system*.

We define recursively

$$f^{n+1} = f \circ f^n$$

for each $n \in \mathbb{N}$. We also write $f^0 = \text{Id}$, where Id is the identity map. Clearly,

$$f^{m+n} = f^m \circ f^n \tag{2.1}$$

for every $m, n \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. When the map f is invertible, we also define $f^{-n} = (f^{-1})^n$ for each $n \in \mathbb{N}$. In this case, identity (2.1) holds for every $m, n \in \mathbb{Z}$.

Example 2.1 Given dynamical systems $f: X \rightarrow X$ and $g: Y \rightarrow Y$, we define a new dynamical system $h: X \times Y \rightarrow X \times Y$ by

$$h(x, y) = (f(x), g(y)).$$

We note that if f and g are invertible, then the map h is also invertible and its inverse is given by

$$h^{-1}(x, y) = (f^{-1}(x), g^{-1}(y)).$$

Now we consider the case of continuous time.

Definition 2.2 A family of maps $\varphi_t: X \rightarrow X$ for $t \geq 0$ such that $\varphi_0 = \text{Id}$ and

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \text{for every } t, s \geq 0$$

is called a *semiflow*. A family of maps $\varphi_t: X \rightarrow X$ for $t \in \mathbb{R}$ such that $\varphi_0 = \text{Id}$ and

$$\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \text{for every } t, s \in \mathbb{R}$$

is called a *flow*.

We also say that a family of maps φ_t is a *dynamical system with continuous time* or simply a *dynamical system* if it is a flow or a semiflow. We note that if φ_t is a flow, then

$$\varphi_t \circ \varphi_{-t} = \varphi_{-t} \circ \varphi_t = \varphi_0 = \text{Id}$$

and thus, each map φ_t is invertible and its inverse is given by $\varphi_t^{-1} = \varphi_{-t}$.

A simple example of a flow is any movement by translation with constant velocity.

Example 2.2 Given $y \in \mathbb{R}^n$, consider the maps $\varphi_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by

$$\varphi_t(x) = x + ty, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n.$$

Clearly, $\varphi_0 = \text{Id}$ and

$$\begin{aligned} \varphi_{t+s}(x) &= x + (t+s)y \\ &= (x + sy) + ty = (\varphi_t \circ \varphi_s)(x). \end{aligned}$$

In other words, the family of maps φ_t is a flow.

Example 2.3 Given two flows $\varphi_t: X \rightarrow X$ and $\psi_t: Y \rightarrow Y$, for $t \in \mathbb{R}$, the family of maps $\alpha_t: X \times Y \rightarrow X \times Y$ defined for each $t \in \mathbb{R}$ by

$$\alpha_t(x, y) = (\varphi_t(x), \psi_t(y))$$

is also a flow. Moreover,

$$\alpha_t^{-1}(x, y) = (\varphi_{-t}(x), \psi_{-t}(y)).$$

We emphasize that the expression *dynamical system* is used to refer both to dynamical systems with discrete time and to dynamical systems with continuous time.

2.2 Examples with Discrete Time

In this section we describe several examples of dynamical systems with discrete time.

2.2.1 Rotations of the Circle

We first consider the rotations of the circle. The *circle* S^1 is defined to be \mathbb{R}/\mathbb{Z} , that is, the real line with any two points $x, y \in \mathbb{R}$ identified if $x - y \in \mathbb{Z}$. In other words,

$$S^1 = \mathbb{R}/\mathbb{Z} = \mathbb{R}/\sim,$$

where \sim is the equivalence relation on \mathbb{R} defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}$. The corresponding equivalence classes, which are the elements of S^1 , can be written in the form

$$[x] = \{x + m : m \in \mathbb{Z}\}.$$

In particular, one can introduce the operations

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] - [y] = [x - y].$$

One can also identify S^1 with $[0, 1]/\{0, 1\}$, where the endpoints of the interval $[0, 1]$ are identified.

Definition 2.3 Given $\alpha \in \mathbb{R}$, we define the *rotation* $R_\alpha: S^1 \rightarrow S^1$ by

$$R_\alpha([x]) = [x + \alpha]$$

(see Fig. 2.1).

Sometimes, we also write

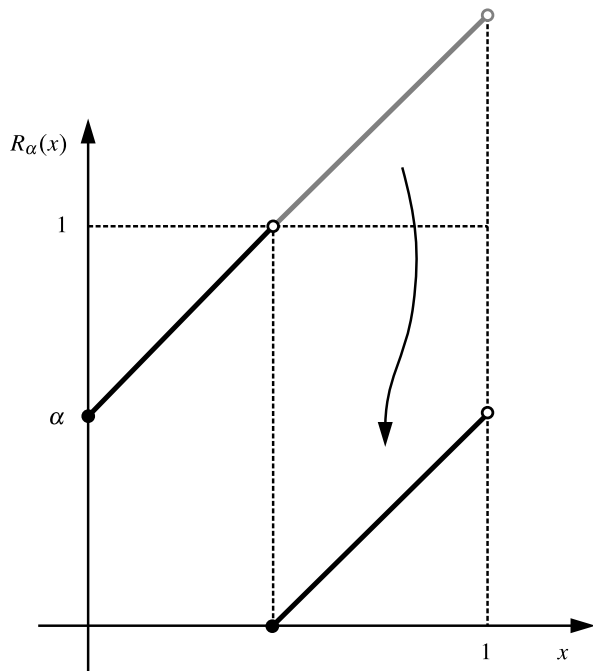
$$R_\alpha(x) = x + \alpha \bmod 1,$$

thus identifying $[x]$ with its representative in the interval $[0, 1)$. The map R_α could also be called a *translation of the interval*. Clearly, $R_\alpha: S^1 \rightarrow S^1$ is invertible and its inverse is given by $R_\alpha^{-1} = R_{-\alpha}$.

Now we introduce the notion of a periodic point.

Definition 2.4 Given $q \in \mathbb{N}$, a point $x \in X$ is said to be a *q-periodic point* of a map $f: X \rightarrow X$ if $f^q(x) = x$. We also say that $x \in X$ is a *periodic point* of f if it is q -periodic for some $q \in \mathbb{N}$.

In particular, the fixed points, that is, the points $x \in X$ such that $f(x) = x$ are q -periodic for any $q \in \mathbb{N}$. Moreover, a q -periodic point is kq -periodic for any $k \in \mathbb{N}$.

Fig. 2.1 The rotation R_α 

Definition 2.5 A periodic point is said to have *period* q if it is q -periodic but is not l -periodic for any $l < q$.

Now we consider the particular case of the rotations R_α of the circle. We verify that their behavior is very different depending on whether α is rational or irrational.

Proposition 2.1 Given $\alpha \in \mathbb{R}$:

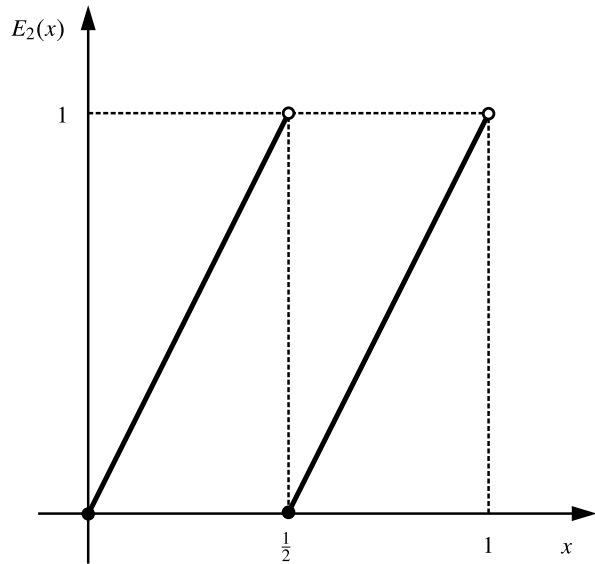
1. if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then R_α has no periodic points;
2. if $\alpha = p/q \in \mathbb{Q}$ with p and q coprime, then all points of S^1 are periodic for R_α and have period q .

Proof We note that $[x] \in S^1$ is q -periodic if and only if $[x + q\alpha] = [x]$, that is, if and only if $q\alpha \in \mathbb{Z}$. The two properties in the proposition follow easily from this observation. \square

2.2.2 Expanding Maps of the Circle

In this section we consider another family of maps of S^1 .

Fig. 2.2 The expanding map E_2



Definition 2.6 Given an integer $m > 1$, the *expanding map* $E_m: S^1 \rightarrow S^1$ is defined by

$$E_m(x) = mx \bmod 1.$$

For example, for $m = 2$, we have

$$E_2(x) = \begin{cases} 2x & \text{if } x \in [0, 1/2), \\ 2x - 1 & \text{if } x \in [1/2, 1) \end{cases}$$

(see Fig. 2.2).

Now we determine the periodic points of the expanding map E_m . Since $E_m^q(x) = m^q x \bmod 1$, a point $x \in S^1$ is q -periodic if and only if

$$m^q x - x = (m^q - 1)x \in \mathbb{Z}.$$

Hence, the q -periodic points of the expanding map E_m are

$$x = \frac{p}{m^q - 1}, \quad \text{for } p = 1, 2, \dots, m^q - 1. \quad (2.2)$$

Moreover, the number $n_m(q)$ of periodic points of E_m with period q can be computed easily for each given q (see Table 2.1 for $q \leq 6$). For example, if q is prime, then

$$n_m(q) = m^q - m.$$

Table 2.1 The number $n_m(q)$ of periodic points of E_m with period q

q	$n_m(q)$
1	$m - 1$
2	$m^2 - m = m^2 - 1 - (m - 1)$
3	$m^3 - m = m^3 - 1 - (m - 1)$
4	$m^4 - m^2 = m^4 - 1 - (m^2 - 1)$
5	$m^5 - m = m^5 - 1 - (m - 1)$
6	$m^6 - m^3 - m^2 + m$

2.2.3 Endomorphisms of the Torus

In this section we consider a third family of dynamical systems with discrete time. Given $n \in \mathbb{N}$, the n -torus or simply the *torus* is defined to be

$$\mathbb{T}^n = \mathbb{R}^n / \mathbb{Z}^n = \mathbb{R}^n / \sim,$$

where \sim is the equivalence relation on \mathbb{R}^n defined by $x \sim y \Leftrightarrow x - y \in \mathbb{Z}^n$. The elements of \mathbb{T}^n are thus the equivalence classes

$$[x] = \{x + y : y \in \mathbb{Z}^n\},$$

with $x \in \mathbb{R}^n$. Now let A be an $n \times n$ matrix with entries in \mathbb{Z} .

Definition 2.7 The *endomorphism of the torus* $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ is defined by

$$T_A([x]) = [Ax] \quad \text{for } [x] \in \mathbb{T}^n.$$

We also say that T_A is the *endomorphism of the torus induced by* A .

Since A is a linear transformation,

$$Ax - Ay \in \mathbb{Z}^n \quad \text{when } x - y \in \mathbb{Z}^n.$$

This shows that $Ay \in [Ax]$ when $y \in [x]$ and hence, T_A is well defined.

In general, the map T_A may not be invertible, even if the matrix A is invertible. When T_A is invertible, we also say that it is the *automorphism of the torus induced by* A . We represent in Fig. 2.3 the automorphism of the torus \mathbb{T}^2 induced by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

Now we determine the periodic points of a class of automorphisms of the torus.

Proposition 2.2 Let $T_A : \mathbb{T}^n \rightarrow \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A without eigenvalues with modulus 1. Then the periodic points of T_A are the points with rational coordinates, that is, the elements of $\mathbb{Q}^n / \mathbb{Z}^n$.

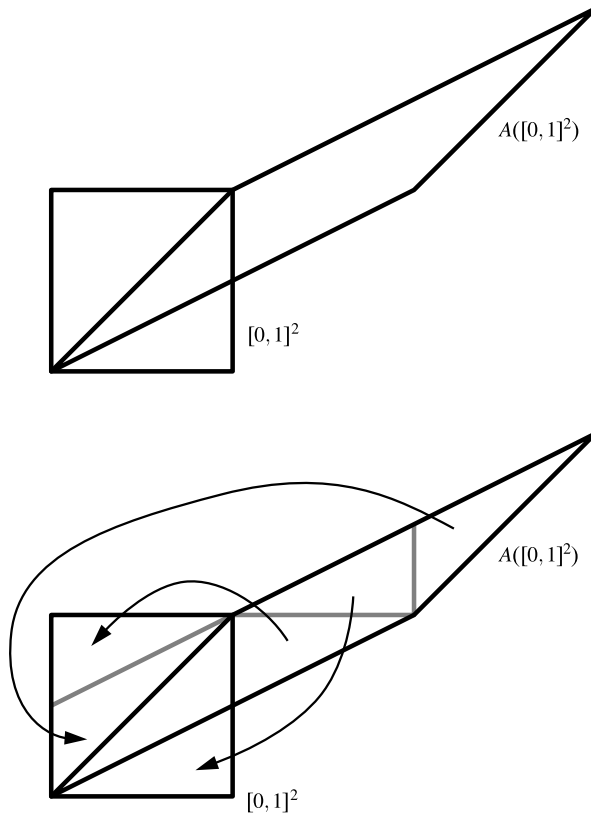


Fig. 2.3 An automorphism of the torus \mathbb{T}^2

Proof Let $[x] = [(x_1, \dots, x_n)] \in \mathbb{T}^n$ be a periodic point. Then there exist $q \in \mathbb{N}$ and $y = (y_1, \dots, y_n) \in \mathbb{Z}^n$ such that $A^q x = x + y$, that is,

$$(A^q - \text{Id})x = y.$$

Since A has no eigenvalues with modulus 1, the matrix $A^q - \text{Id}$ is invertible and one can write

$$x = (A^q - \text{Id})^{-1}y.$$

Moreover, since $A^q - \text{Id}$ has only integer entries, each entry of $(A^q - \text{Id})^{-1}$ is a rational number and thus $x \in \mathbb{Q}^n$.

Now we assume that $[x] = [(x_1, \dots, x_n)] \in \mathbb{Q}^n / \mathbb{Z}^n$ and we write

$$(x_1, \dots, x_n) = \left(\frac{p_1}{r}, \dots, \frac{p_n}{r} \right), \quad (2.3)$$

where $p_1, \dots, p_n \in \{0, 1, \dots, r-1\}$. Since A has only integer entries, for each $q \in \mathbb{N}$ we have

$$A^q(x_1, \dots, x_n) = \left(\frac{p'_1}{r}, \dots, \frac{p'_n}{r} \right) + (y_1, \dots, y_n)$$

for some $p'_1, \dots, p'_n \in \{0, 1, \dots, r-1\}$ and $(y_1, \dots, y_n) \in \mathbb{Z}^n$. But since the number of points of the form (2.3) is r^n , there exist $q_1, q_2 \in \mathbb{N}$ with $q_1 \neq q_2$ such that

$$A^{q_1}(x_1, \dots, x_n) - A^{q_2}(x_1, \dots, x_n) \in \mathbb{Z}^n.$$

Assuming, without loss of generality, that $q_1 > q_2$, we obtain

$$A^{q_1 - q_2}(x_1, \dots, x_n) - (x_1, \dots, x_n) \in \mathbb{Z}^n$$

(see Exercise 2.12) and thus $T_A^{q_1 - q_2}([x]) = [x]$. □

The following example shows that Proposition 2.2 cannot be extended to arbitrary endomorphisms of the torus.

Example 2.4 Consider the endomorphism of the torus $T_A : \mathbb{T}^2 \rightarrow \mathbb{T}^2$ induced by the matrix

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}.$$

We note that T_A is not an automorphism since $\det A = 2$ (see Exercise 2.12). Now we observe that

$$T_A\left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad T_A\left(\frac{1}{2}, \frac{1}{2}\right) = (0, 0) \quad \text{and} \quad T_A(0, 0) = (0, 0).$$

This shows that the points with rational coordinates $(0, 1/2)$ and $(1/2, 1/2)$ are not periodic. On the other hand, the eigenvalues of A are $2 + \sqrt{2}$ and $2 - \sqrt{2}$, both without modulus 1.

2.3 Examples with Continuous Time

In this section we give some examples of dynamical systems with continuous time.

2.3.1 Autonomous Differential Equations

We first consider autonomous (ordinary) differential equations, that is, differential equations not depending explicitly on time. We verify that they give rise naturally to the concept of a flow.

Proposition 2.3 *Let $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function such that, given $x_0 \in \mathbb{R}^n$, the initial value problem*

$$\begin{cases} x' = f(x), \\ x(0) = x_0 \end{cases} \quad (2.4)$$

has a unique solution $x(t, x_0)$ defined for $t \in \mathbb{R}$. Then the family of maps $\varphi_t: \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined for each $t \in \mathbb{R}$ by

$$\varphi_t(x_0) = x(t, x_0)$$

is a flow.

Proof Given $s \in \mathbb{R}$, consider the function $y: \mathbb{R} \rightarrow \mathbb{R}^n$ defined by

$$y(t) = x(t + s, x_0).$$

We have $y(0) = x(s, x_0)$ and

$$y'(t) = x'(t + s, x_0) = f(x(t + s, x_0)) = f(y(t))$$

for $t \in \mathbb{R}$. In other words, the function y is also a solution of the equation $x' = f(x)$. Since by hypothesis the initial value problem (2.4) has a unique solution, we obtain

$$y(t) = x(t, y(0)) = x(t, x(s, x_0)),$$

or equivalently,

$$x(t + s, x_0) = x(t, x(s, x_0)) \quad (2.5)$$

for $t, s \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. It follows from (2.5) that $\varphi_{t+s} = \varphi_t \circ \varphi_s$. Moreover,

$$\varphi_0(x_0) = x(0, x_0) = x_0,$$

that is, $\varphi_0 = \text{Id}$. This shows that the family of maps φ_t is a flow. \square

Now we consider two specific examples of autonomous differential equations and we describe the flows that they determine.

Example 2.5 Consider the differential equation

$$\begin{cases} x' = -y, \\ y' = x. \end{cases}$$

If $(x, y) = (x(t), y(t))$ is a solution, then

$$(x^2 + y^2)' = 2xx' + 2yy' = -2xy + 2yx = 0.$$

Thus, there exists a constant $r \geq 0$ such that

$$x(t)^2 + y(t)^2 = r^2.$$

Writing

$$x(t) = r \cos \theta(t) \quad \text{and} \quad y(t) = r \sin \theta(t),$$

where θ is some differentiable function, it follows from the identity $x' = -y$ that

$$-r\theta'(t) \sin \theta(t) = -r \sin \theta(t).$$

Hence, $\theta'(t) = 1$ and there exists a constant $c \in \mathbb{R}$ such that $\theta(t) = t + c$. Thus, writing

$$(x_0, y_0) = (r \cos c, r \sin c) \in \mathbb{R}^2,$$

we obtain

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} r \cos(t + c) \\ r \sin(t + c) \end{pmatrix} \\ &= \begin{pmatrix} \cos t \cdot r \cos c - \sin t \cdot r \sin c \\ \sin t \cdot r \cos c + \cos t \cdot r \sin c \end{pmatrix} \\ &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \end{aligned}$$

Notice that

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

is a rotation matrix for each $t \in \mathbb{R}$. Since $R(0) = \text{Id}$, it follows from Proposition 2.3 that the family of maps $\varphi_t: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\varphi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = R(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a flow. Incidentally, the identity $\varphi_{t+s} = \varphi_t \circ \varphi_s$ is equivalent to the identity between rotation matrices

$$R(t+s) = R(t)R(s).$$

Example 2.6 Now we consider the differential equation

$$\begin{cases} x' = y, \\ y' = x. \end{cases}$$

If $(x, y) = (x(t), y(t))$ is a solution, then

$$(x^2 - y^2)' = 2xx' - 2yy' = 2xy - 2yx = 0.$$

Thus, there exists a constant $r \geq 0$ such that

$$x(t)^2 - y(t)^2 = r^2 \quad \text{or} \quad x(t)^2 - y(t)^2 = -r^2. \quad (2.6)$$

In the first case, one can write

$$x(t) = r \cosh \theta(t) \quad \text{and} \quad y(t) = r \sinh \theta(t),$$

where θ is some differentiable function. Since $x' = y$, we have

$$r\theta'(t) \sinh \theta(t) = r \sinh \theta(t)$$

and hence, $\theta(t) = t + c$ for some constant $c \in \mathbb{R}$. Thus, writing

$$(x_0, y_0) = (r \cosh c, r \sinh c) \in \mathbb{R}^2,$$

we obtain

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} r \cosh(t + c) \\ r \sinh(t + c) \end{pmatrix} \\ &= \begin{pmatrix} \cosh t \cdot r \cosh c + \sinh t \cdot r \sinh c \\ \sinh t \cdot r \cosh c + \cosh t \cdot r \sinh c \end{pmatrix} \\ &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \end{aligned}$$

where

$$S(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}.$$

In the second case in (2.6), one can write

$$x(t) = r \sinh \theta(t) \quad \text{and} \quad y(t) = r \cosh \theta(t).$$

Proceeding analogously, we find that $\theta(t) = t + c$ for some constant $c \in \mathbb{R}$. Thus, writing

$$(x_0, y_0) = (r \sinh c, r \cosh c) \in \mathbb{R}^2,$$

we obtain

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} r \sinh(t + c) \\ r \cosh(t + c) \end{pmatrix} \\ &= \begin{pmatrix} \sinh t \cdot r \cosh c + \cosh t \cdot r \sinh c \\ \cosh t \cdot r \cosh c + \sinh t \cdot r \sinh c \end{pmatrix} \\ &= S(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \end{aligned}$$

Notice that $S(0) = \text{Id}$. It follows from Proposition 2.3 that the family of maps $\psi_t : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$\psi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a flow. In particular, it follows from the identity $\psi_{t+s} = \psi_t \circ \psi_s$ that

$$S(t+s) = S(t)S(s) \quad \text{for } t, s \in \mathbb{R}.$$

2.3.2 Discrete Time Versus Continuous Time

In this section we describe some relations between dynamical systems with discrete time and dynamical systems with continuous time.

Example 2.7 Given a flow $\varphi_t : X \rightarrow X$, for each $T \in \mathbb{R}$, the map $f = \varphi_T : X \rightarrow X$ is a dynamical system with discrete time. We note that f is invertible and that its inverse is given by $f^{-1} = \varphi_{-T}$. Similarly, given a semiflow $\varphi_t : X \rightarrow X$, for each $T \geq 0$, the map $f = \varphi_T : X \rightarrow X$ is a dynamical system with discrete time.

Now we describe a class of semiflows obtained from a dynamical system with discrete time $f : X \rightarrow X$. Given a function $\tau : X \rightarrow \mathbb{R}^+$, consider the set Y obtained from

$$Z = \{(x, t) \in X \times \mathbb{R} : 0 \leq t \leq \tau(x)\}$$

identifying the points $(x, \tau(x))$ and $(f(x), 0)$, for each $x \in X$. More precisely, we define $Y = Z/\sim$, where \sim is the equivalence relation on Z defined by

$$(x, t) \sim (y, s) \iff y = f(x), t = \tau(x) \text{ and } s = 0$$

(see Fig. 2.4).

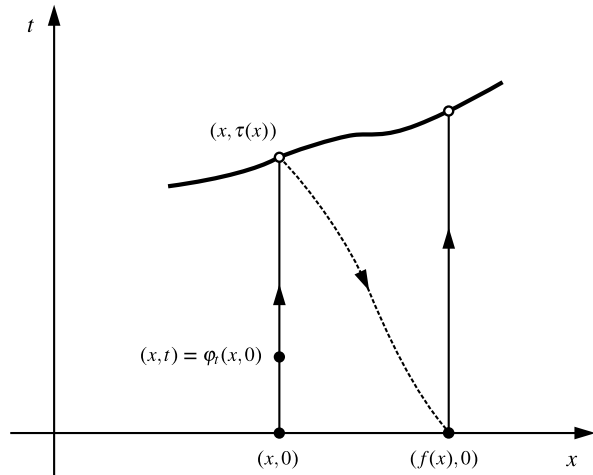
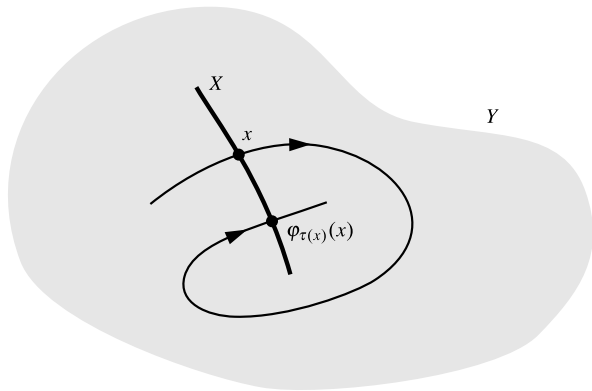
Definition 2.8 The *suspension semiflow* $\varphi_t : Y \rightarrow Y$ over f with *height* τ is defined for each $t \geq 0$ by

$$\varphi_t(x, s) = (x, s + t) \quad \text{when } s + t \in [0, \tau(x)] \quad (2.7)$$

(see Fig. 2.4).

One can easily verify that each suspension semiflow is indeed a semiflow. Moreover, when f is invertible, the family of maps φ_t in (2.7), for $t \in \mathbb{R}$, is a flow. It is called the *suspension flow* over f with height τ .

Conversely, given a semiflow $\varphi_t : Y \rightarrow Y$, sometimes one can construct a dynamical system with discrete time $f : X \rightarrow X$ such that the semiflow can be seen as a suspension semiflow over f .

Fig. 2.4 A suspension flow**Fig. 2.5** A Poincaré section

Definition 2.9 A set $X \subset Y$ is said to be a *Poincaré section* for a semiflow φ_t if

$$\tau(x) := \inf\{t > 0 : \varphi_t(x) \in X\} \in \mathbb{R}^+ \quad (2.8)$$

for each $x \in X$ (see Fig. 2.5), with the convention that $\inf \emptyset = +\infty$. The number $\tau(x)$ is called the *first return time* of x to the set X .

Thus, the first return time to X is a function $\tau : X \rightarrow \mathbb{R}^+$. We observe that (2.8) includes the hypothesis that each point of X returns to X . In fact, each point of X returns infinitely often to X .

Given a Poincaré section, one can introduce a corresponding Poincaré map.

Definition 2.10 Given a Poincaré section X for a semiflow φ_t , we define its *Poincaré map* $f : X \rightarrow X$ by

$$f(x) = \varphi_{\tau(x)}(x).$$

2.3.3 Differential Equations on the Torus \mathbb{T}^2

We also consider a class of differential equations on \mathbb{T}^2 . We recall that two vectors $x, y \in \mathbb{R}^2$ represent the same point of the torus \mathbb{T}^2 if and only if $x - y \in \mathbb{Z}^2$.

Example 2.8 Let $f, g: \mathbb{R}^2 \rightarrow \mathbb{R}$ be C^1 functions such that

$$f(x + k, y + l) = f(x, y) \quad \text{and} \quad g(x + k, y + l) = g(x, y)$$

for any $x, y \in \mathbb{R}$ and $k, l \in \mathbb{Z}$. Then the differential equation in the plane \mathbb{R}^2 given by

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y) \end{cases} \quad (2.9)$$

can be seen as a differential equation on \mathbb{T}^2 . Clearly, Eq. (2.9) has unique solutions (that are global, that is, they are defined for $t \in \mathbb{R}$ since the torus is compact). Let $\varphi_t: \mathbb{T}^2 \rightarrow \mathbb{T}^2$ be the corresponding flow (see Proposition 2.3).

Now we assume that f takes only positive values. Then each solution $\varphi_t(0, z) = (x(t), y(t))$ of Eq. (2.9) crosses infinitely often the line segment $x = 0$, which is thus a *Poincaré section* for φ_t (see Definition 2.9). The first intersection (for $t > 0$) occurs at the time

$$T_z = \inf\{t > 0 : x(t) = 1\}.$$

We also consider the map $h: S^1 \rightarrow S^1$ defined by

$$h(z) = y(T_z) \quad (2.10)$$

(see Fig. 2.6). One can use the C^1 dependence of the solutions of a differential equation on the initial conditions to show that h is a diffeomorphism, that is, a bijective (one-to-one and onto) differentiable map with differentiable inverse (see Exercise 2.20).

For example, if $f = 1$ and $g = \alpha \in \mathbb{R}$, then

$$\varphi_t(0, z) = (t, z + t\alpha) \bmod 1.$$

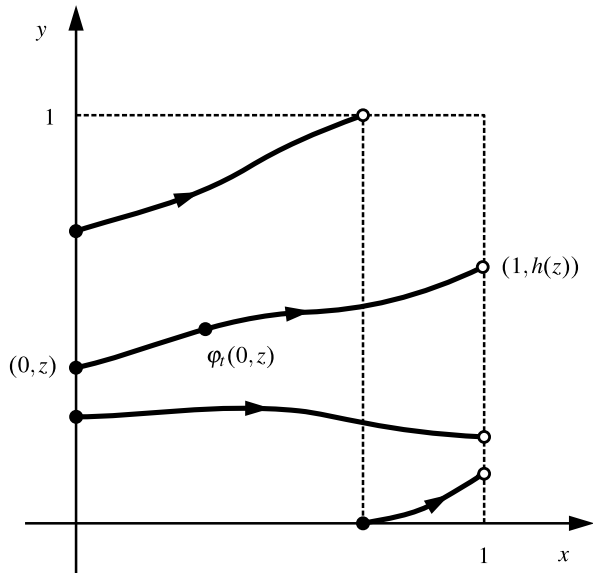
Thus, $T_z = 1$ for each $z \in \mathbb{R}$ and

$$h(z) = z + \alpha \bmod 1 = R_\alpha(z).$$

2.4 Invariant Sets

In this section we introduce the notion of an invariant set with respect to a dynamical system.

Fig. 2.6 The Poincaré map determined by the Poincaré section $x = 0$



Definition 2.11 Given a map $f: X \rightarrow X$, a set $A \subset X$ is said to be:

1. *f*-invariant if $f^{-1}A = A$, where

$$f^{-1}A = \{x \in X : f(x) \in A\};$$

2. *forward f*-invariant if $f(A) \subset A$;
3. *backward f*-invariant if $f^{-1}A \subset A$.

Example 2.9 Consider the rotation $R_\alpha: S^1 \rightarrow S^1$. For $\alpha \in \mathbb{Q}$, each set

$$\gamma(x) = \{R_\alpha^n(x) : n \in \mathbb{Z}\}$$

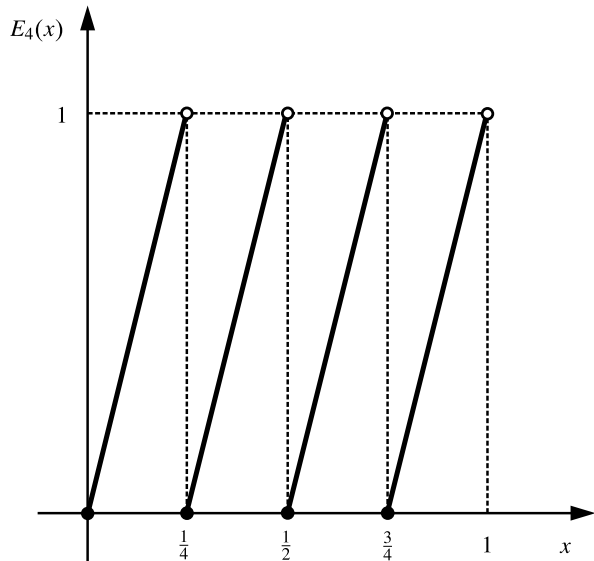
is finite and R_α -invariant. More generally, if $\alpha \in \mathbb{Q}$, then a nonempty set $A \subset X$ is R_α -invariant if and only if it is a union of sets of the form $\gamma(x)$ (see the discussion after Definition 2.12). For example, the set \mathbb{Q}/\mathbb{Z} is R_α -invariant.

On the other hand, for $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, each set $\gamma(x)$ is also R_α -invariant, but now it is infinite. Again, a nonempty set $A \subset X$ is R_α -invariant if and only if it is a union of sets of the form $\gamma(x)$. One can show that each set $\gamma(x)$ is dense in S^1 (see Example 3.2) and thus, the closed R_α -invariant sets are \emptyset and S^1 .

Example 2.10 Now we consider the expanding map $E_4: S^1 \rightarrow S^1$, given by

$$E_4(x) = \begin{cases} 4x & \text{if } x \in [0, 1/4), \\ 4x - 1 & \text{if } x \in [1/4, 2/4), \\ 4x - 2 & \text{if } x \in [2/4, 3/4), \\ 4x - 3 & \text{if } x \in [3/4, 1) \end{cases}$$

Fig. 2.7 The expanding map E_4



(see Fig. 2.7). For example, the set

$$A = \bigcap_{n \geq 0} E_4^{-n}([0, 1/4] \cup [2/4, 3/4]) \quad (2.11)$$

is forward E_4 -invariant. We note that A is a *Cantor set*, that is, A is a closed set without isolated points and containing no intervals.

We also introduce the notions of orbit and semiorbit.

Definition 2.12 For a map $f: X \rightarrow X$, given a point $x \in X$, the set

$$\gamma^+(x) = \gamma_f^+(x) = \{f^n(x) : n \in \mathbb{N}_0\}$$

is called the *positive semiorbit* of x . Moreover, when f is invertible,

$$\gamma^-(x) = \gamma_f^-(x) = \{f^{-n}(x) : n \in \mathbb{N}_0\}$$

is called the *negative semiorbit* of x and

$$\gamma(x) = \gamma_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$$

is called the *orbit* of x .

We note that when f is invertible, a nonempty set $A \subset X$ is f -invariant if and only if it is a union of orbits. Indeed, $A \subset X$ is f -invariant if and only if

$$x \in A \quad \Leftrightarrow \quad x \in f^{-1}A \quad \Leftrightarrow \quad f(x) \in A.$$

By induction, this is equivalent to

$$x \in A \Leftrightarrow \{f^n(x) : n \in \mathbb{Z}\} \subset A \Leftrightarrow \gamma(x) \in A$$

since f is invertible. Thus, a nonempty set $A \subset X$ is f -invariant if and only if

$$A = \bigcup_{x \in A} \gamma(x).$$

Now we introduce the notion of an invariant set with respect to a flow or a semiflow.

Definition 2.13 Given a flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ of X , a set $A \subset X$ is said to be Φ -invariant if

$$\varphi_t^{-1}A = A \quad \text{for } t \in \mathbb{R}.$$

Given a semiflow $\Phi = (\varphi_t)_{t \geq 0}$ of X , a set $A \subset X$ is said to be Φ -invariant if

$$\varphi_t^{-1}A = A \quad \text{for } t \geq 0.$$

In the case of flows, since $\varphi_t^{-1} = \varphi_{-t}$ for $t \in \mathbb{R}$, a set $A \subset X$ is Φ -invariant if and only if

$$\varphi_t(A) = A \quad \text{for } t \in \mathbb{R}.$$

Example 2.11 Consider the differential equation

$$\begin{cases} x' = 2y^3, \\ y' = -3x. \end{cases} \quad (2.12)$$

Each solution $(x, y) = (x(t), y(t))$ satisfies

$$\begin{aligned} (3x^2 + y^4)' &= 6xx' + 4y^3y' \\ &= 12xy^3 - 12y^3x = 0. \end{aligned}$$

Thus, for each set $I \subset \mathbb{R}^+$, the union

$$A = \bigcup_{a \in I} \{(x, y) \in \mathbb{R}^2 : 3x^2 + y^4 = a\}$$

is invariant with respect to the flow determined by Eq. (2.12).

We also introduce the notions of orbit and semiorbit for a semiflow.

Definition 2.14 For a semiflow $\Phi = (\varphi_t)_{t \geq 0}$ of X , given a point $x \in X$, the set

$$\gamma^+(x) = \gamma_\Phi^+(x) = \{\varphi_t(x) : t \geq 0\}$$

is called the *positive semiorbit* of x . Moreover, for a flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ of X ,

$$\gamma^-(x) = \gamma_\Phi^-(x) = \{\varphi_{-t}(x) : t \geq 0\}$$

is called the *negative semiorbit* of x and

$$\gamma(x) = \gamma_\Phi(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$$

is called the *orbit* of x .

2.5 Exercises

Exercise 2.1 Determine whether the map $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 3x - 3x^2$ has periodic points with period 2.

Exercise 2.2 Determine whether the map $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2 + 1$ has periodic points with period 5.

Exercise 2.3 Given a continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$, show that:

1. if $[a, b] \subset f([a, b])$, then f has a fixed point in $[a, b]$;
2. if $[a, b] \supset f([a, b])$, then f has a fixed point in $[a, b]$.

Exercise 2.4 Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and let $[a, b]$ and $[c, d]$ be intervals in \mathbb{R} such that

$$[c, d] \subset f([a, b]), \quad [a, b] \subset f([c, d]) \quad \text{and} \quad [a, b] \cap [c, d] = \emptyset.$$

Show that f has a periodic point with period 2.

Exercise 2.5 Show that if $f: [a, b] \rightarrow [a, b]$ is a homeomorphism (that is, a continuous bijective function with continuous inverse), then f has no periodic points with period 3 or larger.

Exercise 2.6 Determine whether there exists a homeomorphism $f: \mathbb{R} \rightarrow \mathbb{R}$ with:

1. a periodic point with period 2;
2. a periodic point with period 3.

Exercise 2.7 Show that any power of an expanding map is still an expanding map.

Exercise 2.8 Show that the set of periodic points of the expanding map E_m is dense in S^1 .

Exercise 2.9 For each $q \in \mathbb{N}$, find the number of q -periodic points of the map $f: R \rightarrow R$ defined by $f(z) = z^2$ in the set

$$R = \{z \in \mathbb{C} : |z| = 1\}.$$

Exercise 2.10 Show that the number of periodic points of the expanding map E_m with period $p = q^r$, for q prime and $r \in \mathbb{N}$, is given by

$$n_m(p) = m^p - m^{p/q}.$$

Exercise 2.11 Find the smallest E_3 -invariant set containing $[0, 1/3] \cup [2/3, 1]$.

Exercise 2.12 Show that the following properties are equivalent:

1. the endomorphism of the torus $T_A: \mathbb{T}^n \rightarrow \mathbb{T}^n$ is invertible;
2. $x \in \mathbb{Z}^n$ if and only if $Ax \in \mathbb{Z}^n$;
3. $|\det A| = 1$.

Exercise 2.13 Let $T_A: \mathbb{T}^n \rightarrow \mathbb{T}^n$ be an endomorphism of the torus. Show that for each $x \in \mathbb{Q}^n / \mathbb{Z}^n$, there exists an $m \in \mathbb{N}$ such that $T_A^m(x)$ is a periodic point of T_A .

Exercise 2.14 Show that the complement of a forward f -invariant set is backward f -invariant.

Exercise 2.15 Given a map $f: X \rightarrow X$, show that:

1. a set $A \subset X$ is f -invariant if and only if $f^{-1}A \subset A$ and $f(A) \subset A$;
2. a set $A \subset X$ is f -invariant if and only if $X \setminus A$ is f -invariant.

Exercise 2.16 Show that if X is a Poincaré section for a semiflow φ_t , then:

1. φ_t has no fixed points in X ;
2. f is invertible when φ_t is a flow.

Exercise 2.17 Find the flow determined by the equation $x'' + 4x = 0$.

Exercise 2.18 Find the flow determined by the equation $x'' - 5x' + 6x = 0$.

Exercise 2.19 Show that the equation $x' = x^2$ does not determine a flow.

Exercise 2.20 Use the C^1 dependence of the solutions of a differential equation on the initial conditions¹ together with the implicit function theorem to show that the map h defined by (2.10) is a diffeomorphism.

¹**Theorem** (See for example [12]) If $f: D \rightarrow \mathbb{R}^n$ is a C^1 function in an open set $D \subset \mathbb{R}^n$ and $\varphi(\cdot, x_0)$ is the solution of the initial value problem (2.4), then the function $(t, x) \mapsto \varphi(t, x)$ is of class C^1 .

Dynamical Systems

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