Chapter 2
Multivariate Concave and Convex Stochastic Dominance

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2.1 Introduction

One of the big challenges in decision analysis is the assessment of a decision maker's utility function. To the extent that the alternatives under consideration in a decision-making problem can be partially ordered based on less-than-full information about the utility function, the problem can be simplified somewhat by eliminating dominated alternatives. At the same time, partial orders can help in the creation of alternatives by providing an indication of the types of strategies that might be most promising. Stochastic dominance has been studied extensively in the univariate case, particularly in the finance and economics literature; early papers are Hadar and Russell (1969) and Hanoch and Levy (1969). For example, assuming that util-

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ity for money is increasing and concave can simplify many problems in finance and economics.

Moreover, stochastic dominance can be even more helpful in group decision making, where the challenge is amplified by divergent preferences. Even though the group members can be expected to have different utility functions, these utility functions may share some common characteristics. Thus, if an alternative can be eliminated based on an individual's utility function being risk averse, then all group members will agree that it can be eliminated if each member of the group is risk averse, even though the degree of risk aversion may vary considerably within the group.

Multiattribute consequences make the assessment of utility even more difficult, and extensions to multivariate stochastic dominance are tricky because there are many multivariate stochastic orders (Denuit et al. 1999; Müller and Stoyan 2002; Shaked and Shantikumar 2007; Denuit and Mesfioui 2010) on which the dominance can be based. Hazen (1986) investigates multivariate stochastic dominance when simple forms of utility independence (Keeney and Raiffa 1976) can be assumed. If utility independence cannot be assumed, the potential benefits of stochastic dominance are even greater. Studies of multivariate stochastic dominance include Levy and Paroush (1974), Levhari et al. (1975), Mosler (1984), Scarsini (1988), and Denuit and Eeckhoudt (2010). In this paper we use a stochastic order that can be related to characteristics such as risk aversion and correlation aversion, is consistent with a basic preference assumption, and is a natural extension of the standard order typically used for univariate stochastic dominance. We also consider a stochastic order that is consistent with characteristics such as risk taking and correlation loving by reversing the basic preference assumption.

The objective of this paper is to study multivariate stochastic dominance for the above-mentioned stochastic orders. In Sect. 2.2, we define these stochastic orders, which form the basis for what we call nth-degree multivariate concave and convex stochastic dominance. We extend the concept of nth-degree risk to the multivariate case and show that it is related to multivariate concave and convex stochastic dominance. Then we show a connection with a preference for combining good with bad in the concave case and with the opposite preference for combining good with good and bad with bad in the convex case. We develop some ways to facilitate the comparison of alternatives via multivariate stochastic dominance in Sect. 2.3, focusing on the impact of background risk and on eliminating alternatives from consideration by comparing an alternative with a mixture of other alternatives. A simple hypothetical example is presented to illustrate the concepts from Sects. 2.2–2.3. In Sect. 2.4, we consider infinite-degree concave and convex stochastic dominance, which can be related to utility functions that are mixtures of multiattribute exponential utilities, and present dominance results when the joint probability distribution for the attributes is multivariate normal. In Sect. 2.5, we compare our multivariate stochastic dominance with dominance based on another family of stochastic orders possessing some interesting similarities and differences. A brief summary and concluding comments are given in Sect. 2.6.

2.2 Multivariate Stochastic Dominance

2.2.1 Multivariate Concave and Convex Stochastic Dominance

We begin by defining some notation. A random vector is denoted by a tilde, $\tilde{\mathbf{x}}$, and $\mathbf{0}$ is a vector of zeroes. For two N-dimensional vectors $\mathbf{x}$ and $\mathbf{y}$, $\mathbf{x} > \mathbf{y}$ if $x_j > y_j$ for $j = 1, \ldots, N$ and $\mathbf{x} \gneqq \mathbf{y}$ if $x_j \geq y_j$ for all j and $\mathbf{x} \neq \mathbf{y}$. Also, $\mathbf{x} + \mathbf{y}$ denotes the component-wise sum, $(x_1 + y_1, \ldots, x_N + y_N)$.

Next, we consider a differentiable utility function u for a vector of N attributes and formalize the notion of alternating signs for the partial derivatives of u.

Definition 2.2.1

$$\underline{\mathbb{U}}_n^N = \left\{ u \,\middle|\, (-1)^{k-1} \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_k}} \geq 0 \text{ for } k = 1, \ldots, n \text{ and } i_j \in \{1, \ldots, N\},\ j = 1, \ldots, k \right\}.$$

$\underline{\mathbb{U}}_n^N$ consists of all N-dimensional real-valued functions for which all partial derivatives of a given degree up to degree n have the same sign, and that sign alternates, being positive for odd degrees and negative for even degrees. Observe that if $u \in \underline{\mathbb{U}}_n^N$, then $u \in \underline{\mathbb{U}}_k^N$ for any $k < n$. Also, if $u \in \underline{\mathbb{U}}_n^N$, then for any $k < n$ and $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$,

$$(-1)^k \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in \underline{\mathbb{U}}_{n-k}^N.$$

Now we use $\underline{\mathbb{U}}_n^N$ to define multivariate concave stochastic dominance.

Definition 2.2.2 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of nth-degree concave stochastic dominance if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all $u \in \underline{\mathbb{U}}_n^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

Next we define multivariate convex stochastic dominance.

Definition 2.2.3

$$\overline{\mathbb{U}}_n^N = \left\{ u \,\middle|\, \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_k}} \geq 0 \text{ for } k = 1, \ldots, n \text{ and } i_j \in \{1, \ldots, N\},\ j = 1, \ldots, k \right\}.$$

$\overline{\mathbb{U}}_n^N$, consisting of all N-dimensional real-valued functions for which all partial derivatives of degree up to n are positive, is called $\mathbb{U}_{s\text{-idircx}}$ by Denuit and Mesfioui (2010) and forms the basis for the s-increasing directionally convex order. Similar to $\underline{\mathbb{U}}_n^N$, if $u \in \overline{\mathbb{U}}_n^N$, then $u \in \overline{\mathbb{U}}_k^N$ for any $k < n$. Also, if $u \in \overline{\mathbb{U}}_n^N$, then for any $k < n$ and $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$,

$$\frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in \overline{\mathbb{U}}_{n-k}^N .$$

Definition 2.2.4 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of nth-degree convex stochastic dominance if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all $u \in \overline{\mathbb{U}}_n^N$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

Remark 2.2.5 The multivariate convex stochastic dominance in Definition 2.2.4 is different from what Fishburn (1974) calls convex stochastic dominance. Fishburn's usage of "convex" does not relate to the utility function. Instead, it refers to dominance results for convex combinations, or mixtures, of probability distributions in the univariate case, which we will extend to the multivariate case in Sect. 2.3.2 and use to eliminate alternatives in decision-making problems in Sect. 2.3.3. To clarify the distinction, we will use the term "mixture dominance" when referring to the type of stochastic dominance developed by Fishburn (1974, 1978). In contrast, our multivariate convex stochastic dominance can be thought of as "risk-seeking stochastic dominance" because $u \in \overline{\mathbb{U}}_n^N$ for any $n > 1$ implies that u is risk seeking with respect to each individual attribute and is multivariate risk seeking in the sense of Richard (1975). Similarly, our multivariate concave stochastic dominance from Definition 2.2.2 can be thought of as "risk-averse stochastic dominance" because $u \in \underline{\mathbb{U}}_n^N$ for any $n > 1$ means that u is risk averse with respect to each attribute and is multivariate risk averse (Richard 1975). The correlation-increasing transformations of Epstein and Tanny (1980) link multivariate risk aversion and multivariate risk seeking to correlation aversion and correlation loving, respectively.

2.2.2 The Notion of nth-Degree Risk in the Multivariate Case

By Definition 2.2.2 (2.2.4), concave (convex) stochastic dominance of degree n implies stochastic dominance of any higher degree. To isolate a higher-degree effect in the univariate case, Ekern (1980) introduced the concept of nth-degree risk. Examples include Rothschild and Stiglitz (1970), who focus on a 2nd-degree effect in terms of a mean-preserving spread, and Menezes et al. (1980), who isolate a 3rd-degree effect via a mean-variance-preserving transformation. This subsection extends that concept to the multivariate case and relates it to concave and convex stochastic dominance.

Definition 2.2.6 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$, $\tilde{\mathbf{y}}$ has more nth-degree risk than $\tilde{\mathbf{x}}$ if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ such that

$$(-1)^{n-1}\frac{\partial^n u(\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_n}} \geq 0$$

for any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, n$.

Theorem 2.2.7 *The random vector $\tilde{\mathbf{y}}$ has more nth-degree risk than the random vector $\tilde{\mathbf{x}}$ if and only if*

(1) *$\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of nth-degree concave stochastic dominance, and*
(2) *the kth moments of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ are identical for $k = 1, \ldots, n-1$:*

$$\mathbb{E}[\tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_k}] = \mathbb{E}[\tilde{y}_{i_1}\tilde{y}_{i_2}\cdots\tilde{y}_{i_k}]$$

for any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$.

Proof For the "only if" part, (1) holds by the definition of $\underline{\mathbb{U}}_n^N$. For (2), consider $u(\mathbf{x}) = x_{i_1}x_{i_2}\cdots x_{i_k}$ for any $i_j \in \{1, \ldots, N\}$ and $k < n$. For this $u(\mathbf{x})$,

$$(-1)^{n-1}\frac{\partial^n u(\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_n}} = 0$$

for any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, n$. Therefore,

$$\mathbb{E}[\tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_k}] \geq \mathbb{E}[\tilde{y}_{i_1}\tilde{y}_{i_2}\cdots\tilde{y}_{i_k}].$$

Similarly, for $u(\mathbf{x}) = -x_{i_1}x_{i_2}\cdots x_{i_k}$,

$$\mathbb{E}[\tilde{y}_{i_1}\tilde{y}_{i_2}\cdots\tilde{y}_{i_k}] \geq \mathbb{E}[\tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_k}].$$

Thus,

$$\mathbb{E}[\tilde{x}_{i_1}\tilde{x}_{i_2}\cdots\tilde{x}_{i_k}] = \mathbb{E}[\tilde{y}_{i_1}\tilde{y}_{i_2}\cdots\tilde{y}_{i_k}].$$

Now, suppose that (1) and (2) hold. We need to prove that for any u such that

$$(-1)^{n-1}\frac{\partial^n u(\mathbf{x})}{\partial x_{i_1}\cdots\partial x_{i_n}} \geq 0$$

for any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, n$, $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$. Since u is differentiable at least n times, all lower-degree derivatives exist and are bounded on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$. Therefore, there exist coefficients $c_{i_1,\dots,i_k}$ for $k = 1, \dots, n-1$ and any $i_j \in \{1, \dots, N\}$, $j = 1, \dots, k$, such that

$$v(\mathbf{x}) = u(\mathbf{x}) + \sum c_{i_1,\dots,i_k} x_{i_1} x_{i_2} \cdots x_{i_k}$$

and $v \in \underline{\mathbb{U}}_n^N$, where the summation is over all possible combinations of $i_1, \dots, i_k$. By (1), $\mathbb{E}[v(\tilde{\mathbf{x}})] \geq \mathbb{E}[v(\tilde{\mathbf{y}})]$, and by (2), $\mathbb{E}[v(\tilde{\mathbf{x}})] - \mathbb{E}[v(\tilde{\mathbf{y}})] = \mathbb{E}[u(\tilde{\mathbf{x}})] - \mathbb{E}[u(\tilde{\mathbf{y}})]$. Therefore, $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$. □

Remark 2.2.8 In the univariate case, Ekern (1980) defines a person as being "nth-degree risk averse" if the nth derivative of her utility function is positive (negative) when n is odd (even). Our interpretation of multivariate concave stochastic dominance as risk-averse stochastic dominance is consistent with the extension of the notion of being nth-degree risk averse to the multivariate case.

Theorem 2.2.9 *The random vector $\tilde{\mathbf{y}}$ has more nth-degree risk than the random vector $\tilde{\mathbf{x}}$ if and only if*

(1) $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ ($\tilde{\mathbf{y}}$ *dominates* $\tilde{\mathbf{x}}$) *in the sense of nth-degree convex stochastic dominance when n is odd* (*even*), *and*
(2) *the kth moments of* $\tilde{\mathbf{x}}$ *and* $\tilde{\mathbf{y}}$ *are identical for* $k = 1, \dots, n-1$.

The proof of Theorem 2.2.9 is similar to the proof of Theorem 2.2.7.

Corollary 2.2.10 (to Theorems 2.2.7 and 2.2.9) *If n is odd* (*even*) *and the kth moments of* $\tilde{\mathbf{x}}$ *and* $\tilde{\mathbf{y}}$ *are identical for* $k = 1, \dots, n-1$, *then* $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of nth-degree concave stochastic dominance if and only if* $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ ($\tilde{\mathbf{y}}$ *dominates* $\tilde{\mathbf{x}}$) *in the sense of nth-degree convex stochastic dominance*.

Thus, if all moments of degree less than n are identical, convex dominance goes along with higher nth moments for both odd and even n. With concave dominance, this holds only for odd n. For even n, concave dominance goes along with lower nth moments. These results relate stochastic dominance to ordering by moments, in the sense that convex dominance likes all moments to be higher, whereas concave dominance likes odd moments to be higher and even moments to be lower.

2.2.3 Connections with Preferences for Combining Good with Bad or Good with Good and Bad with Bad

Next, we show a connection between our definition of multivariate concave stochastic dominance and a preference for combining good lotteries with bad lotteries as opposed to combining good lotteries with good and bad lotteries with bad. This preference can be thought of as a type of risk aversion, so it is similar in spirit to the assumption of risk aversion in the single-attribute case. We let $\langle \tilde{\mathbf{x}}, \tilde{\mathbf{y}} \rangle$ denote a lottery with equal chances of getting $\tilde{\mathbf{x}}$ or $\tilde{\mathbf{y}}$.

Theorem 2.2.11 *Let $\tilde{\mathbf{x}}_m$, $\tilde{\mathbf{y}}_m$, $\tilde{\mathbf{x}}_n$, and $\tilde{\mathbf{y}}_n$ be mutually independent N-dimensional random vectors with $\tilde{\mathbf{x}}_i$ dominating $\tilde{\mathbf{y}}_i$ in the sense of ith-degree concave stochastic dominance, $i = m, n$. Then $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n \rangle$ dominates $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n \rangle$ in the sense of $(n+m)$th-degree concave stochastic dominance.*

Proof Consider any $u \in \underline{\mathbb{U}}_{n+m}^N$, and denote

$$v(\mathbf{z}) = \mathbb{E}\big[u(\tilde{\mathbf{y}}_m + \mathbf{z})\big] - \mathbb{E}\big[u(\tilde{\mathbf{x}}_m + \mathbf{z})\big].$$

Now

$$0.5\,\mathbb{E}\big[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n)\big] + 0.5\,\mathbb{E}\big[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n)\big] \geq 0.5\,\mathbb{E}\big[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n)\big] + 0.5\,\mathbb{E}\big[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n)\big]$$

is equivalent to

$$\mathbb{E}\big[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n)\big] - \mathbb{E}\big[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n)\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n)\big] - \mathbb{E}\big[u(\tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n)\big],$$

or $\mathbb{E}[v(\tilde{\mathbf{x}}_n)] \geq \mathbb{E}[v(\tilde{\mathbf{y}}_n)]$. It remains to show that $v(\mathbf{z}) \in \underline{\mathbb{U}}_n^N$. For any $k = 1, \ldots, n$ and any $i_j \in \{1, \ldots, N\}$, $j = 1, \ldots, k$,

$$(-1)^{k-1} \frac{\partial^k v(\mathbf{z})}{\partial z_{i_1} \cdots \partial z_{i_k}} = (-1)^{k-1} \left(\mathbb{E}\left[\frac{\partial^k u(\tilde{\mathbf{y}}_m + \mathbf{z})}{\partial z_{i_1} \cdots \partial z_{i_k}} \right] - \mathbb{E}\left[\frac{\partial^k u(\tilde{\mathbf{x}}_m + \mathbf{z})}{\partial z_{i_1} \cdots \partial z_{i_k}} \right] \right),$$

and

$$(-1)^k \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}} \in \underline{\mathbb{U}}_{m+n-k}^N \subset \underline{\mathbb{U}}_m^N .$$

Therefore, $(-1)^{k-1} \frac{\partial^k v(\mathbf{z})}{\partial z_{i_1} \cdots \partial z_{i_k}} \geq 0$, so $v(\mathbf{z}) \in \underline{\mathbb{U}}_n^N$. □

Theorem 2.2.11 shows that concave stochastic dominance from Definition 2.2.2 is consistent with a preference for combining good with bad (up to degree n), where good and bad are understood in terms of lower-degree concave stochastic dominance. What if a decision maker prefers to combine good with good and bad with bad, as opposed to combining good with bad?

Theorem 2.2.12 *Let $\tilde{\mathbf{x}}_m$, $\tilde{\mathbf{y}}_m$, $\tilde{\mathbf{x}}_n$, and $\tilde{\mathbf{y}}_n$ be mutually independent N-dimensional random vectors with $\tilde{\mathbf{x}}_i$ dominating $\tilde{\mathbf{y}}_i$ in the sense of ith-degree convex stochastic dominance, $i = m, n$. Then $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n \rangle$ dominates $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n \rangle$ in the sense of $(n+m)$th-degree convex stochastic dominance.*

Proof This is, essentially, a corollary to Theorem 2.2.11. Observe that $u(\mathbf{x}) \in \underline{\mathbb{U}}_n^N$ if and only if $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \overline{\mathbb{U}}_n^N$. Therefore, $\tilde{\mathbf{x}}_i$ dominates $\tilde{\mathbf{y}}_i$ in the sense of ith-degree convex stochastic dominance if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_i$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_i$ in the sense of ith-degree concave stochastic dominance. By Theorem 2.2.11, $\langle \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_n, \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_n \rangle$ dominates $\langle \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} -$

$\tilde{\mathbf{x}}_n, \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_m + \underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}_n\rangle$ in the sense of $(n+m)$th-degree concave stochastic dominance, and thus $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{x}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{y}}_n\rangle$ dominates $\langle \tilde{\mathbf{x}}_m + \tilde{\mathbf{y}}_n, \tilde{\mathbf{y}}_m + \tilde{\mathbf{x}}_n\rangle$ in the sense of $(n+m)$th-degree convex stochastic dominance. □

Definition 2.2.2 extends the standard definition of univariate stochastic dominance to the multivariate case. As Theorem 2.2.11 shows, it preserves a preference for combining good with bad (Eeckhoudt and Schlesinger 2006; Eeckhoudt et al. 2009). The preference for combining good with bad associated with $u \in \underline{\mathbb{U}}_n^N$ can be viewed as a form of risk aversion. For example, it implies that u is correlation averse (Epstein and Tanny 1980; Eeckhoudt et al. 2007, Denuit et al. 2010), which can be interpreted as a form of risk aversion. Definition 2.2.4 and Theorem 2.2.12 develop similar orderings based on the opposite preference for combining good with good and bad with bad, and show the connection between convex and concave stochastic dominance that follows from the fact that $u(\mathbf{x}) \in \underline{\mathbb{U}}_n^N$ if and only if $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \overline{\mathbb{U}}_n^N$. The preference for combining good with good and bad with bad associated with $u \in \overline{\mathbb{U}}_n^N$ implies that u is correlation loving, a form of risk taking.

2.3 Comparing Alternatives via Multivariate Stochastic Dominance

Here we present several results that are useful for comparing alternatives according to the stochastic dominance relations from Sect. 2.2. In Sect. 2.3.1 we show conditions under which dominance orderings remain unchanged in the presence of background risk, with independence playing an important role. In Sect. 2.3.2 we use mixture dominance to show that an alternative, even if not dominated by any single alternative, can be eliminated from consideration if it is dominated by a mixture of other alternatives. A simple example is presented in Sect. 2.3.3 to illustrate the concepts from Sects. 2.2–2.3.

2.3.1 Stochastic Dominance with Additive and Multiplicative Background Risk

When one faces a choice between two (or more) risky alternatives, this decision is often not made in isolation, in the sense that there are other risks that affect the decision maker but are outside of the decision maker's control. Therefore, it is important to know whether a stochastic dominance ordering established in the absence of background risk will remain the same when background risk is present.

Consider a choice between two projects, with consequences characterized by random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. In the presence of additive background risk, represented by the random vector $\tilde{\mathbf{a}}$, we are interested in comparing $\tilde{\mathbf{a}} + \tilde{\mathbf{x}}$ and $\tilde{\mathbf{a}} + \tilde{\mathbf{y}}$. In the presence of multiplicative background risk, represented by the random vector $\tilde{\mathbf{m}}$, the appropriate comparison is between $\tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and $\tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$, where $\mathbf{m} \otimes \mathbf{x}$ denotes the component-wise product, $(m_1 x_1, \ldots, m_N x_N)$. If both additive and multiplicative background risks are present, $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ are compared.

Theorem 2.3.1 *Let* $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{a}}$, *and* $\tilde{\mathbf{m}}$, $\tilde{\mathbf{m}} \gneqq \mathbf{0}$, *be* N*-dimensional random vectors such that for any fixed* $\mathbf{a}$ *and* $\mathbf{m}$, $\tilde{\mathbf{x}}|\mathbf{m}, \mathbf{a}$ *dominates* $\tilde{\mathbf{y}}|\mathbf{m}, \mathbf{a}$ *in the sense of* n*th-degree concave* (*convex*) *stochastic dominance. Then* $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ *in the sense of* n*th-degree concave* (*convex*) *stochastic dominance.*

Proof Consider any $u \in \underline{\mathbb{U}}_n^N$ $(u \in \overline{\mathbb{U}}_n^N)$. For any fixed $\mathbf{a}$ and $\mathbf{m}$, $v(\mathbf{x} \mid \mathbf{a}, \mathbf{m}) = u(\mathbf{a} + \mathbf{m} \otimes \mathbf{x})$, as a function of $\mathbf{x}$, belongs to $\underline{\mathbb{U}}_n^N$ $(\overline{\mathbb{U}}_n^N)$. Therefore, $\mathbb{E}[v(\tilde{\mathbf{x}} \mid \mathbf{a}, \mathbf{m})] \geq \mathbb{E}[v(\tilde{\mathbf{y}} \mid \mathbf{a}, \mathbf{m})]$. Taking expectations with respect to $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$ yields $\mathbb{E}[u(\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}})]$. □

The result of Theorem 2.3.1 is quite intuitive. If $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$ for each possible value of $\mathbf{a}$ and $\mathbf{m}$, then $\tilde{\mathbf{x}}$ is preferred to $\tilde{\mathbf{y}}$ even if we are uncertain about the exact values of $\mathbf{a}$ and $\mathbf{m}$. If the project risk is independent of the background risk, the situation is further simplified.

Corollary 2.3.2 (to Theorem 2.3.1) *Let* $\tilde{\mathbf{x}}$, $\tilde{\mathbf{y}}$, $\tilde{\mathbf{a}}$, *and* $\tilde{\mathbf{m}}$, $\tilde{\mathbf{m}} \gneqq \mathbf{0}$, *be* N*-dimensional random vectors such that* $\tilde{\mathbf{x}}$ *and* $\tilde{\mathbf{y}}$ *are independent of* $\tilde{\mathbf{a}}$ *and* $\tilde{\mathbf{m}}$. *If* $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of* n*th-degree concave* (*convex*) *stochastic dominance, then* $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{y}}$ *in the sense of* n*th-degree concave* (*convex*) *stochastic dominance.*

Thus, independent background risk preserves stochastic dominance orderings. Note that no assumption is made about the relationship between the background risks $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{m}}$; they can be dependent. The assumption of independence of the project risk and the background risk is crucial, however. If background risk is not independent of project risk, preferences with and without background risk might be the opposite (Tsetlin and Winkler 2005). For example, suppose that a manager is considering adding a new project to an existing portfolio of projects. Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ represent the consequences of two potential new projects, and let $\tilde{\mathbf{a}}$ represent the consequences of the existing portfolio. Even if the manager is multivariate risk averse and $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in terms of multivariate concave stochastic dominance, she might prefer the new project associated with $\tilde{\mathbf{y}}$ (i.e., prefer $\tilde{\mathbf{a}} + \tilde{\mathbf{y}}$ to $\tilde{\mathbf{a}} + \tilde{\mathbf{x}}$) if the correlations between the components of $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{y}}$ are smaller than those for $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{x}}$.

Theorem 2.3.1 and its Corollary 2.3.2 can also be used to compare random vectors that are functions of other random vectors, which can be ordered by stochastic dominance. For instance, if the consequences of a particular alternative can be represented as $\tilde{\mathbf{a}} + \tilde{\mathbf{m}} \otimes \tilde{\mathbf{x}}$ and any of the mutually independent random vectors $\tilde{\mathbf{x}}$, $\tilde{\mathbf{a}}$, and $\tilde{\mathbf{m}}$ is improved in the sense of stochastic dominance, what can we say about the resulting changes to this alternative?

Corollary 2.3.3 (to Theorem 2.3.1) *Let* $\tilde{\mathbf{x}}_1, \tilde{\mathbf{y}}_1, \tilde{\mathbf{x}}_2$, *and* $\tilde{\mathbf{y}}_2$ *be mutually independent* N*-dimensional random vectors with* $\tilde{\mathbf{x}}_i$ *dominating* $\tilde{\mathbf{y}}_i$ *in the sense of* n*th-degree concave (convex) stochastic dominance,* $i = 1, 2$. *Then* $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{x}}_2$ *dominates* $\tilde{\mathbf{y}}_1 + \tilde{\mathbf{y}}_2$ *in the sense of* n*th-degree concave (convex) stochastic dominance. If* $\tilde{\mathbf{x}}_1 \gneqq \mathbf{0}$, $\tilde{\mathbf{y}}_1 \gneqq \mathbf{0}$, $\tilde{\mathbf{x}}_2 \gneqq \mathbf{0}$, *and* $\tilde{\mathbf{y}}_2 \gneqq \mathbf{0}$, *then* $\tilde{\mathbf{x}}_1 \otimes \tilde{\mathbf{x}}_2$ *dominates* $\tilde{\mathbf{y}}_1 \otimes \tilde{\mathbf{y}}_2$ *in the sense of* n*th-degree concave (convex) stochastic dominance.*

Remark 2.3.4 It might be that, e.g., $\tilde{\mathbf{x}}_1 + \tilde{\mathbf{x}}_2$ dominates $\tilde{\mathbf{y}}_1 + \tilde{\mathbf{y}}_2$ in the sense of stochastic dominance of degree lower than n. For example, consider the univariate case (i.e., $N = 1$) with $\tilde{x}_1 = 1$, $\tilde{x}_2 = \tilde{y}_1 = 0$, and $\tilde{y}_2 = [-c, c]$. Then $\tilde{x}_i$ dominates $\tilde{y}_i$ in the sense of second-degree concave stochastic dominance for $i = 1, 2$, but also note that $\tilde{x}_1$ dominates $\tilde{y}_1$ in the sense of first-degree stochastic dominance. In this case $\tilde{x}_1 + \tilde{x}_2 = 1$ and $\tilde{y}_1 + \tilde{y}_2 = [-c, c]$. For $c \leq 1$, $\tilde{x}_1 + \tilde{x}_2$ dominates $\tilde{y}_1 + \tilde{y}_2$ in the sense of first-degree stochastic dominance, but for $c > 1$, $\tilde{x}_1 + \tilde{x}_2$ dominates $\tilde{y}_1 + \tilde{y}_2$ only in the sense of second-degree concave stochastic dominance.

Theorem 2.3.1 and its corollaries show that, e.g., adding a nonnegative random vector improves a multivariate distribution in the sense of first-degree concave and convex stochastic dominance. They also imply that if a set of N variables can be divided into two stochastically independent subgroups and one of these groups is improved in the sense of nth-degree concave (convex) stochastic dominance, then the joint distribution over all N variables is improved in the sense of nth-degree concave (convex) stochastic dominance. In particular, if N random variables are independent, then their joint distribution is improved in the sense of nth-degree concave (convex) stochastic dominance whenever the marginal distribution of any of the variables is improved in the sense of nth-degree concave (convex) stochastic dominance.

2.3.2 Elimination by Mixtures

If an alternative (represented by a random vector) is dominated by some other alternative when the decision maker's utility falls in a particular class (e.g., $u \in \underline{\mathbb{U}}_n^N$ for concave stochastic dominance and $u \in \overline{\mathbb{U}}_n^N$ for convex stochastic dominance), then the dominated alternative can be eliminated from further consideration, thereby simplifying the decision-making problem. Mixture dominance, developed by Fishburn (1974) as "convex stochastic dominance" for the univariate case, allows us to eliminate an alternative even if it is not dominated by any other single alternative, as long as it is dominated by a mixture of other alternatives, which is a weaker condition (Fishburn 1978). We define mixture dominance for the multivariate case and then extend Fishburn's (1978) result regarding elimination by mixtures.

Definition 2.3.5 For the random vectors $\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_k$ and utility class $\mathbb{U}^*$, $\tilde{\mathbf{x}}_{-k} = (\tilde{\mathbf{x}}_1, \ldots, \tilde{\mathbf{x}}_{k-1})$ dominates $\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to $\mathbb{U}^*$

if there exists $\mathbf{p} = (p_1, \dots, p_{k-1}) \geq 0$, $\sum_{i=1}^{k-1} p_i = 1$, such that

$$\sum_{i=1}^{k-1} p_i \, \mathbb{E}\big[u(\tilde{\mathbf{x}}_i)\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{x}}_k)\big]$$

for all $u \in \mathbb{U}^*$.

From Definition 2.3.5, the mixture can be thought of as a two-step process. In the first step, an alternative (a random vector $\tilde{\mathbf{x}}_i$) is chosen from $\tilde{\mathbf{x}}_{-k}$ where p_i represents the probability of choosing $\tilde{\mathbf{x}}_i$. Then at the second step, the uncertainty about $\tilde{\mathbf{x}}_i$ is resolved. Mixture dominance means that this mixture has a higher expected utility than $\tilde{\mathbf{x}}_k$ for all $u \in \mathbb{U}^*$.

Theorem 2.3.6 *If $\tilde{\mathbf{x}}_{-k}$ dominates $\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to $\mathbb{U}^*$, then for every $u \in \mathbb{U}^*$, there is an $i \in \{1, \dots, k-1\}$ such that*

$$\mathbb{E}\big[u(\tilde{\mathbf{x}}_i)\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{x}}_k)\big].$$

Proof For any $u \in \mathbb{U}^*$, there is a $\mathbf{p}$ such that

$$\sum_{i=1}^{k-1} p_i \, \mathbb{E}\big[u(\tilde{\mathbf{x}}_i)\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{x}}_k)\big].$$

This is impossible unless $\mathbb{E}[u(\tilde{\mathbf{x}}_i)] \geq \mathbb{E}[u(\tilde{\mathbf{x}}_k)]$ for some $i \in \{1, \dots, k-1\}$. □

Note that the $\tilde{\mathbf{x}}_i$ in Theorem 2.3.6 can be different for different $u \in \mathbb{U}^*$. The importance of Theorem 2.3.6 is that if $u \in \mathbb{U}^*$ and $\tilde{\mathbf{x}}_{-k}$ dominates $\tilde{\mathbf{x}}_k$ in the sense of mixture dominance with respect to the utility class of interest, then we can eliminate $\tilde{\mathbf{x}}_k$ from consideration even if none of $\tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{k-1}$ dominates $\tilde{\mathbf{x}}_k$ individually. Reducing the set of alternatives that need to be considered seriously is always helpful. Since some of the mixing probabilities can be zero, we can eliminate an alternative if it is dominated in the sense of mixture dominance by any subset of the other alternatives. Of course, mixture dominance with respect to $\underline{\mathbb{U}}_n^N$ or $\overline{\mathbb{U}}_n^N$ is of particular interest because it invokes concave or convex stochastic dominance and relates to a preference for combining good with bad or the opposite preference for combining good with good and bad with bad.

2.3.3 Example

A decision-making task is somewhat simplified if some potential alternatives can be eliminated from consideration without having to assess the full utility function, and that is where multivariate stochastic dominance can be helpful. In this section, we present a simple hypothetical example to illustrate the concepts from Sects. 2.2–2.3 without getting distracted by complicating details.

Suppose that a telecom company is entering a new market and deciding among different entry strategies. For simplicity, assume that a decision maker (DM) focuses on two attributes, x_1 (the net present value (NPV) of profits for the first five years, in millions of dollars) and x_2 (the market share in percentage terms at the end of the five-year period). To begin, it is not surprising to find that the DM prefers more of each of these attributes to less. For example, she prefers $(x_1, x_2) = (300, 40)$ to $(200, 30)$. This is simple first-degree multivariate stochastic dominance.

Next, if the DM concludes that she is risk averse with respect to NPV, then $(250, 30)$ would be preferred to $\langle(300, 30), (200, 30)\rangle$, a risky alternative that yields $(300, 30)$ or $(200, 30)$ with equal probabilities. Similarly, if she is risk averse with respect to market share, then $(250, 35)$ would be preferred to $\langle(250, 30), (250, 40)\rangle$. These two choices are consistent with second-degree concave stochastic dominance but not sufficient to indicate that she would always want to behave in accordance with second-degree concave stochastic dominance. For example, the risk aversion with respect to NPV and market share is not sufficient to dictate her choice between the two risky alternatives $\langle(300, 40), (200, 30)\rangle$ and $\langle(300, 30), (200, 40)\rangle$. She states a preference for the latter and decides after some thought that she is, in general, correlation averse. Thus, her preferences are consistent with second-degree concave stochastic dominance.

In practice, most comparisons between competing alternatives are not as clear-cut as the above examples. In other words, once obviously inferior alternatives have been eliminated, it may be hard to find many cases where one alternative dominates another. However, by looking at three or more alternatives, we may still be able to eliminate alternatives via mixture dominance, as discussed in Sect. 2.3.2.

For a simple example, consider the choice among three alternatives: $(300, 30)$, $(200, 40)$, and $\langle(300, 40), (200, 30)\rangle$. The first alternative gives a higher NPV, the second alternative gives a higher market share, and the third alternative is risky, with equal chances of either the high NPV and the high market share or the low NPV and the low market share. Note that a 50–50 mixture of the first two alternatives, $\langle(300, 30), (200, 40)\rangle$ dominates the third alternative by second-degree concave stochastic dominance, consistent with the DM's preference for combining good with bad. By Theorem 2.3.6, then, we can eliminate the third alternative.

Of course, if the DM has the opposite preference for combining good with good and bad with bad, then convex stochastic dominance is relevant, and the second-degree dominance orderings in the above examples will be reversed. For example, $\langle(300, 30), (200, 30)\rangle$ dominates $(250, 30)$ by second-degree convex stochastic dominance. Similarly, $\langle(300, 40), (200, 30)\rangle$ dominates $\langle(300, 30), (200, 40)\rangle$ by second-degree convex stochastic dominance, reflecting the fact that the DM is correlation loving.

The above comparisons among alternatives might have to be made in the presence of background risk. For example, the DM might be uncertain about the financial results of other ongoing projects of the telecom company, implying additive background risk with respect to the first attribute (NPV). She might also be uncertain about competitors' moves, which could translate into additive background risk with respect to the second attribute (market share). Finally, suppose that the company operates internationally and wants to express its NPV in another currency. In this case, the appropriate exchange rate, in the absence of hedging, would operate as multiplicative background risk with respect to the first attribute. As shown in Sect. 2.3.1, if the consequences of each alternative are independent of the background risk, then any stochastic dominance orderings are preserved, and any resulting elimination of alternatives remains optimal under such background risk.

2.4 Infinite-Degree Dominance

Now we explore what emerges if a preference between combining good with bad, or combining good with good and bad with bad, holds for any n. In this case dominance relations are defined via $\underline{\mathbb{U}}_\infty^N$ and $\overline{\mathbb{U}}_\infty^N$ that extend $\underline{\mathbb{U}}_n^N$ and $\overline{\mathbb{U}}_n^N$.

Definition 2.4.1

$$\underline{\mathbb{U}}_\infty^N = \left\{ u \left| (-1)^{k-1} \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_k}} \geq 0 \text{ for } k = 1, 2, \ldots \text{ and } i_j \in \{1, \ldots, N\}, j = 1, \ldots, k \right. \right\},$$

and

$$\overline{\mathbb{U}}_\infty^N = \left\{ u \left| \frac{\partial^k u(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_k}} \geq 0 \text{ for } k = 1, 2, \ldots \text{ and } i_j \in \{1, \ldots, N\}, j = 1, \ldots, k \right. \right\}.$$

Definition 2.4.2 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of infinite-degree concave (convex) stochastic dominance if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all $u \in \underline{\mathbb{U}}_\infty^N$ ($u \in \overline{\mathbb{U}}_\infty^N$), u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

Increasing the degree of dominance (n) restricts the set of utility functions with respect to which two random vectors are compared. Similarly, expanding the domain of definition of u (i.e., decreasing $\underline{\mathbf{x}}$ and/or increasing $\bar{\mathbf{x}}$) also restricts the set of utility functions and thus increases the set of random vectors that can be ordered by stochastic dominance.

2.4.1 Infinite-Degree Dominance and Mixtures of Multiattribute Exponential Utilities

We show in Theorem 2.4.3 that any $u \in \underline{\mathbb{U}}_\infty^N$, u defined on $[\underline{\mathbf{x}}, \infty)$, or $u \in \overline{\mathbb{U}}_\infty^N$, u defined on $(-\infty, \bar{\mathbf{x}}]$, is a mixture of multiattribute exponential utilities. Theorem 2.4.4

then shows that infinite-order dominance can be operationalized via multiattribute exponential utilities.

Theorem 2.4.3 *Consider a function* $u(\mathbf{x})$ *defined on* $[\underline{\mathbf{x}}, \infty)$. *Then* $u \in \underline{\mathbb{U}}^N_\infty$ *if and only if there exists a (not necessarily finite) measure* F *on* $[\mathbf{0}, \infty)$ *and constants* $b_1, \ldots, b_N$ *with* $b_i \geq 0$, $i = 1, \ldots, N$, *such that*

$$\begin{aligned} u(\mathbf{x}) = u(\underline{\mathbf{x}}) & \\ &+ \int_0^\infty \cdots \int_0^\infty \big(1 - \exp\big(-\big(r_1(x_1 - \underline{x}_1) + \cdots \\ &+ r_N(x_N - \underline{x}_N)\big)\big)\big)\, dF(r_1, \ldots, r_N) + \sum_{i=1}^N b_i(x_i - \underline{x}_i). \end{aligned} \tag{2.1}$$

Viewing the linear terms in (2.1) *as limiting forms of exponential utilities (as* $r_i \to 0$ *with* $r_j = 0$ *for* $j \neq i$) *and rescaling, we can express any* $u \in \underline{\mathbb{U}}^N_\infty$, u *defined on* $[\underline{\mathbf{x}}, \infty)$, *as a mixture of multiattribute exponential utilities*,

$$u(\mathbf{x}) = -\int_0^\infty \cdots \int_0^\infty \exp(-r_1 x_1 - \cdots - r_N x_N)\, dF(r_1, \ldots, r_N). \tag{2.2}$$

Similarly, any $u \in \overline{\mathbb{U}}^N_\infty$, u *defined on* $(-\infty, \bar{\mathbf{x}}]$, *can be expressed as*

$$u(\mathbf{x}) = \int_0^\infty \cdots \int_0^\infty \exp(r_1 x_1 + \cdots + r_N x_N)\, dF(r_1, \ldots, r_N). \tag{2.3}$$

A proof for the concave case in Theorem 2.4.3 is given in Tsetlin and Winkler (2009), and the proof for the convex case is similar. From Theorem 2.4.3 we can state the following result without a proof.

Theorem 2.4.4 *The random vector* $\tilde{\mathbf{x}}$ *dominates the random vector* $\tilde{\mathbf{y}}$ *in the sense of infinite-degree concave stochastic dominance for* u *defined on* $[\underline{\mathbf{x}}, \infty)$ *if and only if*

$$\mathbb{E}\big[\exp(-r_1\tilde{y}_1 - \cdots - r_N\tilde{y}_N)\big] \geq \mathbb{E}\big[\exp(-r_1\tilde{x}_1 - \cdots - r_N\tilde{x}_N)\big]$$

for all $\mathbf{r} \in [\mathbf{0}, \infty)$, *and* $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of infinite-degree convex stochastic dominance for* u *defined on* $(-\infty, \bar{\mathbf{x}}]$ *if and only if*

$$\mathbb{E}\big[\exp(r_1\tilde{x}_1 + \cdots + r_N\tilde{x}_N)\big] \geq \mathbb{E}\big[\exp(r_1\tilde{y}_1 + \cdots + r_N\tilde{y}_N)\big]$$

for all $\mathbf{r} \in [\mathbf{0}, \infty)$.

Theorem 2.4.4 provides a convenient criterion for comparing multivariate probability distributions. Note that the expectations in Theorem 2.4.4 correspond to moment generating functions for distributions of $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$. If we define $M_{\tilde{\mathbf{x}}}(\mathbf{r}) = \mathbb{E}[\exp(r_1\tilde{x}_1 + \cdots + r_N\tilde{x}_N)]$, then for concave stochastic dominance, we need

$M_{\tilde{\mathbf{x}}}(\mathbf{r}) \le M_{\tilde{\mathbf{y}}}(\mathbf{r})$ for all $\mathbf{r} \in (-\infty, \mathbf{0}]$, and for convex stochastic dominance, we need $M_{\tilde{\mathbf{x}}}(\mathbf{r}) \ge M_{\tilde{\mathbf{y}}}(\mathbf{r})$ for all $\mathbf{r} \in [\mathbf{0}, \infty)$.

Remark 2.4.5 The domain of definition of u is crucial for the result stated in Theorem 2.4.4. For instance, if $\tilde{\mathbf{x}} = (x_1, x_2) = (0.5, 0.5)$ and $\tilde{\mathbf{y}} = \langle(0, 1), (1, 0)\rangle$, then by examining the expectations in Theorem 2.4.4 we can show that $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ by infinite-degree concave stochastic dominance for u defined on $[\underline{\mathbf{x}}, \infty)$ (e.g., on $[\mathbf{0}, \infty)$). However, consider $u(\mathbf{x}) = x_1 + x_2 - x_1x_2$, u defined on $[\mathbf{0}, \mathbf{1}]$. Theorem 2.4.4 does not apply here, and taking expectations with respect to u yields $\mathbb{E}[u(\tilde{\mathbf{x}})] = 0.75 < \mathbb{E}[u(\tilde{\mathbf{y}})] = 1$. Therefore, $\tilde{\mathbf{x}}$ does not dominate $\tilde{\mathbf{y}}$ by infinite-degree concave stochastic dominance. If we increase the upper limit of the domain of this u above $\mathbf{1}$, then $u \notin \underline{\mathbb{U}}^2_\infty$ because $\frac{\partial u(\mathbf{x})}{\partial x_i} < 0$, $i = 1, 2$, when $\mathbf{x} > \mathbf{1}$. A similar situation can occur for any N, including the univariate case ($N = 1$). As noted previously, expanding the domain of definition of u restricts the set of utility functions with respect to which random vectors are compared. In the example, the set of utility functions $u \in \underline{\mathbb{U}}^2_\infty$ defined on $[\mathbf{0}, \mathbf{1}]$ is larger than the set of utility functions $u \in \underline{\mathbb{U}}^2_\infty$ defined on $[\mathbf{0}, \infty)$. The former set includes $u(\mathbf{x}) = x_1 + x_2 - x_1x_2$, whereas the latter does not.

2.4.2 Comparison of Multivariate Normal Distributions via Infinite-Degree Dominance

The multivariate normal distribution is the most commonly encountered multivariate distribution, is very tractable, and is a reasonable representation of uncertainty in many situations. Müller (2001) provides several results on the stochastic ordering of multivariate normal distributions. The expectations appearing in Theorem 2.4.4 are especially tractable in this case, and thus the comparison of two multivariate normal distributions based on infinite-degree (concave and convex) stochastic dominance is greatly simplified. If the random vector $\tilde{\mathbf{x}}$ is multivariate normal with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_N)$ and covariance matrix $\boldsymbol{\Sigma} = (\rho_{ij}\sigma_i\sigma_j)$, then

$$\mathbb{E}\big[\exp(r_1\tilde{x}_1 + \cdots + r_N\tilde{x}_N)\big] = \exp\left(\mathbf{r}\boldsymbol{\mu}^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t}{2}\right)\right),$$

where a superscript t denotes transposition, and

$$\mathbf{r}\boldsymbol{\mu}^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}\mathbf{r}^t}{2}\right) = \sum_{i=1}^{N} r_i\mu_i + \left(\sum_{i=1}^{N}\sum_{j=1}^{N} r_ir_j\frac{\rho_{ij}\sigma_i\sigma_j}{2}\right).$$

Thus, we have the following corollary to Theorem 2.4.4.

Corollary 2.4.6 (to Theorem 2.4.4) *Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be multivariate normal vectors with mean vectors $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_y$, and covariance matrices $\boldsymbol{\Sigma}_x$ and $\boldsymbol{\Sigma}_y$. Then $\tilde{\mathbf{x}}$ dominates*

$\tilde{\mathbf{y}}$ *in the sense of infinite-degree concave stochastic dominance if and only if*

$$-\mathbf{r}\boldsymbol{\mu}_y^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}_y\mathbf{r}^t}{2}\right) \geq -\mathbf{r}\boldsymbol{\mu}_x^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}_x\mathbf{r}^t}{2}\right)$$

for all $\mathbf{r} \in [\mathbf{0}, \infty)$, *and* $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of infinite-degree convex stochastic dominance if and only if*

$$\mathbf{r}\boldsymbol{\mu}_x^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}_x\mathbf{r}^t}{2}\right) \geq \mathbf{r}\boldsymbol{\mu}_y^t + \left(\frac{\mathbf{r}\boldsymbol{\Sigma}_y\mathbf{r}^t}{2}\right)$$

for all $\mathbf{r} \in [\mathbf{0}, \infty)$.

Thus, increasing any mean μ_i leads to stochastic dominance improvement (both concave and convex). Decreasing any correlation ρ_{ij} leads to concave (convex) stochastic dominance improvement (deterioration). Decreasing any standard deviation σ_i leads to concave (convex) stochastic dominance improvement (deterioration) if $\rho_{ij} \geq 0$ for all j. However, if $\rho_{ij} < 0$ for some j, things are more complicated. Overall, adding independent noise to attribute i leads to the increase of σ_i and to the decrease of the absolute value of correlations ρ_{ij}. Thus, increasing σ_i without changing correlations is equivalent to adding independent noise to attribute i and then to adjusting the correlations ρ_{ij} up (if ρ_{ij} is positive) or down (if ρ_{ij} is negative). For concave (convex) stochastic dominance, adding independent noise is bad (good), and adjusting correlations up (down) is bad (good). If all correlations are positive, increasing any standard deviation leads to convex (concave) stochastic dominance improvement (deterioration). If some correlations are negative, the effect might go either way. Tsetlin and Winkler (2007) established similar confounding effects of increasing standard deviations in target-oriented situations.

2.5 Comparisons with Other Multivariate Stochastic Orders

Many multivariate stochastic orders have been studied, and the appropriate order upon which to base multivariate stochastic dominance is not as obvious as it is in the univariate case. Once we move from $N = 1$ to $N > 1$, the relationship among the attributes complicates matters both in terms of the joint probability distribution and in terms of the utility function.

Two commonly used multivariate stochastic orders are the lower and upper orthant orders, based on lower orthants $\{\mathbf{x} \mid \mathbf{x} \leq \mathbf{c}\}$ and upper orthants $\{\mathbf{x} \mid \mathbf{x} > \mathbf{c}\}$ for a given $\mathbf{c}$ (Müller and Stoyan 2002). By definition, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ via the lower orthant order if

$$\mathbb{P}(\tilde{\mathbf{x}} \leq \mathbf{c}) \leq \mathbb{P}(\tilde{\mathbf{y}} \leq \mathbf{c})$$

for all $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, and $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ via the upper orthant order if

$$\mathbb{P}(\tilde{\mathbf{x}} > \mathbf{c}) \geq \mathbb{P}(\tilde{\mathbf{y}} > \mathbf{c})$$

for all $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$ These orders highlight an important way in which moving from the univariate to the multivariate case makes stochastic orders and stochastic dominance more complex. In the univariate case, $\mathbb{P}(\tilde{x} \leq c) + \mathbb{P}(\tilde{x} > c) = 1$ for any c. When $N \geq 2$, $\mathbb{P}(\tilde{\mathbf{x}} \leq \mathbf{c}) + \mathbb{P}(\tilde{\mathbf{x}} > \mathbf{c}) \leq 1$ for any $\mathbf{c} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, and this becomes more of an issue as N increases because the lower and upper orthants for a given $\mathbf{c}$ represent only 2 of the 2^N orthants associated with $\mathbf{c}$.

We focus here on multivariate $\mathbf{s}$-increasing orders, a family of stochastic orders for which some interesting connections and comparisons with our multivariate concave and convex stochastic dominance can be drawn. This helps to highlight potential advantages and disadvantages of our approach.

We begin by presenting the multivariate $\mathbf{s}$-increasing concave order, where $\mathbf{s} = (s_1, \ldots, s_N)$ is a vector of positive integers, and defining stochastic dominance in terms of this order. This is a natural generalization of the bivariate (s_1, s_2)-increasing concave orders introduced by Denuit et al. (1999) and studied by Denuit and Eeckhoudt (2010) and Denuit et al. (2010).

Definition 2.5.1

$$\mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icv}} = \left\{ u \,\middle|\, (-1)^{\sum_{i=1}^N k_i - 1} \frac{\partial^{\sum_{i=1}^N k_i} u(\mathbf{x})}{\partial x_1^{k_1} \cdots \partial x_N^{k_N}} \geq 0 \text{ for } k_i = 0, 1, \ldots, s_i, \right.$$
$$\left. i = 1, \ldots, N, \sum_{i=1}^N k_i \geq 1 \right\}.$$

Definition 2.5.2 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing concave order if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all $u \in \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icv}}$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

If $s_1 = \cdots = s_N = s$, we say that the order is an s-increasing concave order. Special cases of this are the lower orthant order when $s = 1$ and the lower orthant concave order when $s = 2$ (Mosler 1984).

Our multivariate concave stochastic dominance, based on $\underline{\mathbb{U}}_n^N$, has a convex counterpart, based on $\overline{\mathbb{U}}_n^N$. Similarly, $\mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icv}}$ and dominance in terms of the $\mathbf{s}$-increasing concave order have convex counterparts (Denuit and Mesfioui 2010).

Definition 2.5.3

$$\mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icx}} = \left\{ u \,\middle|\, \frac{\partial^{\sum_{i=1}^N k_i} u(\mathbf{x})}{\partial x_1^{k_1} \cdots \partial x_N^{k_N}} \geq 0 \text{ for } k_i = 0, 1, \ldots, s_i, i = 1, \ldots, N, \sum_{i=1}^N k_i \geq 1 \right\}.$$

Definition 2.5.4 For random vectors $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing convex order if

$$\mathbb{E}\big[u(\tilde{\mathbf{x}})\big] \geq \mathbb{E}\big[u(\tilde{\mathbf{y}})\big]$$

for all $u \in \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icx}}$, u defined on $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$.

The $\mathbf{s}$-increasing concave order and the $\mathbf{s}$-increasing convex order are closely related, because $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the $\mathbf{s}$-increasing concave order if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}$ in the $\mathbf{s}$-increasing convex order. This follows from the fact that if $u \in \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icv}}$, then $-u(\underline{\mathbf{x}} + \bar{\mathbf{x}} - \mathbf{x}) \in \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icx}}$. An $\mathbf{s}$-increasing convex order with $s_1 = \cdots = s_N = s$ is an s-increasing convex order. Analogous to the concave case, the s-increasing convex order with $s = 1$ is the upper orthant order.

Theorem 2.5.5 provides conditions characterizing stochastic dominance in the sense of the multivariate $\mathbf{s}$-increasing concave and convex orders via partial moments, without reference to utilities. The following remark indicates an alternative characterization in terms of integral conditions.

Theorem 2.5.5 *Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be random vectors with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$, and denote $x_+ = \max\{x, 0\}$. Then*

(1) *$\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing concave order if and only if*

$$\mathbb{E}\left[\prod_{i=1}^{N} (c_i - \tilde{x}_i)_+^{k_i - 1}\right] \leq \mathbb{E}\left[\prod_{i=1}^{N} (c_i - \tilde{y}_i)_+^{k_i - 1}\right]$$

for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \bar{x}_i$ if $k_i = 1, \ldots, s_i - 1$, $i = 1, \ldots, N$.

(2) *$\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing convex order if and only if*

$$\mathbb{E}\left[\prod_{i=1}^{N} (\tilde{x}_i - c_i)_+^{k_i - 1}\right] \geq \mathbb{E}\left[\prod_{i=1}^{N} (\tilde{y}_i - c_i)_+^{k_i - 1}\right]$$

for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \underline{x}_i$ if $k_i = 1, \ldots, s_i - 1$, $i = 1, \ldots, N$.

Proof Statement (2) is proven in Denuit and Mesfioui (2010) (Proposition 3.1). Statement (1) follows from (2) and the duality between the concave and convex orders: $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing concave order if and only if $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{y}}$ dominates $\underline{\mathbf{x}} + \bar{\mathbf{x}} - \tilde{\mathbf{x}}$ in the multivariate $\mathbf{s}$-increasing convex order. Therefore, from (2),

$$\mathbb{E}\left[\prod_{i=1}^{N} (\underline{x}_i + \bar{x}_i - \tilde{y}_i - c_i)_+^{k_i - 1}\right] \geq \mathbb{E}\left[\prod_{i=1}^{N} (\underline{x}_i + \bar{x}_i - \tilde{x}_i - c_i)_+^{k_i - 1}\right]$$

with $c_i = \underline{x}_i$ if $k_i < s_i$ and $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$, which is equivalent to (1). $\square$

Remark 2.5.6 Alternative necessary and sufficient conditions for dominance in the multivariate **s**-increasing concave and convex orders involve integral conditions. Let $F_{\tilde{\mathbf{x}}}$ be the cumulative distribution function $\mathbb{P}(\tilde{\mathbf{x}} \leq \mathbf{x})$ for $\tilde{\mathbf{x}}$. Starting with $F_{\tilde{\mathbf{x}}}^{(1,\ldots,1)} = F_{\tilde{\mathbf{x}}}$, define recursively the integrated left tails of $\tilde{\mathbf{x}}$ as

$$F_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_i+1,\ldots,k_N)}(\mathbf{x}) = \int_{\underline{x}_i}^{x_i} F_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_i,\ldots,k_N)}(x_1,\ldots,z_i,\ldots,x_N)\,dz_i \tag{2.4}$$

for $k_1,\ldots,k_N \geq 1$. The lower partial moments in Theorem 2.5.5(1) can be expressed via integrated left tails:

$$\mathbb{E}\left[\prod_{i=1}^{N}(c_i - \tilde{x}_i)_+^{k_i-1}\right] = \left[\prod_{i=1}^{N}(k_i-1)!\right] F_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_N)}(\mathbf{c}).$$

Then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate **s**-increasing concave order if and only if $F_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_N)}(\mathbf{c}) \leq F_{\tilde{\mathbf{y}}}^{(k_1,\ldots,k_N)}(\mathbf{c})$ for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \bar{x}_i$ if $k_i = 1,\ldots,s_i-1$, $i = 1,\ldots,N$. When $N = 1$, (2.4) is the standard integral condition for univariate stochastic dominance.

An expression similar to (2.4), involving integrated right tails of $\tilde{\mathbf{x}}$, holds for the multivariate **s**-increasing convex order (Denuit and Mesfioui 2010). If $G_{\tilde{\mathbf{x}}}(\mathbf{x}) = \mathbb{P}(\tilde{\mathbf{x}} > \mathbf{x})$ and $G_{\tilde{\mathbf{x}}}^{(1,\ldots,1)} = G_{\tilde{\mathbf{x}}}$, define recursively

$$G_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_i+1,\ldots,k_N)}(\mathbf{x}) = \int_{x_i}^{\bar{x}_i} G_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_i,\ldots,k_N)}(x_1,\ldots,z_i,\ldots,x_N)\,dz_i \tag{2.5}$$

for $k_1,\ldots,k_N \geq 1$. Then $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate **s**-increasing convex order if and only if $G_{\tilde{\mathbf{x}}}^{(k_1,\ldots,k_N)}(\mathbf{c}) \geq G_{\tilde{\mathbf{y}}}^{(k_1,\ldots,k_N)}(\mathbf{c})$ for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \underline{x}_i$ if $k_i = 1,\ldots,s_i-1$, $i = 1,\ldots,N$.

Mosler (1984) showed that stochastic dominance in terms of two special cases of the multivariate s-increasing concave order is related to multiplicative utilities. First, $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in terms of the lower orthant order ($s = 1$) if and only if $\mathbb{E}[u(\tilde{\mathbf{x}})] \geq \mathbb{E}[u(\tilde{\mathbf{y}})]$ for all multiplicative utilities of the form $u(\mathbf{x}) = -\prod_{i=1}^{N}(-u_i(x_i))$, where $u_i(x_i) \leq 0$ and $\frac{du_i(x_i)}{dx_i} \geq 0$ for all x_i, $i = 1,\ldots,N$. Second, this dominance extends to the lower orthant concave order ($s = 2$) if each $u_i(x_i)$ is also concave. Theorem 2.5.7 extends these results to the multivariate **s**-increasing concave and convex orders for any **s**, showing that this order corresponds to the preferences of decision makers having utility functions consistent with mutual utility independence (Keeney and Raiffa 1976).

Theorem 2.5.7 *Let $\tilde{\mathbf{x}}$ and $\tilde{\mathbf{y}}$ be random vectors with support contained in $[\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, $-\infty < \underline{\mathbf{x}} < \bar{\mathbf{x}} < \infty$. Then*

(1) $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of the multivariate* $\mathbf{s}$*-increasing concave order if and only if*

$$(-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \leq (-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right]$$

for all $u_i \leq 0$, $u_i \in \underline{\mathbb{U}}_{s_i}^1$, $i = 1, \ldots, N$.

(2) $\tilde{\mathbf{x}}$ *dominates* $\tilde{\mathbf{y}}$ *in the sense of the multivariate* $\mathbf{s}$*-increasing convex order if and only if*

$$\mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \geq \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right]$$

for all $u_i \geq 0$, $u_i \in \overline{\mathbb{U}}_{s_i}^1$, $i = 1, \ldots, N$.

Proof For (1), suppose that $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing concave order, and let

$$v(\mathbf{x}) = -\prod_{i=1}^N \big(-u_i(x_i)\big).$$

If $u_i \leq 0$ and $u_i \in \underline{\mathbb{U}}_{s_i}^1$, $i = 1, \ldots, N$, then $v \in \mathbb{U}_{\mathbf{s}\text{-icv}}^{\mathrm{N}}$. Therefore, $\mathbb{E}[v(\tilde{\mathbf{x}})] \geq \mathbb{E}[v(\tilde{\mathbf{y}})]$, so that

$$(-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \leq (-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right].$$

For the converse, suppose that

$$(-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{x}_i)\right] \leq (-1)^N \mathbb{E}\left[\prod_{i=1}^N u_i(\tilde{y}_i)\right]$$

for all $u_i \leq 0$, $u_i \in \underline{\mathbb{U}}_{s_i}^1$, $i = 1, \ldots, N$. For $i = 1, \ldots, N$ and $k = 1, \ldots, s_i - 1$, let $u_i(x_i) = -(c_i - x_i)_+^{k_i+1}$ with $c_i = \bar{x}_i$ if $k_i < s_i$ and $c_i \in [\underline{x}, \bar{x}]$ if $k_i = s_i$. Thus, $u_i \leq 0$ and $u_i \in \underline{\mathbb{U}}_{s_i}^1$ if $k_i < s_i$. For $k_i = s_i$, u_i belongs to the closure of $\underline{\mathbb{U}}_{s_i}^1$ (i.e., there exists a sequence of functions $v_j \in \underline{\mathbb{U}}_{s_i}^1$, $j = 1, 2, \ldots$ with $v_j \to u_i$). Thus,

$$\mathbb{E}\left[\prod_{i=1}^N (c_i - \tilde{x}_i)_+^{k_i-1}\right] \leq \mathbb{E}\left[\prod_{i=1}^N (c_i - \tilde{y}_i)_+^{k_i-1}\right],$$

and by Theorem 2.5.5(1), $\tilde{\mathbf{x}}$ dominates $\tilde{\mathbf{y}}$ in the sense of the multivariate $\mathbf{s}$-increasing concave order. Statement (2) follows from (1) and the duality between the concave and convex orders, as in the proof of Theorem 2.5.5. □

We now compare our multivariate dominance with dominance for the multivariate **s**-increasing orders. There are some close similarities between the two approaches and some important differences. In terms of infinite-degree stochastic dominance, the two approaches are equivalent, because

$$\lim_{\min\{s_i\}\to\infty} \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icv}} = \underline{\mathbb{U}}^N_\infty$$

and

$$\lim_{\min\{s_i\}\to\infty} \mathbb{U}^{\mathrm{N}}_{\mathbf{s}\text{-icx}} = \overline{\mathbb{U}}^N_\infty .$$

However, this equivalence does not hold for finite n and **s**.

For finite n, nth-degree concave (convex) stochastic dominance is stronger than the n-increasing concave (convex) order, while the s-increasing concave (convex) order is stronger than (sN)th-degree concave (convex) stochastic dominance. In other words, (sN)th-degree concave (convex) stochastic dominance is between the s- and (sN)-increasing concave (convex) orders.

At a very basic level, our multivariate stochastic dominance is a natural extension of standard univariate stochastic dominance in that both are based on a preference between combining good with bad and combining good with good and bad with bad. A preference for combining good with bad leads to multivariate concave dominance and the most common univariate dominance. The opposite preference leads to multivariate convex dominance and a risk-taking version of univariate dominance. The preference condition is easy for decision makers to understand and therefore easy to check. If the decision maker has a consistent preference one way or the other, this implies corresponding constraints on the utility function via $\underline{\mathbb{U}}^N_\infty$ and $\overline{\mathbb{U}}^N_\infty$, but the discussion about preferences does not require direct consideration of utility.

Dominance in the sense of the **s**-increasing orders cannot be related to a simple preference assumption, but it can be characterized in terms of integral conditions that are extensions of the integral conditions for standard univariate dominance. In contrast, our multivariate dominance admits no such integral conditions. From a practical standpoint, however, the integral conditions in (2.4) and (2.5) might be difficult to verify as N increases or $\sum_{i=1}^N s_i$ increases.

Of course, not all decision makers share the same preferences. Thus, the preferences of different decision makers can be consistent with different classes of utility functions and therefore with different definitions of dominance. The approach to multivariate stochastic dominance developed here is intuitively appealing and should fit the preferences of some decision makers. As such, it is a useful addition to the stochastic dominance toolbox.

2.6 Summary and Conclusions

The concept of stochastic dominance has been widely studied in the univariate case, and there is widespread agreement on an underlying stochastic order for such dominance. This standard order is consistent with a basic preference condition, a preference for combining good with bad, as opposed to combining good with good and bad with bad. Many multivariate stochastic orders have been studied. However, most lack sufficient connections with the standard univariate stochastic dominance order and are not based on an intuitive preference condition that is easy to explain to decision makers. We fill this gap by defining multivariate nth-degree concave stochastic dominance and nth-degree risk in a way that naturally extends the univariate case because it is consistent with the same basic preference assumption. As in the univariate case, multivariate infinite-degree stochastic dominance is equivalent to an exponential ordering. We also develop the notion of multivariate convex stochastic dominance, which is consistent with a preference for combining good with good and bad with bad, as opposed to combining good with bad.

After developing our notion of multivariate stochastic dominance, we present some results that are useful in applying our multivariate stochastic dominance relations to rank alternatives. We show that independent additive or multiplicative background risk does not change stochastic dominance orderings and show how stochastic dominance can be applied to the choice among several alternatives using elimination by mixtures. We consider multivariate infinite-degree stochastic dominance, which is equivalent to an exponential ordering, as in the univariate case, and discuss the ordering of multivariate normal distributions. Finally, we discuss the connection of our approach with one based on a family of multivariate orders having some similarities to the order we use.

Many situations involve multiple decision makers, and somewhat divergent preferences can make decision making challenging. Even if each member of the group assesses a utility function (a challenging task itself, particularly in a multiattribute setting), it would be surprising for all members of the group to have identical utilities. However, the preferences of group members might be somewhat similar, especially when they are making a decision for their company and not a personal decision. They most likely will agree on a preference for more of each attribute to less or can define the attributes in such a way as to guarantee that preference, so that first-order stochastic dominance is applicable. They might also agree that the company's situation makes it prudent to be risk averse and that in general, a preference for combining good with bad is reasonable. This implies that they all should be willing to use a utility function $u \in \underline{\mathbb{U}}_n^N$ for any $n > 1$ and therefore to use multiattribute concave stochastic dominance to eliminate some alternatives from consideration.

Making a decision in a multiattribute situation is likely to be a multistage process. Some alternatives might be eliminated using stochastic dominance; choice among other alternatives might require more careful preference assessments, with emphasis on particular tradeoffs. That in turn might lead to clarification of objectives and attributes and generation of new promising alternatives (Keeney 1992). The results of our paper can be useful in that kind of decision process.

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