Chapter 2
Moments, Couples, Equipollent Systems

2.1 Moment of a Vector About a Point

The moment of a vector $\mathbf{v}$, whose line of action passes through a point B, about a point A is the vector

$$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v}, \tag{2.1}$$

where $\mathbf{r}_{AB}$ is the position vector of B relative to A, and B is any point of line of action, Δ, of the vector $\mathbf{v}$ (Fig. 2.1). The moment vector $\mathbf{M}_A^{\mathbf{v}} = \mathbf{0}$ if and only if the line of action of $\mathbf{v}$ passes through A or $\mathbf{v} = \mathbf{0}$. The magnitude of $\mathbf{M}_A^{\mathbf{v}}$ is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = |\mathbf{r}_{AB}|\,|\mathbf{v}|\,\sin\theta = r_{AB}\,v\,\sin\theta,$$

where θ is the angle between $\mathbf{r}_{AB}$ and $\mathbf{v}$. The perpendicular distance from A to the line of action of $\mathbf{v}$ is

$$d = |\mathbf{r}_{AB}|\,\sin\theta = r_{AB}\,\sin\theta.$$

The moment vector is zero if the vectors $\mathbf{v}$ and $\mathbf{r}_{AB}$ are parallel. The magnitude of the $\mathbf{M}_A^{\mathbf{v}}$ is

$$|\mathbf{M}_A^{\mathbf{v}}| = M_A^{\mathbf{v}} = |\mathbf{v}|\,d = v\,d.$$

The moment vector $\mathbf{M}_A^{\mathbf{v}}$ is perpendicular to both $\mathbf{r}_{AB}$ and $\mathbf{v}$: $\mathbf{M}_A^{\mathbf{v}} \perp \mathbf{r}_{AB}$ and $\mathbf{M}_A^{\mathbf{v}} \perp \mathbf{v}$. If the moment vector is non-zero then it is perpendicular to the plane defined by the distinct directions of $\mathbf{r}_{AB}$ and $\mathbf{v}$. The moment given by Eq. (2.1) does not depend upon the choice of point on the line of action of $\mathbf{v}$. Instead of using the point B, the point B', $B' \in \Delta$ (Fig. 2.1), can be used. The position vector of B relative to A is $\mathbf{r}_{AB} = \mathbf{r}_{AB'} + \mathbf{r}_{B'B}$ where the vector $\mathbf{r}_{B'B}$ is parallel to $\mathbf{v}$, $\mathbf{r}_{B'B}||\mathbf{v}$. Therefore,

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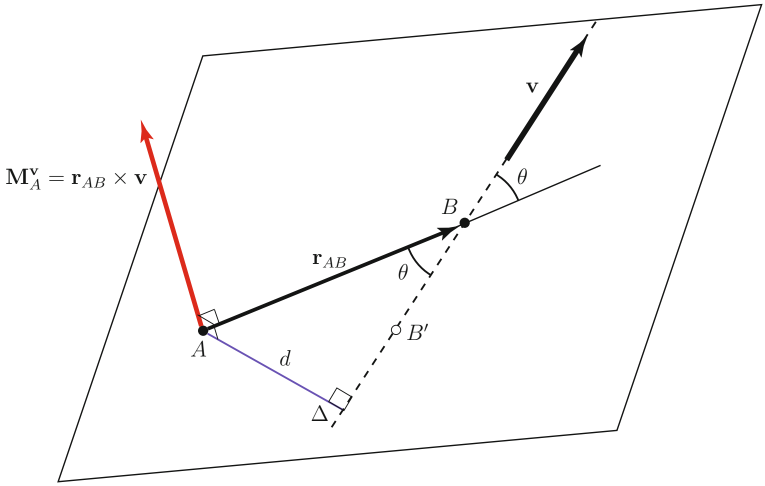


Fig. 2.1 Moment of a vector **v** about a point A

$$\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v} = (\mathbf{r}_{AB'} + \mathbf{r}_{B'B}) \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v} + \mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v}, \quad (2.2)$$

because $\mathbf{r}_{B'B} \times \mathbf{v} = \mathbf{0}$. The moment about a point is a vector in a particular direction. This moment vector is a sliding vector along that direction.

Next, using MATLAB®, the validity of Eq. (2.2) is shown. Three points A, B, and C are defined by three symbolic position vectors `rA_`, `rB_`, and `rC_`:

```
syms xA yA zA xB yB zB xC yC zC real
rA_ = [xA yA zA];
rB_ = [xB yB zB];
rC_ = [xC yC zC];
rBC_ = rC_ - rB_;
```

The vector **v** is selected as $\mathbf{v} = \mathbf{r}_C - \mathbf{r}_B$, or in MATLAB:

```
v_ = rC_ - rB_;
```

The line of action of the vector **v** is defined as should be by the line segment BC. A generic point B' (in MATLAB `Bp`) divides the line segment joining two given points B and C in a given ratio. The position vector of the point `Bp` is `rBp_`:

```
syms k real % k is a given real number
rBp_ = rB_ + k*(rC_-rB_);
```

The moment of the vector **v** with respect to A is calculated as $\mathbf{r}_{AB} \times \mathbf{v}$, $\mathbf{r}_{AB'} \times \mathbf{v}$, and $\mathbf{r}_{AC} \times \mathbf{v}$, or with MATLAB:

```
MB_ = cross(rB_-rA_, v_);      % rAB_ x v_
MBp_ = cross(rBp_-rA_, v_);    % rABp_ x v_
MC_ = cross(rC_-rA_, v_);      % rAC_ x v_
```

To prove that $\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB} \times \mathbf{v} = \mathbf{r}_{AB'} \times \mathbf{v} = \mathbf{r}_{AC} \times \mathbf{v}$ the following MATLAB commands are used:

```
% rAB_ x v_ = rABp_ x v_ = rAC_ x v_
fprintf('1=TRUE 0=FALSE\n')
T1=expand(MB_) == expand(MBp_);
fprintf('rAB_ x v_ == rABp_ x v_ => [%d %d %d]\n',T1)
T2=expand(MB_) == expand(MC_);
fprintf('rAB_ x v_ == rAC_ x v_ => [%d %d %d]\n',T2)
```

If `T1 = [1 1 1]` then `rAB_ x v_ == rABp_ x v_` is true and if `T2 = [1 1 1]` then `rAB_ x v_ == rAC_ x v_` is true.

As an example consider the vectors $\mathbf{r}_{AB}$, $\mathbf{r}_{AB'}$, $\mathbf{r}_{AC}$, $\mathbf{v}$ and $\mathbf{M}_A^{\mathbf{v}}$ where the following numerical data are used:$x_A = y_A = z_A = 0$, $x_B = 1$, $y_B = 2$, $z_B = 0$, $x_C = 3$, $y_C = 3$, $z_C = 0$, and $k = 0.75$. The numerical values for the vectors `rA_`, `rB_`, `rC_`, `rBp_`, `v_`, `MB_`, `MBp_`, and `MC_` are calculated in MATLAB with:

```
% A = O origin
slist={xA,yA,zA,...
       xB,yB,zB,...
       xC,yC,zC,k};
nlist={0,0,0,1,2,0,3,3,0,.75};

rB_ = subs(rB_,slist,nlist);
rC_ = subs(rC_,slist,nlist);
rBp_ = subs(rBp_,slist,nlist);
V_ = subs(v_,slist,nlist);

MB_ = subs(MB_,slist,nlist);
MBp_ = subs(MBp_,slist,nlist);
MC_ = subs(MC_,slist,nlist);
```

The numerical values are:

```
rB_ = [1.0 2.0 0]
rC_ = [3.0 3.0 0]
rBp_ = [2.5 2.8 0]
V_ = [2.0 1.0 0]

MB_ = [0 0 -3]
MBp_ = [0 0 -3]
MC_ = [0 0 -3]
```

The MATLAB commands for the current axes and for the Cartesian reference with the origin at A are:

```
a=3;
axis([0 a  0 a  -a a])
```

```
grid on, hold on
% Cartesian axes A=O origin
quiver3(0,0,0,a-.5,0,0,1, ...
    'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $x$',...
    'Position',[a-.5,0,0],'FontSize',12)
quiver3(0,0,0,0,a-.5,0,1, ...
    'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $y$',...
    'Position',[0,a-.5,0],'FontSize',12)
quiver3(0,0,0,0,0,a-.5,1, ...
    'Color','k','LineWidth',1)
text('Interpreter','latex','String','  $z$',...
    'Position',[0,0,a-.5],'FontSize',12)
```

The fonts for the labels x, y, and z are LaTex fonts. The vectors `rB_`, `rC_`, `rBp_`, `V_`, and the line `BC` are plotted with:

```
quiver3(0,0,0, rB_(1),rB_(2),rB_(3),1,...
    'Color','k','LineWidth',1)
quiver3(0,0,0, rC_(1),rC_(2),rC_(3),1,...
    'Color','k','LineWidth',1)
quiver3(0,0,0, rBp_(1),rBp_(2),rBp_(3),1,...
    'Color','k','LineWidth',1)
quiver3(rB_(1),rB_(2),rB_(3), V_(1),V_(2),V_(3),1,...
    'Color','g','LineWidth',1)
line...
 ([rB_(1) rC_(1)],[rB_(2) rC_(2)],[rB_(3) rC_(3)],...
    'LineStyle','--','LineWidth',2)
```

The vectors `MB_`, `MBp_`, and `MC_` are plotted with:

```
quiver3(0,0,0, MB_(1),MB_(2),MB_(3),1,...
    'Color','r','LineWidth',2)
quiver3(0,0,0, MBp_(1),MBp_(2),MBp_(3),1,...
    'Color','g','LineWidth',2)
quiver3(0,0,0, MC_(1),MC_(2),MC_(3),1,...
    'Color','r','LineWidth',2)
```

The labels for the vectors are printed with

```
text('Interpreter','latex','String','  $A=O$',...
text('Interpreter','latex','String','  $A=O$',...
    'Position',[0,0,0],'FontSize',12)
text('Interpreter','latex','String','  $B$',...
    'Position',[rB_(1),rB_(2),rB_(3)],'FontSize',12)
text('Interpreter','latex','String',...
```

```
'$B^\prime$','Position',[rBp_(1),rBp_(2),rBp_(3)],...
'FontSize',12)
text('Interpreter','latex','String','  $C$',...
    'Position',[rC_(1),rC_(2),rC_(3)],'FontSize',12)
text('Interpreter','latex','String',...
'  ${\bf M}_A^{\bf v}$','Position',...
[MB_(1),MB_(2),MB_(3)+.5],'FontSize',12)
```

The MATLAB representation of the vectors is shown in Fig. 2.2.

Moment of a Vector About a Line

The moment $\mathbf{M}_{\Omega}^{\mathbf{v}}$ of a vector $\mathbf{v}$ about a line Ω is the Ω resolute (Ω component) of the moment $\mathbf{v}$ about any point on Ω as shown in Fig. 2.3a. The moment of the vector $\mathbf{v}$ about the line Ω is

$$\mathbf{M}_{\Omega}^{\mathbf{v}} = \mathbf{n}\cdot\mathbf{M}_A^{\mathbf{v}}\,\mathbf{n} = \mathbf{n}\cdot(\mathbf{r}\times\mathbf{v})\,\mathbf{n} = [\mathbf{n},\mathbf{r},\mathbf{v}]\,\mathbf{n},$$

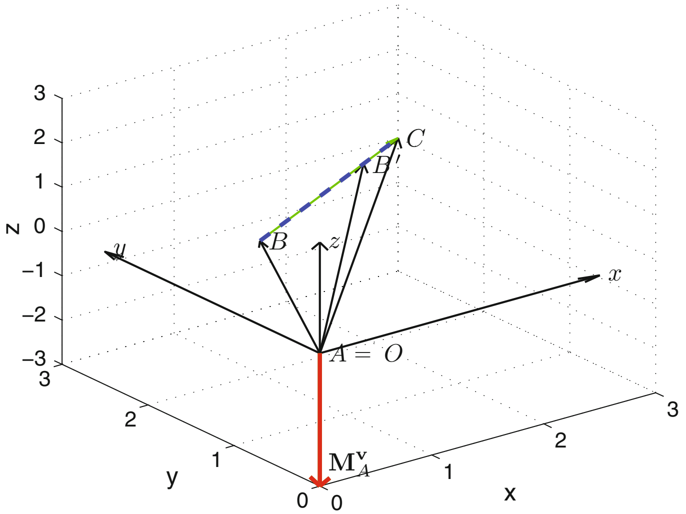


Fig. 2.2 Moment of $\mathbf{v} = \mathbf{r}_{BC}$ about A: $\mathbf{M}_A^{\mathbf{v}} = \mathbf{r}_{AB}\times\mathbf{v} = \mathbf{r}_{AB'}\times\mathbf{v} = \mathbf{r}_{AC}\times\mathbf{v}$

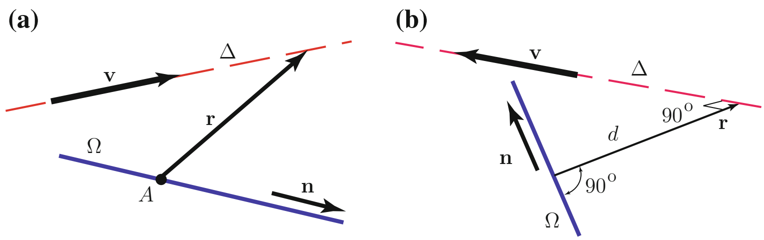


Fig. 2.3 Moment of a vector $\mathbf{v}$ about a line Ω; the line of action of $\mathbf{v}$ does not intersect the line Ω

where $\mathbf{n}$ is a unit vector parallel to Ω, and $\mathbf{r}$ is the position vector of a point on the line of action of $\mathbf{v}$ relative to a point on Ω. The magnitude of $\mathbf{M}_{\Omega}^{\mathbf{v}}$ is given by

$$\left|\mathbf{M}_{\Omega}^{\mathbf{v}}\right| = M_{\Omega}^{\mathbf{v}} = |[\mathbf{n}, \mathbf{r}, \mathbf{v}]|.$$

The moment of a vector about a line is a free vector. If a line Ω is parallel to the line of action Δ of a vector $\mathbf{v}$, then $[\mathbf{n}, \mathbf{r}, \mathbf{v}]\mathbf{n} = \mathbf{0}$ and $\mathbf{M}_{\Omega}^{\mathbf{v}} = \mathbf{0}$. If a line Ω intersects the line of action Δ of $\mathbf{v}$, then $\mathbf{r}$ can be chosen in such a way that $\mathbf{r} = \mathbf{0}$ and $\mathbf{M}_{\Omega}^{\mathbf{v}} = \mathbf{0}$. If a line Ω is perpendicular to the line of action Δ of a vector $\mathbf{v}$, and d is the shortest distance between these two lines, Fig. 2.3b, then

$$\left|\mathbf{M}_{\Omega}^{\mathbf{v}}\right| = |[\mathbf{n}, \mathbf{r}, \mathbf{v}]| = |\mathbf{n}\cdot(\mathbf{r} \times \mathbf{v})| = |\mathbf{n}\cdot(|\mathbf{r}||\mathbf{v}| \sin(\mathbf{r}, \mathbf{v})\mathbf{n})| = |\mathbf{r}||\mathbf{v}| = d|\mathbf{v}|.$$

Moment of a System of Vectors

The moment of a system $\{S\}$ of vectors $\mathbf{v}_i$, $\{S\} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\mathbf{v}_i\}_{i=1,2,\ldots,n}$ about a point A is

$$\mathbf{M}_{A}^{\{S\}} = \sum_{i=1}^{n} \mathbf{M}_{A}^{\mathbf{v}_i}.$$

The moment of a system $\{S\}$ of vectors $\mathbf{v}_i$, $\{S\} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \{\mathbf{v}_i\}_{i=1,2,\ldots,n}$ about a line Ω is

$$\mathbf{M}_{\Omega}^{\{S\}} = \sum_{i=1}^{n} \mathbf{M}_{\Omega}^{\mathbf{v}_i}.$$

The moments $\mathbf{M}_{A}^{\{S\}}$ and $\mathbf{M}_{P}^{\{S\}}$ of a system $\{S\}$, $\{S\} = \{\mathbf{v}_i\}_{i=1,2,\ldots,n}$, of vectors, $\mathbf{v}_i$, about two points A and P, are related to each other as follows,

$$\mathbf{M}_{A}^{\{S\}} = \mathbf{M}_{P}^{\{S\}} + \mathbf{r}_{AP} \times \mathbf{R}, \tag{2.3}$$

where $\mathbf{r}_{AP}$ is the position vector of P relative to A, and $\mathbf{R}$ is the resultant of $\{S\}$.

Proof Let B_i a point on the line of action of the vector $\mathbf{v}_i$, $\mathbf{r}_{ABi}$ and $\mathbf{r}_{PBi}$ the position vectors of B_i relative to A and P, Fig. 2.4. Thus,

$$\begin{aligned}\mathbf{M}_{A}^{\{S\}} &= \sum_{i=1}^{n} \mathbf{M}_{A}^{\mathbf{v}_i} = \sum_{i=1}^{n} \mathbf{r}_{ABi} \times \mathbf{v}_i \\ &= \sum_{i=1}^{n} (\mathbf{r}_{AP} + \mathbf{r}_{PBi}) \times \mathbf{v}_i = \sum_{i=1}^{n} (\mathbf{r}_{AP} \times \mathbf{v}_i + \mathbf{r}_{PBi} \times \mathbf{v}_i)\end{aligned}$$

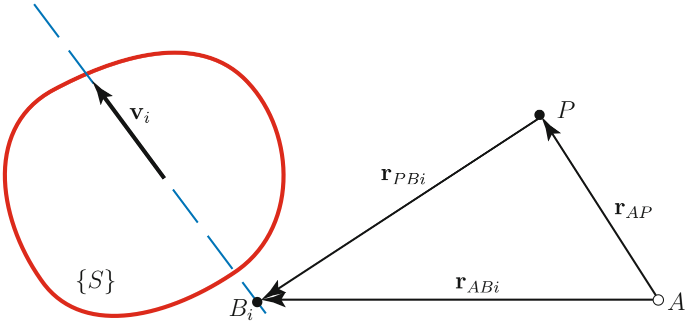


Fig. 2.4 Moments of a system of vectors, $\mathbf{v}_i$ about two points A and P

$$= \sum_{i=1}^{n} \mathbf{r}_{AP} \times \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{r}_{PBi} \times \mathbf{v}_i$$
$$= \mathbf{r}_{AP} \times \sum_{i=1}^{n} \mathbf{v}_i + \sum_{i=1}^{n} \mathbf{r}_{PBi} \times \mathbf{v}_i = \mathbf{r}_{AP} \times \mathbf{R} + \sum_{i=1}^{n} \mathbf{M}_P^{\mathbf{v}_i} = \mathbf{r}_{AP} \times \mathbf{R} + \mathbf{M}_P^{\{S\}}.$$

The proof of Eq. (2.3) for a system of three vectors `v1_`, `v2_`, and `v3_` is given by the following MATLAB commands:

```
% vectors vi_ i=1,2,3
v1_ = sym('[v1x v1y v1z]');
v2_ = sym('[v2x v2y v2z]');
v3_ = sym('[v3x v3y v3z]');

% application points Bi of vi
rB1_ = sym('[xB1 yB1 zB1]');
rB2_ = sym('[xB2 yB2 zB2]');
rB3_ = sym('[xB3 yB3 zB3]');

% any two points A and P
% any two points A and P
rA_ = sym('[xA yA zA]');
rP_ = sym('[xP yP zP]');

rAP_ = rP_-rA_;

rPB1_ = rB1_-rP_;
rPB2_ = rB2_-rP_;
rPB3_ = rB3_-rP_;
```

```
rAB1_ = rAP_+rPB1_;
rAB2_ = rAP_+rPB2_;
rAB3_ = rAP_+rPB3_;

R_ = v1_+v2_+v3_;

% MA_ = sum(ABi_ x vi_) i=1,2,3
MA_ = cross(rAB1_,v1_)+...
      cross(rAB2_,v2_)+...
      cross(rAB3_,v3_);

% MP_ = sum(PBi_ x vi_) i=1,2,3
MP_ = cross(rB1_-rP_,v1_)+...
      cross(rB2_-rP_,v2_)+...
      cross(rB3_-rP_,v3_);

% MA_ = AP_ x R_ + MP_
T1=expand(MA_) == ....
expand(cross(rP_-rA_,R_)+MP_);
fprintf('MA_ == AP_ x R_ + MP_ => [%d %d %d]\n',T1)
fprintf('1=TRUE 0=FALSE\n')
```

The scalar product of the moments $\mathbf{M}_A^{\{S\}}$ and $\mathbf{M}_P^{\{S\}}$, about any points A and P, with the resultant $\mathbf{R}$ of $\{S\}$ are constant

$$\mathbf{M}_A^{\{S\}} \cdot \mathbf{R} = \mathbf{M}_P^{\{S\}} \cdot \mathbf{R}. \tag{2.4}$$

The scalar product $\mathbf{M}_A^{\{S\}} \cdot \mathbf{R}$ is an invariant of the system $\{S\}$. Taking into account the previous MATLAB program the proof for Eq. (2.1) is given below:

```
T2 = expand(MA_*R_.') == expand(MP_*R_.');
fprintf('MA_*R_ == MP_*R_ => %d \n',T2)
```

This invariant is the scalar invariant or the second invariant of the system of vectors. The resultant vector of the system is the vector invariant of that system or the first invariant. The resultant moment $\mathbf{M}_O^{\{S\}}$ with respect to a point O is is not an invariant of the system. The resolution of the moment vector into two components is

$$\mathbf{M}_O^{\{S\}} = \mathbf{M}_R + \mathbf{M}_N,$$

where $\mathbf{M}_R$ is the component along the resultant $\mathbf{R}$ direction and $\mathbf{M}_N$ is perpendicular to the resultant direction. The magnitude of the component along the resultant direction is

$$M_R = \mathbf{M}_O^{\{S\}} \cdot \mathbf{u}_R = \mathbf{M}_O^{\{S\}} \cdot \frac{\mathbf{R}}{R} = \frac{\mathbf{M}_O^{\{S\}} \cdot \mathbf{R}}{R}.$$

The projection of the resultant moment on the resultant of the system, M_R, is an invariant of the system. For the minimum value of the component M_R a corresponding minimum moment, $\mathbf{M}_{min}$, can be defined. The minimum moment is obtained when the normal component is zero, $\mathbf{M}_N = \mathbf{0}$. The minimum moment $\mathbf{M}_{min}$ is given by

$$\mathbf{M}_{min} = \frac{\mathbf{R} \cdot \mathbf{M}_O^{\{S\}}}{\mathbf{R} \cdot \mathbf{R}} \mathbf{R}.$$

If the resultant $\mathbf{R}$ of a system $\{S\}$ of vectors is not equal to zero, $\mathbf{R} \neq \mathbf{0}$, the points about which $\{S\}$ has a minimum moment $\mathbf{M}_{min}$ are on a line called *central axis*, (CA), of $\{S\}$, which is parallel to $\mathbf{R}$ and passes through a point P. The position vector $\mathbf{r}$ of point P relative to an arbitrarily selected reference point O is given by

$$\mathbf{r} = \frac{\mathbf{R} \times \mathbf{M}_O^{\{S\}}}{\mathbf{R} \cdot \mathbf{R}}.$$

The equation of the central axis is obtained from the following program:

```
% resultant force
R_  = sym('[Rx Ry Rz]');
% resultant moment
MO_ = sym('[MOx MOy MOz]');

rA_ = sym('[xA yA zA]');
% O(0,0,0) is the origin
% MA_ = MO_ + AO_ x R_
MA_ = MO_ + cross(-rA_,R_);

% colinearity condition between R_ and MO_
% MA_ = lambda*R_
syms lambda real
eq_ = MA_ - lambda*R_;

% solve for lambda
eqx=solve(eq_(1),'lambda');
eqy=solve(eq_(2),'lambda');
eqz=solve(eq_(3),'lambda');
```

and it results:

```
equation for central axis
(MOx - Rz*yA + Ry*zA)/Rx=
(MOy + Rz*xA - Rx*zA)/Ry=
(MOz - Ry*xA + Rx*yA)/Rz.
```

2.2 Couples

A *couple* is a system of vectors whose resultant is equal to zero and whose moment about some point is not equal to zero. A couple consisting of only two vectors is called a *simple couple*. The vectors of a simple couple have equal magnitudes, parallel lines of action, and opposite senses. The term "couple" can be used to denote the simple couple. In many textbooks the use of the term couple is restricted to the situation in which the contributing vectors are forces and the moment of a couple about a point is called the *torque* of the couple, and is usually denoted by **M** or **T**. The moment of a couple about one point is equal to the moment of the couple about any other point. The moment of a couple is independent of the specific point. The moment of a couple is a free vector.

The torques are vectors and the magnitude of a torque of a simple couple is given by

$$|\mathbf{M}| = d\,|\mathbf{v}| = d\,v,$$

where d is the distance between the lines of action of the two vectors comprising the couple, and **v** is one of these vectors.

Proof In Fig. 2.5, the moment **M** is the sum of the moments of **v** and $-\mathbf{v}$ about any point. The moments about point A are

$$\mathbf{M} = \mathbf{M}_A^{\mathbf{v}} + \mathbf{M}_A^{-\mathbf{v}} = \mathbf{r} \times \mathbf{v} + \mathbf{0}.$$

Hence,

$$|\mathbf{M}| = |\mathbf{r} \times \mathbf{v}| = |\mathbf{r}||\mathbf{v}| \sin(\mathbf{r}, \mathbf{v}) = d|\mathbf{v}|.$$

The direction of the moment of a simple couple can be determined by inspection: **M** is perpendicular to the plane determined by the lines of action of the two vectors comprising the couple, and the sense of **M** is the same as that of $\mathbf{r} \times \mathbf{v}$. The moment of a couple about a line Ω is equal to the Ω resolute of the torque of the couple.

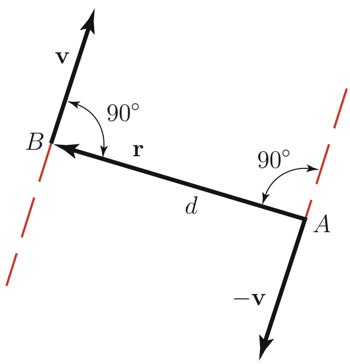


Fig. 2.5 Couple of the vectors **v** and $-\mathbf{v}$, simple couple

2.3 Force Vectors

Force is a vector quantity, having both magnitude and direction. Force is commonly explained in terms of Newton's three laws of motion in *Principia Mathematica*, 1687. Newton's first principle: a body that is at rest or moving at a uniform rate in a straight line will remain in that state until some force is applied to it. Newton's second law of motion: a particle acted on by forces whose resultant is not zero will move in such a way that the time rate of change of its momentum will at any instant be proportional to the resultant force. Newton's third law: when one body exerts a force on another body, the second body exerts an equal force in magnitude, opposite in direction, and collinear, on the first body. This is the principle of action and reaction. The vector representation of forces implies that they are concentrated either at a single point or along a single line.

Force is measured in newtons (N); a force of 1 N will accelerate a mass of one kilogram at a rate of one meter per second. The newton is a unit of the International System (SI) used for measuring force. Using the English system, the force is measured in pounds (lb).

The force vector $\mathbf{F}$ can be expressed in terms of a cartesian reference frame, with the unit vectors $\mathbf{\imath}$, $\mathbf{\jmath}$, and $\mathbf{k}$, Fig. 2.6a

$$\mathbf{F} = F_x\mathbf{\imath} + F_y\mathbf{\jmath} + F_z\mathbf{k}. \tag{2.5}$$

The components of the force in the x, y, and z directions are F_x, F_y, and F_z. The resultant of two forces: $\mathbf{F}_1 = F_{1x}\mathbf{\imath} + F_{1y}\mathbf{\jmath} + F_{1z}\mathbf{k}$ and $\mathbf{F}_2 = F_{2x}\mathbf{\imath} + F_{2y}\mathbf{\jmath} + F_{2z}\mathbf{k}$ is the vector sum of those forces

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (F_{1x} + F_{2x})\mathbf{\imath} + (F_{1y} + F_{2y})\mathbf{\jmath} + (F_{1z} + F_{2z})\mathbf{k}. \tag{2.6}$$

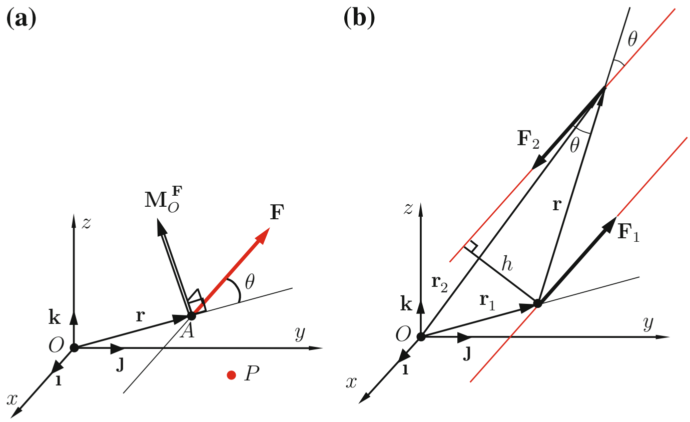


Fig. 2.6 **a** Moment of a force about (with respect to) a point and **b** couple of two forces

A moment is defined as the moment of a force about (with respect to) a point. The moment of the force **F** about the point O is the cross product vector

$$\begin{aligned}
\mathbf{M}_O^{\mathbf{F}} &= \mathbf{r} \times \mathbf{F} \\
&= \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} \\
&= (r_y F_z - r_z F_y)\mathbf{\imath} + (r_z F_x - r_x F_z)\mathbf{\jmath} + (r_x F_y - r_y F_x)\mathbf{k},
\end{aligned} \tag{2.7}$$

where $\mathbf{r} = r_x\mathbf{\imath} + r_y\mathbf{\jmath} + r_z\mathbf{k}$ is a position vector directed from the point about which the moment is taken (O in this case) to any point A on the line of action of the force, see Fig. 2.6a. If the coordinates of O are $x_O,\ y_O,\ z_O$ and the coordinates of A are $x_A,\ y_A,\ z_A$, then $\mathbf{r} = \mathbf{r}_{OA} = (x_A - x_O)\mathbf{\imath} + (y_A - y_O)\mathbf{\jmath} + (z_A - z_O)\mathbf{k}$ and the the moment of the force **F** about the point O is

$$\mathbf{M}_O^{\mathbf{F}} = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A - x_O & y_A - y_O & z_A - z_O \\ F_x & F_y & F_z \end{vmatrix}.$$

The magnitude of $\mathbf{M}_O^{\mathbf{F}}$ is

$$|\mathbf{M}_O^{\mathbf{F}}| = M_O^{\mathbf{F}} = r\,F\,|\sin\theta|,$$

where $\theta = \angle(\mathbf{r},\ \mathbf{F})$ is the angle between vectors **r** and **F**, and $r = |\mathbf{r}|$ and $F = |\mathbf{F}|$ are the magnitudes of the vectors. The line of action of $\mathbf{M}_O^{\mathbf{F}}$ is perpendicular to the plane containing **r** and **F** ($\mathbf{M}_O^{\mathbf{F}} \perp \mathbf{r}\ \&\ \mathbf{M}_O^{\mathbf{F}} \perp \mathbf{F}$) and the sense is given by the right-hand rule. The moment of the force **F** about another point P is

$$\mathbf{M}_P^{\mathbf{F}} = \mathbf{r}_{PA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A - x_P & y_A - y_P & z_A - z_P \\ F_x & F_y & F_z \end{vmatrix},$$

where $x_P,\ y_P,\ z_P$ are the coordinates of the point P.

The system of two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, which have equal magnitudes $|\mathbf{F}_1| = |\mathbf{F}_2|$, opposite senses $\mathbf{F}_1 = -\mathbf{F}_2$, and parallel directions ($\mathbf{F}_1 || \mathbf{F}_2$) is a couple. The resultant force of a couple is zero $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = \mathbf{0}$. The resultant moment $\mathbf{M} \neq \mathbf{0}$ about an arbitrary point is

$$\mathbf{M} = \mathbf{r}_1 \times \mathbf{F}_1 + \mathbf{r}_2 \times \mathbf{F}_2,$$

or

$$\mathbf{M} = \mathbf{r}_1 \times (-\mathbf{F}_2) + \mathbf{r}_2 \times \mathbf{F}_2 = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}_2 = \mathbf{r} \times \mathbf{F}_2, \tag{2.8}$$

where $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$ is a vector from any point on the line of action of $\mathbf{F}_1$ to any point of the line of action of $\mathbf{F}_2$. The direction of the torque of the couple is perpendicular to the plane of the couple and the magnitude is given by, Fig. 2.6b

$$|\mathbf{M}| = M = r\,F_2\,|\sin\theta| = h\,F_2, \tag{2.9}$$

where $h = r\,|\sin\theta|$ is the perpendicular distance between the lines of action. The resultant moment of a couple is independent of the point with respect to which moments are taken.

2.4 Equipollent Force Systems

Two systems $\{S\}$ and $\{S'\}$ of vectors are *equipollent* if and only if

1. the resultant of $\{S\}$, $\mathbf{R}$, is equal to the resultant of $\{S'\}$, $\mathbf{R}'$
$$\mathbf{R} = \mathbf{R}'.$$
2. there exists at least one point about which $\{S\}$ and $\{S'\}$ have equal moments
$$\text{exists } P: \ \mathbf{M}_P^{\{S\}} = \mathbf{M}_P^{\{S'\}}.$$

Figures 2.7a and b show two forces acting on a rod. The two systems of forces are equipollent. The effects on the rod by the two systems are different tension and compression. Here the equipollence is not a physical equivalence.

Transitivity relation If $\{S\}$ is equipollent to $\{S'\}$, and $\{S'\}$ is equipollent to $\{S''\}$, then $\{S\}$ is equipollent to $\{S''\}$.

Every system $\{S\}$ of bound vectors with the resultant $\mathbf{R}$ is equipollent with a system consisting of a couple C and a single vector $\mathbf{v}$ whose line of action passes through a point O. The torque $\mathbf{M}$ of C depends on the choice of the point $\mathbf{M} = \mathbf{M}_O^{\{S\}}$. The vector $\mathbf{v}$ is independent of the choice of base point, $\mathbf{v} = \mathbf{R}$. A couple C can be equipollent with any system of couples, the sum of whose torque is equal to the torque of C. When a system of vectors consists of a couple of torque $\mathbf{M}$ and a single resultant vector parallel to $\mathbf{M}$, it is called a *wrench*. Any system is equipollent to either a null force and null couple, or a single force, or a single couple, or a wrench.

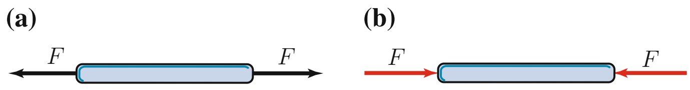


Fig. 2.7 Rod subjected to the action of a pair of forces

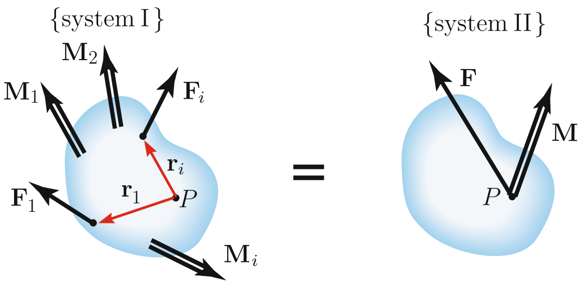


Fig. 2.8 Equipollent systems

To simplify the analysis of forces and moments acting on a given system, the system can be equipollent by a less complicated system. The forces and moments acting on the system can be equipollent with a total force and a total moment system.

Figure 2.8 shows an arbitrary system of forces and moments, {system I}, and a point P. This system is equipollent with a system, {system II}, consisting of a single force $\mathbf{F}$ acting at P and a single couple of torque $\mathbf{M}$. The conditions for equipollence are

$$\sum \mathbf{F}^{\{\text{system II}\}} = \sum \mathbf{F}^{\{\text{system I}\}} \implies \mathbf{F} = \sum \mathbf{F}^{\{\text{system I}\}},$$

and

$$\sum \mathbf{M}_P^{\{\text{system II}\}} = \sum \mathbf{M}_P^{\{\text{system I}\}} \implies \mathbf{M} = \sum \mathbf{M}_P^{\{\text{system I}\}}.$$

These conditions are satisfied if $\mathbf{F}$ equals the sum of the forces in {system I}, and $\mathbf{M}$ equals the sum of the moments about P in {system I}. Thus, no matter how complicated a system of forces and moments may be, it can be represented by a single force acting at a given point and a single couple. Three particular cases occur frequently in practice.

Force Equipollent with a Force and a Couple

A force $\mathbf{F}_I$ acting at a point I {system I} in Fig. 2.9 is equipollent with a force $\mathbf{F}$ acting at a different point P and a couple of torque $\mathbf{M}$, {system II}.The moment of {system I} about point P is $\mathbf{r}_{PI} \times \mathbf{F}_I$, where $\mathbf{r}_{PI}$ is the vector from P to I. The conditions for equipollence are

$$\sum \mathbf{F}^{\{\text{system II}\}} = \sum \mathbf{F}^{\{\text{system I}\}} \implies \mathbf{F} = \mathbf{F}_I,$$

and

$$\sum \mathbf{M}_P^{\{\text{system II}\}} = \sum \mathbf{M}_P^{\{\text{system I}\}} \implies \mathbf{M} = \mathbf{M}_P^{\mathbf{F}_I} = \mathbf{r}_{PI} \times \mathbf{F}_I.$$

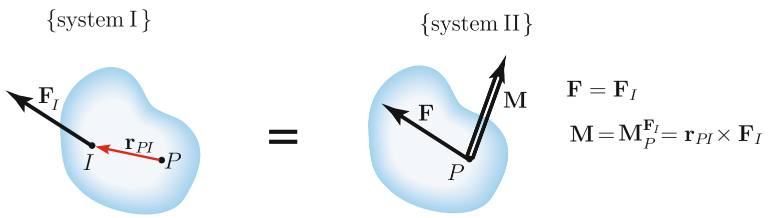


Fig. 2.9 Force $\mathbf{F}_I$ acting on {system I} and equipollent system {system II}

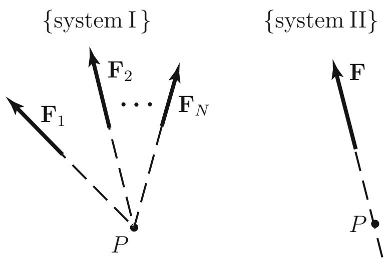


Fig. 2.10 System of concurrent forces and equipollent system

The systems are equipollent if the force $\mathbf{F}$ equals the force $\mathbf{F}_I$ and the couple of torque $\mathbf{M}_P^{\mathbf{F}_I}$ equals the moment of $\mathbf{F}_I$ about P.

Concurrent Forces Equipollent with a Single Force
A system of concurrent forces whose lines of action intersect at a point P {system I} in Fig. 2.10 is equipollent with a single force whose line of action intersects P, {system II}.
The sums of the forces in the two systems are equal if

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n.$$

The sum of the moments about P equals zero for each system, so the systems are equipollent if the force $\mathbf{F}$ equals the sum of the forces in {system I}.

Parallel Forces Equipollent with a Force
A system of parallel forces whose sum is not zero is equipollent with a single force $\mathbf{F}$ shown in Fig. 2.11.

System Equipollent with a Wrench
In general any system of forces and moments is equipollent with a single force acting at a given point and a single couple. Figure 2.12 shows an arbitrary force $\mathbf{F}$ acting at a point I and an arbitrary couple of torque $\mathbf{M}$, {system I}. This system is equipollent with a simpler one where the force $\mathbf{F}$ is acting at a different point P and the component of $\mathbf{M}$ is parallel to $\mathbf{F}$. A coordinate system is chosen so that $\mathbf{F}$ is along the y axis

$$\mathbf{F} = F\mathbf{J},$$

and $\mathbf{M}$ is contained in the xy plane

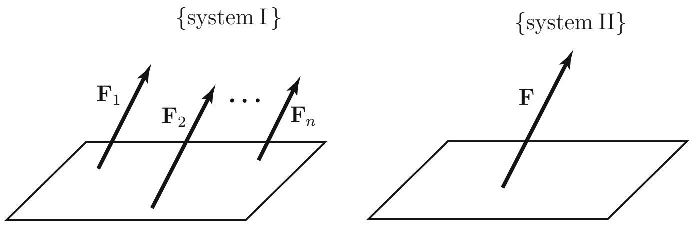


Fig. 2.11 System of parallel forces and equipollent system

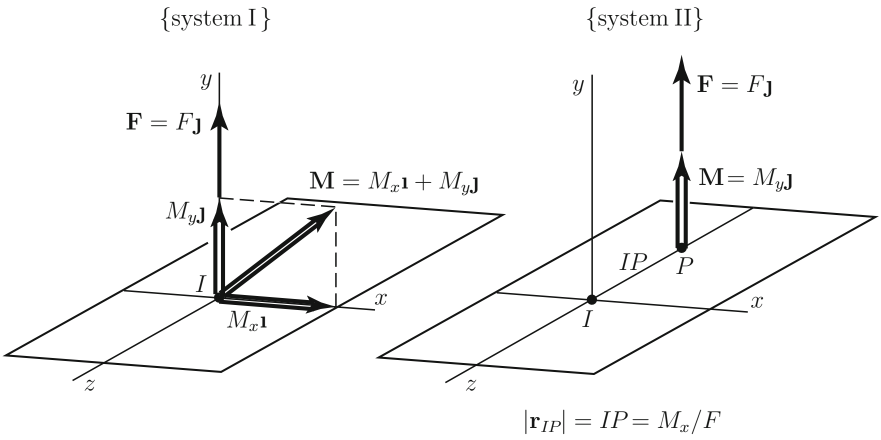


Fig. 2.12 System equipollent with a wrench

$$\mathbf{M} = M_x\mathbf{I} + M_y\mathbf{J}.$$

The equivalent system, {system II}, consists of the force $\mathbf{F}$ acting at a point P on the z axis

$$\mathbf{F} = F\mathbf{J},$$

and the component of $\mathbf{M}$ parallel to $\mathbf{F}$

$$\mathbf{M}_p = M_y\mathbf{J}.$$

The distance IP is chosen so that $|\mathbf{r}_{IP}| = IP = M_x/F$. The {system I} is equipollent to {system II}. The sum of the forces in each system is the same $\mathbf{F}$. The sum of the moments about I in {system I} is $\mathbf{M}$, and the sum of the moments about I in {system II} is

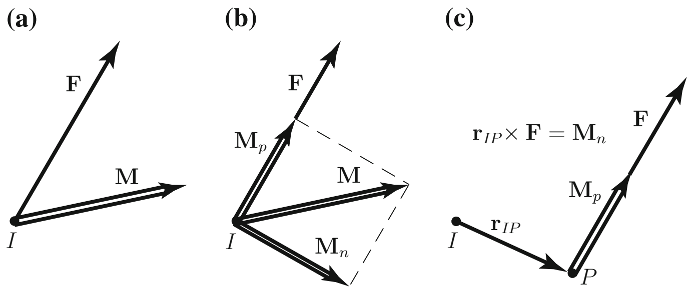


Fig. 2.13 Steps required for a system of forces and moments to be equipollent with wrench

$$\sum \mathbf{M}_I^{\{\text{system II}\}} = \mathbf{r}_{PI} \times \mathbf{F} + M_y\mathbf{J} = [-(IP)\,\mathbf{k}] \times (F\mathbf{J}) + M_y\mathbf{J} = M_x\mathbf{\imath} + M_y\mathbf{J} = \mathbf{M}.$$

The system of the force $\mathbf{F} = F\mathbf{J}$ and the couple $\mathbf{M}_p = M_y\mathbf{J}$ that is parallel to $\mathbf{F}$ is a wrench. A wrench is the simplest system equipollent to an arbitrary system of forces and moments.
A given system of forces and moments is made equipollent with wrench following the steps:

1. Choose a convenient point I the application point of force $\mathbf{F}$ and the moment $\mathbf{M}$, see Fig. 2.13a.
2. Determine the components of $\mathbf{M}$ parallel and normal to $\mathbf{F}$, see Fig. 2.13b:

$$\mathbf{M} = \mathbf{M}_p + \mathbf{M}_n, \quad \text{where} \;\; \mathbf{M}_p || \mathbf{F}.$$

3. The wrench consists of the force $\mathbf{F}$ acting at a point P and the parallel component $\mathbf{M}_p$, see Fig. 2.13c. For equipollence, the following condition must be satisfied:

$$\mathbf{r}_{IP} \times \mathbf{F} = \mathbf{M}_n,$$

where $\mathbf{M}_n$ is the normal component of $\mathbf{M}$.
In general, the {system I} cannot be represented by a force $\mathbf{F}$ alone.

2.5 Examples

Example 2.1 Calculate the moment about the base point O of the the force F, as shown in Fig. 2.14a. For the numerical application use: $F = 500$ N, $\theta = 45°$, $a = 1$ m, and $b = 5$ m.

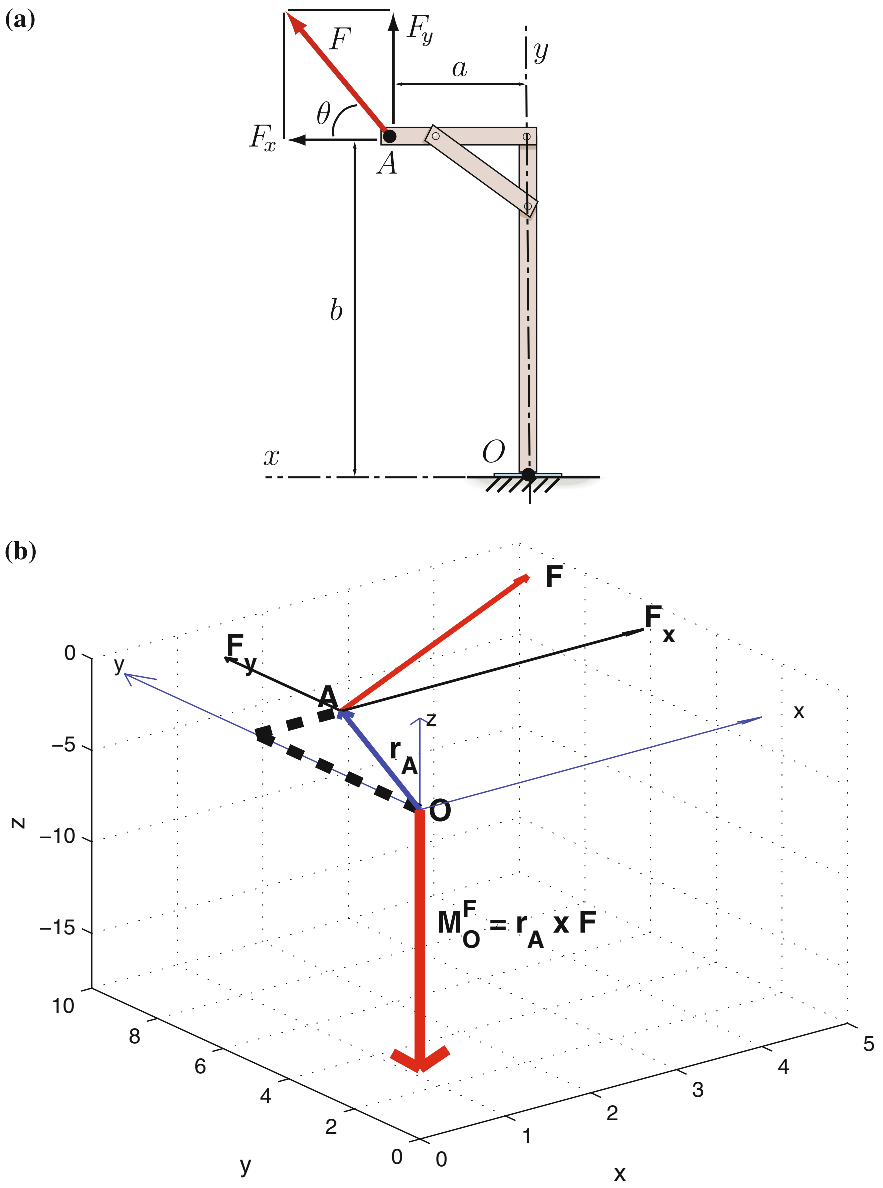


Fig. 2.14 **a** Example 2.1 and **b** MATLAB figure

Solution A cartesian reference frame with the origin at O, as shown in Fig. 2.14a, is selected. The moment of the force $\mathbf{F}$ with respect to the point O is

$$\mathbf{M}_O^{\mathbf{F}} = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A - x_O & y_A - y_O & 0 \\ F_x & F_y & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ a & b & 0 \\ F\cos\theta & F\sin\theta & 0 \end{vmatrix}$$
$$= (aF\cos\theta - bF\sin\theta)\,\mathbf{k} = [(1)\,5\cos 45^\circ - (5)\,500\sin 45^\circ]\,\mathbf{k}$$
$$= -1414.2\,\mathbf{k} \text{ Nm}.$$

The minus sign indicates that the moment vector is in the negative z-direction. The MATLAB program for the the moment of the force **F** about the point O is:

```
syms F theta a b  real
rA_ = [a b 0];
FA_ = [F*cos(theta) F*sin(theta) 0];
MO_ = cross(rA_, FA_);
MOz= MO_(3);
sl = {F, theta, a, b};
nl = {5, pi/4, 1, 5};
fprintf('MOz = %s =',char(MOz))
fprintf('%6.3f (kN m)\n',subs(MOz,sl,nl))
```

and the output of the program is

```
MOz = a*F*sin(theta)-b*F*cos(theta) = -14.142 (kN m)
```

The MATLAB program for plotting the vectors and the figure are:

```
% numerical values for vectors
rAn_ = double(subs(rA_,sl,nl));
Fn_ = double(subs(FA_,sl,nl));
Mn_ = subs(MO_,sl,nl);
% figure plotting
line([0 0],[0 rAn_(2)],[0,0],'LineStyle','--',...
    'Color','k','LineWidth',4)
line...
([0 rAn_(1)],[rAn_(2) rAn_(2)],[0,0],...
'LineStyle','--','Color','k','LineWidth',4)
% vector plotting
% rAn_
quiver3(0,0,0,rAn_(1),rAn_(2),0,1,...
    'Color','b','LineWidth',2)
% rFn_
quiver3(rAn_(1),rAn_(2),0,Fn_(1),Fn_(2),0,1,...
    'Color','r','LineWidth',2)
% rFn_(1)
quiver3(rAn_(1),rAn_(2),0,Fn_(1),0,0,1,...
    'Color','k','LineWidth',1)
% rFn_(2)
```

```
quiver3(rAn_(1),rAn_(2),0,0,Fn_(2),0,1,...
    'Color','k','LineWidth',1)
% Mn_
quiver3(0,0,0,0,0,Mn_(3),1,...
    'Color','r','LineWidth',4)
```

The vector representation with MATLAB is shown in Fig. 2.14b.

Example 2.2 The beam in Fig. 2.15a is subjected to a T tension that is directed from A to B. Find the the moment created by the force about the support at O. For the numerical application use: $T = 10$ kN, $a = 12$ m, $b = 9$ m, and $c = 15$ m.

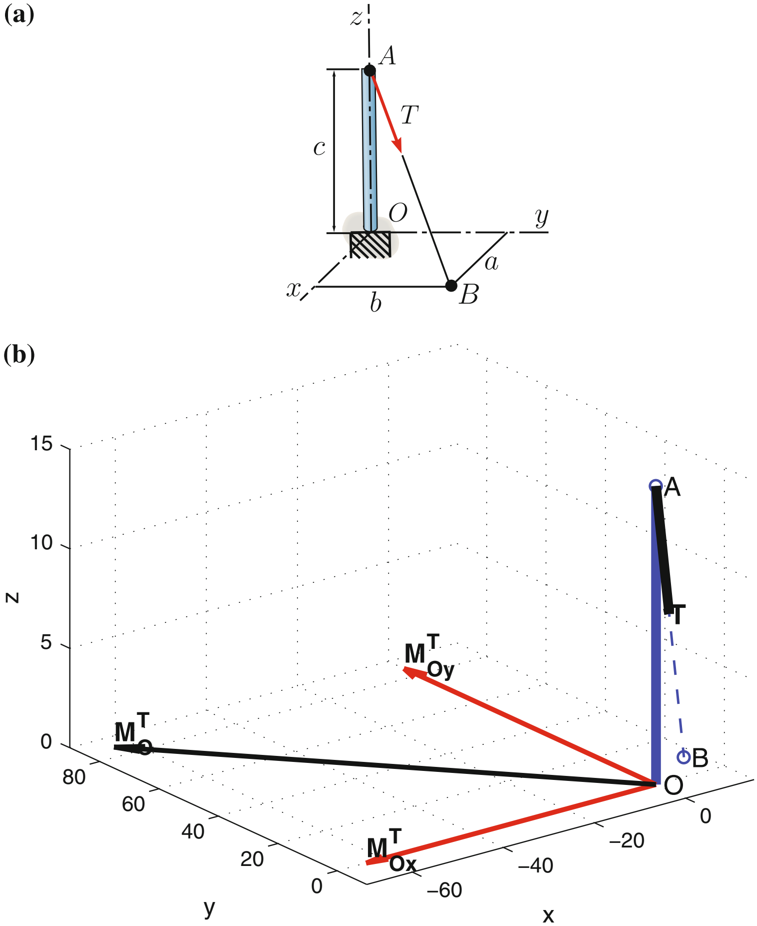


Fig. 2.15 **a** Example 2.2 and **b** MATLAB figure

Solution The vector expression for the tension **T** is

$$\mathbf{T} = T\,\mathbf{u}_{AB} = T\,\frac{\mathbf{r}_{AB}}{|\mathbf{r}_{AB}|} = T\,\frac{(x_B - x_A)\,\boldsymbol{\imath} + (y_B - y_A)\,\boldsymbol{\jmath} + (z_B - z_A)\,\mathbf{k}}{\sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}}$$
$$= T\,\frac{a\,\boldsymbol{\imath} + b\,\boldsymbol{\jmath} - c\,\mathbf{k}}{\sqrt{a^2 + b^2 + c^2}} = (10)\,\frac{12\,\boldsymbol{\imath} + 9\,\boldsymbol{\jmath} - 15\,\mathbf{k}}{\sqrt{12^2 + 9^2 + 15^2}} = 5.657\,\boldsymbol{\imath} + 4.243\,\boldsymbol{\jmath} - 7.071\,\mathbf{k}\ \text{kN},$$

where $r_B = x_B\,\boldsymbol{\imath} + y_B\,\boldsymbol{\jmath} + z_B\,\mathbf{k} = a\,\boldsymbol{\imath} + b\,\boldsymbol{\jmath}$ and $r_C = x_C\,\boldsymbol{\imath} + y_C\,\boldsymbol{\jmath} + z_C\,\mathbf{k} = c\,\mathbf{k}$. The moment of the tension **T** with respect to the point O is

$$\mathbf{M}_O^{\mathbf{T}} = \mathbf{r}_{OA} \times \mathbf{T} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_A & y_A & z_A \\ T_x & T_y & T_z \end{vmatrix} = \frac{T}{\sqrt{a^2 + b^2 + c^2}} \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ 0 & 0 & c \\ a & b & -c \end{vmatrix} = \frac{T\,(-b\,c\,\boldsymbol{\imath} + a\,c\,\boldsymbol{\jmath})}{\sqrt{a^2 + b^2 + c^2}}$$
$$= \frac{10\,[-9\,(15)\,\boldsymbol{\imath} + 12\,(9)\,\boldsymbol{\jmath})}{\sqrt{12^2 + 9^2 + 15^2}} = -63.640\,\boldsymbol{\imath} + 84.853\,\boldsymbol{\jmath}\ \text{kN m},$$

and $|\mathbf{M}_O^{\mathbf{T}}| = 106.066\,\text{kN m}$.
The MATLAB program for the calculation of **T** and $\mathbf{M}_O^{\mathbf{T}}$ and is given by:

```
syms T a b c real
rB_ = [a b 0];
rA_ = [0 0 c];
rAB_= rB_-rA_;
uAB_= rAB_/sqrt(dot(rAB_, rAB_));
TAB_= T*uAB_;
MO_ = cross(rA_, TAB_);
% numerical calculations
sl = {T, a, b, c};
nl = {10, 12, 9, 15};
Tn_ = subs(TAB_,sl,nl);
Mn_ = subs(MO_,sl,nl);
```

and the output is:

```
T = [ 5.657  4.243 -7.071] (kN)
MOx = -c*T*b/(a^2+b^2+c^2)^(1/2) = -63.640 (kN m)
MOy = c*T*a/(a^2+b^2+c^2)^(1/2) = 84.853 (kN m)
MOz = 0 =  0.000 (kN m)
|MO| = 106.066 (kN m)
```

The MATLAB program for plotting the vectors is:

```
rAn_ = double(subs(rA_,sl,nl));
Fn_ = double(subs(FA_,sl,nl));
Mn_ = subs(MO_,sl,nl);
```

```
axis([0 5  0 10  -18 0])
line([0 0],[0 rAn_(2)],[0,0],'LineStyle','--',...
    'Color','k','LineWidth',4)
line...
([0 rAn_(1)],[rAn_(2) rAn_(2)],[0,0],...
'LineStyle','--','Color','k','LineWidth',4)
quiver3(0,0,0,rAn_(1),rAn_(2),0,1,...
    'Color','b','LineWidth',2)
quiver3(rAn_(1),rAn_(2),0,Fn_(1),Fn_(2),0,1,...
    'Color','r','LineWidth',2)
quiver3(rAn_(1),rAn_(2),0,Fn_(1),0,0,1,...
    'Color','k','LineWidth',1)
quiver3(rAn_(1),rAn_(2),0,0,Fn_(2),0,1,...
    'Color','k','LineWidth',1)
quiver3(0,0,0,0,0,Mn_(3),1,...
    'Color','r','LineWidth',4)
```

The vector representation with MATLAB is shown in Fig. 2.15b.

Example 2.3 Determine the moment of the force F about A as shown in Fig. 2.16a. For the numerical application use: $F = 1$ kN, $a = 1$ m, $b = 3$ m, and $c = 2$ m.

Solution The moment of a force about a point is given by the cross product of a position vector with the force vector. The position vector must run from the point about which the moment is being calculated to a point on the line of action of the force. Figure 2.16a shows the location of the point A, the force F, and the line of action of the force. Point B is on the line of action of the force. Thus the position

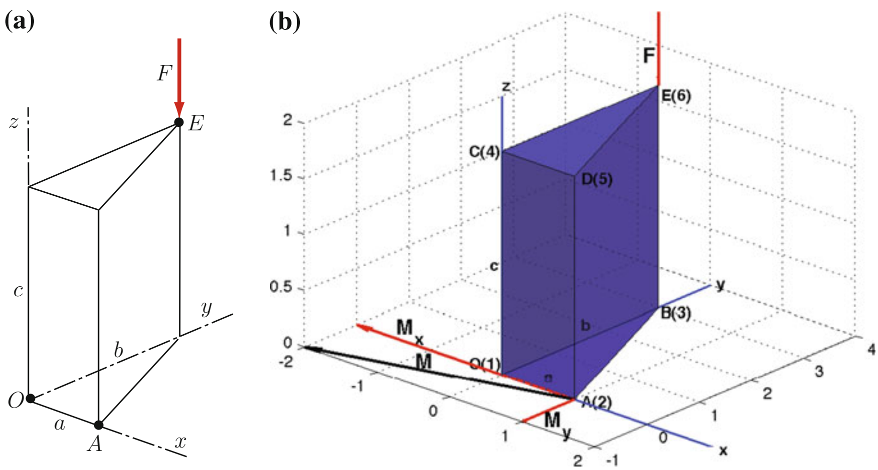


Fig. 2.16 **a** Example 2.3 and **b** MATLAB figure

vector of interest is the vector from point A to point B. From the figure this position vector can be seen to be a units in the -x followed by b units in the positive y.

$$\mathbf{r}_{AB} = -a\,\mathbf{\imath} + b\,\mathbf{\jmath}$$

The force vector is parallel to the z-axis with magnitude F. Thus it can be expressed in vector form as: $\mathbf{F} = -F\,\mathbf{k}$. The desired moment is the cross product of these two vectors

$$\mathbf{M}_A^{\mathbf{F}} = (-a\,\mathbf{\imath} + b\,\mathbf{\jmath}) \times (-F\mathbf{k})\,.$$

Recalling that $\mathbf{\imath} \times \mathbf{k}$ is $-\mathbf{\jmath}$ and $\mathbf{\jmath} \times \mathbf{k}$ is $\mathbf{\imath}$ yields

$$\mathbf{M}_A^{\mathbf{F}} = -b\,F\,\mathbf{\imath} - aF\,\mathbf{\jmath}.$$

The MATLAB program for the moment of the force $\mathbf{F}$ about point A is given by:

```
syms a b c F
rA_ = [a 0 0];
rB_ = [0 b 0];
rE_ = [0 b c];
rAE_ = rE_ - rA_;
rAB_ = rB_ - rA_;
f_ = [0 0 -F];

ME_ = cross(rAE_, f_); % M = rAE x F
MB_ = cross(rAB_, f_); % M = rAB x F
T = ME_ == MB_; % rAB x F = rAE x F
fprintf('ME_ == MB_ => [%d %d %d]\n',T)
fprintf('1=TRUE 0=FALSE\n')

% numerical calculation
sl = {a, b, c, F};
nl = {1, 3, 2, 1};
MEn_ = double(subs(ME_,sl,nl));
MBn_ = double(subs(MB_,sl,nl));
```

The output of the MATLAB program is:

```
ME_ == MB_ => [1 1 1]
1=TRUE 0=FALSE

M_ = rAB_ x F_ = rAE_ x F_
Mx = -F*b; My = -F*a; Mz = 0.
ME_ = [-3.000 -1.000 0] (kN m)
MB_ = [-3.000 -1.000 0] (kN m)
```

The MATLAB program for plotting the vectors and the triangular prism is:

```
F=1; % kN
a=1; b=3; c=2; % m

axis([-2 2 -1 4 0 2])
hold on, grid on

% Cartesian axes
line ...
([0 4],[0 0],[0,0],'Color','b','LineWidth',1.5)
text(3,0,0,'x','fontweight','b')

line ...
([0 0],[0 4],[0,0],'Color','b','LineWidth',1.5)
text(0,4.1,0,'y','fontweight','b')

line ...
([0 0],[0 0],[0,2.5],'Color','b','LineWidth',1.5)
text(0,0,2.6,'z','fontweight','b')

text(-.45,0,0,'O(1)','fontweight','b')
text(a+.1,0,0,'A(2)','fontweight','b')
text(.1,b-.1,0,'B(3)','fontweight','b')
text(-.45,0,c-.1,'C(4)','fontweight','b')
text(a+.1,0,c,'D(5)','fontweight','b')
text(0,b+.05,c-.1,'E(6)','fontweight','b')

text((a+.1)/3,.3,0,'a','fontweight','b')
text(.05,(b-.1)/2,.17,'b','fontweight','b')
text(-.16,0,(c-.1)/2,'c','fontweight','b')

view(42,34);
% view(AZ,EL) set the angle of the view from
% which an observer sees the current 3-D plot
% AZ is the azimuth or horizontal rotation
% EL is the vertical elevation
% (both in degrees)

% Generate data
vert=...
[0 0 0; a 0 0; 0 b 0; 0 0 c; a 0 c; 0 b c];
% define the matrix of the vertices
```

```
% O: 0,0,0 defined as vertex 1
% A: a,0,0 defined as vertex 2
% B: 0,b,0 defined as vertex 3
% C: 0,0,c defined as vertex 4
% D: a,0,c defined as vertex 5
% E: 0,b,c defined as vertex 6

face_up=[1 2 3; 4 5 6];
% define the lower and upper face of
% the triangular prism
% lower face is defined by vertices
% 1, 2, 3 (O, A, B)
% upper face is defined by vertices
% 4, 5, 6 (C, D, E)

face_l=[1 2 5 4; 2 3 6 5; 1 3 6 4];
% generate the lateral faces
% lateral face 1 is defined by 1, 2, 5, 4
% lateral face 2 is defined by 2, 3, 6, 5
% lateral face 3 is defined by 1, 3, 6, 4
% when defined a face the order of the vertices
% has to be given clockwise or counterclockwise

% draw the lower and upper triangular patches
patch...
('Vertices',vert,'Faces',face_up,'facecolor','b')
% patch(x,y,C) adds the "patch" or
% filled 2-D polygon defined by
% vectors x and y to the current axes.
% C specifies the color of the face(s)
% X represents the matrix vert
% Y represents the matrix face_up

% draw the lateral rectangular patches
patch...
('Vertices',vert,'Faces',face_l,'facecolor','b')

quiver3 ...
(0,b,F+c,0,0,-F,1,'Color','r','LineWidth',1.75)
text ...
(-.3,b,c+.2,' F','fontsize',14,'fontweight','b')

quiver3(a,0,0,MBn_(1),MBn_(2),MBn_(3),1,...
    'Color','k','LineWidth',2)
text((a+MBn_(1))/2,MBn_(2)/2,MBn_(3)/2,...
```

```
    '  M','fontsize',14,'fontweight','b')

quiver3 ...
(a,0,0,MBn_(1),0,0,1,'Color','r','LineWidth',2)
text((a+MBn_(1))/1.3,0,0,...
    ' M_x','fontsize',14,'fontweight','b')

quiver3 ...
(a,0,0,0,MBn_(2),0,1,'Color','r','LineWidth',2)
text(a+.3,MBn_(2),0,...
    'M_y','fontsize',14,'fontweight','b')

light('Position',[1 2 3]);
% light('PropertyName',propertyvalue,...)
% light creates a light object in current axes
% Lights affect only patch and surface objects

% light the peaks surface plot with a light source
% located at infinity and oriented along the
% direction defined by the vector [1 2 3]

material shiny

% material shiny makes the objects shiny

alpha('color');
% alpha get or set alpha properties for
% objects in the current axis
% alpha('color') set the alphadata to be
% the same as the color data
```

The vector representation with MATLAB is shown in Fig. 2.16b.

Example 2.4 A force F acts on a link at the point A as shown in Fig. 2.17a. Find an equivalent system consisting of a force at O and a couple. Numerical application: $F = 100$ lb, $OA = l = 1$ ft, $\theta = 45°$, and $\alpha = 100°$.

Solution The original F force is equivalent to the force at O as shown in Fig. 2.17b

$$\begin{aligned}\mathbf{R} &= \mathbf{F} = -F\,\cos(\alpha-\theta)\,\imath + F\,\sin(\alpha-\theta)\,\jmath\\ &= -100\,\cos(100°-45°)\,\imath + 100\,\sin(100°-45°)\,\jmath = -57.358\,\imath + 81.915\,\jmath\ \text{lb}.\end{aligned}$$

The moment of the force $\mathbf{F}$ with respect to the point O, as shown in Fig. 2.17b, is

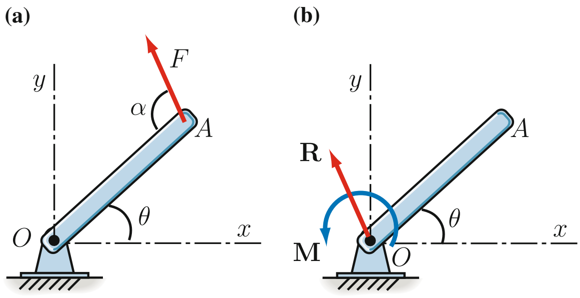


Fig. 2.17 Example 2.4

$$\mathbf{M} = \mathbf{M}_O^{\mathbf{F}} = \mathbf{r}_{OA} \times \mathbf{F} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ x_A & y_A & 0 \\ F_x & F_y & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{\imath} & \mathbf{\jmath} & \mathbf{k} \\ l\cos\theta & l\sin\theta & 0 \\ -F\cos(\alpha-\theta) & F\sin(\alpha-\theta) & 0 \end{vmatrix}$$
$$= [l\,F\,(\cos\theta)\sin(\alpha-\theta) + l\,F\,(\sin\theta)\cos(\alpha-\theta)]\,\mathbf{k}$$
$$= [1(100)\,(\cos 45^\circ)\sin(100^\circ - 45^\circ) + 1(100)\,(\sin 45^\circ)\cos(100^\circ - 45^\circ)]\,\mathbf{k}$$
$$= 98.481\,\mathbf{k}\ \text{lb ft}.$$

The MATLAB program is:

```
syms F l theta alfa real
sl = {F, l, theta, alfa};
nl = {100, 1, pi/4, pi/1.8};
FA_ = [-F*cos(alfa-theta), F*sin(alfa-theta), 0];
rA_ = [l*cos(theta), l*sin(theta), 0];
FAn_= subs(FA_, sl, nl);
MO_ = cross(rA_, FA_);
MOz= simplify(MO_(3));
MOzn= subs(MOz, sl, nl);
```

and the results are:

```
R_ = [-57.358 81.915 0](lb)
MOz = F*l*sin(alfa) = 98.481 (lb.ft)
```

Example 2.5 Three forces $\mathbf{F}_A$, $\mathbf{F}_B$, and $\mathbf{F}_C$, as shown in Fig. 2.18, are acting on a rectangular planar plate ($\mathbf{F}_A||Oz$, $\mathbf{F}_B||Oy$, $\mathbf{F}_C||Ox$). The three forces acting on the plate are replaced by a wrench. Find: (a) the resultant force for the wrench; (b) the magnitude of couple moment, M, for the wrench and the point $T(x, z)$ where its line of action intersects the plate. For the numerical application use: $F_A = 900$ lb, $F_B = 500$ lb, $F_C = 300$ lb, $a = BC = 4$ ft, and $b = OC = 6$ ft.

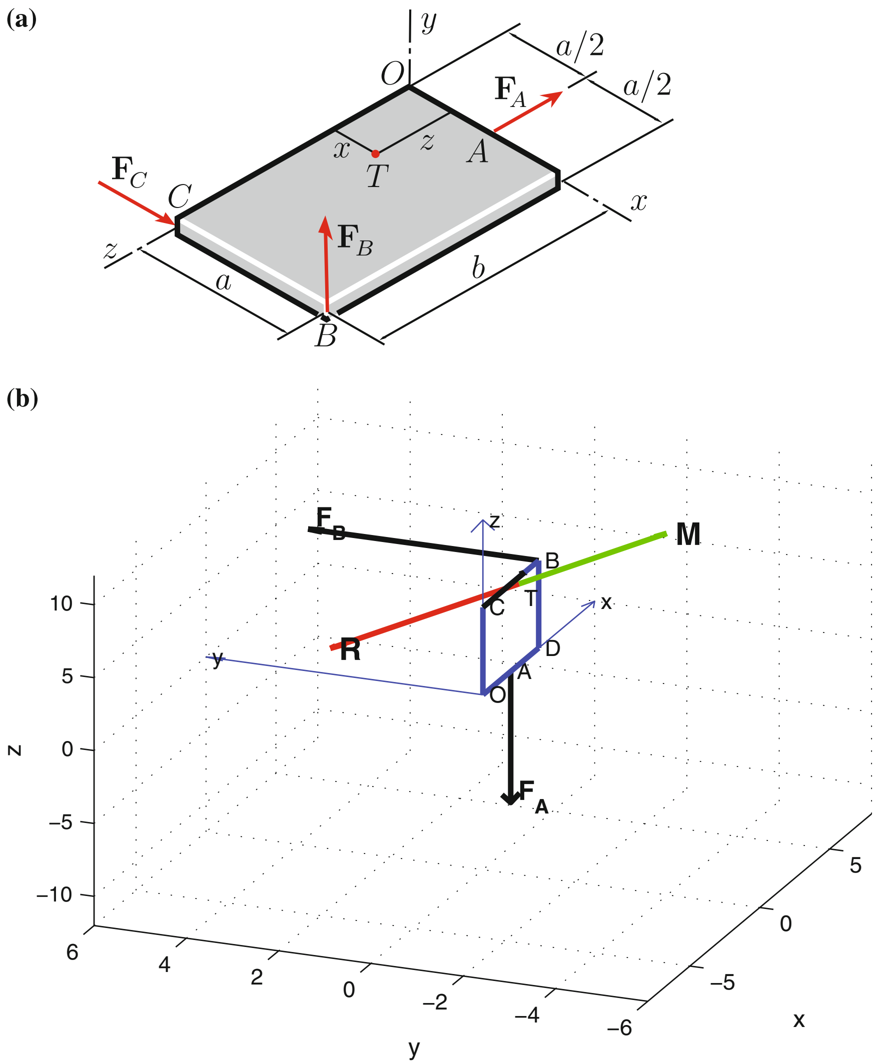


Fig. 2.18 **a** Example 2.5 and **b** MATLAB figure

Solution (a) The direction cosines of the resultant force **R**, are the same as those of the moment **M** of the couple of the wrench, assuming that the wrench is positive. The resultant force is

$$\mathbf{R} = \mathbf{F}_A + \mathbf{F}_B + \mathbf{F}_C = F_C\,\mathbf{\imath} + F_B\,\mathbf{\jmath} - F_A\,\mathbf{k} = 300\,\mathbf{\imath} + 500\,\mathbf{\jmath} - 900\,\mathbf{k} \ \text{ lb}$$

$$R = |\mathbf{R}| = \sqrt{F_A^2 + F_B^2 + F_C^2} = \sqrt{300^2 + 500^2 + 900^2} = 1072.381 \ \text{ lb} = 1.072 \ \text{ kip}.$$

The direction cosines of the resultant force are

$$\cos\theta_x = \frac{F_C}{R} = 0.280, \quad \cos\theta_y = \frac{F_B}{R} = 0.466, \quad \cos\theta_z = \frac{-F_A}{R} = -0.839.$$

The MATLAB program for calculating the direction cosines or the components of the unit vector of the resultant force are:

```
syms a b FA FB FC x z M
sl = {a, b, FA, FB, FC};
nl = {4, 6, 0.9, 0.5, 0.3};
FA_ = [0 0 -FA]; rA_ = [a/2 0 0];
FB_ = [0 FB  0]; rB_ = [a 0 b];
FC_ = [FC 0  0]; rC_ = [0 0 b];
R_ = FA_+FB_+FC_;
Rn_ = subs(R_, sl, nl);
uR_ = R_/magn(R_);
uRn_ = subs(uR_, sl, nl);
```

The function `magn` is:

```
function val = magn(v)
% The symbolic magnitude function of a vector
% v = [v(1) v(2) v(3)]
% The function accepts sym as the input argument
val=sqrt(v(1)*v(1)+v(2)*v(2)+v(3)*v(3));
```

(b) The moment of the wrench couple must equal the sum of the moments of the given forces about point T through which the resultant passes. The moments about $T(x,\ 0,\ z)$ of the three forces are

$$\mathbf{M}_T = \mathbf{M}_T^{\mathbf{F}_A} + \mathbf{M}_T^{\mathbf{F}_B} + \mathbf{M}_T^{\mathbf{F}_C},$$

where

$$\mathbf{M}_T^{\mathbf{F}_A} = \mathbf{r}_{TA} \times \mathbf{F}_A = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_A - x & y_A & z_A - z \\ 0 & 0 & -F_A \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ a - x & 0 & -z \\ 0 & 0 & -F_A \end{vmatrix} = (a-x)\,F_A\,\boldsymbol{\jmath}.$$

$$\mathbf{M}_T^{\mathbf{F}_B} = \mathbf{r}_{TB} \times \mathbf{F}_B = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_B - x & y_B & z_B - z \\ 0 & F_B & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ a - x & 0 & b - z \\ 0 & F_B & 0 \end{vmatrix} = (z-b)\,F_B\,\boldsymbol{\imath}$$
$$+ (a-x)\,F_B\,\mathbf{k}.$$

$$\mathbf{M}_T^{\mathbf{F}_C} = \mathbf{r}_{TC} \times \mathbf{F}_C = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ x_C - x & y_C & z_C - z \\ F_C & 0 & 0 \end{vmatrix} = \begin{vmatrix} \boldsymbol{\imath} & \boldsymbol{\jmath} & \mathbf{k} \\ -x & 0 & b - z \\ F_C & 0 & 0 \end{vmatrix} = (b-z)\,F_C\,\boldsymbol{\jmath}.$$

The total moment about the point T of the forces is

$$\mathbf{M} = (z - b)\, F_B\, \mathbf{\imath} + [(a - x)\, F_A + (b - z)\, F_C]\, \mathbf{\jmath} + (a - x)\, F_B\, \mathbf{k}.$$

The direction cosines of the moment $\mathbf{M}$, of magnitude M, are the same as the direction cosines of the resultant $\mathbf{R}$ and three scalar equations can be written

$$\cos\theta_x = \frac{M_x}{M},\ \cos\theta_y = \frac{M_y}{M},\ \cos\theta_z = \frac{M_z}{M}, \quad \text{or}$$
$$\frac{F_C}{R} = \frac{(z-b)\, F_B}{M},\ \frac{F_B}{R} = \frac{(a-x)\, F_A + (b-z)\, F_C}{M},\ \frac{-F_A}{R} = \frac{(a-x)\, F_B}{M} \quad \text{or}$$

$$\begin{aligned} -3000 + 500\, z &= 0.280\, M, \\ 3600 - 900\, x - 300\, z &= 0.465\, M, \\ 2000 - 500\, x &= -0.839\, M. \end{aligned}$$

There are three scalar equations with three unknowns M, x, and z. The solution of the equations is obtained using the MATLAB function `solve`:

```
rT_ = [x 0 z];
MTA_ = cross(rA_-rT_, FA_);
MTB_ = cross(rB_-rT_, FB_);
MTC_ = cross(rC_-rT_, FC_);
MT_ = MTA_ + MTB_ + MTC_;
eq1 = MT_(1)/M - uR_(1);
eq2 = MT_(2)/M - uR_(2);
eq3 = MT_(3)/M - uR_(3);
eq1n = subs(eq1, sl, nl);
eq2n = subs(eq2, sl, nl);
eq3n = subs(eq3, sl, nl);
digits(3)
fprintf('first equation:\n')
pretty(eq1)
fprintf('%s = 0 \n\n',char(vpa(eq1n)))
fprintf('second equation:\n')
pretty(eq2)
fprintf('%s = 0 \n\n',char(vpa(eq2n)))
fprintf('third equation:\n')
pretty(eq3)
fprintf('%s = 0 \n\n',char(vpa(eq3n)))
sol = solve(eq1, eq2, eq3,'x, z, M');
Ms = sol.M; Mn = subs(Ms, sl, nl);
xs = sol.x; xn = subs(xs, sl, nl);
zs = sol.z; zn = subs(zs, sl, nl);
```

```
fprintf('M = ')
pretty(Ms)
fprintf('M = %6.3f (kip ft)\n', Mn)
fprintf('x = ')
pretty(xs)
fprintf('x = %6.3f (ft)\n', xn)
fprintf('z = ')
pretty(zs)
fprintf('z = %6.3f (ft)\n', zn)
```

The function `pretty(x)` prints the symbolic expression `x` in a format that looks like type-set mathematics. The results obtained with MATLAB are:

```
first equation:
             FC              FB (b - z)
  - -------------------- - ----------
        2     2     2 1/2        M
    (FA  + FB  + FC )
(0.5*(z - 6.0))/M - 0.28 = 0

second equation:
                     / a     \
  FC (b - z) + FA | - - x |
                     \ 2     /              FB
  ------------------------- - --------------------
              M                    2     2     2 1/2
                                (FA  + FB  + FC )
- (1.0*(0.9*x + 0.3*z - 3.6))/M - 0.466 = 0

third equation:
           FA              FB (a - x)
  -------------------- + ----------
      2     2     2 1/2        M
  (FA  + FB  + FC )
0.839 - (0.5*(x - 4.0))/M = 0

M =
                2     2     2 1/2
    FA FB a (FA  + FB  + FC )
  - ----------------------------
            2       2       2
        2 FA  + 2 FB  + 2 FC
M = -0.839 (kip ft)
x =
      2         2         2
```

```
  a FA  + 2 a FB  + 2 a FC
  -------------------------
        2       2       2
    2 FA  + 2 FB  + 2 FC
x =  2.591 (ft)
z =
        2                   2         2
  2 b FA  - a FA FC + 2 b FB  + 2 b FC
  -------------------------------------
             2       2       2
          2 FA  + 2 FB  + 2 FC
z =  5.530 (ft)
```

The moment $M = -839.254$ lb ft = -0.839 kip ft is negative, and that is why the couple vector is pointing in the direction opposite to **R**, which makes the wrench negative. The MATLAB program for plotting the vectors and the figure are:

```
a=4; b=6;
axis([-2*a 2*a  -b b -2*b 2*b])

xA=a/2; yA=0; zA=0;
xB=a; yB=0; zB=b;
xC=0; yC=0; zC=b;
xD=a; yD=0; zD=0;
xT=xn; yT=0; zT=zn;

line([0 xC],[0 yC],[0,zC],...
    'Color','b','LineWidth',2)
line([0 xD],[0 yD],[0,zD],...
    'Color','b','LineWidth',2)
line([xD xB],[yD yB],[zD,zB],...
    'Color','b','LineWidth',2)
line([xC xB],[yC yB],[zC,zB],...
    'Color','b','LineWidth',2)

fs=10; % force scale
FAn_ = fs*subs(FA_, sl, nl);
FBn_ = fs*subs(FB_, sl, nl);
FCn_ = fs*subs(FC_, sl, nl);
Rtn_ = fs*Rn_;
Mtn_ = fs*Mn*uRn_;

quiver3...
(xA,yA,zA,FAn_(1),FAn_(2),FAn_(3),1,...
'Color','k','LineWidth',2)
```

```
quiver3...
(xB,yB,zB,FBn_(1),FBn_(2),FBn_(3),1,...
'Color','k','LineWidth',2)
quiver3...
(xC,yC,zC,FCn_(1),FCn_(2),FCn_(3),1,...
'Color','k','LineWidth',2)
quiver3...
(xT,yT,zT,Rtn_(1),Rtn_(2),Rtn_(3),1,...
'Color','r','LineWidth',2)
quiver3...
(xT,yT,zT,Mtn_(1),Mtn_(2),Mtn_(3),1,...
'Color','G','LineWidth',2)
```

The vector representation with MATLAB is shown in Fig. 2.18b.

2.6 Problems

2.1 (a) Determine the resultant of the forces $\mathbf{F}_1 = F_{1x}\,\mathbf{\imath} + F_{1y}\,\mathbf{\jmath} + F_{1z}\,\mathbf{k}$, $\mathbf{F}_2 = F_{2x}\,\mathbf{\imath} + F_{2y}\,\mathbf{\jmath} + F_{2z}\,\mathbf{k}$, and $\mathbf{F}_3 = F_{3x}\,\mathbf{\imath} + F_{3y}\,\mathbf{\jmath} + F_{3z}\,\mathbf{k}$, which are concurrent at the point $P(x_P,\ y_P,\ z_P)$, where $F_{1x} = 2$, $F_{1y} = 3.5$, $F_{1z} = -3$, $F_{2x} = -1.5$, $F_{2y} = 4.5$, $F_{2z} = -3$, $F_{3x} = 7$, $F_{3y} = -6$, $F_{3z} = 5$, $x_P = 1$, $y_P = 2$, and $z_P = 3$. (b) Find the total moment of the given forces about the origin $O(0,\ 0,\ 0)$. The units for the forces are in Newtons and for the coordinates are given in meters.

2.2 (a) Determine the resultant of the three forces shown in Fig. 2.19. The force $\mathbf{F}_1$ acts along the x-axis, the force $\mathbf{F}_2$ acts along the z-axis, and the direction of the force $\mathbf{F}_3$ is given by the line O_3P_3, where $O_3 = O(x_{O_3},\ y_{O_3},\ z_{O_3})$ and $P_3 = P(x_{P_3},\ y_{P_3},\ z_{P_3})$. The application point of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is the origin $O(0,\ 0,\ 0)$ of the reference frame as shown in Fig. 2.19. (b) Find the total moment of the given forces about the point P_3. Numerical application: $|\mathbf{F}_1| = F_1 = 250$ N, $|\mathbf{F}_2| = F_2 = 300$ N, $|\mathbf{F}_3| = F_3 = 300$ N, $O_3 = O_3(1,\ 2,\ 3)$ and $P_3 = P_3(5,\ 7,\ 9)$. The coordinates are given in meters.

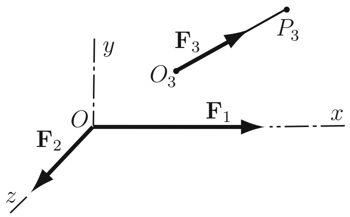


Fig. 2.19 Problem 2.2

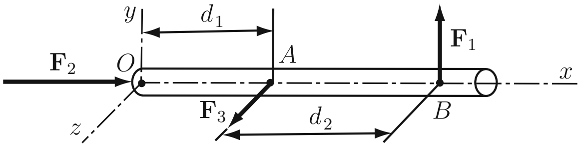


Fig. 2.20 Problem 2.3

2.3 Replace the three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, shown in Fig. 2.20, by a resultant force, $\mathbf{R}$, through O and a couple. The force $\mathbf{F}_2$ acts along the x-axis, the force $\mathbf{F}_1$ is parallel to the y-axis, and the force $\mathbf{F}_3$ is parallel to the z-axis. The application point of the forces $\mathbf{F}_2$ is O, the application point of the forces $\mathbf{F}_1$ is B, and the application points of the force $\mathbf{F}_3$ is A. The distance between O and A is d_1 and the distance between A and B is d_2 as shown in Fig. 2.20. Numerical application: $|\mathbf{F}_1| = F_1 = 250$ N, $|\mathbf{F}_2| = F_2 = 300$ N, $|\mathbf{F}_3| = F_3 = 400$ N, $d_1 = 1.5$ m and $d_2 = 2$ m.

2.4 Two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ and a couple of moment M in the xy plane are given. The force $\mathbf{F}_1 = F_{1x}\,\mathbf{ı} + F_{1y}\,\mathbf{ȷ} + F_{1z}\,\mathbf{k}$ acts at the point $P_1 = P_1(x_1,\ y_1,\ z_1)$ and the force $\mathbf{F}_2 = F_{2x}\,\mathbf{ı} + F_{2y}\,\mathbf{ȷ} + F_{2z}\,\mathbf{k}$ acts at the point $P_2 = P_2(x_2,\ y_2,\ z_2)$. Find the resultant force-couple system at the origin $O(0,\ 0,\ 0)$. Numerical application: $F_{1x} = 10$, $F_{1y} = 5$, $F_{1z} = 40$, $F_{2x} = 30$, $F_{2y} = 10$, $F_{2z} = -30$, $F_{3x} = 7$, $F_{3y} = -6$, $F_{3z} = 5$, $P_1 = P_1(0,\ 1,\ -1)$, $P_2 = P_2(1,\ 1,\ 1)$ and $M = -30$ N·m. The units for the forces are in Newtons and for the coordinates are given in meters.

2.5 Replace the three forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, shown in Fig. 2.21, by a resultant force at the origin O of the reference frame and a couple. The force $\mathbf{F}_1$ acts along the x-axis, the force $\mathbf{F}_2$ is parallel with the z-axis, and the force $\mathbf{F}_3$ is parallel with the y-axis. The application point of the force $\mathbf{F}_1$ is at O, the application point of the forces $\mathbf{F}_2$ is at A, and the application points of the force $\mathbf{F}_3$ is at B. The distance between the origin O and the point A is d_1 and the distance between the point A and the point B is d_2. The line AB is parallel with the z-axis. Numerical application: $|\mathbf{F}_1| = F_1 = 50$ N, $|\mathbf{F}_2| = F_2 = 30$ N, $|\mathbf{F}_3| = F_3 = 60$ N, $d_1 = 1$ m, and $d_2 = 0.7$ m

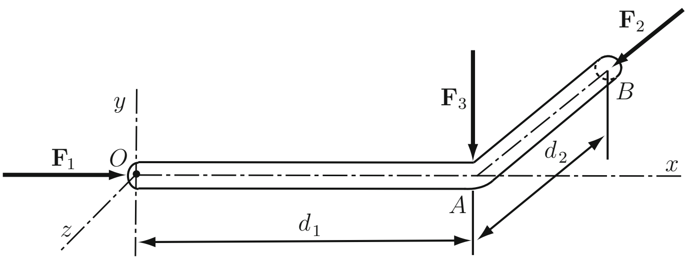


Fig. 2.21 Problem 2.5

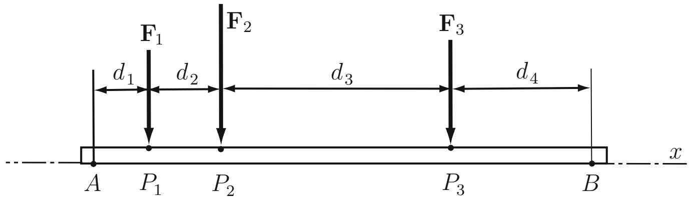


Fig. 2.22 Problem 2.6

2.6 Three forces $\mathbf{F}_1$, $\mathbf{F}_2$ and $\mathbf{F}_3$ act on a beam as shown in Fig. 2.22. The directions of the forces are parallel with y-axis. The application points of the forces are P_1, P_2, and P_3, and the distances $AP_1 = d_1$, $P_1P_2 = d_2$, $P_2P_3 = d_3$ and $P_3B = d_4$ are given. (a) Find the resultant of the system. (b) Resolve this resultant into two components at the points A and B. Numerical application: $|\mathbf{F}_1| = F_1 = 30$ N, $|\mathbf{F}_2| = F_2 = 60$ N, $|\mathbf{F}_3| = F_3 = 50$ N, $d_1 = 0.1$ m, $d_2 = 0.3$ m, $d_3 = 0.4$ m and $d_4 = 0.4$ m.

2.7 A force $\mathbf{F}$ acts vertically downward, parallel to the y-axis, and intersects the xz plane at the point $P_1(x_1, y_1, z_1)$. Resolve this force into three components acting through the points $P_2 = P_2(x_2, y_2, z_2)$, $P_3 = P_3(x_3, y_3, z_3)$ and $P_4 = P_4(x_4, y_4, z_4)$. Numerical application: $|\mathbf{F}| = F = 50$ N, $P_1 = P_1(2, 0, 4)$, $P_2 = P_2(1, 1, 1)$, $P_3 = P_3(6, 0, 0)$, and $P_4 = P_4(0, 0, 3)$. The coordinates are given in meters.

2.8 Determine the resultant of the given system of forces $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$, shown in the Fig. 2.23. The angle between the direction of the force $\mathbf{F}_1$ and the Ox axis is θ_1 and the angle between the direction of the force $\mathbf{F}_2$ with the x-axis is θ_2. The x and y components of the force $\mathbf{F}_3 = |\mathbf{F}_{3x}|\,\mathbf{i} + |\mathbf{F}_{3y}|\,\mathbf{j} = F_{3x}\,\mathbf{i} + F_{3y}\,\mathbf{j}$ are given. Numerical application: $|\mathbf{F}_1| = F_1 = 250$ N, $|\mathbf{F}_2| = F_2 = 220$ N, $|\mathbf{F}_{3x}| = F_{3x} = 50$ N, $|\mathbf{F}_{3y}| = F_{3y} = 120$ N, $\theta_1 = 30°$, and $\theta_2 = 45°$.

2.9 The rectangular plate in Fig. 2.24 is subjected to four parallel forces. Determine the magnitude and direction of a resultant force equivalent to the given force system and locate its point of application on the plate. Numerical application: $F_O = 700$ lb, $F_A = 600$ lb, $F_B = 500$ lb, $F_C = 100$ lb, $a = 8$ ft, and $b = 10$ ft. Hint: the moments about the x-axis and y-axis of the resultant force, are equal

Fig. 2.23 Problem 2.8

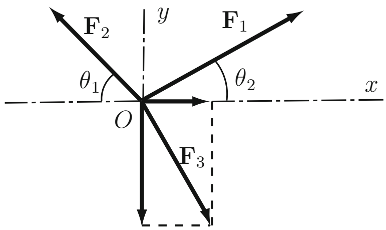


Fig. 2.24 Problem 2.9

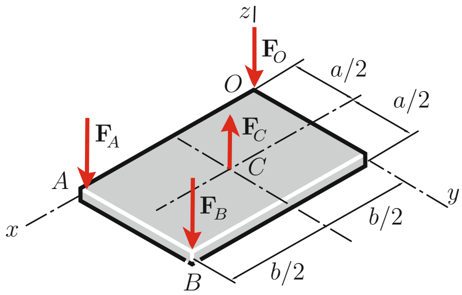
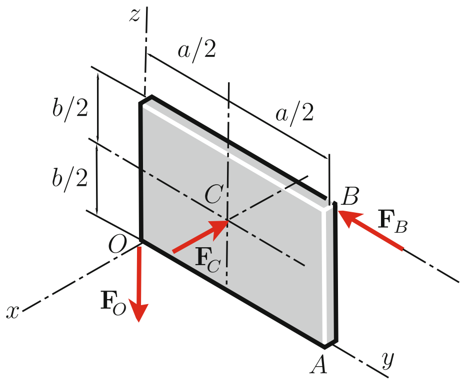


Fig. 2.25 Problem 2.10



to the sum of the moments about the x-axis and y-axis of all the forces in the system.

2.10 Three forces $\mathbf{F}_O$, $\mathbf{F}_B$, and $\mathbf{F}_C$, as shown in Fig. 2.25 , are acting on a rectangular planar plate ($\mathbf{F}_O || Oz$, $\mathbf{F}_B || Oy$, $\mathbf{F}_C || Ox$). The three forces acting on the plate are replaced by a wrench. Find: (a) the resultant force for the wrench; (b) the magnitude of couple moment, M, for the wrench and the point $Q(y, z)$ where its line of action intersects the plate. Numerical application: $F_O = 800$ lb, $F_B = F_C = 500$ lb, $a = OA = 6$ ft, and $b = AB = 5$ ft.

2.7 Programs

2.7.1 Program 2.1

```
% example 2.1
clear all; clc; close all
syms F theta a b real
rA_ = [a b 0];
```

```
FA_ = [F*cos(theta) F*sin(theta) 0];
MO_ = cross(rA_, FA_);
MOz= MO_(3);
sl = {F, theta, a, b};
nl = {5, pi/4, 1, 5};
fprintf('MOz = %s =',char(MOz))
fprintf('%6.3f (kN m)\n',subs(MOz,sl,nl))

% numerical values
rAn_ = double(subs(rA_,sl,nl));
Fn_ = double(subs(FA_,sl,nl));
Mn_ = subs(MO_,sl,nl);

% vector plotting
axis([0 5  0 10  -18 0])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on

% Cartesian axes
text(0,0,0,' O','fontsize',14,'fontweight','b')
quiver3(0,0,0,4,0,0,1,'Color','b')
text(4.1,0,0,'x')
quiver3(0,0,0,0,9,0,1,'Color','b')
text(0,9.4,0,'y')
quiver3(0,0,0,0,0,5,1,'Color','b')
text(0,0,5.5,' z')

line([0 0],[0 rAn_(2)],[0,0],'LineStyle','--',...
    'Color','k','LineWidth',4)
line...
([0 rAn_(1)],[rAn_(2) rAn_(2)],[0,0],...
'LineStyle','--','Color','k','LineWidth',4)
text(rAn_(1),rAn_(2),0,'   A',...
     'fontsize',14,'fontweight','b')

quiver3(0,0,0,rAn_(1),rAn_(2),0,1,...
    'Color','b','LineWidth',2)
text(rAn_(1)/2,rAn_(2)/2,0,...
    ' r_A','fontsize',14,'fontweight','b')

quiver3(rAn_(1),rAn_(2),0,Fn_(1),Fn_(2),0,1,...
    'Color','r','LineWidth',2)
quiver3(rAn_(1),rAn_(2),0,Fn_(1),0,0,1,...
    'Color','k','LineWidth',1)
quiver3(rAn_(1),rAn_(2),0,0,Fn_(2),0,1,...
```

```
    'Color','k','LineWidth',1)
text(rAn_(1)+Fn_(1),rAn_(2),0,...
    'F_x','fontsize',14,'fontweight','b')
text(rAn_(1),rAn_(2)+Fn_(2),0,...
    'F_y','fontsize',14,'fontweight','b')
text(rAn_(1)+Fn_(1),rAn_(2)+Fn_(2),0,...
'  F','fontsize',14,'fontweight','b')

quiver3(0,0,0,0,0,Mn_(3),1,...
    'Color','r','LineWidth',4)
text(Mn_(1)/2,Mn_(2)/2,Mn_(3)/2,...
    '  M_O^F = r_A x F',...
    'fontsize',14,'fontweight','b')

% end of program
```

2.7.2 Program 2.2

```
% example 2.1
clear all; clc; close all
syms F theta a b real
rA_ = [a b 0];
FA_ = [F*cos(theta) F*sin(theta) 0];
MO_ = cross(rA_, FA_);
MOz= MO_(3);
sl = {F, theta, a, b};
nl = {5, pi/4, 1, 5};
fprintf('MOz = %s =',char(MOz))
fprintf('%6.3f (kN m)\n',subs(MOz,sl,nl))

% numerical values
rAn_ = double(subs(rA_,sl,nl));
Fn_ = double(subs(FA_,sl,nl));
Mn_ = subs(MO_,sl,nl);

% vector plotting
axis([0 5  0 10  -18 0])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on

% Cartesian axes
text(0,0,0,' O','fontsize',14,'fontweight','b')
quiver3(0,0,0,4,0,0,1,'Color','b')
```

```
text(4.1,0,0,'x')
quiver3(0,0,0,0,9,0,1,'Color','b')
text(0,9.4,0,'y')
quiver3(0,0,0,0,0,5,1,'Color','b')
text(0,0,5.5,' z')

line([0 0],[0 rAn_(2)],[0,0],'LineStyle','--',...
    'Color','k','LineWidth',4)
line...
([0 rAn_(1)],[rAn_(2) rAn_(2)],[0,0],...
'LineStyle','--','Color','k','LineWidth',4)
text(rAn_(1),rAn_(2),0,'   A',...
     'fontsize',14,'fontweight','b')

quiver3(0,0,0,rAn_(1),rAn_(2),0,1,...
    'Color','b','LineWidth',2)
text(rAn_(1)/2,rAn_(2)/2,0,...
    ' r_A','fontsize',14,'fontweight','b')

quiver3(rAn_(1),rAn_(2),0,Fn_(1),Fn_(2),0,1,...
    'Color','r','LineWidth',2)
quiver3(rAn_(1),rAn_(2),0,Fn_(1),0,0,1,...
    'Color','k','LineWidth',1)
quiver3(rAn_(1),rAn_(2),0,0,Fn_(2),0,1,...
    'Color','k','LineWidth',1)
text(rAn_(1)+Fn_(1),rAn_(2),0,...
    'F_x','fontsize',14,'fontweight','b')
text(rAn_(1),rAn_(2)+Fn_(2),0,...
    'F_y','fontsize',14,'fontweight','b')
text(rAn_(1)+Fn_(1),rAn_(2)+Fn_(2),0,...
'  F','fontsize',14,'fontweight','b')

quiver3(0,0,0,0,0,Mn_(3),1,...
    'Color','r','LineWidth',4)
text(Mn_(1)/2,Mn_(2)/2,Mn_(3)/2,...
    '  M_O^F = r_A x F',...
    'fontsize',14,'fontweight','b')

% end of program
```

2.7.3 Program 2.3

```
% example 2.3
clear all; clc; close all
syms a b c F
rA_ = [a 0 0];
rB_ = [0 b 0];
rE_ = [0 b c];
rAE_ = rE_ - rA_;
rAB_ = rB_ - rA_;
f_ = [0 0 -F];

ME_ = cross(rAE_, f_); % M = rAE x F
MB_ = cross(rAB_, f_); % M = rAB x F
T = ME_ == MB_; % rAB x F = rAE x F
fprintf('ME_ == MB_ => [%d %d %d]\n',T)
fprintf('1=TRUE 0=FALSE\n')
fprintf('\n')
fprintf('M_ = rAB_ x F_ = rAE_ x F_ \n')
fprintf('Mx = %s; ',char(ME_(1)))
fprintf('My = %s; ',char(ME_(2)))
fprintf('Mz = %s.\n',char(ME_(3)))

% numerical calculation
sl = {a, b, c, F};
nl = {1, 3, 2, 1};
MEn_ = double(subs(ME_,sl,nl));
MBn_ = double(subs(MB_,sl,nl));

fprintf('ME_ = [%6.3f %6.3f %d] (kN m)\n',MEn_)
fprintf('MB_ = [%6.3f %6.3f %d] (kN m)\n',MBn_)

% graphical representation
F=1; % kN
a=1; b=3; c=2; % m

axis([-2 2 -1 4 0 2])
hold on, grid on

% Cartesian axes
line ...
([0 4],[0 0],[0,0],'Color','b','LineWidth',1.5)
text(3,0,0,'x','fontweight','b')

line ...
```

```
([0 0],[0 4],[0,0],'Color','b','LineWidth',1.5)
text(0,4.1,0,'y','fontweight','b')

line ...
([0 0],[0 0],[0,2.5],'Color','b','LineWidth',1.5)
text(0,0,2.6,'z','fontweight','b')

text(-.45,0,0,'O(1)','fontweight','b')
text(a+.1,0,0,'A(2)','fontweight','b')
text(.1,b-.1,0,'B(3)','fontweight','b')
text(-.45,0,c-.1,'C(4)','fontweight','b')
text(a+.1,0,c,'D(5)','fontweight','b')
text(0,b+.05,c-.1,'E(6)','fontweight','b')

text((a+.1)/3,.3,0,'a','fontweight','b')
text(.05,(b-.1)/2,.17,'b','fontweight','b')
text(-.16,0,(c-.1)/2,'c','fontweight','b')

view(42,34);
% view(AZ,EL) set the angle of the view from
% which an observer sees the current 3-D plot
% AZ is the azimuth or horizontal rotation
% EL is the vertical elevation
% (both in degrees)

% Generate data
vert=...
[0 0 0; a 0 0; 0 b 0; 0 0 c; a 0 c; 0 b c];
% define the matrix of the vertices
% O: 0,0,0 defined as vertex 1
% A: a,0,0 defined as vertex 2
% B: 0,b,0 defined as vertex 3
% C: 0,0,c defined as vertex 4
% D: a,0,c defined as vertex 5
% E: 0,b,c defined as vertex 6

face_up=[1 2 3; 4 5 6];
% define the lower and upper face of
% the triangular prism
% lower face is defined by vertices
% 1, 2, 3 (O, A, B)
% upper face is defined by vertices
% 4, 5, 6 (C, D, E)

face_l=[1 2 5 4; 2 3 6 5; 1 3 6 4];
```

```
% generate the lateral faces
% lateral face 1 is defined by 1, 2, 5, 4
% lateral face 2 is defined by 2, 3, 6, 5
% lateral face 3 is defined by 1, 3, 6, 4
% when defined a face the order of the vertices
% has to be given clockwise or counterclockwise

% draw the lower and upper triangular patches
patch...
('Vertices',vert,'Faces',face_up,'facecolor','b')
% patch(x,y,C) adds the "patch" or
% filled 2-D polygon defined by
% vectors x and y to the current axes.
% C specifies the color of the face(s)
% X represents the matrix vert
% Y represents the matrix face_up

% draw the lateral rectangular patches
patch...
('Vertices',vert,'Faces',face_l,'facecolor','b')

quiver3 ...
(0,b,F+c,0,0,-F,1,'Color','r','LineWidth',1.75)
text ...
(-.3,b,c+.2,' F','fontsize',14,'fontweight','b')

quiver3(a,0,0,MBn_(1),MBn_(2),MBn_(3),1,...
    'Color','k','LineWidth',2)
text((a+MBn_(1))/2,MBn_(2)/2,MBn_(3)/2,...
    '  M','fontsize',14,'fontweight','b')

quiver3 ...
(a,0,0,MBn_(1),0,0,1,'Color','r','LineWidth',2)
text((a+MBn_(1))/1.3,0,0,...
    ' M_x','fontsize',14,'fontweight','b')

quiver3 ...
(a,0,0,0,MBn_(2),0,1,'Color','r','LineWidth',2)
text(a+.3,MBn_(2),0,...
    'M_y','fontsize',14,'fontweight','b')

light('Position',[1 2 3]);
% light('PropertyName',propertyvalue,...)
% light creates a light object in current axes
% Lights affect only patch and surface objects
```

```
% light the peaks surface plot with a light source
% located at infinity and oriented along the
% direction defined by the vector [1 2 3]


material shiny

% material shiny makes the objects shiny

alpha('color');
% alpha get or set alpha properties for
% objects in the current axis
% alpha('color') set the alphadata to be
% the same as the color data.

% end of program
```

2.7.4 Program 2.4

```
% example 2.4
clear all; clc; close all
syms F l theta alfa real
sl = {F, l, theta, alfa};
nl = {100, 1, pi/4, pi/1.8};
FA_ = [-F*cos(alfa-theta), F*sin(alfa-theta), 0];
rA_ = [l*cos(theta), l*sin(theta), 0];
FAn_= subs(FA_, sl, nl);
fprintf('R_ = [%6.3f %6.3f %g](lb)\n', FAn_)
MO_ = cross(rA_, FA_);
MOz= simplify(MO_(3));
MOzn= subs(MOz, sl, nl);
fprintf('MOz = %s ',char(MOz))
fprintf('= %6.3f (lb ft)\n',MOzn)

% end of program
```

2.7.5 Program 2.5

```
% example 2.5
clear all; clc; close all
```

```
syms a b FA FB FC x z M
% a)
sl = {a, b, FA, FB, FC};
nl = {4, 6, 0.9, 0.5, 0.3};
FA_ = [0 0 -FA]; rA_ = [a/2 0 0];
FB_ = [0 FB  0]; rB_ = [a 0 b];
FC_ = [FC 0  0]; rC_ = [0 0 b];
R_ = FA_+FB_+FC_;
Rn_ = subs(R_, sl, nl);
uR_ = R_/magn(R_);
uRn_ = subs(uR_, sl, nl);
fprintf('R_ = [%6.3f %6.3f %6.3f] (kip)\n', Rn_)
fprintf('|R_| = %6.3f (kip)\n', magn(Rn_))
fprintf('uR_ = [%6.3f %6.3f %6.3f]\n\n', uRn_)

% b)
rT_ = [x 0 z];
MTA_ = cross(rA_-rT_, FA_);
MTB_ = cross(rB_-rT_, FB_);
MTC_ = cross(rC_-rT_, FC_);
MT_ = MTA_ + MTB_ + MTC_;
eq1 = MT_(1)/M - uR_(1);
eq2 = MT_(2)/M - uR_(2);
eq3 = MT_(3)/M - uR_(3);
eq1n = subs(eq1, sl, nl);
eq2n = subs(eq2, sl, nl);
eq3n = subs(eq3, sl, nl);
digits(3)
fprintf('first equation:\n')
pretty(eq1)
fprintf('%s = 0 \n\n',char(vpa(eq1n)))
fprintf('second equation:\n')
pretty(eq2)
fprintf('%s = 0 \n\n',char(vpa(eq2n)))
fprintf('third equation:\n')
pretty(eq3)
fprintf('%s = 0 \n\n',char(vpa(eq3n)))
sol = solve(eq1, eq2, eq3,'x, z, M');
Ms = sol.M; Mn = subs(Ms, sl, nl);
xs = sol.x; xn = subs(xs, sl, nl);
zs = sol.z; zn = subs(zs, sl, nl);
fprintf('M = ')
pretty(Ms)
fprintf('M = %6.3f (kip ft)\n', Mn)
fprintf('x = ')
```

```
pretty(xs)
fprintf('x = %6.3f (ft)\n', xn)
fprintf('z = ')
pretty(zs)
fprintf('z = %6.3f (ft)\n', zn)

a=4; b=6;

axis([-2*a 2*a  -b b -2*b 2*b])
xlabel('x'), ylabel('y'), zlabel('z')
hold on, grid on

% Cartesian axes
quiver3(0,0,0,2*a,0,0,1,'Color','b')
text(2*a,0,0,' x')
quiver3(0,0,0,0,b,0,1,'Color','b')
text(0,b,0,' y')
quiver3(0,0,0,0,0,2*b,1,'Color','b')
text(0,0,2*b,' z')

xA=a/2; yA=0; zA=0;
xB=a; yB=0; zB=b;
xC=0; yC=0; zC=b;
xD=a; yD=0; zD=0;
xT=xn; yT=0; zT=zn;

line([0 xC],[0 yC],[0,zC],...
    'Color','b','LineWidth',2)
line([0 xD],[0 yD],[0,zD],...
    'Color','b','LineWidth',2)
line([xD xB],[yD yB],[zD,zB],...
    'Color','b','LineWidth',2)
line([xC xB],[yC yB],[zC,zB],...
    'Color','b','LineWidth',2)

text(0,0,0,' O')
text(xA,yA,zA,' A')
text(xB,yB,zB,' B')
text(xC,yC,zC,' C')
text(xD,yD,zD,' D')
text(xT,yT,zT-1,' T')

fs=10; % force scale
FAn_ = fs*subs(FA_, sl, nl);
FBn_ = fs*subs(FB_, sl, nl);
```

```
FCn_ = fs*subs(FC_, sl, nl);
Rtn_ = fs*Rn_;
Mtn_ = fs*Mn*uRn_;

quiver3...
(xA,yA,zA,FAn_(1),FAn_(2),FAn_(3),1,...
'Color','k','LineWidth',2)
quiver3...
(xB,yB,zB,FBn_(1),FBn_(2),FBn_(3),1,...
'Color','k','LineWidth',2)
quiver3...
(xC,yC,zC,FCn_(1),FCn_(2),FCn_(3),1,...
'Color','k','LineWidth',2)
quiver3...
(xT,yT,zT,Rtn_(1),Rtn_(2),Rtn_(3),1,...
'Color','r','LineWidth',2)
quiver3...
(xT,yT,zT,Mtn_(1),Mtn_(2),Mtn_(3),1,...
'Color','G','LineWidth',2)

text(xA+FAn_(1),yA+FAn_(2),zA+FAn_(3),...
   ' F_A','fontsize',12,'fontweight','b')
text(xB+FBn_(1),yB+FBn_(2),zB+FBn_(3),...
   ' F_B','fontsize',12,'fontweight','b')
text(xT+Rtn_(1),yT+Rtn_(2),zT+Rtn_(3),...
   ' R','fontsize',14,'fontweight','b')
text(xT+Mtn_(1),yT+Mtn_(2),zT+Mtn_(3),...
   ' M','fontsize',14,'fontweight','b')

view(-68,30);

% end of program
```

References

1. P. Appell, *Traité de mécanique rationnelle* (Gauthier-Villars, Paris, 1955)
2. M. Atanasiu, *Mechanics* (EDP, Bucharest, 1973)
3. H. Baruh, *Analytical Dynamics* (WCB/McGraw-Hill, Boston, 1999)
4. F.P. Beer, E.R. Johnston, Vector Mechanics for Engineers: Statics and Dynamics 5/e (McGraw-Hill Publishing Company, New York, 1988)
5. F.P. Beer, E.R. Johnston, D.F. Mazurek, Vector Mechanics for Engineers: Statics, 10/e (McGraw-Hill Publishing Company, New York, 2012)
6. A.M. Bedford, W. Fowler, K.M. Liechti, *Statics and Mechanics of Materials* (Prentice Hall, Inc., Upper Saddle River, 2002)

7. A.M. Bedford, W. Fowler, *Engineering Mechanics: Statics, 5/e* (Prentice Hall, Inc., Upper Saddle River, 2007)
8. A.P. Boresi, R.J. Schmidt, Engineering Mechanics: Statics (PWS Publishing Company, Boston, 2000)
9. M.I. Buculei, *Mechanics* (University of Craiova Press, Craiova, 1974)
10. M.I. Buculei, D. Bagnaru, G. Nanu, D.B. Marghitu, *Analysis of Mechanisms with Bars* (Scrisul romanesc, Craiova, 1986)
11. I. Stroe et al., Analytical Mechanics Problems (University Politehnica of Bucharest, Bucharest, 1997)
12. V. Ceausu, N. Enescu, F. Ceausu, Mechanics Problems, (Printech, Bucharest, 1999)
13. S.J. Chapman, MATLAB Programming for Engineers (Thomson Learning, Pacific Grove, CA, 2002)
14. S. Attaway, MATLAB: A Practical Introduction to Programming and Problem Solving, Butterworth-Heinemann (Elsevier, Amsterdam, 2012)
15. D.M. Etter, D.C. Kuncicky, *Introduction to MATLAB for Engineers and Scientists* (Prentice Hall, Inc., Upper Saddle River, 1996)
16. C. Iacob, *Theoretical Mechanics* (EDP, Bucharest, 1980)
17. J.H. Ginsberg, *Advanced Engineering Dynamics* (Cambridge University Press, Cambridge, 1995)
18. D.T. Greenwood, *Principles of Dynamics* (Prentice-Hall, Inc., Englewood Cliffs, 1998)
19. L.E. Goodman, W.H. Warner, *Statics* (Dover Publications, Inc, 2001)
20. R.C. Hibbeler, *Engineering Mechanics: Statics and Dynamics 13/e* (Prentice-Hall, Inc., Upper Saddle River, 2013)
21. T.R. Kane, *Analytical Elements of Mechanics*, vol. 1 (Academic Press, New York, 1959)
22. T.R. Kane, *Analytical Elements of Mechanics*, vol. 2 (Academic Press, New York, 1961)
23. T.R. Kane, D.A. Levinson, *Dynamics* (McGraw-Hill, New York, 1985)
24. R. Maeder, *Programming in Mathematica* (Addison-Wesley Publishing Company, Redwood City, 1990)
25. N.H. Madsen, Statics and Dynamics, class notes, available at http://www.eng.auburn.edu/users/nmadsen/
26. D.B. Marghitu, *Mechanical Engineer's Handbook* (Academic Press, San Diego, 2001)
27. D.B. Marghitu, M.J. Crocker, *Analytical Elements of Mechanisms* (Cambridge University Press, Cambridge, 2001)
28. D.B. Marghitu, *Kinematic Chains and Machine Component Design* (Elsevier, Amsterdam, 2005)
29. D.B. Marghitu, Mechanisms and Robots Analysis with MATLAB (Springer, London, 2009)
30. D.B. Marghitu, M. Dupac, Advanced Dynamics: Analytical and Numerical Calculations with MATLAB (Springer, New York, 2012)
31. D.B. Marghitu, Statics and Dynamics, class notes, available at http://www.eng.auburn.edu/users/marghitu/
32. D.J. McGill, W.W. King, *Engineering Mechanics: Statics and an Introduction to Dynamics* (PWS Publishing Company, Boston, 1995)
33. J.L. Meriam, L.G. Kraige, *Engineering Mechanics: Statics, 7/e* (Wiley, New York, 2011)
34. R.L. Mott, *Machine Elements in Mechanical Design* (Prentice Hall, Upper Saddle River, 1999)
35. R.L. Norton, *Machine Design* (Prentice-Hall, Upper Saddle River, 1996)
36. M. Plesha, G. Gray, F. Costanzo, Engineering Mechanics: Statics, 2/e, McGraw-Hill Publishing Company (2012)
37. M. Radoi, E. Deciu, *Mechanics* (EDP, Bucharest, 1981)
38. W.F. Riley, L.D. Sturges, Engineering Mechanics: Statics, 2/e (Wiley, New York, 1995)
39. A. Ruina, R. Pratap, Introduction to Statics and Dynamics (Oxford University Press, Oxford, 2002)
40. A. Ripianu, P. Popescu, B. Balan, *Technical Mechanics* (EDP, Bucharest, 1979)
41. I.H. Shames, Engineering Mechanics Statics, 4/e, Prentice Hall, Inc. (1996)

42. S.D. Sheppard, B.H. Tongue, *Statics: Analysis and Design of Systems in Equilibrium* (Wiley, New York, 2005)
43. D. Smith, *Engineering Computation with MATLAB* (Pearson Education, Upper Saddle River, 2008)
44. R.W. Soutas-Little, D.J. Inman, *Engineering Mechanics: Statics and Dynamics* (Prentice-Hall, Upper Saddle River, 1999)
45. R.W. Soutas-Little, D.J. Inman, D. Balint, Engineering Mechanics (Statics, Cengage, Learning, 2007)
46. S. Staicu, *Theoretical Mechanics* (EDP, Bucharest, 1998)
47. A. Stan, M. Grumazescu, *Mechanics Problems* (EDP, Bucharest, 1973)
48. J. Sticklen, M.T. Eskil, *An Introduction to Technical Problem Solving with MATLAB* (Great Lakes Press, Wildwood, 2006)
49. A. Stoenescu, G. Silas, *Theoretical Mechanics* (ET, Bucharest, 1957)
50. J.H. Jackson, H.G. Wirtz, Schaum's Outline of Theory and Problems of Statics and Strength of Materials (McGraw-Hill Publishing Company, New York, 1983)
51. The MathWorks: https://www.mathworks.com/
52. Statics eBook : https://ecourses.ou.edu/
53. R. Voinea, D. Voiculescu, V. Ceausu, *Mechanics* (EDP, Bucharest, 1983)
54. V. Valcovici, S. Balan, R. Voinea, *Theoretical Mechanics* (ET, Bucharest, 1959)
55. K.J. Waldron, G.L. Kinzel, *Kinematics, Dynamics, and Design of Machinery* (Wiley, New York, 1999)
56. H.B. Wilson, L.H. Turcotte, *D* Halpern, Advanced Mathematics and Mechanics Applications Using MATLAB (Chapman & Hall/CRC, Boca Raton, FL, 2003)
57. J.H. Williams, *Fundamentals of Applied Dynamics* (Wiley, New York, 1996)
58. S. Wolfram, *Mathematica* (Wolfram Media/Cambridge University Press, Cambridge, 1999)

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