Chapter 2
Signal and System Norms

Many control objectives can be stated in terms of the size of some particular signals. Therefore, a quantitative treatment of the performance of control systems requires the introduction of appropriate norms, which give measurements of the sizes of the signals considered. Another concept closely related to the size of a signal is the size of a LTI system. The latter concept is of great practical importance because it is the basis of technical control $\mathbf{H}_\infty$ and the study of robustness (see Chap. 6). These different concepts are detailed hereafter.

2.1 Signal Norms

We consider real valued signals[1] that are piecewise continuous functions of time $t \in [0, \infty)$. In this section we introduce some different norms for these signals.

Definition 2.1 (Norm on a Vector Space) Let $\mathbf{V}$ be a vector space, a given non-negative function $\phi : \mathbf{V} \to \mathbf{R}^+$ is a norm on $\mathbf{V}$ if it satisfies

$$\begin{aligned} &\phi(v) \geq 0, \quad \phi(v) = 0 \iff v = 0 \\ &\phi(\alpha v) = |\alpha|\phi(v) \\ &\phi(v + w) \leq \phi(v) + \phi(w) \end{aligned} \tag{2.1}$$

for all $\alpha \in \mathbf{R}$ and $v, w \in \mathbf{V}$.

A norm is defined on a vector space. To apply this concept to the case of signals, it is necessary to define sets of signals that are vector spaces. This is the case of the signal spaces described below.

[1] In the case of a stochastic signal, we will always assume that it is modeled as an ergodic stationary stochastic process. For a comprehensive description of stochastic signals see the references given in the section Notes and References.

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2.1.1 $\mathbf{L}_1$-*Space and* $\mathbf{L}_1$-*Norm*

The $\mathbf{L}_1$-space is defined as the set of absolute-value integrable signals, i.e., $\mathbf{L}_1 = \{u(t) \in \mathbf{R} : \int_0^{+\infty} |u(t)|\, dt < \infty\}$. The $\mathbf{L}_1$-norm of a signal $u \in \mathbf{L}_1$, denoted $\|u\|_1$, is given by

$$\|u\|_1 = \int_0^{+\infty} \big|u(t)\big|\, dt \tag{2.2}$$

this norm can be used, for instance, to measure a consumption. In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T \in \mathbf{L}_1^{n_u}$ with $u_i(t) \in \mathbf{L}_1$ $i = 1, \ldots, n_u$, the norm is given by

$$\|u\|_1 = \int_0^{+\infty} \sum_{i=1}^{n_u} \big|u_i(t)\big|\, dt = \sum_{i=1}^{n_u} \big\|u_i(t)\big\|_1 \tag{2.3}$$

2.1.2 $\mathbf{L}_2$-*Space and* $\mathbf{L}_2$-*Norm*

The $\mathbf{L}_2$-space is defined as the set of square integrable signals, i.e., we have $\mathbf{L}_2 = \{u(t) \in \mathbf{R} : \int_0^{+\infty} u(t)^2\, dt < \infty\}$. The $\mathbf{L}_2$-norm of a signal $u \in \mathbf{L}_2$, denoted $\|u\|_2$, is given by

$$\|u\|_2 = \left(\int_0^{+\infty} u(t)^2\, dt\right)^{1/2} \tag{2.4}$$

the square of this norm represents the total energy contained in the signal. According to *Parseval's theorem*,[2] the $\mathbf{L}_2$-norm of a signal $u \in \mathbf{L}_2$ can be calculated in the frequency-domain as follows:

$$\|u\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \big|U(j\omega)\big|^2\, d\omega\right)^{1/2} \tag{2.5}$$

where $U(j\omega)$ is the *Fourier transform* of the signal $u(t)$.

[2] Parseval's theorem states that for a causal signal $u \in \mathbf{L}_2$, we have

$$\int_0^{+\infty} u(t)^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U^*(j\omega) U(j\omega)\, d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \big|U(j\omega)\big|^2\, d\omega$$

where $U(j\omega)$ represents the *Fourier transform* of $u(t)$

$$U(j\omega) = \mathcal{F}\big(u(t)\big) = \int_{-\infty}^{+\infty} u(t) e^{-j\omega t}\, dt$$

In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T \in \mathbf{L}_2^{n_u}$ with $u_i(t) \in \mathbf{L}_2$ $i = 1, \ldots, n_u$, the norm is given by

$$\|u\|_2 = \left(\int_0^{+\infty} u(t)^T u(t)\, dt\right)^{\frac{1}{2}} = \left(\int_0^{+\infty} \sum_{i=1}^{n_u} u_i(t)^2\, dt\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n_u} \|u_i\|_2^2\right)^{\frac{1}{2}} \tag{2.6}$$

2.1.3 $\mathbf{L}_\infty$*-Space and* $\mathbf{L}_\infty$*-Norm*

The $\mathbf{L}_\infty$-space is defined as the set of signals bounded in amplitude, i.e., $\mathbf{L}_\infty = \{u(t) \in \mathbf{R} : \sup_{t\geq 0} |u(t)| < \infty\}$. The $\mathbf{L}_\infty$-norm of a signal $u \in \mathbf{L}_\infty$, denoted $\|u\|_\infty$, is given by

$$\|u\|_\infty = \sup_{t\geq 0} \big|u(t)\big| \tag{2.7}$$

this norm represents the maximum value that the signal can take. In the case of multidimensional signals $u(t) \in \mathbf{L}_\infty^{n_u}$ $(u(t) = (u_1(t), \ldots, u_{n_u}(t))^T$ with $u_i(t) \in \mathbf{L}_\infty)$, the norm is given by

$$\|u\|_\infty = \max_{1\leq i\leq n_u} \left(\sup_{t\geq 0} \big|u_i(t)\big|\right) = \max_{1\leq i\leq n_u} \|u_i\|_\infty \tag{2.8}$$

2.1.4 Extended $\mathbf{L}_p$*-Space*

The $\mathbf{L}_p$-space, $p = 1, 2, \infty$, only includes bounded signals. For instance, the $\mathbf{L}_2$-space only includes signals with bounded energy. In order to also include in our study unbounded signals as well, it is necessary to introduce extended versions of the standard $\mathbf{L}_p$-spaces. For this purpose, consider the *projection function* denoted $P_T(.)$ defined as

$$P_T\big(u(t)\big) = u_T(t) = \begin{cases} u(t), & t \leq T \\ 0, & t > T \end{cases} \tag{2.9}$$

where T is a given time interval over which the signal is considered. The *extended* $\mathbf{L}_{pe}$*-space*, $p = 1, 2, \infty$, is then defined as the space of piecewise continuous signals $u : \mathbf{R}_+ \to \mathbf{R}^m$ such that $u_T \in \mathbf{L}_p$.

2.1.5 RMS-Value

Some signals are of special interest for system analysis and synthesis. This is the case, for instance, of the sinusoidal signal $u(t) = A\sin(\omega t + \varphi)$. However, this signal is not square integrable and is often called an *infinite energy signal*. A very common measurement of the size of an infinite energy signal is the root-mean-square (RMS) value. The *RMS-value* of a given signal $u(t)$ is defined as

$$u_{\mathrm{rms}} = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)^2 \, dt \right)^{1/2} \tag{2.10}$$

The square of this quantity represents the average power of the signal. The RMS-value of a given signal $u(t)$ can be also computed in the frequency domain as follows:

$$u_{\mathrm{rms}} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega) \, d\omega \right)^{1/2} \tag{2.11}$$

where $S_u(\omega)$ is the *power spectral density*[3] (PSD), which represents the way in which the average power of the signal $u(t)$ is distributed over the frequency range. In the case of multidimensional signals $u(t) = (u_1(t), \ldots, u_{n_u}(t))^T$, the RMS-value of the vector signal $u(t)$ is given by

$$u_{\mathrm{rms}} = \left(\lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)^T u(t) \, dt \right)^{1/2} \tag{2.12}$$

The RMS-value of a given signal vector $u(t)$ can also be computed in the frequency domain as follows:

$$u_{\mathrm{rms}} = \left(\mathrm{Trace}\left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega) \, d\omega \right) \right)^{1/2} \tag{2.13}$$

where $S_u(\omega)$ is the power spectral density matrix[4] of the signal vector $u(t)$.

2.2 LTI Systems

Broadly speaking, a system can be seen as a device that associates to a given input signal $u(t)$, an output signal $y(t)$. In this book, for tractability reasons, we consider the particular class of linear time invariant finite-dimensional systems or LTI-system

[3]The PSD of a signal $u(t)$ is defined as $S_u(\omega) = \int_{-\infty}^{+\infty} r_u(\tau) e^{-j\omega\tau} \, d\tau$, where $r_u(\tau)$ is the autocorrelation function of the signal $u(t)$: $r_u(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)u(t+\tau) \, dt$. Note that the square of the RMS-value of the signal $u(t)$ is nothing but $u_{\mathrm{rms}}^2 = r_u(0)$.

[4]The PSD matrix of the vector signal $u(t)$ is defined as $S_u(\omega) = \int_{-\infty}^{+\infty} R_u(\tau) e^{-j\omega\tau} \, d\tau$, where $R_u(\tau)$ is the correlation matrix of the signal vector $u(t)$: $R_u(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_0^T u(t)u(t+\tau)^T \, dt$. Note that the square of the RMS-value of the signal vector $u(t)$ is nothing but $u_{\mathrm{rms}}^2 = \mathrm{Trace}(R_u(0))$. The matrix $R_u(0)$ is often referred to as the *covariance matrix* of the signal vector $u(t)$.

for short. The so-called *state-space representation* of this kind of system is defined as follows:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.14}$$

where $u \in \mathbf{R}^{n_u}$ is the input vector, $y \in \mathbf{R}^{n_y}$ is the output vector, $x \in \mathbf{R}^{n_x}$ is the state vector, and A, B, C, D are constant matrices of appropriate dimension. It can be established that the solution of the state equation in (2.14), for a given initial state vector $x(t_0)$, is as follows:

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau \tag{2.15}$$

Note that this solution is the superposition of two terms, the first term $e^{A(t-t_0)}x(t_0)$ represents the state evolution of the *autonomous system*, i.e., for $u = 0$, whereas the second term $\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$ represents the state evolution of the system for zero initial condition. This last term is written as the *convolution product* of the quantity $e^{At}B$, called the *input-to-state impulse matrix*,[5] by the input $u(t)$. From (2.14) and (2.15) we can see that the response $y(t)$ of the system to a given input vector $u(t)$ is then given by

$$y(t) = Ce^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t) \tag{2.16}$$

An important question is to determine in which conditions the state remains bounded (and therefore the output as well) when the system is driven by a bounded input signal. This question is closely related to the ability of the autonomous system to recover its *equilibrium point*[6] starting from any initial state. This is the problem of stability, which is briefly considered in the next section.

[5]Recall that the *Dirac delta function* (or Dirac impulse), denoted $\delta(t)$, is the neutral element of the convolution product. Therefore, when the input is a Dirac impulse $u(t) = \mathbf{1}_k\delta(t)$, where $\mathbf{1}_k$ is a unit vector (e.g., $\mathbf{1}_3 = (0, 0, 1, 0, \ldots, 0)$), the state response is given by $e^{At}B\mathbf{1}_k$, this is why $e^{At}B$ is called input-to-state impulse matrix.

[6]Consider a nonlinear autonomous system described by $\dot{x}(t) = f(x(t))$. A point x_e is said to be an equilibrium point (or a stationary point) for this system if $f(x_e) = 0$. In other words the equilibrium points are those from which the system does not evolve anymore. In the case of a LTI system the equilibrium points are the solutions of the equation $Ax_e = 0$. If A is of full rank, then $x_e = 0$, we have a single equilibrium point which is the origin of the state space. Otherwise, the solutions lie in the null space of A.

2.2.1 System Stability

A fundamental property of any system is its *stability*. Stability is the ability of an autonomous system[7] to recover its equilibrium point after being disturbed from it. More formally, the system described by (2.14) is stable if for every initial condition $x(t_0)$ the following limit:

$$\lim_{t\to\infty} x(t) = 0 \tag{2.17}$$

holds when $u = 0$. From (2.15) we can see that the state vector solution of the autonomous system is given by $x(t) = e^{A(t-t_0)}x(t_0)$. Therefore, the limit (2.17) holds if and only if the matrix A, also called state matrix, has all its eigenvalues in the open left-half plane $\mathbf{C}^-$. The eigenvalues of the matrix $A \in \mathbf{R}^{n_x \times n_x}$ are the n_x roots of the *polynomial characteristic* defined by

$$\det(\lambda I - A) = \lambda^{n_x} + a_{n_x-1}\lambda^{n_x-1} + \cdots + a_1 s + a_0 \tag{2.18}$$

If the n_x roots of the polynomial characteristic (2.18) are all in the open left-half plane, then the matrix A is said to be *Hurwitz*. The set of n-by-n Hurwitz matrices is defined as

$$\mathcal{H} = \left\{H \in \mathbf{R}^{n\times n} : \lambda_i(H) \in \mathbf{C}^-,\ i = 1, \ldots, n\right\} \tag{2.19}$$

where $\lambda_i(H)$ represents the ith eigenvalue of H. Therefore, the autonomous system $\dot{x}(t) = Ax(t)$ is stable if and only if $A \in \mathbf{R}^{n_x\times n_x}$ is Hurwitz, i.e., $A \in \mathcal{H}$. At this point, it is important to note that the set of Hurwitz matrices is not convex.

- **Non-convexity of the Set of Hurwitz Matrices.** *Given two Hurwitz matrices $A_1,\ A_2 \in \mathcal{H}$, the convex combination $A(\alpha) = \alpha A_1 + (1-\alpha)A_2$, $\alpha \in [0,\ 1]$ does not necessarily belong to $\mathcal{H}$ for any α. To observe this, consider for instance the matrices*

$$A_1 = \begin{bmatrix} a & 2b \\ 0 & b \end{bmatrix}, \qquad A_2 = \begin{bmatrix} a & 0 \\ 2a & b \end{bmatrix}$$

with $a,\ b < 0$, the convex combination of A_1 and A_2 is given by

$$A(\alpha) = \begin{bmatrix} a & 2\alpha b \\ 2(1-\alpha)a & b \end{bmatrix}$$

It can be easily seen that $A_1,\ A_2 \in \mathcal{H}$ whereas $A(\frac{1}{2}) \notin \mathcal{H}$.

[7]Recall that the autonomous system is defined by $e^{A(t-t_0)}x(t_0)$ which represents the state evolution of the system for $u = 0$, see relation (2.15).

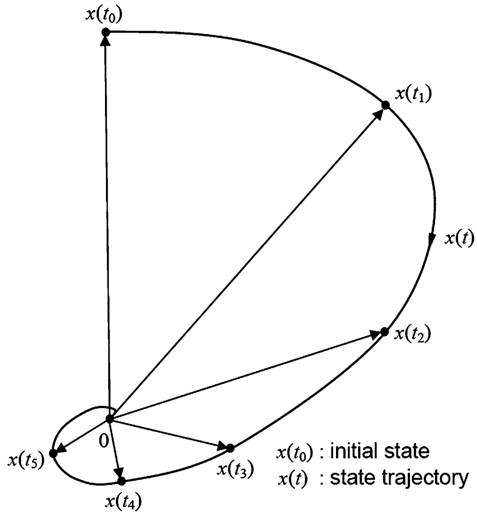

Fig. 2.1 Example of a state trajectory of a stable system. From any initial state $x(t_0)$, the state trajectory converges to the equilibrium point of the autonomous system i.e., the origin of the state space

Lyapunov Method Another way to establish the stability of a given LTI autonomous system

$$\dot{x}(t) = Ax(t) \tag{2.20}$$

is the *Lyapunov method*. Consider a quantity related to the distance of the current state vector $x(t)$ to the origin of the state space[8] e.g., its squared quadratic norm: $V(x(t)) = \|x\|_P^2 = x(t)^T P x(t)$, where P is a *symmetric positive definite matrix*.[9] Under these conditions, it is clear that the limit (2.17) holds if and only if the distance of $x(t)$ to the origin decreases as time increases (see Fig. 2.1). Therefore, we can conclude that the system is stable if and only if there is a matrix $P = P^T \succ 0$ such that $V(x(t)) = x^T P x$ is a strictly decreasing function of time, i.e., $\dot{V}(x(t)) < 0$ for all $x \neq 0$. The time derivative of V is given by

$$\begin{aligned}
\dot{V}\big(x(t)\big) &= \dot{x}(t)^T P x(t) + x(t)^T P \dot{x}(t) \\
&= x(t)^T A^T P x(t) + x(t)^T P A x(t) \\
&= x(t)^T \big(A^T P + P A\big) x(t)
\end{aligned} \tag{2.21}$$

The quadratic form $x(t)^T(A^T P + PA)x(t)$ is negative definite for all $x \neq 0$ if and only if the symmetric matrix $A^T P + PA$ is negative definite, i.e., all its eigenvalues

[8]The origin represents the unique equilibrium point of a LTI autonomous system for which $\det(A) \neq 0$. If $\det(A) = 0$, the system is necessarily unstable in the sense of (2.17).

[9]A matrix P is symmetric if $P = P^T$. The eigenvalues of a symmetric matrix are real. A symmetric matrix P is said to be positive definite if the associated quadratic form is always positive: $x^T P x > 0$ for all $x \in \mathbf{R}^{n_x}$. This last condition is satisfied if and only if all the eigenvalues of P are positive. We denote by $P \succ 0$ a positive definite matrix.

are negative, which is denoted by

$$A^T P + PA \prec 0 \tag{2.22}$$

This expression is called a *Lyapunov inequality* on P, which is also a *linear matrix inequality* (LMI). This LMI can be solved by taking any matrix $Q = Q^T \succ 0$ and by solving the following linear equation, also called the *Lyapunov equation*:

$$A^T P + PA = -Q \tag{2.23}$$

of unknown P. Thus, if the autonomous system (2.20) is stable then the matrix P solution of the Lyapunov equation is definite positive.

The system stability issue can then be summarized as follows.

- **System Stability Condition.** *The LTI system* (2.14) *is said to be stable if and only if the state matrix A has all its eigenvalues in the open left-half plane* $\mathbf{C}^-$*, i.e., the eigenvalues of A have a negative real part. In this case, the state matrix A is said to be a Hurwitz matrix.*

  Equivalently, the LTI system (2.14) *is said to be stable if and only if there exists a positive definite symmetric matrix P satisfying the Lyapunov inequality* $A^T P + PA \prec 0$.

Remark 2.1 The stability result given above is often referred to as the *internal stability*. The notion of internal stability must be distinguished from the so-called *BIBO-stability*. The LTI-system (2.14) is said to be BIBO-stable if a bounded input produces a bounded output. From relation (2.16) it is clear that an internally stable system is also BIBO-stable, the converse is false in general. This is because between the input and output there can be unstable *hidden modes*, i.e., some unbounded state variables which are not influenced by the inputs or have no influence to the outputs. Therefore since these unstable modes are not input/output visible, the system can be input/output stable but not internally stable. In this book, the notion of stability always refers to internal stability.

2.2.2 Controllability, Observability

In Remark 2.1 we have introduced the notion of hidden modes. To illustrate this notion, consider the LTI system (2.14) with

$$A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ 0 & 0 \\ b_{31} & b_{32} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & 0 \\ c_{21} & c_{22} & 0 \end{bmatrix}, \qquad D = 0$$

Relation (2.15) makes it possible to calculate the evolution of the state vector for a given initial state $x(0) = (x_1(0),\ x_2(0),\ x_3(0))$, and for a given input vector $u(t) = (u_1(t),\ u_2(t))$, we have

$$\begin{aligned}
x_1(t) &= e^{\lambda_1 t}x_1(0) + b_{11}\int_0^t e^{\lambda_1(t-\tau)}u_1(\tau)\,d\tau + b_{12}\int_0^t e^{\lambda_1(t-\tau)}u_2(\tau)\,d\tau \\
x_2(t) &= e^{\lambda_2 t}x_2(0) \\
x_3(t) &= e^{\lambda_3 t}x_3(0) + b_{31}\int_0^t e^{\lambda_3(t-\tau)}u_1(\tau)\,d\tau + b_{32}\int_0^t e^{\lambda_3(t-\tau)}u_2(\tau)\,d\tau
\end{aligned}$$

we can see that the input vector $u(t)$ has no influence on the evolution of the state variable x_2. In this case we say that λ_2 is an uncontrollable mode. The evolution of the output vector is given by $y(t) = Cx(t)$, we have

$$y_1(t) = c_{11}x_1(t) + c_{12}x_2(t), \qquad y_2(t) = c_{21}x_1(t) + c_{22}x_2(t)$$

we can see that the state variable x_3 has no influence to the output vector $y(t)$. In this case we say that λ_3 is a unobservable mode. Consider now the input/output relation calculated for zero initial conditions; we have

$$\begin{aligned}
y_1(t) &= c_{11}\left(b_{11}\int_0^t e^{\lambda_1(t-\tau)}u_1(\tau)\,d\tau + b_{12}\int_0^t e^{\lambda_1(t-\tau)}u_2(\tau)\,d\tau\right) \\
y_2(t) &= c_{21}\left(b_{11}\int_0^t e^{\lambda_1(t-\tau)}u_1(\tau)\,d\tau + b_{12}\int_0^t e^{\lambda_1(t-\tau)}u_2(\tau)\,d\tau\right)
\end{aligned}$$

we can see that the input/output relation, evaluated for zero initial conditions, only involves modes that are both controllable and observable, in this example λ_1. Note also that in the case where $\lambda_1 \in \mathbf{C}^-$ and $\lambda_2, \lambda_3 \in \mathbf{C}^+$, the system is BIBO-stable but internally unstable.

The example given above suggests the following definitions about the notions of controllability and observability of an LTI system.

Definition 2.2 (Controllability) An LTI system is controllable if every mode of A is connected to the input vector u.

Definition 2.3 (Observability) An LTI system is observable if every mode of A is connected to the output vector y.

The following results can be used to test the controllability and the observability of a given LTI system. The notions of stabilizability and detectability are also specified.

- **Controllability, Stabilizability.** *The LTI system* (2.14) *is controllable if and only if the controllability matrix*

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & \cdots & A^{n_x-1}B \end{bmatrix} \tag{2.24}$$

is of full rank, i.e., $\mathrm{rank}(\mathcal{C}) = n_x$. *In this case, the pair* (A, B) *is said to be controllable.*

In the case where $\mathrm{rank}(\mathcal{C}) = n < n_x$*, the rank defect* $n_x - n$ *represents the number of uncontrollable modes. The uncontrollable modes are the eigenvalues* λ *of the state matrix* A *satisfying* $\mathrm{rank}([\lambda I - A\ B]) < n_x$. *The LTI system* (2.14) *is said to be stabilizable if and only if all uncontrollable modes are stable.*

- **Observability, Detectability.** *The LTI system* (2.14) *is observable if and only if the observability matrix*

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n_x-1} \end{bmatrix} \tag{2.25}$$

is of full rank, i.e., $\mathrm{rank}(\mathcal{O}) = n_x$. *In this case, the pair* (A, C) *is said to be observable.*

In the case where $\mathrm{rank}(\mathcal{O}) = n < n_x$*, the rank defect* $n_x - n$ *represents the number of unobservable modes. The unobservable modes are the eigenvalues* λ *of the state matrix* A *satisfying* $\mathrm{rank}\left[\begin{smallmatrix} \lambda I - A \\ C \end{smallmatrix}\right] < n_x$. *The LTI system* (2.14) *is said to be detectable if and only if all unobservable modes are stable.*

Physical Meaning of the Controllability and Observability Controllability is related to the ability of a system to attain a given state under the action of an appropriate control signal. If a state is not controllable, then it not possible to move this state to another one by acting on the control input. If the dynamics of a non-controllable state is stable, then the state is said to be stabilizable.

Observability is linked to the possibility of evaluating the state of a system through output measurements. If a state is not observable there is no way to determine its evolution. If the dynamics of a non-observable state is stable, then the state is said to be detectable.

2.2.3 Transfer Matrix

The state space representation is often referred as an internal representation because it involves the state variables which are internal variables of the system. The input/output representation, also called external representation, is obtained by eliminating the *Laplace transform*[10] of the state vector, between the state equation and the output equation for zero initial conditions. Taking the Laplace transform of the

[10] The Laplace transform of a given signal $u(t)$ is defined as $U(s) = \mathcal{L}(u(t)) = \int_0^\infty x(t)e^{-st}\,dt$. From this definition, it is easy to show that the *Laplace transform* of the derivative of a signal is given by $\mathcal{L}(\dot{u}(t)) = s\mathcal{L}(u(t)) - u(0)$.

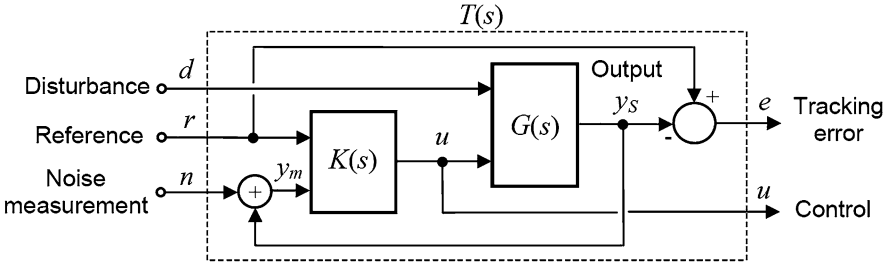

Fig. 2.2 Block diagram of a closed-loop system, $G(s)$ is the transfer matrix of the system to be controlled, and $K(s)$ is the controller which must be designed to obtain a low tracking error and a control signal compatible with the possibility of the plant despite the external influences r, d, and n

state equation in (2.14), we get $X(s) = (sI - A)^{-1}BU(s)$. By substituting $X(s)$ in the output equation, we obtain the input/output relation

$$Y(s) = G(s)U(s), \quad G(s) = C(sI - A)^{-1}B + D \tag{2.26}$$

where $G(s)$ is called the transfer matrix of the system. This transfer matrix represents the Laplace transform of the input to output impulse matrix. The elements of the matrix $G(s)$ are real rational transfer functions (i.e., ratios of polynomials in s with real coefficients). A transfer matrix $G(s)$ is *proper* if $G(\infty) = D$, and *strictly proper* if $G(\infty) = 0$. We have seen that the input/output representation only involves the eigenvalues that are both controllable and observable. These are called the poles of the system. A proper transfer matrix $G(s)$ is stable if the poles lie in the open left-half plane $\mathbf{C}^-$. The set of proper and stable transfer matrices of size $n_y \times n_u$ is denoted $\mathbf{RH}_\infty^{n_y \times n_u}$. The set of strictly proper and stable transfer matrices of size $n_y \times n_u$ is denoted $\mathbf{RH}_2^{n_y \times n_u}$. It can be easily shown that these sets are convex. This is in contrast to the non-convexity of the set of Hurwitz matrices (see Sect. 2.2.1).

2.3 System Norms

Given an LTI-system an important issue is to characterize, in some sense, the amplification (or attenuation) introduced by the system for a given input signal. To emphasize the importance of this issue, consider the control problem shown in Fig. 2.2 where $K(s)$ is the controller to be designed and $G(s)$ is the transfer matrix of the system to be controlled.

The objective is to determine the controller $K(s)$ to obtain a low tracking error and a control signal compatible with the possibility of the plant (i.e. the control signal must be admissible by the system) despite the external influences r, d, and n. One way to evaluate the performance of the closed-loop system is to measure the gain provided by the system T, between the inputs (r, d and n) and the outputs e

and u:

$$\begin{bmatrix} e \\ u \end{bmatrix} = T(s) \begin{bmatrix} r \\ d \\ n \end{bmatrix}$$

Good performance is then obtained if the transfer matrix $T(s)$ is small or, more specifically, if the gain of $T(s)$ is small. The word "gain" must be understood here as a measurement of the size of the matrix $T(s)$.

The gain of a system quantifies the amplification provided by the system between the inputs and the outputs. This notion of gain needs to be defined more accurately, this is the subject of the next section on $\mathbf{H}_2$ and $\mathbf{H}_\infty$ norms of a system.

2.3.1 Definition of the H_2-Norm and H_∞-Norm of a System

Let $G(s)$ be the transfer function of a stable single input single output (SISO) LTI-system of input $u(t)$ and output $y(t)$. We know that $G(s)$ is the Laplace transform of the impulse response $g(t)$ of the system, we define the $\mathbf{H}_2$-norm of $G(s)$ as the $\mathbf{L}_2$-norm of its impulse response:

$$\|G\|_2 = \left(\int_0^\infty g(t)^2 \, dt \right)^{1/2} = \|g\|_2 \tag{2.27}$$

Note that the previous norm is defined for a particular signal which is here the Dirac impulse $\delta(t)$. According to Parseval's theorem the $\mathbf{H}_2$ norm is defined in the frequency domain as follows:

$$\|G\|_2 = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|G(j\omega)\right|^2 d\omega \right)^{1/2} \tag{2.28}$$

Remark 2.2 It is interesting to give an interpretation of the $\mathbf{H}_2$-norm of a system. To this end, recall that if $S_u(\omega)$ is the power spectral density (DSP) of the signal applied to the input of a stable system of transfer function $G(s)$, the DSP of the signal output $S_y(\omega)$ is given by $S_y(\omega) = |G(j\omega)|^2 S_u(\omega)$. Now, assume that the input $u(t)$ is a *white noise* signal, i.e. $S_u(\omega) = 1$ for all ω, in this case, the DSP of the signal output is nothing but the square of the frequency gain of the system: $S_y(\omega) = |G(j\omega)|^2$. Using (2.11) the RMS-value of the signal output is given by

$$y_{\mathrm{rms}} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_y(\omega) \, d\omega \right)^{1/2} = \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \left|G(j\omega)\right|^2 d\omega \right)^{1/2} \tag{2.29}$$

which coincides with the definition of the $\mathbf{H}_2$-norm of the system (see relation (2.28)). In other words, the $\mathbf{H}_2$-norm of a system represents the RMS-value of the system response to a white noise input.

We can define the gain provided by the system for a given particular input as the ratio of the $\mathbf{L}_2$-norm of the output signal to the $\mathbf{L}_2$-norm of the input signal $\|G\|_{\text{gain}} = \|Gu\|_2/\|u\|_2$, with $\|u\|_2 \neq 0$. For obvious reason, this gain is often referred to as the $\mathbf{L}_2$-*gain* of the system. Instead of evaluating the $\mathbf{L}_2$-gain for a particular input, one can also determine the greatest possible $\mathbf{L}_2$-gain over the set of square integrable signals, this is the definition of the $\mathbf{H}_\infty$-norm of a system

$$\|G\|_\infty = \sup_{\substack{u \in \mathbf{L}_2 \\ \|u\|_2 \neq 0}} \frac{\|Gu\|_2}{\|u\|_2} \tag{2.30}$$

This quantity represents the largest possible $\mathbf{L}_2$-gain provided by the system. For a MIMO system with n_u inputs and n_y outputs, the $\mathbf{H}_\infty$-norm is defined as

$$\|G\|_\infty = \sup_{\substack{u \in \mathbf{L}_2^{n_u} \\ \|u\|_2 \neq 0}} \frac{\|Gu\|_2}{\|u\|_2} \quad \text{with } y \in \mathbf{L}_2^{n_y} \tag{2.31}$$

2.3.2 Singular Values of a Transfer Matrix

The usual notion of the frequency gain of a SISO system can be extended to the MIMO case by considering the *singular values* of the transfer matrix $G(s)$ of the system. Let $y(t)$ be the system response to a causal input $u(t)$. In the frequency domain, this response is written

$$Y(j\omega) = G(j\omega)U(j\omega) \tag{2.32}$$

where $Y(j\omega) = (Y(s))_{s=j\omega}$, $U(j\omega) = (U(s))_{s=j\omega}$, $Y(s) = \mathcal{L}(y(t))$, $U(s) = \mathcal{L}(u(t))$, and the notation $\mathcal{L}(.)$ stands for the Laplace transform of the signal passed in argument. In the SISO case, the gain of the system at frequency ω is given by $|G(j\omega)|$. This notion of frequency gain can be extended to the MIMO case by using the singular values, denoted σ_i, of the matrix $G(j\omega) = (G(s))_{s=j\omega}$. The singular values of the matrix $G(j\omega)$ are defined as the square roots of eigenvalues of $G(j\omega)G(-j\omega)^T$

$$\sigma_i\big(G(j\omega)\big) = \sqrt{\lambda_i\big(G(j\omega)G(-j\omega)^T\big)} = \sqrt{\lambda_i\big(G(-j\omega)^T G(j\omega)\big)} \tag{2.33}$$

with $i = 1, \ldots, \min(n_u, n_y)$. The matrix $G(-j\omega)^T$ represents the conjugate transpose of $G(j\omega)$, and is usually denoted $G(j\omega)^*$ i.e., $G(j\omega)^* = G(-j\omega)^T$. The matrices $G(j\omega)G(j\omega)^*$ and $G(j\omega)^*G(j\omega)$ are *Hermitian*[11] positive semi-definite, their eigenvalues are therefore non-negative.

[11] A complex matrix is said to be Hermitian if it is equal to its conjugate transpose.

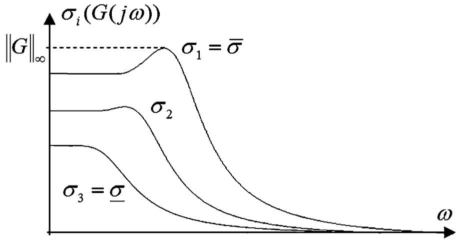

Fig. 2.3 Singular values and $\mathbf{H}_\infty$-norm of a transfer matrix. The frequency gain of the MIMO system lies between the smallest and the largest singular values. The maximum over ω of the largest singular value represents the $\mathbf{H}_\infty$-norm of the LTI system

We denote $\bar{\sigma}(G(j\omega))$ the largest singular value of G and $\underline{\sigma}(G(j\omega))$ the smallest

$$\bar{\sigma}\big(G(j\omega)\big) = \sigma_1\big(G(j\omega)\big) \geq \sigma_2\big(G(j\omega)\big) \geq \cdots \geq \underline{\sigma}\big(G(j\omega)\big) \geq 0 \quad \forall\omega \tag{2.34}$$

we then have[12]

$$\underline{\sigma}\big(G(j\omega)\big) \leq \big\|G(j\omega)U(j\omega)\big\|_2 / \big\|U(j\omega)\big\|_2 \leq \bar{\sigma}\big(G(j\omega)\big) \tag{2.35}$$

This means that the frequency gain of the system lies between the smallest and the largest singular values. Therefore, the singular values can be used to extend to the MIMO case the usual notion of gain. The singular values are positive functions of ω and can be represented in the frequency domain as shown Fig. 2.3.

In the case of a SISO system, $G(s)$ is scalar, it is then easy to see that we have only one singular value which is equal to the modulus of $G(j\omega)$

$$\sigma\big(G(j\omega)\big) = \big|G(j\omega)\big| \tag{2.36}$$

It is worth noting that any complex matrix $M \in \mathbf{C}^{n_y \times n_u}$ has a *singular value decomposition*, see the Notes and References.

---

[12]Indeed, it can be shown that for a complex matrix $A \in \mathbf{C}^{p\times m}$ and a complex vector $x \in \mathbf{C}^m$, we have

$$\bar{\sigma}(A) = \max_{\substack{x\in\mathbf{C}^m \\ \|x\|_2\neq 0}} \frac{\|Ax\|_2}{\|x\|_2} \quad \text{and} \quad \underline{\sigma}(A) = \min_{\substack{x\in\mathbf{C}^m \\ \|x\|_2\neq 0}} \frac{\|Ax\|_2}{\|x\|_2}$$

To observe this, consider the first-order optimality condition of $\lambda = \|Ax\|_2^2/\|x\|_2^2 = (x^*A^*Ax)/(x^*x)$. We have

$$\frac{\partial\lambda}{\partial x} = \big(A^*A - \lambda I\big)x = 0$$

thus, λ represents the eigenvalues of the matrix A^*A. Therefore, since $\lambda = \|Ax\|_2^2/\|x\|_2^2$, the maximum of $\|Ax\|_2/\|x\|_2$ is given by the square root of the largest eigenvalue of A^*A i.e., $\bar{\sigma}(A) = \sqrt{\bar{\lambda}(A^*A)}$, and the minimum of $\|Ax\|_2/\|x\|_2$ is given by the square root of the smallest eigenvalue of A^*A i.e., $\underline{\sigma}(A) = \sqrt{\underline{\lambda}(A^*A)}$. Note that the input vector for which the gain is maximal (respectively, minimal) is given by the eigenvector associated to the largest (respectively, smallest) eigenvalue of A^*A.

2.3.3 *Singular Values and* $\mathbf{H}_2$, $\mathbf{H}_\infty$*-Norms*

Let $G(s)$ be a stable and strictly proper transfer matrix[13] of dimension $p \times m$. The set of stable and strictly proper transfer matrices is denoted $\mathbf{RH}_2^{n_y \times n_u}$. For any transfer matrix $G(s) \in \mathbf{RH}_2^{n_y \times n_u}$, we define the $\mathbf{H}_2$-norm as[14]

$$\|G(s)\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Trace}\big(G(j\omega)G^*(j\omega)\big)\, d\omega\right)^{1/2} \tag{2.37}$$

this norm can be also expressed using the singular values:[15]

$$\|G(s)\|_2 = \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \sum_{i=1}^{\min(m,p)} \sigma_i^2\big(G(j\omega)\big)\, d\omega\right)^{1/2} \tag{2.38}$$

The square of the $\mathbf{H}_2$-norm represents the area under the curve of the sum of squared singular values.

Now, consider a stable and proper transfer matrix $G(s)$. The set of stable and proper transfer matrices is noted $\mathbf{RH}_\infty^{n_y \times n_u}$. For any transfer matrix $G(s) \in \mathbf{RH}_\infty^{n_y \times n_u}$ the $\mathbf{H}_\infty$-norm is defined as

$$\|G(s)\|_\infty = \sup_\omega \bar{\sigma}\big(G(j\omega)\big) \tag{2.39}$$

This norm represents the largest possible frequency gain, which corresponds to the maximum of the largest singular value of $G(j\omega)$ (see relation (2.35) and Fig. 2.3). In the case of a SISO system, $\|G(s)\|_\infty$ is the maximum of $|G(j\omega)|$

$$\|G\|_\infty = \max_\omega \big|G(j\omega)\big| \tag{2.40}$$

2.3.4 *Computing Norms from the State Space Equation*

The calculation of the $\mathbf{H}_2$ and $\mathbf{H}_\infty$-norms by a direct application of the definitions (2.38) and (2.39) can only be done in the simplest cases. We will see that they can be evaluated more easily from the state space representation of the system.

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[13] A transfer matrix is called proper (respectively, strictly proper) if for each of the component transfer function matrix, the degree of the numerator is less than or equal (respectively, strictly less than) the degree of the denominator.

[14] From (2.13), the RMS-value of the system response is given by $y_{\text{rms}} = (\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Trace}(S_y(\omega))\, d\omega)^{1/2}$. Using (2.29) we deduce that $y_{\text{rms}} = (\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{Trace}(G(j\omega) \times G(j\omega)^*)\, d\omega)^{1/2}$, which is by definition the $\mathbf{H}_2$-norm of the system.

[15] It can be shown that for a complex matrix $M \in \mathbf{C}^{n_y \times n_u}$, we have $\mathrm{Trace}(MM^*) = \sum_{i=1}^{\min(n_y,n_u)} \sigma_i^2(M)$.

Computing the $\mathbf{H_2}$-Norm Let $G(s)$ be a stable and strictly proper transfer matrix ($G(s) \in \mathbf{RH}_2^{n_y \times n_u}$). The state space representation of such a system is given by (2.14) with $D = 0$ (strictly proper system). In view of (2.26), the *input-to-output impulse matrix* of the system for zero initial conditions is given by

$$G(t) = \left[g_{ij}(t)\right] = \mathcal{L}^{-1}\left(C(sI - A)^{-1}B\right) = Ce^{At}B \tag{2.41}$$

where $g_{ij}(t)$ represents the impulse response from the input $u_j(t)$ to the output $y_i(t)$.

According to Parseval's theorem, the square of the $\mathbf{H}_2$-norm, defined by (2.37), can be also written as

$$\|G(s)\|_2^2 = \text{Trace}\left(\int_0^{+\infty} G(t)^T G(t)\,dt\right) \tag{2.42}$$

therefore, substituting $G(t)$ by its expression

$$\begin{aligned}\|G(s)\|_2^2 &= \text{Trace}\left(\int_0^{+\infty} \left(Ce^{At}B\right)^T \left(Ce^{At}B\right)dt\right)\\ &= \text{Trace}\left(B^T \int_0^{+\infty} e^{A^T t} C^T C e^{At}\,dt\, B\right)\end{aligned}$$

giving

$$\|G(s)\|_2^2 = \text{Trace}\left(B^T G_o B\right), \quad \text{with } G_o = \int_0^{+\infty} e^{A^T t} C^T C e^{At}\,dt \tag{2.43}$$

where G_o represents the *observability Gramian*.[16] The matrix G_o is a solution of the Lyapunov equation

$$A^T G_o + G_o A + C^T C = 0 \tag{2.44}$$

Similarly, we have

$$\|G(s)\|_2^2 = \text{Trace}\left(C G_c C^T\right), \quad \text{with } G_c = \int_0^{\infty} e^{At} B B^T e^{A^T t}\,dt \tag{2.45}$$

where G_c represents the *controllability Gramian*.[17] The matrix G_c is a solution of the Lyapunov equation

$$A G_c + G_c A^T + B B^T = 0 \tag{2.46}$$

[16] The observability Gramian is related to the total output energy of the autonomous system when it evolves from a given initial state x_0, we have $x_0^T G_o x_0 = \int_0^{\infty} y(t)^T y(t)\,dt$.

[17] The controllability Gramian makes it possible to determine the set of the state-space points that can be reached with an input of unit-energy; Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$, $x(0) = 0$. A point x_d can be reached at time T with a unit-energy signal (i.e. $\int_0^T u^T(t)u(t)\,dt \le 1$) if and only if $x_d^T G_c^{-1} x_d \le 1$.

Thus, the $\mathbf{H}_2$-norm of a system is obtained via the solution of a Lyapunov equation.

Computing the $\mathbf{H}_\infty$-Norm Let $G(s)$ be a stable and proper transfer matrix ($G(s) \in \mathbf{RH}_\infty^{n_y \times n_u}$). The state space representation of such a system is given by (2.14). If we can find $V(x(t)) = x^T(t)Px(t)$ with $P = P^T \succ 0$ and a positive real number γ, such that

$$\dot{V}\big(x(t)\big) + y^T(t)y(t) - \gamma^2 u^T(t)u(t) < 0 \tag{2.47}$$

then the $\mathbf{H}_\infty$-norm of the transfer matrix $G(s)$ is bounded by γ, i.e., $\|G(s)\|_\infty < \gamma$. Indeed, integrating (2.47) from 0 to T with $x(0) = 0$, we have

$$V\big(x(T)\big) + \int_0^T y^T(t)y(t)\,dt - \gamma^2 \int_0^T u^T(t)u(t)\,dt < 0$$

since $V(x(t))$ is positive, this implies that

$$\int_0^T y^T(t)y(t)\,dt \Big/ \int_0^T u^T(t)u(t)\,dt < \gamma^2$$

this relation holds for all T and $u \in \mathcal{L}_2^{n_u}$, consequently

$$\sup_{\substack{u \in \mathcal{L}_2^m \\ \|u\|_2 \neq 0}} \frac{\|y\|_2^2}{\|u\|_2^2} < \gamma^2 \tag{2.48}$$

The left hand side of (2.48) represents the square of the greatest $\mathbf{L}_2$-gain of the system which is the $\mathbf{H}_\infty$-norm of the system, hence: $\|G(s)\|_\infty < \gamma$. Since $\dot{V}(x(t)) = \dot{x}^T P x + x^T P \dot{x}$, with $\dot{x} = Ax + Bu$, the relation (2.47) can also be written as

$$\begin{aligned} &x^T\big(A^T P + PA + C^T C\big)x + x^T\big(PB + C^T D\big)u + \cdots \\ &\quad + u^T\big(B^T P + D^T C\big)x + u^T\big(D^T D - \gamma^2 I\big)u < 0 \end{aligned}$$

or equivalently

$$\begin{bmatrix} x^T & u^T \end{bmatrix} \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} < 0$$

Therefore, the $\mathbf{H}_\infty$-norm of $G(s)$ is such that $\|G(s)\| < \gamma$, where γ is a positive number, if one can find $P = P^T \succ 0$ satisfying the linear matrix inequality (LMI)

$$\begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \gamma^2 I \end{bmatrix} \prec 0 \tag{2.49}$$

We can then assess the $\mathbf{H}_\infty$-norm of the transfer matrix $G(s)$ by finding the smallest possible value of $\gamma > 0$ satisfying the LMI (2.49). In other words, the $\mathbf{H}_\infty$-norm of the transfer matrix $G(s)$ is a solution of the following convex optimization problem:

$$\begin{aligned} &\text{minimize } \nu \\ &\text{subject to } P = P^T \succ 0 \\ &\qquad\qquad \begin{bmatrix} A^T P + PA + C^T C & PB + C^T D \\ B^T P + D^T C & D^T D - \nu I \end{bmatrix} \prec 0 \end{aligned} \tag{2.50}$$

where $\nu = \gamma^2$.

2.4 Notes and References

A comprehensive presentation of the signal theory both in the deterministic and stochastic case can be found in the book by Kwakernaak and Sivan [82]. The theory of linear systems is covered in the books by Chen [34], Rugh [115]. A very nice presentation of the norms of signals and systems can be found in the book by Boyd and Barratt [20].

2.4.1 LMI Formulation for the Computation of the $\mathbf{H}_2$*-Norm*

Let $G(s)$ be a strictly proper transfer matrix ($G(s) \in \mathbf{RH}_2^{n_y \times n_u}$). The state space representation of the underlying system is given by (2.14) with $D = 0$. We have seen in Sect. 2.3.4 that the $\mathbf{H}_2$-norm is given by $\|G(s)\| = \text{Trace}(CG_cC^T) = \text{Trace}(B^T G_o B)$, where G_c (respectively, G_o) is a solution of the Lyapunov equation $AG_c + G_cA^T + BB^T = 0$ (respectively, $A^T G_o + G_o A + C^T C = 0$). Let P be a symmetric and positive definite matrix ($P = P^T \succ 0$) satisfying the following LMI:

$$AP + PA^T + BB^T \prec 0 \tag{2.51}$$

then we have $P \succ G_c$. Indeed, since G_c is a solution of $AG_c + G_cA^T + BB^T = 0$, we have $AP + PA^T + BB^T \prec AG_c + G_cA^T + BB^T$, hence $A(P - G_c) + (P - G_c)A^T \prec 0$. The matrix A being Hurwitz (stable system), we have $P \succ G_c$. Under these conditions, P is such that $\text{Trace}(CPC^T) > \text{Trace}(CG_cC^T) = \|G(s)\|_2^2$. Therefore, if P satisfy $\text{Trace}(CPC^T) < \gamma^2$ where γ is a given positive number, then $\|G(s)\|_2^2 < \gamma^2$.

To conclude, let γ be a given positive number, and P a symmetric and positive definite matrix solution of the LMI $AP + PA^T + BB^T \prec 0$. If we have $\text{Trace}(CPC^T) < \gamma^2$, then the $\mathbf{H}_2$-norm of the LTI system satisfy $\|G(s)\|_2 < \gamma$. Equivalently, if we have $\text{Trace}(B^T PB) < \gamma^2$, where P is a solution of $AP + PA^T + C^T C \prec 0$, then $\|G(s)\|_2 < \gamma$.

The smallest possible value of the upper bound of the $\mathbf{H}_2$-norm of the transfer matrix $G(s)$ can then be calculated by finding the matrix $P = P^T \succ 0$

that minimizes $\mathrm{Trace}(CPC^T)$ (respectively, $\mathrm{Trace}(B^T PB)$) under the constraints $P = P^T \succ 0$ and $AP + PA^T + BB^T \prec 0$ (respectively, $AP + PA^T + C^T C \prec 0$), i.e.,

$$\begin{array}{ll} \text{min. } \mathrm{Trace}(CPC^T) & \\ \text{s.t.} & P = P^T \succ 0 \\ & AP + PA^T + BB^T \prec 0 \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \text{min. } \mathrm{Trace}(B^T PB) & \\ \text{s.t.} & P = P^T \succ 0 \\ & AP + PA^T + C^T C \prec 0 \end{array} \tag{2.52}$$

2.4.2 *Computing the* $\mathbf{H}_2$ *and* $\mathbf{H}_\infty$*-Norms of a Given LTI System*

The following example uses the formulation (2.52) and (2.50) for the calculation of the $\mathbf{H}_2$ and $\mathbf{H}_\infty$-norms of a given LTI system.

Example 2.1 Consider the LTI system defined by

$$A = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 4 & -3 \\ 1 & -3 & -1 & -3 \\ 0 & 4 & 2 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad D = 0$$

The following MatLab program, written with the commands of the cvx solver [62], make it possible to determine the $\mathbf{H}_2$ by solving the optimization problem (2.52).

MatLab-cvx Code 1—Calculation of the $\mathbf{H}_2$-Norm

```
function G2=H2norm(A,B,C)
n=length(A);
cvx_begin sdp quiet
    variable P(n,n) symmetric;
    minimize trace(C*P*C');
    subject to
        P>=0;
        A*P+P*A'+B*B'<0;
cvx_end
G2=sqrt(trace(C*P*C'));
```

With this program we get $\|G(s)\|_2 = 1.1751$.

Example 2.2 Consider the system of Example 2.1 with $D = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. The following MatLab program, written with the commands of the cvx solver [62], gives the smallest upper bound of the $\mathbf{H}_\infty$-norm, which is a solution of the optimization problem (2.50).

MatLab-cvx code 2—Calculation of the $\mathbf{H}_\infty$-Norm

```
function Ginf=Hinfnorm(A,B,C,D)
dim=size(B);
n=dim(1);
m=dim(2);
cvx_begin sdp quiet
    variable P(n,n) symmetric;
    variable g;
    minimize g;
    subject to
        P>=0;
        [A'*P+P*A+C'*C    P*B+C'*D;
          (P*B+C'*D)'     D'*D-g*eye(m)]<=0;
cvx_end
Ginf=sqrt(g);
```

With this program we get $\|G(s)\|_\infty < 1.379$.

2.4.3 Singular Value Decomposition

Any complex matrix $M \in \mathbf{C}^{n_y \times n_u}$ admits a singular value decomposition defined as follows:

$$M = V\Sigma W^* \quad \text{with} \begin{cases} \Sigma = \operatorname{diag}\{\sigma_1, \dots, \sigma_{n_u}\} = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_{n_u} \end{bmatrix} & \text{if } n_y = n_u \\ \Sigma = \left[\, \operatorname{diag}\{\sigma_1, \dots, \sigma_{n_y}\} \; 0_{(n_u - n_y)\times n_y} \,\right] & \text{if } n_y < n_u \\ \Sigma = \begin{bmatrix} \operatorname{diag}\{\sigma_1, \dots, \sigma_{n_u}\} \\ 0_{(n_y - n_u)\times n_u} \end{bmatrix} & \text{if } n_y > n_u \end{cases} \tag{2.53}$$

where V and W are unitary matrices, i.e., $VV^* = V^*V = I_{n_y}$ and $WW^* = W^*W = I_{n_u}$.

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