

Contents

1	Discrete Spaces of Elementary Events	1
1.1	Probability Space	1
1.2	The Classical Scheme	4
1.3	The Bernoulli Scheme	6
1.4	The Probability of the Union of Events. Examples	9
2	An Arbitrary Space of Elementary Events	13
2.1	The Axioms of Probability Theory. A Probability Space	13
2.2	Properties of Probability	20
2.3	Conditional Probability. Independence of Events and Trials	21
2.4	The Total Probability Formula. The Bayes Formula	25
3	Random Variables and Distribution Functions	31
3.1	Definitions and Examples	31
3.2	Properties of Distribution Functions. Examples	33
3.2.1	The Basic Properties of Distribution Functions	33
3.2.2	The Most Common Distributions	37
3.2.3	The Three Distribution Types	39
3.2.4	Distributions of Functions of Random Variables	42
3.3	Multivariate Random Variables	44
3.4	Independence of Random Variables and Classes of Events	48
3.4.1	Independence of Random Vectors	48
3.4.2	Independence of Classes of Events	50
3.4.3	Relations Between the Introduced Notions	52
3.5	On Infinite Sequences of Random Variables	56
3.6	Integrals	56
3.6.1	Integral with Respect to Measure	56
3.6.2	The Stieltjes Integral	57
3.6.3	Integrals of Multivariate Random Variables. The Distribution of the Sum of Independent Random Variables	59

4	Numerical Characteristics of Random Variables	65
4.1	Expectation	65
4.2	Conditional Distribution Functions and Conditional Expectations	70
4.3	Expectations of Functions of Independent Random Variables	74
4.4	Expectations of Sums of a Random Number of Random Variables	75
4.5	Variance	83
4.6	The Correlation Coefficient and Other Numerical Characteristics	85
4.7	Inequalities	87
4.7.1	Moment Inequalities	87
4.7.2	Inequalities for Probabilities	89
4.8	Extension of the Notion of Conditional Expectation	91
4.8.1	Definition of Conditional Expectation	91
4.8.2	Properties of Conditional Expectations	95
4.9	Conditional Distributions	99
5	Sequences of Independent Trials with Two Outcomes	107
5.1	Laws of Large Numbers	107
5.2	The Local Limit Theorem and Its Refinements	109
5.2.1	The Local Limit Theorem	109
5.2.2	Refinements of the Local Theorem	111
5.2.3	The Local Limit Theorem for the Polynomial Distributions	114
5.3	The de Moivre–Laplace Theorem and Its Refinements	114
5.4	The Poisson Theorem and Its Refinements	117
5.4.1	Quantifying the Closeness of Poisson Distributions to Those of the Sums S_n	117
5.4.2	The Triangular Array Scheme. The Poisson Theorem	120
5.5	Inequalities for Large Deviation Probabilities in the Bernoulli Scheme	125
6	On Convergence of Random Variables and Distributions	129
6.1	Convergence of Random Variables	129
6.1.1	Types of Convergence	129
6.1.2	The Continuity Theorem	134
6.1.3	Uniform Integrability and Its Consequences	134
6.2	Convergence of Distributions	140
6.3	Conditions for Weak Convergence	147
7	Characteristic Functions	153
7.1	Definition and Properties of Characteristic Functions	153
7.1.1	Properties of Characteristic Functions	154
7.1.2	The Properties of Ch.F.s Related to the Structure of the Distribution of ξ	159
7.2	Inversion Formulas	161

7.2.1	The Inversion Formula for Densities	161
7.2.2	The Inversion Formula for Distributions	163
7.2.3	The Inversion Formula in L_2 . The Class of Functions that Are Both Densities and Ch.F.s	164
7.3	The Continuity (Convergence) Theorem	167
7.4	The Application of Characteristic Functions in the Proof of the Poisson Theorem	169
7.5	Characteristic Functions of Multivariate Distributions. The Multivariate Normal Distribution	171
7.6	Other Applications of Characteristic Functions. The Properties of the Gamma Distribution	175
7.6.1	Stability of the Distributions Φ_{α, σ^2} and $K_{\alpha, \sigma}$	175
7.6.2	The Γ -distribution and its properties	176
7.7	Generating Functions. Application to Branching Processes. A Problem on Extinction	180
7.7.1	Generating Functions	180
7.7.2	The Simplest Branching Processes	180
8	Sequences of Independent Random Variables. Limit Theorems . . .	185
8.1	The Law of Large Numbers	185
8.2	The Central Limit Theorem for Identically Distributed Random Variables	187
8.3	The Law of Large Numbers for Arbitrary Independent Random Variables	188
8.4	The Central Limit Theorem for Sums of Arbitrary Independent Random Variables	199
8.5	Another Approach to Proving Limit Theorems. Estimating Approximation Rates	209
8.6	The Law of Large Numbers and the Central Limit Theorem in the Multivariate Case	214
8.7	Integro-Local and Local Limit Theorems for Sums of Identically Distributed Random Variables with Finite Variance	216
8.7.1	Integro-Local Theorems	216
8.7.2	Local Theorems	219
8.7.3	The Proof of Theorem 8.7.1 in the General Case	222
8.7.4	Uniform Versions of Theorems 8.7.1–8.7.3 for Random Variables Depending on a Parameter	225
8.8	Convergence to Other Limiting Laws	227
8.8.1	The Integral Theorem	230
8.8.2	The Integro-Local and Local Theorems	235
8.8.3	An Example	236
9	Large Deviation Probabilities for Sums of Independent Random Variables	239
9.1	Laplace's and Cramér's Transforms. The Rate Function	240

9.1.1	The Cramér Condition. Laplace's and Cramér's Transforms	240
9.1.2	The Large Deviation Rate Function	243
9.2	A Relationship Between Large Deviation Probabilities for Sums of Random Variables and Those for Sums of Their Cramér Transforms. The Probabilistic Meaning of the Rate Function	250
9.2.1	A Relationship Between Large Deviation Probabilities for Sums of Random Variables and Those for Sums of Their Cramér Transforms	250
9.2.2	The Probabilistic Meaning of the Rate Function	251
9.2.3	The Large Deviations Principle	254
9.3	Integro-Local, Integral and Local Theorems on Large Deviation Probabilities in the Cramér Range	256
9.3.1	Integro-Local and Integral Theorems	256
9.3.2	Local Theorems	261
9.4	Integro-Local Theorems at the Boundary of the Cramér Range . . .	264
9.4.1	Introduction	264
9.4.2	The Probabilities of Large Deviations of S_n in an $o(n)$ -Vicinity of the Point α_+n ; the Case $\psi''(\lambda_+) < \infty$. . .	264
9.4.3	The Class of Distributions \mathcal{ER} . The Probability of Large Deviations of S_n in an $o(n)$ -Vicinity of the Point α_+n for Distributions \mathbf{F} from the Class \mathcal{ER} in Case $\psi''(\lambda_+) = \infty$. .	266
9.4.4	On the Large Deviation Probabilities in the Range $\alpha > \alpha_+$ for Distributions from the Class \mathcal{ER}	269
9.5	Integral and Integro-Local Theorems on Large Deviation Probabilities for Sums S_n when the Cramér Condition Is not Met .	269
9.5.1	Integral Theorems	270
9.5.2	Integro-Local Theorems	271
9.6	Integro-Local Theorems on the Probabilities of Large Deviations of S_n Outside the Cramér Range (Under the Cramér Condition) . .	274
10	Renewal Processes	277
10.1	Renewal Processes. Renewal Functions	277
10.1.1	Introduction	277
10.1.2	The Integral Renewal Theorem for Non-identically Distributed Summands	280
10.2	The Key Renewal Theorem in the Arithmetic Case	285
10.3	The Excess and Defect of a Random Walk. Their Limiting Distribution in the Arithmetic Case	290
10.4	The Renewal Theorem and the Limiting Behaviour of the Excess and Defect in the Non-arithmetic Case	293
10.5	The Law of Large Numbers and the Central Limit Theorem for Renewal Processes	298
10.5.1	The Law of Large Numbers	298
10.5.2	The Central Limit Theorem	299

10.5.3	A Theorem on the Finiteness of the Infimum of the Cumulative Sums	300
10.5.4	Stochastic Inequalities. The Law of Large Numbers and the Central Limit Theorem for the Maximum of Sums of Non-identically Distributed Random Variables Taking Values of Both Signs	302
10.5.5	Extension of Theorems 10.5.1 and 10.5.2 to Random Variables Assuming Values of Both Signs	304
10.5.6	The Local Limit Theorem	306
10.6	Generalised Renewal Processes	307
10.6.1	Definition and Some Properties	307
10.6.2	The Central Limit Theorem	309
10.6.3	The Integro-Local Theorem	311
11	Properties of the Trajectories of Random Walks. Zero-One Laws . .	315
11.1	Zero-One Laws. Upper and Lower Functions	315
11.1.1	Zero-One Laws	315
11.1.2	Lower and Upper Functions	318
11.2	Convergence of Series of Independent Random Variables	320
11.3	The Strong Law of Large Numbers	323
11.4	The Strong Law of Large Numbers for Arbitrary Independent Variables	326
11.5	The Strong Law of Large Numbers for Generalised Renewal Processes	330
11.5.1	The Strong Law of Large Numbers for Renewal Processes .	330
11.5.2	The Strong Law of Large Numbers for Generalised Renewal Processes	331
12	Random Walks and Factorisation Identities	333
12.1	Factorisation Identities	333
12.1.1	Factorisation	333
12.1.2	The Canonical Factorisation of the Function $f_z(\lambda) = 1 - z\varphi(\lambda)$	335
12.1.3	The Second Factorisation Identity	336
12.2	Some Consequences of Theorems 12.1.1–12.1.3	340
12.2.1	Direct Consequences	340
12.2.2	A Generalisation of the Strong Law of Large Numbers . .	343
12.3	Pollaczek–Spitzer’s Identity. An Identity for $S = \sup_{k \geq 0} S_k$. . .	344
12.3.1	Pollaczek–Spitzer’s Identity	345
12.3.2	An Identity for $S = \sup_{k \geq 0} S_k$	347
12.4	The Distribution of S in Insurance Problems and Queueing Theory	348
12.4.1	Random Walks in Risk Theory	348
12.4.2	Queueing Systems	349
12.4.3	Stochastic Models in Continuous Time	350

12.5	Cases Where Factorisation Components Can Be Found in an Explicit Form. The Non-lattice Case	351
12.5.1	Preliminary Notes on the Uniqueness of Factorisation . . .	351
12.5.2	Classes of Distributions on the Positive Half-Line with Rational Ch.F.s	354
12.5.3	Explicit Canonical Factorisation of the Function $\mathfrak{v}(\lambda)$ in the Case when the Right Tail of the Distribution \mathbf{F} Is an Exponential Polynomial	355
12.5.4	Explicit Factorisation of the Function $\mathfrak{v}(\lambda)$ when the Left Tail of the Distribution \mathbf{F} Is an Exponential Polynomial . . .	361
12.5.5	Explicit Canonical Factorisation for the Function $\mathfrak{v}^0(\lambda)$. . .	362
12.6	Explicit Form of Factorisation in the Arithmetic Case	364
12.6.1	Preliminary Remarks on the Uniqueness of Factorisation . . .	365
12.6.2	The Classes of Distributions on the Positive Half-Line with Rational Generating Functions	366
12.6.3	Explicit Canonical Factorisation of the Function $\mathfrak{v}(z)$ in the Case when the Right Tail of the Distribution \mathbf{F} Is an Exponential Polynomial	367
12.6.4	Explicit Canonical Factorisation of the Function $\mathfrak{v}(z)$ when the Left Tail of the Distribution \mathbf{F} Is an Exponential Polynomial	370
12.6.5	Explicit Factorisation of the Function $\mathfrak{v}^0(z)$	371
12.7	Asymptotic Properties of the Distributions of χ_{\pm} and S	372
12.7.1	The Asymptotics of $\mathbf{P}(\chi_+ > x \mid \eta_+ < \infty)$ and $\mathbf{P}(\chi_-^0 < -x)$ in the Case $\mathbf{E}\xi \leq 0$	373
12.7.2	The Asymptotics of $\mathbf{P}(S > x)$	376
12.7.3	The Distribution of the Maximal Values of Generalised Renewal Processes	380
12.8	On the Distribution of the First Passage Time	381
12.8.1	The Properties of the Distributions of the Times η_{\pm}	381
12.8.2	The Distribution of the First Passage Time of an Arbitrary Level x by Arithmetic Skip-Free Walks	384
13	Sequences of Dependent Trials. Markov Chains	389
13.1	Countable Markov Chains. Definitions and Examples. Classification of States	389
13.1.1	Definition and Examples	389
13.1.2	Classification of States	392
13.2	Necessary and Sufficient Conditions for Recurrence of States. Types of States in an Irreducible Chain. The Structure of a Periodic Chain	395
13.3	Theorems on Random Walks on a Lattice	398
13.3.1	Symmetric Random Walks in \mathbb{R}^k , $k \geq 2$	400
13.3.2	Arbitrary Symmetric Random Walks on the Line	401
13.4	Limit Theorems for Countable Homogeneous Chains	404

13.4.1	Ergodic Theorems	404
13.4.2	The Law of Large Numbers and the Central Limit Theorem for the Number of Visits to a Given State	412
13.5	The Behaviour of Transition Probabilities for Reducible Chains	412
13.6	Markov Chains with Arbitrary State Spaces. Ergodicity of Chains with Positive Atoms	414
13.6.1	Markov Chains with Arbitrary State Spaces	414
13.6.2	Markov Chains Having a Positive Atom	420
13.7	Ergodicity of Harris Markov Chains	423
13.7.1	The Ergodic Theorem	423
13.7.2	On Conditions (I) and (II)	429
13.8	Laws of Large Numbers and the Central Limit Theorem for Sums of Random Variables Defined on a Markov Chain	436
13.8.1	Random Variables Defined on a Markov Chain	436
13.8.2	Laws of Large Numbers	437
13.8.3	The Central Limit Theorem	443
14	Information and Entropy	447
14.1	The Definitions and Properties of Information and Entropy	447
14.2	The Entropy of a Finite Markov Chain. A Theorem on the Asymptotic Behaviour of the Information Contained in a Long Message; Its Applications	452
14.2.1	The Entropy of a Sequence of Trials Forming a Stationary Markov Chain	452
14.2.2	The Law of Large Numbers for the Amount of Information Contained in a Message	453
14.2.3	The Asymptotic Behaviour of the Number of the Most Common Outcomes in a Sequence of Trials	454
15	Martingales	457
15.1	Definitions, Simplest Properties, and Examples	457
15.2	The Martingale Property and Random Change of Time. Wald's Identity	462
15.3	Inequalities	477
15.3.1	Inequalities for Martingales	477
15.3.2	Inequalities for the Number of Crossings of a Strip	481
15.4	Convergence Theorems	482
15.5	Boundedness of the Moments of Stochastic Sequences	487
16	Stationary Sequences	493
16.1	Basic Notions	493
16.2	Ergodicity (Metric Transitivity), Mixing and Weak Dependence	497
16.3	The Ergodic Theorem	502
17	Stochastic Recursive Sequences	507
17.1	Basic Concepts	507
17.2	Ergodicity and Renovating Events. Boundedness Conditions	508

17.2.1	Ergodicity of Stochastic Recursive Sequences	508
17.2.2	Boundedness of Random Sequences	514
17.3	Ergodicity Conditions Related to the Monotonicity of f	516
17.4	Ergodicity Conditions for Contracting in Mean Lipschitz Transformations	518
18	Continuous Time Random Processes	527
18.1	General Definitions	527
18.2	Criteria of Regularity of Processes	532
19	Processes with Independent Increments	539
19.1	General Properties	539
19.2	Wiener Processes. The Properties of Trajectories	542
19.3	The Laws of the Iterated Logarithm	545
19.4	The Poisson Process	549
19.5	Description of the Class of Processes with Independent Increments	552
20	Functional Limit Theorems	559
20.1	Convergence to the Wiener Process	559
20.2	The Law of the Iterated Logarithm	568
20.3	Convergence to the Poisson Process	572
20.3.1	Convergence of the Processes of Cumulative Sums	572
20.3.2	Convergence of Sums of Thinning Renewal Processes	575
21	Markov Processes	579
21.1	Definitions and General Properties	579
21.1.1	Definition and Basic Properties	579
21.1.2	Transition Probability	581
21.2	Markov Processes with Countable State Spaces. Examples	583
21.2.1	Basic Properties of the Process	583
21.2.2	Examples	589
21.3	Branching Processes	591
21.4	Semi-Markov Processes	593
21.4.1	Semi-Markov Processes on the States of a Chain	593
21.4.2	The Ergodic Theorem	594
21.4.3	Semi-Markov Processes on Chain Transitions	597
21.5	Regenerative Processes	600
21.5.1	Regenerative Processes. The Ergodic Theorem	600
21.5.2	The Laws of Large Numbers and Central Limit Theorem for Integrals of Regenerative Processes	601
21.6	Diffusion Processes	603
22	Processes with Finite Second Moments. Gaussian Processes	611
22.1	Processes with Finite Second Moments	611
22.2	Gaussian Processes	614
22.3	Prediction Problem	616

Appendix 1	Extension of a Probability Measure	619
Appendix 2	Kolmogorov's Theorem on Consistent Distributions	625
Appendix 3	Elements of Measure Theory and Integration	629
3.1	Measure Spaces	629
3.2	The Integral with Respect to a Probability Measure	630
3.2.1	The Integrals of a Simple Function	630
3.2.2	The Integrals of an Arbitrary Function	631
3.2.3	Properties of Integrals	634
3.3	Further Properties of Integrals	635
3.3.1	Convergence Theorems	635
3.3.2	Connection to Integration with Respect to a Measure on the Real Line	636
3.3.3	Product Measures and Iterated Integrals	638
3.4	The Integral with Respect to an Arbitrary Measure	640
3.5	The Lebesgue Decomposition Theorem and the Radon–Nikodym Theorem	643
3.6	Weak Convergence and Convergence in Total Variation of Distributions in Arbitrary Spaces	649
3.6.1	Weak Convergence	649
3.6.2	Convergence in Total Variation	652
Appendix 4	The Helly and Arzelà–Ascoli Theorems	655
Appendix 5	The Proof of the Berry–Esseen Theorem	659
Appendix 6	The Basic Properties of Regularly Varying Functions and Subexponential Distributions	665
6.1	General Properties of Regularly Varying Functions	665
6.2	The Basic Asymptotic Properties	668
6.3	The Asymptotic Properties of the Transforms of R.V.F.s (Abel-Type Theorems)	672
6.4	Subexponential Distributions and Their Properties	674
Appendix 7	The Proofs of Theorems on Convergence to Stable Laws . .	687
7.1	The Integral Limit Theorem	687
7.2	The Integro-Local and Local Limit Theorems	699
Appendix 8	Upper and Lower Bounds for the Distributions of the Sums and the Maxima of the Sums of Independent Random Variables	703
8.1	Upper Bounds Under the Cramér Condition	703
8.2	Upper Bounds when the Cramér Condition Is Not Met	704
8.3	Lower Bounds	713
Appendix 9	Renewal Theorems	715
References	723

Index of Basic Notation 725

Subject Index 727

Probability Theory

Borovkov, A.A.

2013, XXVIII, 733 p. 22 illus., Softcover

ISBN: 978-1-4471-5200-2