Identification Without Exogeneity Under Equiconfounding in Linear Recursive Structural Systems

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Abstract This chapter obtains identification of structural coefficients in linear recursive systems of structural equations without requiring that observable variables are exogenous or conditionally exogenous. In particular, standard instrumental variables and control variables need not be available in these systems. Instead, we demonstrate that the availability of one or two variables that are equally affected by the unobserved confounder as is the response of interest, along with exclusion restrictions, permits the identification of all the system's structural coefficients. We provide conditions under which *equiconfounding* supports either full identification of structural coefficients or partial identification in a set consisting of two points.

Keywords Causality · Confounding · Covariance Restrictions · Identification · Structural systems

1 Introduction

This chapter obtains identification of structural coefficients in fully endogenous linear recursive systems of structural equations. In particular, standard exogenous instruments and control variables may be absent in these systems.[1] Instead, identification obtains under *equiconfounding* that is to say in the presence of (one or two) observable variables that are equally directly affected by the unobserved confounder as is the response. Examples of equiconfounding include cases in which the unobserved confounder directly affects the response and one or two observables by an equal

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[1] Standard instruments are uncorrelated with the unobserved confounder whereas conditioning on control variables renders the causes of interest uncorrelated with the confounder.

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proportion (proportional confounding) or an equal standard deviation shift. We show that the availability of one or two variables that are equally (e.g., proportionally) confounded in relation to the response of interest, along with exclusion restrictions, permits the identification of all the system's structural coefficients. We provide conditions under which we obtain either full identification of structural coefficients or partial identification in a set consisting of two points.

The results of this chapter echo a key insight in Halbert White's work regarding the importance of specifying causal relations governing the unobservables for the identification and estimation of causal effects (e.g., White and Chalak 2010, 2011; Chalak and White 2011; White and Lu 2011a,b; Hoderlein et al. 2011). A single chapter can do little justice addressing Hal's prolific and groundbreaking contributions to asymptotic theory, specification analysis, neural networks, time series analysis, and causal inference, to list a few areas, across several disciplines including economics, statistics, finance, and computer and cognitive sciences. Instead, here, we focus on one insight of Hal's recent work and build on it to introduce the notion of equiconfounding and demonstrate how it supports identification in structural systems.

To illustrate this chapter's results, consider the classic structural equation for the return to education (e.g., Mincer 1974; Griliches 1977)

$$Y = \beta_o X + \alpha_u U + \alpha_y U_y, \tag{1}$$

where Y denotes the logarithm of hourly wage, X determinants of wage with observed realizations, and U and U_y determinants of wage whose realizations are not observed by the econometrician. Elements of X may include years of education, experience, and tenure. Interest attaches to the causal effect of X on Y, assumed to be the constant β_o. Here, U denotes an index of unobserved personal characteristics that may determine wage and be correlated with X, such as cognitive and noncognitive skills, and U_y denote other unobserved determinants assumed to be uncorrelated with X and U. Endogeneity arises because of the correlation between X and $\alpha_u U$, leading to bias in the coefficient of a linear regression of Y on X. The method of instrumental variables (IV) permits identification of the structural coefficients under the assumption that a "valid" (i.e. uncorrelated with $\alpha_u U + \alpha_y U_y$) and "relevant" (i.e. $E(XZ')$ is full raw rank) vector Z excluded from Eq. (1) and whose dimension is at least as large as that of X is available (e.g., Wooldridge 2002, pp. 83–84). Alternatively, the presence of key covariates may ensure "conditional exogeneity" or "unconfoundedness" supporting identification (see e.g., White and Chalak 2011 and the citations therein). We do not assume the availability of standard instruments or control variables here, so these routes for identification are foreclosed.

Nevertheless, as we show, a variety of shape restrictions[2] on confounding can secure identification of β_o. To illustrate, begin by considering the simplest such

[2] Shape restrictions have been employed in a variety of different contexts. For example, Matzkin (1992) employs shape restrictions to secure identification in nonparametric binary threshold crossing models with exogeneity.

possibility in which data on a proxy for $\alpha_u U$, such as IQ score, is available. Let Z denote the logarithm of IQ and assume that the predictive proxy Z for U does not directly cause Y, and that Z and Y are equiconfounded. In particular, suppose that Z is structurally generated by

$$Z = \alpha_u U + \alpha_z U_z,$$

with U_z as a source of variation uncorrelated with other unobservables. Then, under this proportional confounding, a one unit increase in U leads to an approximate $100\alpha_u\%$ increase in wage and IQ ceteris paribus. It is straightforward to see that, by substitution, β_o is identified from a regression of $Y - Z$ on X. Note, however, that Z is not a valid instrument here ($E(Z\alpha_u U) \neq 0$) since Z is driven by U.

The above simple structure excludes IQ from the equation for Y to ensure that β_o is identified. Suppose instead that $X = (X_1, X_2, X_3')'$ and that the two variables X_1 and X_2 are structurally generated as follows

$$X_1 = \alpha_u U + \alpha_{x_1} U_{x_1} \quad \text{and} \quad X_2 = \alpha_u U + \alpha_{x_2} U_{x_2},$$

with U_{x_1} and U_{x_2} sources of variation, each uncorrelated with other unobservables. We maintain that the other elements of X are generally endogenous but we restrict X_1 and X_2 to be *equiconfounded joint causes* of Y. For example, X_1 may denote the logarithm of another test score, such as the Knowledge of World of Work (KWW) score (see e.g., Blackburn and Neumark 1992), and we relabel $\log(IQ)$ to X_2. Here, wage, KWW, and IQ are proportionally confounded by U. Substituting for $\alpha_u U = X_1 - \alpha_{x_1} U_{x_1}$ in (1) gives

$$Y - X_1 = \beta_o X - \alpha_{x_1} U_{x_1} + \alpha_y U_y,$$

and thus a regression of $Y - X_1$ on X does not identify β_o since X_1 is correlated with $\alpha_{x_1} U_{x_1}$. Further, although X_2 and X_3 are exogenous in this equation, they are not excluded from it and thus they cannot serve as instruments for X_1. Nevertheless, we demonstrate that in this case β_o is fully (over) identified.

In the previous example, two joint causes and a response that are equiconfounded secure identification. Similarly, one *cause* and two *joint responses* that are *equiconfounded* can ensure that β_o is identified. For example, let Y_1 and Y_2 denote two responses of interest (e.g., two measures of the logarithm of wage, one reported by the employer and another by the employee). In particular, suppose that

$$Y_1 = \beta_{1o} X + \alpha_u U + \alpha_{y_1} U_{y_1} \quad \text{and} \quad Y_2 = \beta_{2o} X + \alpha_u U + \alpha_{y_2} U_{y_2}.$$

Note that β_{1o} and β_{2o} need not be equal. As before, we maintain that an element X_1 (e.g., $\log(IQ)$) of X is structurally generated by

$$X_1 = \alpha_u U + \alpha_{x_1} U_{x_1},$$

with the remaining elements of X generally endogenous. We demonstrate that here $(\beta_{1o}', \beta_{2o}')'$ is partially identified in a set consisting of two points.

Various other exclusion restrictions can secure identification of structural coefficients in the presence of equiconfounding. Consider the classic triangular structure:

$$Y = \beta_o X + \alpha_u U + \alpha_y U_y,$$
$$X = \gamma_o Z + \eta_u U + \alpha_x U_x.$$

As before, U_y and U_x denote exogenous sources of variation. The method of IV identifies β_o provided that the excluded vector Z is valid ($E(\alpha_u U Z') = 0$) and relevant ($E(XZ')$ full raw rank) and thus has dimension at least as large as that of X. Suppose instead that Y, Z, and an element X_1 of X are equiconfounded by U:

$$X_1 = \gamma_{1o} Z + \alpha_u U + \alpha_{x_1} U_{x_1} \quad \text{and} \quad Z = \alpha_u U + \alpha_z U_z,$$

where U_{x_1} and U_z are each uncorrelated with other unobservables. The remaining elements of X are generally endogenous. For example, a researcher may wish to allow IQ to be a structural determinant of the subsequently administered KWW test, in order to capture learning effects, and to exclude IQ from the equation for Y if this test's information is unavailable to employers. Then Z denotes $\log(IQ)$ and X_1 denotes $\log(KWW)$. In this structure we refer to Z and X_1 as *equiconfounded pre-cause* and *intermediate-cause*, respectively. We demonstrate that $(\beta_o', \gamma_o')'$ is either fully identified or partially identified in a set consisting of two points. Importantly, in contrast to the method of IV, here Z is a *scalar endogenous* variable.

This chapter is organized as follows. Section 2 introduces notation. Formal identification results, including for the examples above, are discussed in Sects. 3 to 6. Often we present the identification results as adjustments to standard regression coefficients thereby revealing the regression bias arising due to endogeneity. Section 7 contains a discussion and Sect. 8 concludes. All mathematical proofs as well as constructive arguments for identification are gathered in the appendix.

2 Notation

Throughout, we let the random $k \times 1$ vector X and $p \times 1$ vector Y denote the observed direct causes and responses of interest, respectively.[3] If there are observed variables excluded from the equation for Y, we denote these by the $\ell \times 1$ vector Z. We observe independent and identically distributed realizations $\{Z_i, X_i, Y_i\}_{i=1}^n$ for

[3] This chapter considers linear recursive structural systems. Recursiveness rules out "simultaneity" permitting distinguishing the vectors of primary interest X and Y as the observed direct causes and responses, respectively. In particular, elements of Y are assumed to not cause elements of X. While mutual causality is absent here, endogeneity arises due to the confounder U jointly driving the causes X and responses Y.

Z, X, and Y and stack these into the $n \times \ell$, $n \times k$, and $n \times p$ matrices $\mathbf{Z}$, $\mathbf{X}$, and $\mathbf{Y}$, respectively. The matrices (or vectors) of structural coefficients γ_o and β_o denote finite causal effects determined by theory as encoded in a linear structural system of equations. The scalar index U denotes an unobserved confounder of X, Z, and Y and the vectors U_z, U_x, and U_y of positive dimensions denote unobserved causes of elements of Z, X and Y, respectively. Without loss of generality, we normalize the expectations of U, U_z, U_x, and U_y to zero. The structural coefficients matrices α_z, α_x and α_y denote the effects of elements of U_z, U_x and U_y on elements of Z, X and Y, respectively. Equiconfounding restricts the effect of the confounder U on Y and certain elements of X and Z to be equal; we denote this restricted effect by α_u and we denote unrestricted effects of U on elements of X by ϕ_u.

We employ the following notation for regression coefficients and residuals. Let Y, X, and Z be generic random vectors. We denote the coefficient and residual from a regression of Y on X by

$$\pi_{y.x} \equiv E(YX')E(XX')^{-1} \quad \text{and} \quad \epsilon_{y.x} \equiv Y - \pi_{y.x}X.$$

Similarly, we denote the coefficient associated with X from a regression of Y on X and Z by

$$\pi_{y.x|z} \equiv E(\epsilon_{y.z}\epsilon'_{x.z})E(\epsilon_{x.z}\epsilon'_{x.z})^{-1}.$$

This representation obtains from the Frisch-Waugh-Lovell theorem (Frisch and Waugh 1993; Lovell 1963; see e.g., Baltagi 1999, p. 159). Noting that

$$\begin{aligned} E(\epsilon_{y.z}\epsilon'_{x.z}) &= E(Y\epsilon'_{x.z}) - E(YZ')E(ZZ')^{-1}E(Z\epsilon'_{x.z}) = E(Y\epsilon'_{x.z}) \\ &= E(YX') - E(YZ')E(ZZ')^{-1}E(ZX') = E(\epsilon_{y.z}X'), \end{aligned}$$

we can rewrite $\pi_{y.x|z}$ as

$$\pi_{y.x|z} = E(Y\epsilon'_{x.z})E(X\epsilon'_{x.z})^{-1} = E(\epsilon_{y.z}X')E(\epsilon_{x.z}X')^{-1}.$$

Last, we denote sample regression coefficients by $\hat{\pi}_{y.x} \equiv (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ and residuals by $\hat{\epsilon}_{y.x,i} \equiv Y_i - \hat{\pi}_{y.x}X_i$, which we stack into the $n \times p$ vector $\hat{\epsilon}_{y.x}$. Similarly, we let $\hat{\pi}_{y.x|z} \equiv (\hat{\epsilon}'_{x.z}\mathbf{X})^{-1}\hat{\epsilon}'_{x.z}\mathbf{Y}$.

Throughout, we illustrate a structural system using a directed acyclic graph as in Chalak and White (2011). A graph $G_{\mathcal{S}}$ associated with a structural system $\mathcal{S}$ consists of a set of vertices (nodes) $\{V_g\}$, one for each variable in $\mathcal{S}$, and a set of arrows $\{a_{gh}\}$, corresponding to ordered pairs of distinct vertices. An arrow a_{gh} denotes that V_g is a potential direct cause for V_h, i.e., it appears directly in the structural equation for V_h with a corresponding possibly nonzero coefficient. We use solid nodes for observables and dashed nodes for unobservables. For convenience, we sometimes use vector nodes to represent vectors generated by structural system $\mathcal{S}$. In this case, an arrow from vector node Z to vector node X indicates that at least one element of Z is a direct cause of at least one element of X. We use solid nodes for observable vectors and dashed nodes for vectors with at least one unobservable element. For simplicity, we omit nodes for the exogenous vectors U_z, U_x,and U_y. Lastly, we use dashed arrows emanating from U to Y, X_1, Z , and possibly X_2 to denote equiconfounding.

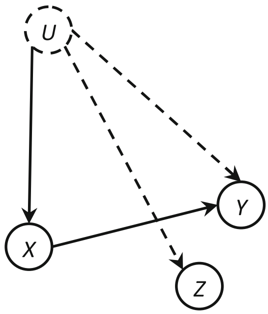
3 Equiconfounded Predictive Proxy and Response

The simplest possibility arises when the response Y and a scalar predictive proxy Z for the unobserved confounder U are equiconfounded. The predictive proxy Z is excluded from the equation for Y. In particular, consider the structural system of equations $\mathcal{S}_1$ with causal graph G_1:

(1) $Z \overset{s}{=} \alpha_u U + \alpha_z U_z$,

(2) $X_1 \overset{s}{=} \phi_u U + \alpha_x U_x$

(3) $Y \overset{s}{=} \beta_o X + \alpha_u U + \alpha_y U_y$
with $U, U_z, U_x,$ and U_y
pairwise uncorrelated
and with $X = (X_1', 1)'$.



Graph 1 (G_1)
Equiconfounded Predictive Proxy and Response

Similar to Chalak and White (2011), we use the "$\overset{s}{=}$" notation instead of "$=$" to emphasize structural equations. We let $\ell = p = 1$ in $\mathcal{S}_1$ as this suffices for identification. Here and in what follows, we let the last element of X be degenerate at 1. The next result shows that the structural vector β_o is point identified. This is obtained straightforwardly by substituting $\alpha_u U$ with $Z - \alpha_z U_z$ in the equation for Y.

Theorem 3.1 *Consider structural system $\mathcal{S}_1$ with $k > 0$, $\ell = p = 1$, and expected values of U, U_z, U_x, U_y normalized to zero. Suppose that $E(U^2)$ and $E(U_x U_x')$ exist and are finite. Then (i) $E(XX')$, $E(ZX')$, and $E(YX')$ exist and are finite. Suppose further that $E(XX')$ is nonsingular. Then (ii) β_o is fully identified as*

$$\beta_o = \pi_{y-z.x}.$$

Under standard conditions (e.g., White 2001) the estimator $\hat{\pi}_{y-z.x} \equiv (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{Y} - \mathbf{Z})$ is a consistent and asymptotically normal estimator for β_o. A heteroskedasticity robust estimator (White 1980) for the asymptotic covariance matrix for $\hat{\pi}_{y-z.x}$ is given by $(\mathbf{X}'\mathbf{X})^{-1}(\sum_{i=1}^{n} \hat{\epsilon}^2_{y-z.x,i} X_i X_i')(\mathbf{X}'\mathbf{X})^{-1}$.

4 Equiconfounded Joint Causes and Response

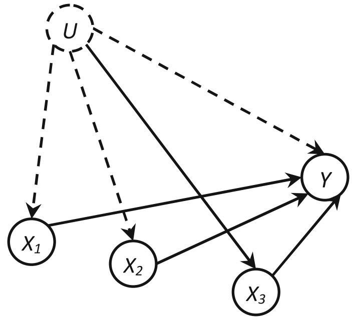
Identification in $\mathcal{S}_1$ requires the predictive proxy Z to be excluded from the equation for Y. However, β_o is also identified if two causes X_1 and X_2 and the response Y are equiconfounded. In particular, consider structural system $\mathcal{S}_2$ with causal graph G_2:

$(1a)$ $X_1 \overset{s}{=} \alpha_u U + \alpha_{x_1} U_{x_1}$,

$(1b)$ $X_2 \overset{s}{=} \alpha_u U + \alpha_{x_2} U_{x_2}$

$(1c)$ $X_{31} \overset{s}{=} \phi_u U + \alpha_{x_3} U_{x_3}$

(2) $Y \overset{s}{=} \beta_o X + \alpha_u U + \alpha_y U_y$
with $U, U_{x_1}, U_{x_2}, U_{x_3},$ and U_y
pairwise uncorrelated and
$X = (X_1', X_2', X_{31}', 1)' = (X_1', X_2', X_3')'$.



Graph 2 (G_2)
Equiconfounded Joint Causes and Response

We can rewrite $1(a, b, c)$ as

$$(1) \quad (X_1', X_2', X_{31}')' \overset{s}{=} \eta_u U + \alpha_x U_x,$$

with $\eta_u = (\alpha_u', \alpha_u', \phi_u')'$, $U_x = (U_{x_1}', U_{x_2}', U_{x_3}')'$, and α_x a block diagonal matrix with α_{x_1}, α_{x_2}, and α_{x_3} at the diagonal entries and zeros at the off-diagonal entries. Here, we let X_1 and X_2 be scalars, $k_1 = k_2 = 1$, as this suffices for identification. The next theorem shows that the structural vector β_o is point identified.

Theorem 4.1 *Consider structural system* $\mathcal{S}_2$ *with* $\dim(X_3) \equiv k_3 \geq 0$, *and* $k_1 = k_2 = p = 1$, *and expected values of* U, U_z, U_x, U_y *normalized to zero. Suppose that* $E(U^2)$ *and* $E(U_x U_x')$ *exist and are finite. Then (i)* $E(XX')$ *and* $E(YX')$ *exist and are finite. Suppose further that* $E(XX')$ *is nonsingular. Then (ii) the vector* β_o *is fully (over-)identified by:*

$$\begin{aligned}\beta_o = \beta_{JC}^* &\equiv \pi_{y.x} - [E(X_2X_1'), E(X_2X_1'), E(X_1X_3')]E(XX')^{-1} \\ = \beta_{JC}^\dagger &\equiv \pi_{y.x} - [E(X_2X_1'), E(X_2X_1'), E(X_2X_3')]E(XX')^{-1}.\end{aligned}$$

The above result obtains by noting that the moment $E(YX')$ identifies β_o when $E(XX')$ is nonsingular provided that $\alpha_u E(UX')$ is identified. But this holds since, $E(X_1X_3') = E(X_2X_3') = (\mathrm{Cov}(\phi_u U, \alpha_u U)', 0)$ and $E(X_1X_2) = \mathrm{Var}(\alpha_u U)$. The expressions for β_{JC}^* and $\beta_{JC}^\dagger$ emphasize the bias $\pi_{y.x} - \beta_{JC}^*$ (or $\pi_{y.x} - \beta_{JC}^\dagger$) in a regression of Y on X arising due to endogeneity. The plug-in estimators $\hat{\beta}_{JC}^*$ and $\hat{\beta}_{JC}^\dagger$ for β_{JC}^* and $\beta_{JC}^\dagger$, respectively:

$$\hat{\beta}^*_{JC} \equiv \hat{\pi}_{y.x} - \sum_{i=1}^{n}[\, X_{2i}X'_{1i},\ X_{2i}X'_{1i},\ X_{1i}X'_{31i},\ 0\,](\mathbf{X}'\mathbf{X})^{-1}, \text{ and}$$

$$\hat{\beta}^*_{JC} \equiv \hat{\pi}_{y.x} - \sum_{i=1}^{n}[\, X_{2i}X'_{1i},\ X_{2i}X'_{1i},\ X_{2i}X'_{31i},\ 0\,](\mathbf{X}'\mathbf{X})^{-1},$$

are consistent estimators under conditions sufficient to invoke the laws of large numbers.

A testable restriction of structure $\mathcal{S}_2$ is that $\text{Cov}(X_1, X_3) = \text{Cov}(X_2, X_3) = (\alpha_u E(U^2)\phi'_u, 0)$. Thus, $\mathcal{S}_2$ can be falsified by rejecting this null. In particular, one can reject the equiconfounding restrictions in equations $1(a, b, c)$ if $E(X_1X'_3) \neq E(X_2X'_3)$. For this, one can employ a standard F-statistic for the overall significance of the regression of $\mathbf{X}_1 - \mathbf{X}_2$ on $\mathbf{X}_3$.

5 Equiconfounded Cause and Joint Responses

The availability of a single cause and two responses that are equiconfounded also ensures the identification of causal coefficients. Specifically, consider structural system $\mathcal{S}_3$ given by:

$(1a)$ $X_1 \overset{s}{=} \alpha_u U + \alpha_{x_1} U_{x_1}$

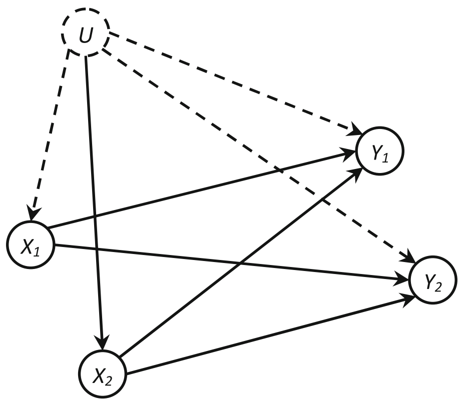
$(1b)$ $X_{21} \overset{s}{=} \phi_u U + \alpha_{x_2} U_{x_2}$

$(2a)$ $Y_1 \overset{s}{=} \beta_{1o} X + \alpha_u U + \alpha_{y_1} U_{y_1}$

$(2b)$ $Y_2 \overset{s}{=} \beta_{2o} X + \alpha_u U + \alpha_{y_2} U_{y_2}$

with $U, U_{x_1}, U_{x_2}, U_{y_1},$ and U_{y_2} pairwise uncorrelated and

$X = (X'_1, X'_{21}, 1)' = (X'_1, X'_2)'$.



Graph 3 (G_3)
Equiconfounded Cause and Joint Responses

Letting $Y = (Y'_1, Y'_2)'$, $\beta_o = (\beta'_{1o}, \beta'_{2o})'$, $U_x = (U'_{x_1}, U'_{x_2})'$, and $U_y = (U'_{y_1}, U'_{y_2})'$, and letting α_x be a block diagonal matrix with diagonal entries α_{x_1} and α_{x_2} and zero off-diagonal entries, and similarly for α_y, we can write $1(a, b)$ and $2(a, b)$ more compactly as

$$(1)\ (X'_1, X'_{21})' \overset{s}{=} \eta_u U + \alpha_x U_x$$

$$(2)\ Y \overset{s}{=} \beta_o X + \alpha_u \iota_p U + \alpha_y U_y,$$

with ι_p a $p \times 1$ vector with each element equal to 1 and $\eta_u = (\alpha_u', \phi_u')'$. Here it suffices for identification that $\dim(X_1) \equiv k_1 = 1$ and $p = 2$. The next theorem demonstrates that the structural matrix β_o is partially identified in a set consisting of two points.

Theorem 5.1 *Consider structural system* $\mathcal{S}_3$ *with* $\dim(X_2) \equiv k_2 \geq 0$, $k_1 = 1$, $p = 2$, *and expected values of* U, U_z, U_x, *and* U_y *normalized to zero. Suppose that* $E(U^2)$ *and* $E(U_x U_x')$ *exist and are finite, then (i)* $E(XX')$ *and* (YX') *exist and are finite. Suppose further that* $E(X_1 X_1')$ *and* $E(X_2 X_2')$ *are nonsingular then (ii.a)* $P_{x_1} \equiv E(\epsilon_{x1.x2}\epsilon_{x1.x2}')$ *and* $P_{x_2} \equiv E(\epsilon_{x2.x1}\epsilon_{x2.x1}')$ *exist and are finite. If also* P_{x_1} *and* P_{x_2} *are nonsingular then (ii.b)* $E(XX')$ *is nonsingular,* $\pi_{y.x}$ *and* $E(\epsilon_{y_1.x} Y_2')$ *exist and are finite, and (ii.c)*

$$\Delta_{JR} = \left[2P_{x_1}^{-1} E(X_1 X_1') - 1\right]^2 - 4P_{x_1}^{-1}\left[E(X_1 X_2') P_{x_2}^{-1} E(X_2 X_1') + E(\epsilon_{y_1.x} Y_2')\right],$$

exists, is finite, and is nonnegative.

(iii) β_o *is partially identified in a set consisting of two points. In particular, (iii.a) if*

$$\begin{aligned}&\mathrm{Var}(\alpha_{x_1} U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)' \\ &\quad [\mathrm{Var}(\phi_u U) + \mathrm{Var}(\alpha_{x_2} U_{x_2})]^{-1} \mathrm{Cov}(\phi_u U, \alpha_u U) - \mathrm{Var}(\alpha_u U) < 0,\end{aligned}$$

then

$$\begin{aligned} 0 \leq \sigma_{JR}^{\dagger} &\equiv E(X_1 X_1') + \frac{1}{2} P_{x_1}(-1 - \sqrt{\Delta_{JR}}) < \alpha_u^2 E(U^2), \ \textit{and} \\ \sigma_{JR}^{*} &\equiv E(X_1 X_1') + \frac{1}{2} P_{x_1}(-1 + \sqrt{\Delta_{JR}}) = \alpha_u^2 E(U^2), \end{aligned}$$

and thus

$$\beta_o = \beta_{JR}^{*} \equiv \pi_{y.x} - \iota_p [\sigma_{JR}^{*}, E(X_1 X_2')] E(XX')^{-1}.$$

(iii.b) If instead the expression in (iii) is nonnegative then

$$\sigma_{JR}^{\dagger} = \alpha_u^2 E(U^2) \quad \textit{and} \quad 0 \leq \alpha_u^2 E(U^2) \leq \sigma_{JR}^{*},$$

and thus

$$\beta_o = \beta_{JR}^{\dagger} \equiv \pi_{y.x} - \iota_p [\, \sigma_{JR}^{\dagger}, \ E(X_1 X_2')\,] E(XX')^{-1}.$$

Observe here that, unlike for the case of equiconfounded joint causes, β_o is not point identified but is partially identified in a set consisting of two points. Also, observe that $\beta_{1o} - \beta_{2o}$ is identified from a regression of $Y_1 - Y_2$ on X. However, $\beta_{1,JR}^{*} - \beta_{2,JR}^{*} = \beta_{1,JR}^{\dagger} - \beta_{2,JR}^{\dagger}$ and thus this does not help in fully identifying β_o. Similar to $\mathcal{S}_2$, with $E(XX')$ nonsingular, the moment $E(YX')$ identifies

β_o provided $\text{Cov}(\phi_u U, \alpha_u U)$ and $\text{Var}(\alpha_u U)$ are identified. While $E(X_{21} X_1) = \text{Cov}(\phi_u U, \alpha_u U)$, identification of $\text{Var}(\alpha_u U)$ is more involved here than in $\mathcal{S}_2$. Appendix B.1 presents a constructive argument showing that the moment $E(Y_1 Y_2)$ delivers a quadratic equation in $\text{Var}(\alpha_u U)$ with two positive roots, $\sigma^{\dagger}_{JR}$ and σ^{*}_{JR}.

Under suitable conditions sufficient to invoke the law of large numbers, the following plug-in estimators are consistent for Δ_{JR}, σ^*_{JR}, $\sigma^{\dagger}_{JR}$, β^*_{JR}, and $\beta^{\dagger}_{JR}$ respectively. To express these, let $\hat{P}_{x_1} = \frac{1}{n}\hat{\epsilon}'_{x_1.x_2}\mathbf{X}_1$ and $\hat{P}_{x_2} \equiv \frac{1}{n}\hat{\epsilon}'_{x_2.x_1}\mathbf{X}_2$. Then

$$\hat{\Delta}_{JR} \equiv \left[2\hat{P}_{x_1}^{-1}\,\frac{1}{n}\mathbf{X}_1'\mathbf{X}_1 - 1\right]^2 - 4\hat{P}_{x_1}^{-1}\left[\frac{1}{n}\mathbf{X}_1'\mathbf{X}_2\;\hat{P}_{x_2}^{-1}\,\frac{1}{n}\mathbf{X}_2'\mathbf{X}_1 + \frac{1}{n}\hat{\epsilon}'_{y_1.x}\mathbf{Y}_2\right],$$

$$\hat{\sigma}^*_{JR} \equiv \frac{1}{n}\mathbf{X}_1'\mathbf{X}_1 + \frac{1}{2}\hat{P}_{x_1}\left(-1 + \sqrt{\hat{\Delta}_{JR}}\right) \quad \text{and}$$

$$\hat{\sigma}^{\dagger}_{JR} \equiv \frac{1}{n}\mathbf{X}_1'\mathbf{X}_1 + \frac{1}{2}\hat{P}_{x_1}\left(-1 - \sqrt{\hat{\Delta}_{JR}}\right),$$

$$\hat{\beta}^*_{JR} \equiv \hat{\pi}_{y.x} - \iota_p\left[\hat{\sigma}^*_{JR}, \frac{1}{n}\mathbf{X}_1'\mathbf{X}_2\right]\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}, \text{ and}$$

$$\hat{\beta}^{\dagger}_{JR} \equiv \hat{\pi}_{y.x} - \iota_p\left[\hat{\sigma}^{\dagger}_{JR}, \frac{1}{n}\mathbf{X}_1'\mathbf{X}_2\right]\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}.$$

Thus, under suitable statistical assumptions, $\hat{\beta}^*_{JR}$ and $\hat{\beta}^{\dagger}_{JR}$ converge to β^*_{JR} and $\beta^{\dagger}_{JR}$, respectively; under the structural assumptions of $\mathcal{S}_3$, either β^*_{JR} or $\beta^{\dagger}_{JR}$ identifies the structural coefficient vector β_o.

6 Equiconfounding in Triangular Structures

Next, we consider the classic triangular structure discussed in the Introduction and show that if one excluded variable Z_1, one element X_1 of the direct causes X, and the response Y are equally confounded by U then all the system's structural coefficients are identified. Consider structural system $\mathcal{S}_4$ with causal graph G_4:

(1) $Z_1 \overset{s}{=} \alpha_u U + \alpha_z U_z$

(2a) $X_1 \overset{s}{=} \gamma_{1o} Z + \alpha_u U + \alpha_{x_1} U_{x_1}$

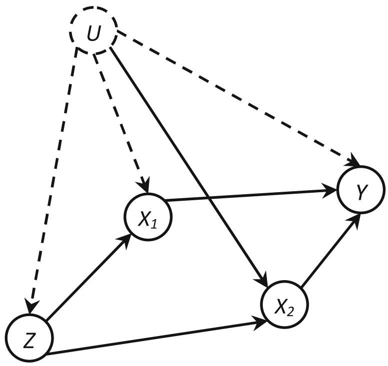
(2b) $X_{21} \overset{s}{=} \gamma_{2o} Z + \phi_u U + \alpha_{x_2} U_{x_2}$

(3) $Y \overset{s}{=} \beta_o X + \alpha_u U + \alpha_y U_y$,

with U_z, U, U_{x_1}, U_{x_2}, and U_y pairwise uncorrelated,

and with $Z = (Z_1', 1')' = (Z_1', Z_2')'$,

and $X = (X_1', X_{21}', 1) = (X_1', X_2')'$.



Graph 4 (G_4)
Equiconfounded Pre-Cause,
Intermediate-Cause, and Response

To rewrite 2(a, b) more compactly, let $\gamma_o = (\gamma_{1o}', \gamma_{2o}')'$ and $\eta_u = (\alpha_u', \phi_u')'$, and write $U_x' = (U_{x_1}', U_{x_2}')'$, with α_{x_1} and α_{x_2} the diagonals entries of the block diagonal matrix α_x with zero off-diagonal entries. Then

$$(2) \quad (X_1', X_{21}')' \overset{s}{=} \gamma_o Z + \eta_u U + \alpha_x U_x.$$

We sometimes refer to Z_1 as a *pre-cause* variable as it is excluded from the equation for Y and to X_1 as an *intermediate cause* as it mediates the effect of Z_1 on Y. As discussed in the Introduction, necessary conditions for the method of IV to identify β_o are that $E(Z(\alpha_u U + \alpha_y U_y)) = 0$ and that $E(XZ')$ is full raw rank. Both of these conditions can fail in $\mathcal{S}_4$, since $E(Z(\alpha_u U))$ is generally nonzero and only one excluded variable suffices for identification here so that $\dim(Z_1) \equiv \ell_1 = \dim(X_1) \equiv k_1 = p = 1$ and thus $\dim(Z) \equiv \ell \leq \dim(X) \equiv k$. Nevertheless, the next theorem demonstrates that the structural vectors γ_o and β_o are jointly either point identified or partially identified in a set consisting of two points.

Theorem 6.1 *Consider structural system* $\mathcal{S}_4$ *with* $\dim(X_2) = k_2 \geq 0$, $\ell_1 = k_1 = p = 1$, *and expected values of* U, U_z, U_x, U_y *normalized to zero. Suppose that* $E(U^2)$, $E(U_z U_z')$, *and* $E(U_x U_x')$ *exist and are finite. Then (i)* $E(ZZ')$, $E(XZ')$, $E(XX')$, $E(YX')$, *and* $E(YZ')$ *exist and are finite. (ii) Suppose further that* $P_{z_1} \equiv E(\epsilon_{z_1.z_2} Z_1') = E(Z_1 Z_1')$, *and thus* $E(ZZ')$, *and* $E(XX')$ *are nonsingular. Then (ii.a)* $\pi_{x.z}$, $\pi_{z.x}$, $E(\epsilon_{x_1.z} X_2')$, *and* $E(\epsilon_{y.x} Z_1')$ *exist and are finite and (ii.b)*

$$\Delta_{PC} = [-\pi_{x.z_1|z_2}' \, \pi_{z_1.x}' - \pi_{z_1.x_1|x_2}' + 1]^2 + 4 P_{z_1}^{-1} \, \pi_{z_1.x_1|x_2}' [E(\epsilon_{y.x} Z_1') + E(\epsilon_{x_1.z} X_2') \, \pi_{z_1.x_2|x_1}']$$

exists, is finite, and nonnegative.

(iii) β_o *is either point identified or partially identified in a set consisting of two points. In particular, (iii.a) if*

$$\pi'_{x.z_1|z_2}\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 - 2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}\alpha_u^2 E(U^2) < 0, \tag{2}$$

then

$$\sigma^\dagger_{PC} \equiv \frac{\pi'_{x.z_1|z_2}\,\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 - \sqrt{\Delta_{PC}}}{2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}} < \alpha_u^2 E(U^2) \quad \textit{and}$$
$$\sigma^*_{PC} \equiv \frac{\pi'_{x.z_1|z_2}\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 + \sqrt{\Delta_{PC}}}{2P_{z_1}^{-1}\,\pi'_{z_1.x_1|x_2}} = \alpha_u^2 E(U^2),$$

and we have

$$\begin{aligned}
\gamma_{1o} &= \gamma^*_{1,PC} \equiv \pi_{x_1.z} - [\sigma^*_{PC}, 0]E(ZZ')^{-1},\\
\gamma_{2o} &= \gamma^*_{2,PC} \equiv \pi_{x_21.z} - [E(X_{21}\epsilon'_{x_1.z})[1 - \sigma^*_{PC}P_{z_1}^{-1}]^{-1}, 0]E(ZZ')^{-1}, \quad \textit{and}\\
\beta_o &= \beta^*_{PC} \equiv \pi_{y.x} - [\sigma^*_{PC}(\pi'_{x_1.z_1|z_2} - \sigma^*_{PC}\,P_{z_1}^{-1} + 1), E(\epsilon_{x_1.z}X'_2)\\
&\quad + \sigma^*_{PC}\pi'_{x_2.z_1|z_2}]E(XX')^{-1}.
\end{aligned}$$

(iii.b) If instead the expression in (2) *is nonnegative then* $\sigma^\dagger_{PC} = \alpha_u^2 E(U^2)$ *and* $\sigma^*_{PC} \geq \alpha_u^2 E(U^2)$, *and*

$$\begin{aligned}
\gamma_{1o} &= \gamma^\dagger_{1,PC} \equiv \pi_{x_1.z} - [\sigma^\dagger_{PC}, 0]E(ZZ')^{-1},\\
\gamma_{2o} &= \gamma^\dagger_{2,PC} \equiv \pi_{x_21.z} - [\,E(X_{21}\epsilon'_{x_1.z})[1 - \sigma^\dagger_{PC}P_{z_1}^{-1}]^{-1},\,0\,]E(ZZ')^{-1}, \quad \textit{and}\\
\beta_o &= \beta^\dagger_{PC} \equiv \pi_{y.x} - [\sigma^\dagger_{PC}(\pi'_{x_1.z_1|z_2} - \sigma^\dagger_{PC}P_{z_1}^{-1} + 1), E(\epsilon_{x_1.z}X'_2)\\
&\quad + \sigma^\dagger_{PC}\pi'_{x_2.z_1|z_2}]E(XX')^{-1}.
\end{aligned}$$

Similar to $\mathcal{S}_3$, the moment $E(YX')$ identifies β_o provided $\alpha_u E(UX')$ is identified, which involves identifying $\mathrm{Var}(\alpha_u U)$. Appendix B.2 provides a constructive argument showing that the moment $E(YZ')$ delivers a quadratic equation in $\mathrm{Var}(\alpha_u U)$ which admits the two roots $\sigma^\dagger_{PC}$ and σ^*_{PC}. Note that it is possible to give primitive conditions in terms of system coefficients and covariances among unobservables for (2) to hold, similar to the condition given for the case of equiconfounded cause and joint responses. We forego this here but we note that, unlike for the case of equiconfounded cause and joint responses, if (2) holds, it is possible for $\sigma^\dagger_{PC}$ to be negative, leading to $\alpha_u^2 E(U^2)$, and thus (γ_o, β_o), to be point identified.

The following plug in estimators are consistent estimators under conditions suitable for the law of large numbers. First, we let $\hat{P}_{z_1} = \frac{1}{n}\hat{\epsilon}'_{z_1.z_2}\mathbf{Z}_1$, then

$$
\begin{aligned}
\hat{\Delta}_{PC} &= [-\hat{\pi}'_{x.z_1|z_2}\hat{\pi}'_{z_1.x} - \hat{\pi}'_{z_1.x_1|x_2} + 1]^2 \\
&\qquad + 4\hat{P}_{z_1}^{-1}\hat{\pi}'_{z_1.x_1|x_2}\left[\frac{1}{n}\hat{\epsilon}'_{y.x}\mathbf{Z}_1 + \left(\frac{1}{n}\hat{\epsilon}'_{x_1.z}\mathbf{X}_2\right)\hat{\pi}'_{z_1.x_2|x_1}\right], \\
\hat{\sigma}^*_{PC} &\equiv (2\hat{P}_{z_1}^{-1}\hat{\pi}'_{z_1.x_1|x_2})^{-1}\left[\hat{\pi}'_{x.z_1|z_2}\hat{\pi}'_{z_1.x} + \hat{\pi}'_{z_1.x_1|x_2} - 1 + \sqrt{\hat{\Delta}_{PC}}\right], \\
\hat{\sigma}^\dagger_{PC} &\equiv (2\hat{P}_{z_1}^{-1}\hat{\pi}'_{z_1.x_1|x_2})^{-1}\left[\hat{\pi}'_{x.z_1|z_2}\hat{\pi}'_{z_1.x} + \hat{\pi}'_{z_1.x_1|x_2} - 1 - \sqrt{\hat{\Delta}_{PC}}\right], \\
\hat{\gamma}^*_{1,PC} &\equiv \hat{\pi}_{x_1.z} - [\hat{\sigma}^*_{PC}, 0]\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1} \quad \text{and} \\
\hat{\gamma}^\dagger_{1,PC} &\equiv \hat{\pi}_{x_1.z} - [\hat{\sigma}^\dagger_{PC}, 0]\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}, \\
\hat{\gamma}^*_{2,PC} &\equiv \hat{\pi}_{x_{21}.z} - \left[\left(\frac{1}{n}\mathbf{X}'_{21}\hat{\epsilon}_{x_1.z}\right)[1 - \hat{\sigma}^*_{PC}\hat{P}_{z_1}^{-1}]^{-1}, 0\right]\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}, \\
\hat{\gamma}^\dagger_{2,PC} &\equiv \hat{\pi}_{x_{21}.z} - \left[\left(\frac{1}{n}\mathbf{X}'_{21}\hat{\epsilon}_{x_1.z}\right)[1 - \hat{\sigma}^\dagger_{PC}\hat{P}_{z_1}^{-1}]^{-1}, 0\right]\left(\frac{1}{n}\mathbf{Z}'\mathbf{Z}\right)^{-1}, \\
\hat{\beta}^*_{PC} &\equiv \hat{\pi}_{y.x} - [\hat{\sigma}^*_{PC}\,(\hat{\pi}'_{x_1.z_1|z_2} - \hat{\sigma}^*_{PC}\hat{P}_{z_1}^{-1} + 1), \frac{1}{n}\hat{\epsilon}'_{x_1.z}\mathbf{X}_2 \\
&\qquad + \hat{\sigma}^*_{PC}\hat{\pi}'_{x_2.z_1|z_2}]\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}, \text{ and} \\
\hat{\beta}^\dagger_{PC} &\equiv \hat{\pi}_{y.x} - [\hat{\sigma}^\dagger_{PC}(\hat{\pi}'_{x_1.z_1|z_2}x - \hat{\sigma}^\dagger_{PC}\hat{P}_{z_1}^{-1} + 1), \frac{1}{n}\hat{\epsilon}'_{x_1.z}\mathbf{X}_2 \\
&\qquad + \hat{\sigma}^\dagger_{PC}\hat{\pi}'_{x_2.z_1|z_2}]\left(\frac{1}{n}\mathbf{X}'\mathbf{X}\right)^{-1}.
\end{aligned}
$$

7 Discussion

Structures $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$, and $\mathcal{S}_4$ do not exhaust the possibilities for identification under equiconfounding. An example of another linear triangular structure with equiconfounding is one involving equiconfounded cause, response, and a *post-response* variable. For example, assuming that KWW score (a potential cause), hourly wage (a response), and the number of hours worked (a post response directly affected by hourly wage but not by the KWW score) are proportionally confounded, with other determinants of wage generally endogenous, may permit identification of this system's structural coefficients.

Roughly speaking, equiconfounding reduces the number of unknowns thereby permitting identification. In contrast, the method of IV supplies additional moments useful for identification. In general, equiconfounding leads to covariance restrictions (see e.g., Chamberlain 1977; Hausman and Taylor 1983) that, along with exclusion restrictions, permit identification. For example, in $\mathcal{S}_4$, the absence of a direct causal effect among X_1 and elements of X_2 and excluding Z_1 from the equation for Y permits identifying $\text{Cov}(\phi_u U, \alpha_u U)$ and $\text{Var}(\alpha_u U)$ given that Z_1, X_1, and Y are equiconfounded. This then permits identifying $\mathcal{S}_4$'s coefficients. Similar arguments apply to $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$. This is conveniently depicted in the causal graphs by (1) a missing arrow between two nodes, one of which is linked to U by a dashed arrow and the other by a solid arrow (e.g., X_1 and X_2 in $\mathcal{S}_4$) and (2) a missing arrow between two nodes that are both linked to U by a dashed arrow (e.g., Z and Y in $\mathcal{S}_4$). Recent papers which make use of alternative assumptions that lead to covariance restrictions useful for identification include Lewbel (2010); Altonji et al. (2011) and Galvao et al. (2012).

As discussed in Sect. 4, the availability of multiple equiconfounded variables can overidentify structural coefficients, leading to tests for equiconfounding. Further, equiconfounding can be used to conduct tests for hypotheses of interest. For example, one could test for endogeneity under equiconfounding without requiring valid exogenous instruments. To illustrate, consider the triangular structure discussed in structure $\mathcal{S}_4$ of Sect. 6 then Theorem 6.1 gives that under equiconfounding β_o is either fully identified or partially identified in $\{\beta^*_{PC}, \beta^\dagger_{PC}\}$. Theorem 6.1 allows for the possibility $\text{Var}(\alpha_u U) = 0$ of zero confounding or exogeneity. Further, if X is exogenous then clearly the regression coefficient $\pi_{y.x}$ also identifies β_o. This over identification provides the foundation for testing the exogeneity of X without requiring the availability of exogenous instruments with dimension at least as large as that of X. Instead, it suffices that a scalar Z_1 and one element X_1 of X are equally (un)affected by U as is Y. For example, in estimating an Engle curve for a particular commodity, total income Z_1 is often used as an instrument for total expenditures X_1 which may be measured with error. Nevertheless, as Lewbel (2010, Sect. 4) notes, "it is possible for reported consumption and income to have common sources of measurement errors" which could invalidate income as an instrument. One possibility for testing the absence of common sources of measurement error is to assume that the consumption Y of the commodity of interest, total expenditures X_1, and income Z_1 are misreported by an equal proportion. In the absence of common sources of measurement error, $\text{Var}(\alpha_u U) = 0$ and one of the equiconfounding estimands should coincide with the regression coefficient $\pi_{y.x}$, providing the foundation for such a test. A statistic for this test can be based on the difference between the regression estimator $\hat{\pi}_{y.x}$ and the equiconfounding estimators $\hat{\beta}^*_{PC}$ and $\hat{\beta}^\dagger_{PC}$ for β_o or alternatively on the estimators $\hat{\sigma}^*_{PC}$ and $\hat{\sigma}^\dagger_{PC}$ for $\text{Var}(\alpha_u U)$. Such a test statistic must account for $\text{Var}(\alpha_u U)$ being possibly partially identified in $\{\sigma^*_{PC}, \sigma^\dagger_{PC}\}$. We do not study formal properties of such tests here but we note the possibility of a test statistic based on $\min\{\hat{\sigma}^*_{PC}, \hat{\sigma}^\dagger_{PC}\}$. A similar test for exogeneity can be constructed in other structures, e.g., $\mathcal{S}_3$.

A key message of this chapter is that, when exogeneity and conditional exogeneity are not plausible, one can proceed to identify structural coefficients and test hypotheses in linear recursive structures by relying on a parsimonious alternative assumption that restricts the shape of confounding, namely equiconfounding. Here, we begin to study identification via restricting the shape of confounding by focusing on equiconfounding in linear structures but there are several potential extensions of interest. One possibility is to maintain the equiconfounding assumption and relax the constant effect structure, e.g., by allowing for random coefficients across individuals. Another possibility is to maintain the constant effect linear assumption and study bounding the structural coefficients under shape restrictions on confounding weaker than equiconfounding. Relaxing the restriction on the shape of confounding could potentially increase the plausibility of this restriction albeit while possibly leading to wider identification sets.

8 Conclusion

This chapter obtains identification of structural coefficients in linear systems of structural equations with endogenous variables under the assumption of equiconfounding. In particular, standard instrumental variables and control variables need not be available in these systems. Instead, we demonstrate an alternative way in which sufficiently specifying the causal relations among unobservables, as Hal White recommends (e.g., Chalak and White 2011; White and Chalak 2010, 2011; White and Lu 2011a,b; Hoderlein et al. 2011), can support identification of causal effects. In particular, we introduce the notion of *equiconfounding*, where one or two observables are equally affected by the unobserved confounder as is the response, and show that, along with exclusion restrictions, equiconfounding permits the identification of all the system's structural coefficients. We distinguish among several cases by the structural role of the equiconfounded variables. We study the cases of equiconfounded (1) predictive proxy and response, (2) joint causes and response, (3) cause and joint responses, and (4) and pre-cause, intermediate-cause, and response. We provide conditions under which we obtain either full identification of structural coefficients or partial identification in a set consisting of two points.

As discussed in Sect. 7, several extensions of this work are of potential interest including characterizing identification under equiconfounding in linear structural systems, developing the asymptotic distributions and properties for the plug-in estimators suggested here, extending the analysis to structures with heterogenous effects, relaxing the restriction on the shape of confounding, developing tests for equiconfounding and for endogeneity, as well as employing these results in empirical applications. We leave pursuing these extensions to future work.

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Appendix A: Mathematical Proofs

Proof of Theorem 3.1 (i) Given that the structural coefficients of S_1 are finite and that $E(U^2)$ and $E(U_x U_x')$ exist and are finite, the following moments exist and are finite:

$$E(XX') = \begin{bmatrix} \phi_u E(U^2)\phi_u' + \alpha_x E(U_x U_x')\alpha_x, & 0 \\ 0, & 1 \end{bmatrix}$$
$$E(ZX') = \alpha_u E(UX') = [\alpha_u E(U^2)\phi_u', 0]$$
$$E(YX') = \beta_o E(XX') + \alpha_u E(UX') = \beta_o E(XX') + [\alpha_u E(U^2)\phi_u', \quad 0].$$

(ii) Substituting for $\alpha_u U$ in (3) with its expression from (1), $\alpha_u U = Z - \alpha_z U_z$, gives

$$Y - Z = \beta_o X - \alpha_z U_z + \alpha_y U_y, \text{ and thus } E[(Y - Z)X'] = \beta_o E(XX').$$

It follows from the nonsingularity of $E(XX')$ that β_o is point identified as

$$\beta_o = \pi_{y-z.x} \equiv E[(Y - Z)X']E(XX')^{-1}.\square$$

Proof of Theorem 4.1 (i) Given that the structural coefficients of $\mathcal{S}_2$ are finite and that $E(U^2)$ and $E(U_x U_x')$ exist and are finite, we have that

$$E(XX') = \begin{bmatrix} \eta_u E(U^2)\eta_u' + \alpha_x E(U_x U_x')\alpha_x', & 0 \\ 0, & 1 \end{bmatrix}, \quad \text{and}$$
$$\begin{aligned} E(YX') &= \beta_o E(XX') + [\,\alpha_u E(UX_1'),\ \alpha_u E(UX_2'),\ \alpha_u E(UX_{31}'),\ \alpha_u E(U)\,] \\ &= \beta_o E(XX') + [\,\alpha_u^2 E(U^2),\ \alpha_u^2 E(U^2),\ \alpha_u E(U^2)\phi_u',\ 0\,] \end{aligned}$$

exist and are finite. (ii) Further, $\alpha_u^2 E(U^2)$ is identified by $\alpha_u^2 E(U^2) = E(X_2 X_1')$ and $\phi_u E(U^2)\alpha_u$ is overidentified by $\phi_u E(U^2)\alpha_u = E(X_{31}X_1') = E(X_{31}X_2')$. Given that $E(XX')$ is nonsingular, it follows that β_o is fully (over)identified by

$$\begin{aligned} \beta_o = \beta_{JC}^* &\equiv \pi_{y.x} - [\,E(X_2X_1'),\ E(X_2X_1'),\ E(X_1X_3')\,]E(XX')^{-1} \\ = \beta_{JC}^\dagger &\equiv \pi_{y.x} - [\,E(X_2X_1'),\ E(X_2X_1'),\ E(X_2X_3')\,]E(XX')^{-1}.\square \end{aligned}$$

Proof of Theorem 5.1 (i) Given that the structural coefficients of $\mathcal{S}_3$ and $E(U^2)$ and $E(U_x U_x')$ exist and are finite we have

$$E(XX') = \begin{bmatrix} \eta_u E(U^2)\eta_u' + \alpha_x E(U_x U_x')\alpha_x', & 0 \\ 0, & 1 \end{bmatrix}, \quad \text{and}$$
$$E(YX') = \beta_o E(XX') + \alpha_u \iota_p [\,E(UX_1'),\ E(UX_2')\,]$$

$$= \beta_o E(XX') + \iota_p [\, \alpha_u^2 E(U^2),\ [\, \alpha_u E(U^2)\phi_u',\ 0\,]\,]$$

exists and are finite.

$(ii.a)$ Given that $E(X_1 X_1')$ and $E(X_2 X_2')$ are nonsingular, we have

$$P_{x_1} \equiv E(\epsilon_{x_1.x_2}\epsilon_{x_1.x_2}') = E(\epsilon_{x_1.x_2} X_1') = E(X_1 X_1') - \pi_{x_1.x_2} E(X_2 X_1') \quad \text{and}$$
$$P_{x_2} \equiv E(\epsilon_{x_2.x_1}\epsilon_{x_2.x_1}') = E(\epsilon_{x_2.x_1} X_2') = E(X_2 X_2') - \pi_{x_2.x_1} E(X_1 X_2')$$

exist and are finite. $(ii.b)$ If also P_{x_1} and P_{x_2} are nonsingular, then $E(XX')^{-1}$ exists, is finite, and is given by (e.g., Baltagi 1999, p. 185):

$$E(XX')^{-1} = \begin{bmatrix} E(X_1X_1'), & E(X_1X_2') \\ E(X_2X_1'), & E(X_2X_2') \end{bmatrix}^{-1} = \begin{bmatrix} P_{x_1}^{-1}, & -\pi_{x_2.x_1}' P_{x_2}^{-1} \\ -\pi_{x_1.x_2}' P_{x_1}^{-1}, & P_{x_2}^{-1} \end{bmatrix},$$

with $P_{x_1}^{-1}\pi_{x_1.x_2} = \pi_{x_2.x_1}' P_{x_2}^{-1}$. It follows that $\pi_{y.x}$ exists and is finite. To show that

$$E(\epsilon_{y_1.x} Y_2') = E(Y_1 Y_2') - E(Y_1 X') E(XX')^{-1} E(XY_2')$$

exists and is finite, note that

$$\begin{aligned} E(YY') &= E[(\beta_o X + \alpha_u \iota_p U + \alpha_y U_y)(\beta_o X + \alpha_u \iota_p U + \alpha_y U_y)'] \\ &= \beta_o E(XX')\beta_o' + \beta_o E(XU)\iota_p' \alpha_u' + \alpha_u \iota_p E(UX')\beta_o' \\ &\quad + \iota_p \iota_p' \alpha_u^2 E(U^2) + \alpha_y E(U_y U_y')\alpha_y'. \end{aligned}$$

Substituting for the diagonal term $E(Y_1 Y_2')$ in the above expression for $E(\epsilon_{y_1.x} Y_2')$ then gives

$$\begin{aligned} E(\epsilon_{y_1.x} Y_2') &= \beta_{1o} E(XX')\beta_{2o}' + \beta_{1o}\alpha_u E(XU) + \alpha_u E(UX')\beta_{2o}' \\ &\quad + \alpha_u^2 E(U^2) - E(Y_1 X')E(XX')^{-1}E(XY_2'), \end{aligned}$$

and thus $E(\epsilon_{y_1.x} Y_2')$ exists and is finite given that $\alpha_u E(UX') = [\alpha_u^2 E(U^2), [\alpha_u E(U^2)\phi_u', 0]]$.

$(ii.c)$ Next, we have that

$$\Delta_{JR} = [2P_{x_1}^{-1} E(X_1 X_1') - 1]^2 - 4P_{x_1}^{-1}[E(X_1 X_2') P_{x_2}^{-1} E(X_2 X_1') + E(\epsilon_{y_1.x} Y_2')],$$

exists and is finite as it is a function of finite moments and coefficients. We now show that Δ_{JR} is nonnegative. Given the nonsingularity of $E(XX')$, substituting for

$$\beta_o = [E(YX') - \alpha_u \iota_p E(UX')]E(XX')^{-1},$$

in the expression for $E(YY')$ gives

$$\begin{aligned}E(YY') &= [E(YX') - \alpha_u \iota_p E(UX')]E(XX')^{-1}E(XX')E(XX')^{-1}[E(XY') \\ &\quad - E(XU')\iota_p'\alpha_u'] + [E(YX') - \alpha_u \iota_p E(UX')]E(XX')^{-1}E(XU)\iota_p'\alpha_u' \\ &\quad + \alpha_u \iota_p E(UX')E(XX')^{-1}[E(XY') - E(XU)\iota_p'\alpha_u'] \\ &\quad + \iota_p\iota_p'\alpha_u^2 E(U^2) + \alpha_y E(U_y U_y')\alpha_y' \\ &= E(YX')E(XX')^{-1}E(XY') - \alpha_u\iota_p E(UX')E(XX')^{-1}E(XU')\iota_p'\alpha_u' \\ &\quad + \iota_p\iota_p'\alpha_u^2 E(U^2) + \alpha_y E(U_y U_y')\alpha_y'.\end{aligned}$$

The off-diagonal term then gives

$$\begin{aligned}E(\epsilon_{y_1.x} Y_2') &= E(Y_1 Y_2') - E(Y_1 X')E(XX')^{-1}E(XY_2') \\ &= \alpha_u^2 E(U^2) - \alpha_u E(UX')E(XX')^{-1}E(XU')\alpha_u'\end{aligned}$$

Substituting for $\alpha_u E(UX') = [\alpha_u^2 E(U^2), [\alpha_u E(U^2)\phi_u', 0]] = [\alpha_u^2 E(U^2), E(X_1 X_2')]$ gives

$$\begin{aligned}&\alpha_u E(UX')E(XX')^{-1}E(XU)\alpha_u' \\ &\quad = [\alpha_u^2 E(U^2), E(X_1X_2')]\begin{bmatrix} P_{x_1}^{-1}, & -\pi_{x_2.x_1}' P_{x_2}^{-1} \\ -\pi_{x_1.x_2}' P_{x_1}^{-1}, & P_{x_2}^{-1}\end{bmatrix}[\alpha_u^2 E(U^2), E(X_1X_2')]' \\ &\quad = \alpha_u^4 E(U^2)^2 P_{x_1}^{-1} - E(X_1X_2')\pi_{x_1.x_2}' P_{x_1}^{-1}\alpha_u^2 E(U^2) \\ &\qquad - \alpha_u^2 E(U^2)\pi_{x_2.x_1}' P_{x_2}^{-1} E(X_2X_1') + E(X_1X_2')P_{x_2}^{-1}E(X_2X_1').\end{aligned}$$

Thus, we expand the term $E(X_1X_2')P_{x_2}^{-1}E(X_2X_1') + E(\epsilon_{y_1.x}Y_2')$ in Δ_{JR} as:

$$\begin{aligned}&E(X_1X_2')P_{x_2}^{-1}E(X_2X_1') + E(\epsilon_{y_1.x}Y_2') \\ &\quad = E(X_1X_2')P_{x_2}^{-1}E(X_2X_1') + \alpha_u^2 E(U^2) - \alpha_u^4 E(U^2)^2 P_{x_1}^{-1} \\ &\qquad + E(X_1X_2')\pi_{x_1.x_2}' P_{x_1}^{-1}\alpha_u^2 E(U^2) + \alpha_u^2 E(U^2)\pi_{x_2.x_1}' P_{x_2}^{-1}E(X_2X_1') \\ &\qquad - E(X_1X_2')P_{x_2}^{-1}E(X_2X_1') \\ &\quad = -\alpha_u^4 E(U^2)^2 P_{x_1}^{-1} + \alpha_u^2 E(U^2)[2P_{x_1}^{-1}\pi_{x_1.x_2}E(X_2X_1') + 1] \\ &\quad = -\alpha_u^4 E(U^2)^2 P_{x_1}^{-1} + \alpha_u^2 E(U^2)[2P_{x_1}^{-1}[E(X_1X_1') - P_{x_1}] + 1] \\ &\quad = -\alpha_u^4 E(U^2)^2 P_{x_1}^{-1} + \alpha_u^2 E(U^2)[2P_{x_1}^{-1}E(X_1X_1') - 1]\end{aligned}$$

where we use $P_{x_1}^{-1}\pi_{x_1.x_2} = \pi_{x_2.x_1}' P_{x_2}^{-1}$ and $P_{x_1} = E(X_1X_1') - \pi_{x_1.x_2}E(X_2X_1')$. Then

$$\begin{aligned}\Delta_{JR} &\equiv [2P_{x_1}^{-1}E(X_1X_1') - 1]^2 - 4P_{x_1}^{-1}[E(X_1X_2')P_{x_2}^{-1}E(X_2X_1') + E(\epsilon_{y_1.x}Y_2')] \\ &= [2P_{x_1}^{-1}E(X_1X_1') - 1]^2 + 4\alpha_u^4 E(U^2)^2 P_{x_1}^{-2} \\ &\quad - 4P_{x_1}^{-1}\alpha_u^2 E(U^2)[2P_{x_1}^{-1}E(X_1X_1') - 1]\end{aligned}$$

$$= \{[2P_{x_1}^{-1}E(X_1X_1') - 1] - 2P_{x_1}^{-1}\alpha_u^2 E(U^2)\}^2 \geq 0.$$

(iii) We begin by showing that

$$\begin{aligned}&\mathrm{Var}(\alpha_{x_1}U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)' \\ &\quad\times [\mathrm{Var}(\phi_u U) + \mathrm{Var}(\alpha_{x_2}U_{x_2})]^{-1}\mathrm{Cov}(\phi_u U, \alpha_u U) - \mathrm{Var}(\alpha_u U)\end{aligned} \tag{A.1}$$

has the same sign as the expression $2P_{x_1}^{-1}E(X_1X_1') - 1 - 2P_{x_1}^{-1}\alpha_u^2 E(U^2)$ from Δ_{JR}. First, clearly, (A.1) can be negative, zero, or positive (e.g., set $\dim(X_{21}) = 1$, $\mathrm{Var}(\alpha_{x_1}U_{x_1}) = 1$, and $\mathrm{Var}(\alpha_{x_2}U_{x_2}) = \mathrm{Var}(\phi_u U) = \frac{1}{2}$. Then (A.1) reduces to $1-\frac{1}{2}\mathrm{Var}(\alpha_u U)$ with sign depending on $\mathrm{Var}(\alpha_u U)$). Next, multiplying this expression by $P_{x_1} \equiv E(\epsilon_{x_1.x_2}\epsilon_{x_1.x_2}')$ preserves its sign and we obtain

$$\begin{aligned}&2E(X_1X_1') - P_{x_1} - 2\alpha_u^2 E(U^2) \\ &= 2E(X_1X_1') - [E(X_1X_1') - E(X_1X_2')E(X_2X_2')^{-1}E(X_2X_1')] - 2\alpha_u^2 E(U^2) \\ &= E(X_1X_1') + E(X_1X_2')E(X_2X_2')^{-1}E(X_2X_1') - 2\alpha_u^2 E(U^2).\end{aligned}$$

But we have

$$\begin{aligned}E(X_1X_1') &= \alpha_u^2 E(U^2) + \alpha_{x_1}E(U_{x_1}U_{x_1}')\alpha_{x_1}' \text{ and} \\ E(X_2X_2') &= \begin{bmatrix} \phi_u E(UU')\phi_u' + \alpha_{x_2}E(U_{x_2}U_{x_2}')\alpha_{x_2}', & 0 \\ 0, & 1\end{bmatrix}.\end{aligned}$$

Then using $[\,\alpha_u E(U^2)\phi_u',\ 0\,] = E(X_1X_2')$ gives

$$\begin{aligned}&E(X_1X_1') + E(X_1X_2')E(X_2X_2')^{-1}E(X_2X_1') - 2\alpha_u^2 E(U^2) \\ &\quad= \alpha_u^2 E(U^2) + \alpha_{x_1}E(U_{x_1}U_{x_1}')\alpha_{x_1}' + \left[\,\alpha_u E(U^2)\phi_u',\ 0\,\right] \\ &\qquad\times \begin{bmatrix} \phi_u E(UU')\phi_u' + \alpha_{x_2}E(U_{x_2}U_{x_2}')\alpha_{x_2}', & 0 \\ 0, & 1\end{bmatrix}^{-1} \begin{bmatrix} \phi_u E(U^2)\alpha_u \\ 0 \end{bmatrix} - 2\alpha_u^2 E(U^2) \\ &\quad= \mathrm{Var}(\alpha_{x_1}U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)'[\mathrm{Var}(\phi_u U) + \mathrm{Var}(\alpha_{x_2}U_{x_2})]^{-1} \\ &\qquad\times \mathrm{Cov}(\phi_u U, \alpha_u U) - \mathrm{Var}(\alpha_u U).\end{aligned}$$

$(iii.a)$ Now, recall from $(ii.c)$ that

$$\Delta_{JR} = \{[2P_{x_1}^{-1}E(X_1X_1') - 1] - 2P_{x_1}^{-1}\alpha_u^2 E(U^2)\}^2.$$

Suppose that (3) is negative, then

$$\begin{aligned}\sqrt{\Delta_{JR}} &= \left|2P_{x_1}^{-1}E(X_1X_1') - 1 - 2P_{x_1}^{-1}\alpha_u^2 E(U^2)\right| \\ &= -2P_{x_1}^{-1}E(X_1X_1') + 1 + 2P_{x_1}^{-1}\alpha_u^2 E(U^2),\end{aligned}$$

and we have

$$\begin{aligned}\sigma_{JR}^{\dagger} &\equiv E(X_1X_1') + \frac{1}{2}P_{x_1}(-1 - \sqrt{\Delta_{JR}}) \\ &= 2E(X_1X_1') - P_{x_1} - \alpha_u^2 E(U^2) \\ &= \mathrm{Var}(\alpha_{x_1}U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)'[\mathrm{Var}(\phi_u U) + \mathrm{Var}(\alpha_{x_2}U_{x_2})]^{-1} \\ &\quad \times \mathrm{Cov}(\phi_u U, \alpha_u U) \\ &\quad < \alpha_u^2 E(U^2) \text{ (and } \geq 0),\end{aligned}$$

and

$$\sigma_{JR}^{*} \equiv E(X_1X_1') + \frac{1}{2}P_{x_1}(-1 + \sqrt{\Delta_{JR}}) = \alpha_u^2 E(U^2).$$

$(iii.b)$ Suppose instead that (A.1) is nonnegative then

$$\begin{aligned}\sqrt{\Delta_{JR}} &= \left|2P_{x_1}^{-1}E(X_1X_1') - 1 - 2P_{x_1}^{-1}\alpha_u^2 E(U^2)\right| \\ &= 2P_{x_1}^{-1}E(X_1X_1') - 1 - 2P_{x_1}^{-1}\alpha_u^2 E(U^2),\end{aligned}$$

and we have

$$\sigma_{JR}^{\dagger} = \alpha_u^2 E(U^2),$$

and

$$\begin{aligned}\sigma_{JR}^{*} &= \mathrm{Var}(\alpha_{x_1}U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)'[\mathrm{Var}(\phi_u U) \\ &\qquad + \mathrm{Var}(\alpha_{x_2}U_{x_2})]^{-1}\mathrm{Cov}(\phi_u U, \alpha_u U) \\ &\geq \alpha_u^2 E(U^2) \geq 0.\end{aligned}$$

Thus, $\alpha_u^2 E(U^2)$ is partially identified in the set $\{\sigma_{JR}^{\dagger}, \sigma_{JR}^{*}\}$. It follows from the moment

$$E(YX') = \beta_o E(XX') + \iota_p[\alpha_u^2 E(U^2), E(X_1X_2')],$$

and the nonsingularity of $E(XX')$ that β_o is partially identified in the set $\{\beta_{JR}^{*}, \beta_{JR}^{\dagger}\}$. □

Proof of Theorem 6.1 (i) We have that

$$E(ZZ') = \begin{bmatrix} \alpha_u^2 E(U^2), & 0 \\ 0, & 1 \end{bmatrix},$$

$$E(XZ') = E\begin{pmatrix} [X_1', X_{21}']'Z' \\ Z' \end{pmatrix} = \begin{bmatrix} \gamma_o E(ZZ') + \left[\eta_u E(U^2)\alpha_u' \ 0\right] \\ [0, \quad 1] \end{bmatrix},$$

$$\begin{aligned} E(XX') &= \begin{bmatrix} \gamma_o E(ZX') + \eta_u E(UX') + \alpha_x E(U_x X'), & E(X) \\ E(X'), & 1 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_o E(ZX') + [[\eta_u E(U^2)\alpha_u', 0]\gamma_o' & \\ \quad +\eta_u E(U^2)\eta_u', 0] + \left[\alpha_x E(U_x U_x)'\alpha_x', 0\right], & \left[0', 1'\right]' \\ \left[0, 1\right], & 1 \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} E(YX') &= \beta_o E(XX') + \alpha_u E(UX') = \beta_o E(XX') \\ &\quad + [[\alpha_u^2 E(U^2), 0]\gamma_{1o}' + \alpha_u^2 E(U^2), [[\alpha_u^2 E(U^2), 0]\gamma_{2o}' + \alpha_u E(U^2)\phi_u', 0]], \\ E(YZ') &= \beta_o E(XZ') + [\alpha_u^2 E(U^2), 0], \end{aligned}$$

Thus, these moments exist and are finite since they are functions of existing finite coefficients and moments.

($ii.a$) Given that $P_{z_1} \equiv E(\epsilon_{z_1.z_2} Z_1') = E(Z_1 Z_1')$ is nonsingular and $Z_2 = 1$, we have that

$$E(ZZ')^{-1} = \begin{bmatrix} P_{z_1}^{-1}, & -\pi_{z_2.z_1}' P_{z_2}^{-1} \\ -\pi_{z_1.z_2}' P_{z_1}^{-1}, & P_{z_2}^{-1} \end{bmatrix} = \begin{bmatrix} E(Z_1 Z_1')^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

is nonsingular and thus $\pi_{x.z}$ and $E(\epsilon_{x_1.z} X_2') = E(X_1 X_2') - \pi_{x_1.z} E(Z X_2')$ exist and are finite. With $E(XX')$ also nonsingular, $\pi_{z.x}$ exists and is finite. Also,

$$\begin{aligned} E(\epsilon_{y.x} Z_1') &= E(Y\epsilon_{z_1.x}') \\ &= \beta_o E(X\epsilon_{z_1.x}') + \alpha_u E(U\epsilon_{z_1.x}') + \alpha_y E(U_y \epsilon_{z_1.x}') \\ &= \alpha_u E(U\epsilon_{z_1.x}'). \end{aligned}$$

Using $E(X_1 X_2') = \gamma_{1o} E(Z X_2') + \alpha_u E(U X_2')$ then gives

$$\begin{aligned} E(\epsilon_{y.x} Z_1') = \alpha_u E(U\epsilon_{z_1.x}') &= \alpha_u E(U Z_1') - \alpha_u E(UX') E(XX')^{-1} E(X Z_1') \\ &= \alpha_u^2 E(U^2) - [[\alpha_u^2 E(U^2), 0]\gamma_{1o}' \\ &\quad + \alpha_u^2 E(U^2), E(X_1 X_2') - \gamma_{1o} E(Z X_2')]\pi_{z_1.x}' \end{aligned}$$

exists and is finite.

$(ii.b)$ We have that Δ_{PC} exists and is finite as it is a function of finite coefficients and moments. Next, we verify that $\Delta_{PC} \geq 0$. We begin by expanding the term $E(\epsilon_{y.x} Z_1')$ in Δ_{PC}. For this, we substitute for γ_{1o} with

$$\gamma_{1o} = \pi_{x_1.z} - [\,\alpha_u^2 E(U^2),\, 0\,] E(ZZ')^{-1},$$

in $-\alpha_u E(UX')\pi_{z.x}'$ which gives

$$\begin{aligned}
&-\alpha_u E(UX')\pi_{z.x}' \\
&\quad= -[[\alpha_u^2 E(U^2), \quad 0]\gamma_{1o}' + \alpha_u^2 E(U^2), \quad E(X_1X_2') - \gamma_{1o} E(ZX_2')]\pi_{z.x}' \\
&\quad= -[\alpha_u^2 E(U^2), \quad 0]\pi_{x_1.z}'\pi_{z.x_1|x_2}' + [\alpha_u^2 E(U^2), \quad 0] E(ZZ')^{-1}[\alpha_u^2 E(U^2), \quad 0]'\pi_{z.x_1|x_2}' \\
&\qquad - \alpha_u^2 E(U^2)\pi_{z.x_1|x_2}' - E(\epsilon_{x_1.z}X_2')\pi_{z.x_2|x_1}' - [\alpha_u^2 E(U^2), \quad 0]\pi_{x_2.z}'\pi_{z.x_2|x_1}' \\
&\quad= -\alpha_u^2 E(U^2)\pi_{x_1.z_1|z_2}'\pi_{z.x_1|x_2}' + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1}\pi_{z.x_1|x_2}' - \alpha_u^2 E(U^2)\pi_{z.x_1|x_2}' \\
&\qquad - E(\epsilon_{x_1.z}X_2')\pi_{z.x_2|x_1}' - \alpha_u^2 E(U^2)\pi_{x_2.z_1|z_2}'\pi_{z.x_2|x_1}',
\end{aligned}$$

where we make use of $[\,\alpha_u^2 E(U^2),\, 0\,] E(ZZ')^{-1} [\,\alpha_u^2 E(U^2),\, 0\,]' = \alpha_u^4 E(U^2)^2 P_{z_1}^{-1}$. Thus,

$$\begin{aligned}
E(\epsilon_{y.x} Z_1') &= \alpha_u^2 E(U^2) - \alpha_u E(UX')\pi_{z_1.x}' \\
&= \alpha_u^2 E(U^2) - \alpha_u^2 E(U^2)\pi_{x_1.z_1|z_2}'\pi_{z_1.x_1|x_2}' + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1}\pi_{z_1.x_1|x_2}' \\
&\quad - \alpha_u^2 E(U^2)\pi_{z_1.x_1|x_2}' - E(\epsilon_{x_1.z}X_2')\pi_{z_1.x_2|x_1}' - \alpha_u^2 E(U^2)\pi_{x_2.z_1|z_2}'\pi_{z_1.x_2|x_1}' \\
&= \alpha_u^2 E(U^2) - \alpha_u^2 E(U^2)\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1}\pi_{z_1.x_1|x_2}' \\
&\quad - \alpha_u^2 E(U^2)\pi_{z_1.x_1|x_2}' - E(\epsilon_{x_1.z}X_2')\pi_{z_1.x_2|x_1}'.
\end{aligned}$$

Then

$$\begin{aligned}
\Delta_{PC} &\equiv [-\pi_{x.z_1|z_2}'\;\pi_{z_1.x}' - \pi_{z_1.x_1|x_2}' + 1]^2 + 4P_{z_1}^{-1}\;\pi_{z_1.x_1|x_2}'[E(\epsilon_{y.x}Z_1') \\
&\qquad + E(\epsilon_{x_1.z}X_2')\;\pi_{z_1.x_2|x_1}'] \\
&= [-\pi_{x.z_1|z_2}'\pi_{z_1.x}' - \pi_{z_1.x_1|x_2}' + 1]^2 \\
&\qquad + 4P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'[\alpha_u^2 E(U^2) - \alpha_u^2 E(U^2)\pi_{x.z_1|z_2}'\pi_{z_1.x}' \\
&\qquad + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1}\pi_{z_1.x_1|x_2}' - \alpha_u^2 E(U^2)\pi_{z_1.x_1|x_2}' \\
&\qquad - E(\epsilon_{x_1.z}X_2')\pi_{z_1.x_2|x_1}' + E(\epsilon_{x_1.z}X_2')\pi_{z_1.x_2|x_1}'] \\
&= \{[\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1] - 2P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2)\}^2 \geq 0.
\end{aligned}$$

(iii) Suppose that

$$\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1 - 2P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2) < 0.$$

Then

$$\begin{aligned}\sqrt{\Delta_{PC}} &= \left|\pi'_{x.z_1|z_2}\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 - 2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}\alpha_u^2 E(U^2)\right| \\ &= -\pi'_{x.z_1|z_2}\pi'_{z_1.x} - \pi'_{z_1.x_1|x_2} + 1 + 2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}\alpha_u^2 E(U^2),\end{aligned}$$

and thus

$$\begin{aligned}\sigma_{PC}^{\dagger} &\equiv \frac{\pi'_{x.z_1|z_2}\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 - \sqrt{\Delta_{PC}}}{2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}} \\ &= \frac{\pi'_{x.z}\pi'_{z.x} + \pi'_{z.x_1|x_2} - 1 - P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}\alpha_u^2 E(U^2)}{P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}} \\ &< \frac{P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}\alpha_u^2 E(U^2)}{P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}} = \alpha_u^2 E(U^2),\end{aligned}$$

and

$$\sigma_{PC}^{*} \equiv \frac{\pi'_{x.z_1|z_2}\pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 + \sqrt{\Delta_{PC}}}{2P_{z_1}^{-1}\pi'_{z_1.x_1|x_2}} = \alpha_u^2 E(U^2).$$

Now, with $E(ZZ')$ nonsingular, we have

$$\begin{aligned}E(X_1 Z') &= \gamma_{1o} E(ZZ') + [\,\alpha_u^2 E(U^2),\ 0\,],\ \text{ or} \\ \gamma_{1o} &= \pi_{x_1.z} - [\,\sigma_{PC}^{*},\ 0\,] E(ZZ')^{-1}.\end{aligned}$$

Further, with $E(XX')$ nonsingular, we have

$$\begin{aligned}E(YX') &= \beta_o E(XX') + \alpha_u E(UX'), \text{or} \\ \beta_o &= \{E(YX') - [[\alpha_u^2 E(U^2), 0]\gamma'_{1o} \\ &\quad + \alpha_u^2 E(U^2),\ E(X_1X_2') - \gamma_{1o}E(ZX_2')]\}E(XX')^{-1}.\end{aligned}$$

Substituting for γ_{1o} gives

$$\begin{aligned}&\left[\,\alpha_u^2 E(U^2),\ 0\,\right]\gamma'_{1o} + \alpha_u^2 E(U^2) \\ &\quad = \left[\,\alpha_u^2 E(U^2),\ 0\,\right]\pi'_{x_1.z} - \left[\,\alpha_u^2 E(U^2),\ 0\,\right]E(ZZ')^{-1}\left[\,\alpha_u^2 E(U^2),\ 0\,\right]' + \alpha_u^2 E(U^2) \\ &\quad = \left[\,\alpha_u^2 E(U^2),\ 0\,\right]\pi'_{x_1.z} - \alpha_u^4 E(U^2)^2\ P_{z_1}^{-1} + \alpha_u^2 E(U^2) \\ &\quad = \alpha_u^2 E(U^2)\ (\pi'_{x_1.z_1|z_2} - \alpha_u^2 E(U^2)\ P_{z_1}^{-1} + 1),\end{aligned}$$

and

$$
\begin{aligned}
&E(X_1X_2') - \gamma_{1o}E(ZX_2') \\
&\quad = E(X_1X_2') - [\pi_{x_1.z} - [\,\alpha_u^2 E(U^2),\, 0\,]E(ZZ')^{-1}]E(ZX_2') \\
&\quad = E(\epsilon_{x_1.z}X_2') + [\,\alpha_u^2 E(U^2),\, 0\,]\pi_{x_2.z}' = E(\epsilon_{x_1.z}X_2') + \alpha_u^2 E(U^2)\,\pi_{x_2.z_1|z_2}',
\end{aligned}
$$

so that

$$
\begin{aligned}
\beta_o = \pi_{y.x} - [\sigma_{PC}^*\,(\pi_{x_1.z_1|z_2}' - \sigma_{PC}^*\,P_{z_1}^{-1} + 1),\, E(\epsilon_{x_1.z}X_2') \\
+ \sigma_{PC}^*\,\pi_{x_2.z_1|z_2}']E(XX')^{-1}.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
E(X_1X_{21}') &= \gamma_{1o}E(ZX_{21}') + \alpha_u E(UX_{21}') \\
&= \gamma_{1o}E(ZX_{21}') + \alpha_u E(UZ')\gamma_{2o}' + \alpha_u E(U^2)\phi_u' \\
&= \gamma_{1o}E(ZX_{21}') + [\,\alpha_u^2 E(U^2),\, 0\,]\gamma_{2o}' + \alpha_u E(U^2)\phi_u' \text{ and} \\
E(X_{21}Z') &= \gamma_{2o}E(ZZ') + [\,\phi_u E(U^2)\alpha_u',\, 0\,].
\end{aligned}
$$

Substituting for

$$
\gamma_{2o} = \pi_{x_{21}.z} - [\,\phi_u E(U^2)\alpha_u',\, 0\,]E(ZZ')^{-1}
$$

in the expression for $E(X_1X_{21}')$ gives

$$
\begin{aligned}
E(X_1X_{21}') &= \gamma_{1o}E(ZX_{21}') + [\,\alpha_u^2 E(U^2),\, 0\,]\pi_{x_{21}.z}' \\
&\quad - [\,\alpha_u^2 E(U^2),\, 0\,]E(ZZ')^{-1}[\,\phi_u E(U^2)\alpha_u',\, 0\,]' + \alpha_u E(U^2)\phi_u' \\
&= \gamma_{1o}E(ZX_{21}') + [\,\alpha_u^2 E(U^2),\, 0\,]\pi_{x_{21}.z}' \\
&\quad - \alpha_u^2 E(U^2)P_{z_1}^{-1}\alpha_u E(U^2)\phi_u' + \alpha_u E(U^2)\phi_u'.
\end{aligned}
$$

Further substituting for γ_{1o} with $[E(X_1Z') - [\,\alpha_u^2 E(U^2),\, 0\,]]E(ZZ')^{-1}$ gives

$$
\begin{aligned}
&E(X_1X_{21}') - [E(X_1Z') - [\,\alpha_u^2 E(U^2),\, 0\,]]E(ZZ')^{-1}E(ZX_{21}') - [\,\alpha_u^2 E(U^2),\, 0\,]\pi_{x_{21}.z}' \\
&\quad = -\alpha_u^2 E(U^2)P_{z_1}^{-1}\alpha_u E(U^2)\phi_u' + \alpha_u E(U^2)\phi_u',
\end{aligned}
$$

or

$$
E(X_1\epsilon_{x_{21}.z}') = -\alpha_u^2 E(U^2)P_{z_1}^{-1}\alpha_u E(U^2)\phi_u' + \alpha_u E(U^2)\phi_u'.
$$

Substituting for

$$
\phi_u E(U^2)\alpha_u' = E(X_{21}\epsilon_{x_1.z}')[1 - \alpha_u^2 E(U^2)P_{z_1}^{-1}]^{-1}
$$

in the expression for γ_{2o} gives

$$\gamma_{2o} = \pi_{x_{21}.z} - [\, \phi_u E(U^2)\alpha_u', \, 0\,] E(ZZ')^{-1}$$
$$= \pi_{x_{21}.z} - [\, E(X_{21}\epsilon_{x_1.z}')[1 - \sigma_{PC}^* P_{z_1}^{-1}]^{-1}, \, 0\,] E(ZZ')^{-1}.$$

$(iii.b)$ Suppose instead that

$$\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1 - 2P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2) \geq 0.$$

Then

$$\sqrt{\Delta_{PC}} = \left|\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1 - 2P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2)\right|$$
$$= \pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1 - 2P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2),$$

and thus

$$\sigma_{PC}^\dagger = \alpha_u^2 E(U^2),$$

and

$$\sigma_{PC}^* = \frac{\pi_{x.z_1|z_2}'\pi_{z_1.x}' + \pi_{z_1.x_1|x_2}' - 1 - P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2)}{P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'}$$
$$\geq \frac{P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'\alpha_u^2 E(U^2)}{P_{z_1}^{-1}\pi_{z_1.x_1|x_2}'} = \alpha_u^2 E(U^2).$$

It follows that

$$\gamma_{1o} = \gamma_1^\dagger \equiv \pi_{x_1.z} - [\, \sigma_{PC}^\dagger, \, 0\,] E(ZZ')^{-1},$$
$$\gamma_{2o} = \gamma_2^\dagger \equiv \pi_{x_2.z} - [\, E(X_{21}\epsilon_{x_1.z}')[1 - \sigma_{PC}^\dagger \, P_{z_1}^{-1}]^{-1}, \, 0\,] E(ZZ')^{-1}, \text{ and}$$
$$\beta_o = \beta^\dagger \equiv \pi_{y.x} - [\sigma_{PC}^\dagger \, (\pi_{x_1.z_1|z_2}' - \sigma_{PC}^\dagger \, P_{z_1}^{-1} + 1), \, E(\epsilon_{x_1.z}X_2')$$
$$+ \sigma_{PC}^\dagger \, \pi_{x_2.z_1|z_2}'] E(XX')^{-1}. \square$$

Appendix B: Constructive Identification

B.1 Equiconfounded Cause and Joint Responses: Constructive Identification

We present an argument to constructively demonstrate how the expression for Δ_{JR} and the identification of $\alpha_u^2 E(U^2)$, and thus β_o, in the proof of Theorem 5.1 obtain. Recall that in $\mathcal{S}_3$

$$E(YX') = \beta_o E(XX') + \iota_p[\,\alpha_u^2 E(U^2),\, [\,\alpha_u E(U^2)\phi_u',\, 0\,]\,].$$

We have that $\alpha_u E(U^2)\phi_u' = E(X_1 X_2')$. It remains to identify $\alpha_u^2 E(U^2)$. For this, recall that the proof of Theorem 5.1 gives

$$\begin{aligned} E(YY') &= E(YX')E(XX')^{-1}E(XY') - \alpha_u \iota_p E(UX')E(XX')^{-1}\alpha_u E(XU)\iota_p' \\ &\quad + \iota_p \iota_p' \alpha_u^2 E(U^2) + \alpha_y E(U_y U_y')\alpha_y', \end{aligned}$$

which we rewrite as

$$\begin{aligned} &\iota_p \iota_p' \alpha_u^2 E(U^2) - \alpha_u \iota_p E(UX')E(XX')^{-1}E(XU)\iota_p' \alpha_u' \\ &\quad - E(\epsilon_{y.x} Y') + \alpha_y E(U_y U_y')\alpha_y' = 0. \end{aligned} \tag{B.1}$$

From the proof of Theorem 5.1, we also have

$$\begin{aligned} &\alpha_u E(UX')E(XX')^{-1}E(XU)\alpha_u' \\ &= \alpha_u^4 E(U^2)^2 P_{x_1}^{-1} - E(X_1 X_2')\pi_{x_1.x_2}' P_{x_1}^{-1}\alpha_u^2 E(U^2) \\ &\quad - \alpha_u^2 E(U^2)\pi_{x_2.x_1}' P_{x_2}^{-1} E(X_2 X_1') + E(X_1 X_2') P_{x_2}^{-1} E(X_2 X_1'). \end{aligned}$$

Thus, collecting the off-diagonal terms in Eq. (B.1) gives:

$$\begin{aligned} &\alpha_u^2 E(U^2) - \alpha_u^4 E(U^2)^2 P_{x_1}^{-1} + E(X_1 X_2')\pi_{x_1.x_2}' P_{x_1}^{-1}\alpha_u^2 E(U^2) \\ &\quad + \alpha_u^2 E(U^2)\pi_{x_2.x_1}' P_{x_2}^{-1} E(X_2 X_1') - E(X_1 X_2') P_{x_2}^{-1} E(X_2 X_1') - E(\epsilon_{y_1.x} Y_2') = 0. \end{aligned}$$

This is a quadratic equation in $\alpha_u^2 E(U^2)$ of the form

$$a\alpha_u^4 E(U^2)^2 + b\alpha_u^2 E(U^2) + c = 0,$$

with

$$\begin{aligned} a &= P_{x_1}^{-1}, \\ b &= -[1 + E(X_1 X_2')\pi_{x_1.x_2}' P_{x_1}^{-1} + \pi_{x_2.x_1}' P_{x_2}^{-1} E(X_2 X_1')] \\ &= -[1 + E(X_1 X_2')\pi_{x_1.x_2}' P_{x_1}^{-1} + P_{x_1}^{-1}\pi_{x_1.x_2} E(X_2 X_1')] \\ &= -[1 + 2P_{x_1}^{-1}\pi_{x_1.x_2} E(X_2 X_1')] \\ &= -[1 + 2P_{x_1}^{-1}[E(X_1 X_1') - P_{x_1}]] = -[2P_{x_1}^{-1} E(X_1 X_1') - 1], \text{ and} \\ c &= E(X_1 X_2') P_{x_2}^{-1} E(X_2 X_1') + E(\epsilon_{y_1.x} Y_2'), \end{aligned}$$

where we make use of $P_{x_1}^{-1}\pi_{x_1.x_2} = \pi_{x_2.x_1}' P_{x_2}^{-1}$ and $P_{x_1} = E(X_1 X_1') - \pi_{x_1.x_2} E(X_2 X_1')$. The discriminant of this quadratic equation gives the expression for $\Delta_{JR} = b^2 - 4ac$. Theorem 5.1 $(ii.c)$ gives that $\Delta_{JR} \geq 0$ and (iii) gives the two roots $\sigma^{\dagger}_{PC}$ and σ^{*}_{PC} of this quadratic equation

$$\begin{aligned}\frac{-b \pm \sqrt{\Delta_{JR}}}{2a} &= \frac{1}{2} P_{x_1} \left\{ 2P_{x_1}^{-1} E(X_1 X_1') - 1 \pm \sqrt{\Delta_{JR}} \right\} \\ &= E(X_1 X_1') + \frac{1}{2} P_{x_1} \left(-1 \pm \sqrt{\Delta_{JR}} \right),\end{aligned}$$

and shows that these are nonnegative. One of these roots identifies $\alpha_u^2 E(U^2)$, depending on the sign of

$$\begin{aligned}&\mathrm{Var}(\alpha_{x_1}' U_{x_1}) + \mathrm{Cov}(\phi_u U, \alpha_u U)' [\mathrm{Var}(\phi_u U) \\ &\quad + \mathrm{Var}(\alpha_{x_2} U_{x_2})]^{-1} \mathrm{Cov}(\phi_u U, \alpha_u U) - \mathrm{Var}(\alpha_u U).\end{aligned}$$

β_o is then identified from the moment $E(YX') = \beta_o E(XX') + \iota_p [\alpha_u^2 E(U^2), E(X_1 X_2')]$.

B.2 Equiconfounding in Triangular Structures: Constructive Identification

We present an argument to constructively demonstrate how the expression for Δ_{PC} and the identification of $\alpha_u^2 E(U^2)$ in the proof of Theorem 6.1 obtain. From the proof of Theorem 6.1, we have that

$$\beta_o = \{E(YX') - \alpha_u E(UX')\} E(XX')^{-1} = \pi_{y.x} - \alpha_u E(UX') E(XX')^{-1}.$$

Substituting for β_o in the expression for $E(YZ')$ gives

$$\begin{aligned}E(YZ') &= \beta_o E(XZ') + [\,\alpha_u^2 E(U^2),\, 0\,], \\ &= \pi_{y.x} E(XZ') - \alpha_u E(UX') E(XX')^{-1} E(XZ') + [\,\alpha_u^2 E(U^2),\, 0\,], \quad \text{or} \\ &\quad - E(\epsilon_{y.x} Z') - \alpha_u E(UX') \pi_{z.x}' + [\,\alpha_u^2 E(U^2),\, 0\,] = 0.\end{aligned}$$

From the proof of Theorem 6.1, we have

$$\begin{aligned}&- \alpha_u E(UX') \pi_{z.x}' \\ &= -\alpha_u^2 E(U^2) \pi_{x_1.z_1|z_2}' \pi_{z.x_1|x_2}' + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1} \pi_{z.x_1|x_2}' - \alpha_u^2 E(U^2) \pi_{z.x_1|x_2}' \\ &\quad - E(\epsilon_{x_1.z} X_2') \pi_{z.x_2|x_1}' - \alpha_u^2 E(U^2) \pi_{x_2.z_1|z_2}' \pi_{z.x_2|x_1}'.\end{aligned}$$

Substituting for $-\alpha_u E(UX') \pi_{z.x}'$ in the above equality then gives

$$
\begin{aligned}
&- E(\epsilon_{y.x} Z') - \alpha_u^2 E(U^2) \pi'_{x_1.z_1|z_2} \pi'_{z.x_1|x_2} + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1} \pi'_{z.x_1|x_2} \\
&- \alpha_u^2 E(U^2) \pi'_{z.x_1|x_2} - E(\epsilon_{x_1.z} X_2') \pi'_{z.x_2|x_1} - \alpha_u^2 E(U^2) \pi'_{x_2.z_1|z_2} \pi'_{z.x_2|x_1} \\
&+ [\, \alpha_u^2 E(U^2),\ 0\,] = 0.
\end{aligned}
$$

Collecting the first elements of this vector equality gives

$$
\begin{aligned}
&- E(\epsilon_{y.x} Z_1') - \alpha_u^2 E(U^2) \pi'_{x_1.z_1|z_2} \pi'_{z_1.x_1|x_2} + \alpha_u^4 E(U^2)^2 P_{z_1}^{-1} \pi'_{z_1.x_1|x_2} \\
&- \alpha_u^2 E(U^2) \pi'_{z_1.x_1|x_2} - E(\epsilon_{x_1.z} X_2') \pi'_{z_1.x_2|x_1} - \alpha_u^2 E(U^2) \pi'_{x_2.z_1|z_2} \pi'_{z_1.x_2|x_1} \\
&+ \alpha_u^2 E(U^2) = 0.
\end{aligned}
$$

This is a quadratic equation in $\alpha_u^2 E(U^2)$ of the from

$$
a \alpha_u^4 E(U^2)^2 + b \alpha_u^2 E(U^2) + c = 0,
$$

with

$$
\begin{aligned}
a &= P_{z_1}^{-1} \pi'_{z_1.x_1|x_2}, \\
b &= -\pi'_{x.z_1|z_2} \pi'_{z_1.x} - \pi'_{z_1.x_1|x_2} + 1, \text{ and} \\
c &= -E(\epsilon_{y.x} Z_1') - E(\epsilon_{x_1.z} X_2') \pi'_{z_1.x_2|x_1}.
\end{aligned}
$$

The discriminant of this equation gives the expression for $\Delta_{PC} = b^2 - 4ac$ in Theorem 6.1 where it is shown that $\Delta_{PC} \geq 0$ and that the solutions to this quadratic equation are $\sigma^{\dagger}_{PC}$ and σ^{*}_{PC}:

$$
\frac{-b \pm \sqrt{\Delta_{PC}}}{2a} = \frac{\pi'_{x.z_1|z_2} \pi'_{z_1.x} + \pi'_{z_1.x_1|x_2} - 1 \pm \sqrt{\Delta_{PC}}}{2 P_{z_1}^{-1} \pi'_{z_1.x_1|x_2}}.
$$

This then enables the identification of (β_o, γ_o) as shown in the proof of Theorem 6.1.

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