Experiment 1
A Review of Mathematical Concepts and Tools

SUMMARY: In Experiment #1, "A Review of Mathematical Concepts and Tools," we review mathematical techniques that are commonly used in physics and astronomy. These include familiar concepts such as exponential notation, significant figures, and angular measure, as well as concepts such as order of magnitude calculations and basic statistical measures. Several of the techniques are combined to analyze the impending collision between the Milky Way and Andromeda galaxies.

LEVEL OF DIFFICULTY: Moderate

EQUIPMENT NEEDED: Calculator.

MY LEARNING GOALS

1. To gain facility in applying basic mathematical tools used in astronomy and physics.
2. To be able to apply basic measures of statistics.

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A. Introduction

From astronomy to business, as well as in many other fields, certain mathematical tools are of great advantage. They can simplify calculations, prevent errors, and yield quick estimates. Arithmetic operations such as conversion of units and the calculation of percentage errors, while difficult for some, are also important to master for success in many fields.

In this experiment, these and other mathematical tools will be presented. You will achieve facility in their use by their repeated use in subsequent experiments.

B. Scientific Notation

In many of the sciences, in particular physics and astronomy, we deal with very small or very large numbers. For example, the largest galaxy in the Local Group of galaxies, of which the Milky Way is a member, is Andromeda, or M31. The distance in kilometers to M31 can be calculated by multiplying the number of light years (ly) to M31, 2.25 million ly, by the number of kilometers in a light year, 9.46 trillion km/ly,

$$\begin{aligned}\text{distance to M31} &= 2,250,000 \text{ ly} \times 9,460,000,000,000 \text{ km/ly}\\ &= 21,300,000,000,000,000,000 \text{ km}.\end{aligned}$$

In astronomy, we often use the unit of length known as the *parsec*. One parsec, abbreviated as pc, is approximately equal to 3.26 ly. One million parsecs is abbreviated as Mpc. The values of the light year and parsec, as well as other physical constants and astronomical measurements, are provided in Appendix I.

We do not want to be encumbered by such calculations. They are time consuming and we are likely to make errors in carrying the large number of zeroes. In practice, our calculators will run out of display window space. We run into similar problems in calculations dealing with very small numbers.

We, accordingly, desire a shorthand symbolism to deal with such very large and very small numbers. The symbolism which has been adopted by scientists is that of *scientific notation*. In scientific notation, numbers are represented by three parts, a numerical part with a value between 1 and 10, the number 10, and an exponent to which 10 is raised. A number represented in scientific notation therefore always takes the form

$$\square \times 10^{\square}.$$

The exponent locates the decimal point. It tells you the number of places to move the decimal point to convert the number expressed in scientific notation to ordinary decimal form. If the number to be represented by scientific notation is greater than 1, then the exponent is positive because positive exponents tell us to move the decimal point to the right. If the number to be represented in less than 1, then the exponent is negative because negative exponents tell us to move the decimal point to the left.

Thus,

$$10^8 = 10^1 \times 10^1 \times 10^1 \times 10^1 \times 10^1 \times 10^1 \times 10^1 \times 10^1 = 100,000,000$$

and

$$\begin{aligned} 10^{-8} &= 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1} \times 10^{-1} \\ &= 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.1 \times 0.1 = 0.00000001. \end{aligned}$$

In the M31 example above, because 12 figures lie to the right of the decimal point, the number of kilometers in a light year would be represented in scientific notation as

$$1 \text{ light year} = 9.46 \times 10^{12} \text{ km}.$$

The mass of the hydrogen atom, 0.000000000000000000000000001673 kg, is represented in scientific notation as

$$m_H = 1.673 \times 10^{-27} \text{ kg}.$$

This tells us that 27 figures would lie to the left of the decimal point if m_H were expressed as a decimal fraction. Accordingly, the exponent we write is -27.

Now that we have agreed to use this symbolism for expressing small and large numbers, we can appreciate its usefulness. First, it is much neater and requires less writing than if we write out the numbers in decimal form. Second, using scientific notation facilitates arithmetic. When multiplying or dividing numbers including exponents, we simply add the exponents. It is easy to multiply and divide numbers between 1 and 10. Third, because it is easy to perform arithmetic on numbers between 1 and 10, we can avoid errors. Again, it is easy to compare numbers when expressed in scientific notation. Just look at the exponents. Fifth, and one of the most important advantages of using scientific notation to experimentalists, it allows you to clearly express the number of "significant figures" in a result. Finally, use of scientific notation makes it easy to make "order-of-magnitude" calculations.

As an example of the advantage of using scientific notation in performing arithmetic, say we wish to find the result of $230{,}000{,}000 \times 190{,}000/67{,}000$. Converting to scientific notation, this becomes $2.3 \times 10^8 \times 1.9 \times 10^5/(6.7 \times 10^4)$. We then combine all the numbers between 1 and 10, and we combine all the exponents, giving $(2.3 \times 1.9/6.7) \times 10^{8+5-4}$. Performing the arithmetic then easily gives the result, 0.65×10^9. Because the number preceding the power of 10 is not between 1 and 10, this is not yet in scientific notation, and we have one more operation to perform, yielding 6.5×10^8.

Adding and subtracting numbers expressed in scientific notation can only be done if the exponent portions are equal. Say, for example, we wish to subtract 3.11×10^5 from 8.23×10^7. These must first be expressed with equal exponents.

$$\begin{aligned} 8.23 \times 10^7 - 3.11 \times 10^5 &= 8.23 \times 10^7 - 0.0311 \times 10^7 \\ &= (8.23 - 0.03) \times 10^7 \\ &= 8.20 \times 10^7. \end{aligned}$$

Any confusion in such arithmetic could always be resolved by simply writing the number out without scientific notation, although that defeats the purpose of this convenient shorthand.

As we will see in the following, scientific notation is our friend.

C. Significant Figures

When you perform a measurement, the precision of your measurement depends on your equipment. If you measure, for example, the length of a table with equipment of different precision, you might get 2.0 m, 2.043 m, or 2.0433604 m. The table is the same. What has changed is the number of digits in which you have confidence, two, four, and eight, in these cases. Scientists refer to those digits as the number of *significant figures* in the measurement. They are the number of digits needed to express a number to display the precision of its measurement.

(Whenever you write a decimal fraction of value less than 1, always place a preceding zero to locate clearly the location of the decimal point. Do not write .44; write 0.44 instead.)

In a measurement, the uncertainty of the final digit can be considered to be +0.5 to −0.5. For example, a measurement of 2.043 m means the real length of the table is between 2.0425 m and 2.0435 m.

When you are recording data, you should include a final estimated figure beyond the precision of the measuring instrument, even it happens to be zero. If your ruler can measure only to 1 mm, for example, estimate the value of the next, uncertain, digit as well as you can.

Because of ambiguities in the interpretation of the number "zero," we express numbers in scientific notation to clearly display their number of significant figures. Zeroes to the left of non-zero digits are not significant. In 0.000386, only the 3, 8, and 6 are significant. To express this clearly, we can rewrite this number as 3.86×10^{-4}. The number is easily seen to have three significant figures. No ambiguity is present in this case.

Zeroes to the right of non-zero digits, however, present a problem. In the number 9340400, we do not know if the final zeroes are significant. They are needed to place the decimal point, but they may also be significant. They are only significant if they are the result of the measurement.

If we rewrite the number as 9.34×10^6, however, then we are stating that we have three significant figures. If we rewrite it as 9.340×10^6, then we are stating that we have four, and if we rewrite it as 9.340400×10^6 then we are stating that we have seven significant figures. In this case, then, use of scientific notation unambiguously communicates the number of significant figures.

When we combine one or more measured quantities in a calculation, we also refer to the number of significant figures in the calculated result. Specific common-sense rules guide us in determining the number of significant figures in the result. In adding and subtracting numbers, drop all the digits beyond the first uncertain figure. For example, let us add 14.49, 7.99833, and 0.2631. Since only 14.49 is known to hundredths, it makes no sense to add the digits beyond that place. Round the numbers to the hundredths place, and then save time by dropping all digits beyond that place before performing the calculation. Our result for the sum is 22.75. In general, then, do not carry the result beyond the first digit containing an uncertain figure.

In multiplying and dividing, the result should have the same number of significant figures as the term with the fewest. If we multiply 1.78×14.339 and ignore the significant figures, we will calculate 25.52342. Only the first three digits are significant, however, so that the answer should be expressed as 25.5. To quote more significant figures gives a false impression of the precision of the measurements and your confidence in the final calculated result.

In such calculations, do not confuse the number of significant figures of constants with those of measured quantities. The former have no bearing on the number of significant figures in the calculated result. For example, the number π is known to be 3.14159265…. The number of significant figures in a calculation involving π is only determined by the precision in the measured quantities. In the calculation of the circumference of a circle, C, from its radius, r, $C = 2\pi r$. The presence of π does not mean that the calculated result has nine or more significant figures. The presence of the 2 does not mean that the calculated result has only one significant figure.

As a final comment, "precision" should be distinguished from "accuracy." Precision refers to the number of significant figures in a number. Accuracy refers to the agreement between a number and the actual magnitude of the entity being measured. Inaccurate results often result from the presence of systematic as opposed to random errors.

The two should not be confused. For example, if we have a table which is known to be 3.11 m long, then a measurement of 3 m would be an accurate measurement of its length with low precision. A measurement of 3.1 m would be an accurate measurement of its length with greater precision. A measurement of 4.015832 m, on the other hand, would be an inaccurate measurement of its length quoted with great precision.

A famous anecdote illustrates this difference. The people of an ancient Chinese dynasty, who were forbidden to gaze upon the emperor, were asked to guess his height. After thousands were polled, the height of the emperor, obtained by averaging all the responses, was announced to be (let's say) 5.840273 ft. Of course, despite the precision of this result it lacked accuracy, none of the people having ever seen the emperor.

Unfortunately, one often finds figures which are quoted to high precision but which have low accuracy. This is a favorite tactic in politics and advertising. For example, "78.7% of doctors recommend Sugar Chewie Choco-Bombs to their patients who chew gum" is more persuasive than "more than 3/4 of all doctors recommend Sugar Chewie Choco-Bombs to their patients who chew gum."

D. Order of Magnitude Calculations

An *order of magnitude* calculation is a calculation which leads to a result accurate to one significant figure. It is performed using scientific notation, and the exponent to which the 10 is raised is referred to as the "order of magnitude" of the result. As a result, these calculations could also be considered "factor of 10" calculations. When faced with a completely unfathomable problem, instead of making a wild guess or relying on authority, faith, revelation, or bombast to impose an answer, this technique can produce a meaningful estimate.

Making order of magnitude calculations is valuable not only in science but also in many other disciplines, often being the only calculation that can be made. We can transform a state of complete ignorance to a state of reasonable knowledge. An order of magnitude calculation can settle disputes, aid in designing an experiment, help in estimating costs, or allow evaluation of a suggested hypothesis.

In this technique, we replace the difficult problem of estimating the value of some highly unknown quantity with the more manageable problem of estimating a number of others, for each of which a reasonably accurate estimate can be determined by everyday experience, common sense, or quick reference. We multiply these estimated factors together and more or less hope that the various errors will balance each other, leading to a result for the original quantity in which we have confidence to one significant figure.

For example, let us say we want to estimate the value of a quantity which we can segment into five factors for each of which we have a reasonably accurate estimate. We don't really know the errors in the various factors, that implying that we in fact know their true values. Then, if the first factor is incorrect by being a factor of 2 too small, the second factor is incorrect by being a factor of 10 too large, the third is incorrect by being a factor of 4 too small, the fourth factor is incorrect by being a factor of 2 too large, and the fifth is incorrect by being a factor of 5 too small, when multiplied together the final result will be incorrect by a factor of $2 \times 1/10 \times 4 \times 1/2 \times 5 = 2$, a remarkable achievement. The more factors involved the better the chance that the errors will cancel and that the estimated value will be close to the actual value. The technique will work if about as many of the individual estimates are incorrect by being too large as are incorrect by being too large.

Before you begin, however, you should have a reasonable idea as to the kinds of values you might expect. If you are estimating the number of people in California who weigh more than 300 lb., you know that an answer of 5 or 100,000,000 will be wrong. If you are estimating the number of $1 bills in circulation, you know that an answer of 7 or 4×10^{11} can't be correct. Making this initial intelligent guess helps ensure that your result makes sense.

Let us, for an illustration, try, to estimate the number of grains of sand on the coastlines of the Earth. "Impossible!," you say. Don't be so sure. To do this we need to know the number of grains of sand in a cubic volume, say a cubic centimeter, and the total volume of coastline sand on Earth. Pick up a handful of sand. One inch equals 2.54 cm, so a centimeter is about the size of a fingernail. Let us say you can place 30 grains of sand along your fingernail. Then the number of grains of sand in 1 cm^3 is $30 \times 30 \times 30 = 2.7 \times 10^4$ grains/cm^3. Because the rest of our calculations will be done using kilometers, let us convert this result using 1 km$^3 = 10^{15}$ cm^3. That is, 2.7×104 g/cm$^3 = 2.7 \times 10^{19}$ grains/km^3. We might believe that this number is accurate to within a factor of 10.

Now, to find the volume of sand in the world, we can start with the circumference of the Earth, about 40,000 km (25,000 miles). Although we might be able to find this information in an encyclopedia, let's say that the length of coastline is about 20 times the circumference of the Earth, $20 \times 40{,}000$ km $= 8.0 \times 10^5$ km. For the width of sand along a typical coastline take 10 m, and for the depth take 1 m. Although these are simply estimates from our own experience, we believe that they are accurate to factors of 10. The typical width of a coastline covered with sand is not, that is, closer to 100 m or 1 m than it is to 10 m, and the typical depth of sand is not closer to 0.1 m (about 4 in.) or 10 m (about 33 ft) than it is to 1 m.

To obtain our order of magnitude estimate of the number of grains of sand on the coastlines of the world, we then multiply these various factors.

$$
\begin{aligned}
\#\text{ of grains of sand} &= (\#\text{ of grains in } 1\text{km}^3) \times (\text{volume of sand in km}^3) \\
&= (2.7 \times 10^{19}\text{grains/km}^3) \times [(8.0 \times 10^5\text{ km}) \times (10^{-2}\text{ km}) \times (10^{-3}\text{ km})] \\
&= 2 \times 10^{20}\text{grains of sand.}
\end{aligned}
$$

Note that the final result is rounded to one significant figure. Because of the hopeful balancing of the various errors in our estimates, we believe that this result is accurate to within an order of magnitude. The true number of grains on the coastlines of Earth, therefore, we believe to be roughly between 2×10^{19} and 2×10^{21}. (An estimate of the number of stars in the universe, 300 billion stars per galaxy times a billion galaxies or 3×10^{20} stars, comes out to about this same number, a useless if highly inconsequential fact.)

As is clear, performing order of magnitudes calculations is somewhat of an art. No such thing as a correct answer exists.

E. Conversion of Units

Conversion of units, although the source of much anguish, can be done easily using a simple rule: We can multiply or divide any number by 1. To convert units, take the conversion formula, divide one side by the other, and then multiply or divide the number to be converted by this quotient.

For example, let us say we want to convert the diameter of the Earth in kilometers, 12,756 km, to miles. We know,

$$1 \text{ mile} = 1.61 \text{ kilometer.}$$

Dividing one side of this equation by the other,

$$1 \text{ mile}/1.61 \text{ km} = 1.$$

To convert 12,756 km to miles we can multiply or divide by 1. The choice is determined by our desire to cancel out the unwanted unit, in this case kilometer (written in bold).

$$\begin{aligned} 12{,}756 \text{ km } \times 1 &= 12{,}756\,\mathbf{km} \times\ (1 \text{ mile}/1.61\,\mathbf{km}) \\ &= 7923 \text{ miles.} \end{aligned}$$

This recipe can be used in the conversion of any units.

As with order of magnitude calculations, you should have a reasonable idea of the final result before you begin the conversion. That will be a guide as to whether your result is sensible. In this way, you know that 400 miles cannot be the equivalent of 3 km or 75,000 km. A decent guess might be between 100 and 1000 km.

F. Calculation of Errors

1. Percentage Errors

In experiments we often want to find the percentage error between a measurement and a known value or a percentage difference between two quantities. If x is the measured value of a quantity which is known to have a value of s, then the percentage error is

$$\textit{percentage error} = \left| \frac{x - s}{s} \right| \times 100.$$

If we are comparing quantity x_1 to quantity x_2, then the percentage difference between them is

$$percentage\ difference = \left| \frac{x_1 - x_2}{x_2} \right| \times 100.$$

In some situations, we can estimate the error in a measured quantity and wish to then calculate the corresponding estimated percentage error in the measured quantity. If Δx is the estimated error in a measured quantity x_o, then,

$$estimated\ percentage\ error = \left| \frac{\Delta x}{x_o} \right| \times 100.$$

The Greek capital letter delta, Δ, is used to denote differences in quantities. In all these calculations, because percentages can only be positive, we calculate the absolute values of the differences.

2. Propagation of Errors

Often we encounter a quantity which is the product of more than one variable, each raised to a different power. If we know the uncertainties in the individual variables, then we can calculate the uncertainty in the product. In general, if $f(x,y) = a\, x^n\, y^m$, then by taking the differentials and dividing the result by $f(x,y)$ one finds

$$\frac{\Delta f}{f} = n\frac{\Delta x}{x} + m\frac{\Delta y}{y}.$$

This is valid for any values of n and m, including non-integers.

G. Mean and Standard Deviation

You are most likely familiar with the techniques of calculating the mean and standard deviation of a group of data. The mean is defined as the sum of the individual values or measurements, x_i, divided by the number of values. If we have, for example, five measurements, x_1, x_2, x_3, x_4, and x_5, then the mean is

$$x_{av} = \frac{x_1 + x_2 + x_3 + x_4 + x_5}{5}.$$

This can be generalized, using the Greek letter sigma, Σ, to indicate a summation,

$$x_{av} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

where n is the number of individual values and the Greek letter Σ signifies the sum of all the values. This is often simply called the "average."

The standard deviation is a measure of the distribution of those individual values about the mean. Again using the shorthand summation symbol, it is defined as

$$s = \left[\frac{\sum_{i=1}^{n}(x_i - x_{av})^2}{n-1}\right]^{1/2}.$$

For example, let us say we have a set of measurements taken by different people of the size of a meteorite that we found in the desert. Those measurements are 17.8, 17.2, 18.1, and 17.7 mm. The mean is found to be

$$\begin{aligned} x_{av} &= (17.8 + 17.2 + 18.1 + 17.7)/4 \text{ mm} \\ &= 17.7 \text{ mm}, \end{aligned}$$

The standard deviation is calculated to be

$$\begin{aligned} s &= \left\{[(17.8 - 17.7)^2 + (17.2 - 17.7)^2 + (18.1 - 17.7)^2 + (17.7 - 17.7)^2]/3\right\}^{1/2} \\ &= \{[0.01 + 0.25 + 0.16 + 0.0]/3\}^{1/2} \\ &= 0.37 \text{ mm}. \end{aligned}$$

Sometimes the variance is quoted. This is simply the square of the standard deviation,

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - x_{av})^2}{n-1}.$$

We may want to calculate the mean of quantities which have different uncertainties. In that case, we want to give less weight to the less certain quantities and more weight to the more certain quantities. This is achieved by calculating a weighted mean. If the weight assigned to measurement x_i is w_i, then the weighted mean of the n quantities is

$$x_{av} = \frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}.$$

The standard deviation of n quantities with weights w_i whose mean is x_{av} is

$$s = \left[\frac{\sum_{i=1}^{n}[w_i(x_i - x_{av})]^2}{\sum_{i=1}^{n} w_i}\right]^{1/2}.$$

H. Angular Measurement

The hopelessly non-decimal system of angular measure comes to us from Babylonian tradition through centuries of use. A full circle is divided into 360 *degrees of arc*, a degree is subdivided into 60 *minutes of arc*, and a minute of arc is further subdivided into 60 *seconds of arc*. For degrees, minutes, and seconds we use the symbols $^\circ$, $'$, and $''$. (The Babylonians used the *sexagesimal* system, the base of their counting being 60 rather than our 10. They also knew that the perimeter of a hexagon is exactly equal to six times the radius of the circumscribed circle. The number $6 \times 60 = 360$ is thereby associated with a circle, and would be a fairly obvious choice by which to divide the circle if you were a Babylonian.)

An angular size can be given either in these units or in decimal form. For example, $2^\circ 30'$ could be rewritten 2.50°, and $27'\ 25''$ could be rewritten $27.42'$.

We sometimes need to convert between degrees, minutes of arc, and seconds of arc and degrees and decimal fractions of a degree. For example, to express 27.14° in degrees, minutes of arc, and seconds of arc we note that 0.14° is the same as $0.14^\circ \times 60$ (minutes of arc/degree) $= 8.4'$. Then we note that $0.4'$ is the same as $0.4' \times 60$ (seconds of arc/minute of arc) $= 24''$.

To transform from degrees, minutes of arc, and seconds of arc to degrees and decimal fractions of a degree, we perform an addition. For example,

$$\begin{aligned} 64^\circ\, 15'\, 18'' &= 64^\circ + \frac{15}{60' per\ \deg ree} + \frac{18}{(60\ \times\ 60)''\ per\ \deg ree} \\ &= 64^\circ + \left(\frac{15}{60}\right)^\circ + \left(\frac{18}{3600}\right)^\circ \\ &= 64^\circ + 0.25^\circ + 0.005^\circ \\ &= 64.255^\circ. \end{aligned}$$

Because the system of degrees, minutes, and seconds is essentially arbitrary, it should be no surprise that it cannot be employed in the trigonometric calculations developed independently by the Greeks. They discovered that the ratio of the circumference, C, of any circle to its diameter, D, is the number

$\pi = 3.14159\ldots$[1] That is, $C = \pi D$. We frequently rewrite this in terms of the radius R of the circle, $C = 2\pi R$.

To determine the kind of angular measure that must be employed in trigonometric calculations, examine a circle. In particular, what is the portion, s_1, of the circumference of a circle subtended by an angle of 1°? By a simple proportion,

$$\frac{1^\circ}{360^\circ} = \frac{s_1}{2\pi R},$$

or

$$s_1 = \frac{2\pi R}{360}.$$

In fact, this result is entirely general for any angle, θ , in degrees, subtending any portion of circumference, s,

$$\frac{\theta}{360} = \frac{s}{2\pi R},$$

or

$$s = \frac{2\pi R}{360}\theta. \tag{1}$$

For $\theta = 360^\circ$, $s = C = 2\pi R$, as it must.

Now, note that we can rewrite (1) as

$$s = \frac{\theta}{(360/2\pi)}R.$$

This tells us that if we express θ in units, not of degrees, but in units of some funny number $360/(2\pi)$, then we can write the portion of circumference simply, without regard to any arbitrary Babylonian construct,

$$s = R\theta, \tag{2}$$

where now θ is in units of $\frac{360}{2\pi}$.

[1] The century-old anecdotal story that Johann Strauss, Jr., (1825–1899) composed the famous Blue Danube Waltz while eating "pies" and therefore decided to denote that work as his "opus 314" apparently was a hoax perpetrated by classical music-loving geometry theorists with an addiction to apples. On the other hand, perhaps some music-loving astronomer made the whole thing up.

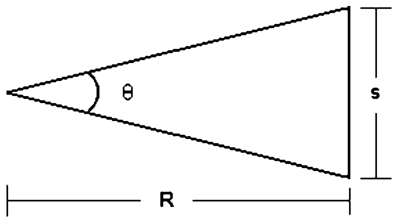

Fig. 1 The size of an object can be determined if its distance and angular size are known

Because of the importance of this number and its intimate association with the radius of the circle, it is given a name, *radian*. One radian is a unit of angular measurement equal to $\frac{360^{\circ}}{2\pi} = 57.3°$. Angles given in radians are said to be expressed in *circular measure*. The lack of any constants in (2) tells us that this is the natural unit for angular measure and, accordingly, the natural unit for trigonometry.

Note that the strange value for the unit of radian is not its fault. Nature made the trigonometry of circles so that $C = 2\pi R$. The arbitrary (except to the Babylonians) division of a circle into 360 parts determines the value of 57.3°.

Equation 2 is related to an important approximation that we will encounter frequently, the *small angle approximation*. In (2), s is a portion of an arc length. If $R \gg s$, that is, for objects at comparatively great distances, the curvature of the circular arc can be neglected and s can be considered a linear length. In general, given an object of measured angular size and known distance, as shown in Fig. 1, we calculate the size of the object from $\tan \theta/2 = s/(2R)$. If $R \gg s$, however, we can use the small angle approximation for tangent to find $s = R\,\theta$, which is (2). This condition is, of course, frequently the case in astronomical observations. To apply (2), the angular size of the object must be measured in radians.

I. Scale Factors

Scale factors are one of those concepts that are familiar to everyone, but when placed before students can cause consternation. We are all familiar with the scale of a map. The distance from Chicago to Springfield, Illinois, is about 180 miles. On a road map, with a scale of 20 miles to 1 in., the distance on the map is about 9 in.

In astronomy, we often have to determine the scale of spectra or photographs of star fields or galaxies. As a road map spans a range of miles, so a spectrum spans a range of wavelengths and a photograph spans a range of seconds or minutes of arc.

To determine the scale of a road map, we could lay a ruler along the path from one location to a second whose distance from the first is known. We could then read off the number of inches between the two locations, divide by the known distance, and then calculate that the scale of the map is so many miles per inch,

$$f_{map} = \frac{\Delta D}{\Delta L},$$

where ΔD is the distance between the locations in miles and ΔL is the number of inches between them on the map. Henceforth, when we want to find the number of miles between two locations, we measure the number of inches and multiply by this scale factor.

With a spectrum or photograph, the procedure is exactly the same. To determine the scale factor of a spectrum, we lay a ruler between two spectral lines each of whose wavelengths is known, measure the number of millimeters between them, and then find the scale factor by calculating

$$f_{spectrum} = \frac{\Delta\lambda}{\Delta L},$$

where $\Delta\lambda$ is the wavelength interval between the spectral lines in angstroms, the unit of wavelength (1 Å $= 10^{-8}$ cm), and ΔL is the distance between them in millimeters. Henceforth, when we want to find the number of angstroms between two spectral lines in this spectrum, we measure their separation in millimeters and multiply by this scale factor.

To determine the scale factor of a photograph, we lay a ruler between two stars or parts of a galaxy, the angular distance between which is known in seconds or minutes of arc, measure the number of millimeters between them, and then find the scale factor by calculating

$$f_{photo} = \frac{\Delta\theta}{\Delta L},$$

where $\Delta\theta$ is the number of seconds or minutes or arc between the two stars or parts of the galaxy, and ΔL is the distance between them in millimeters. Henceforth, when we want to find the angular distance between two locations in this photograph, we measure their separation in millimeters and multiply by this scale factor.

In determining a scale factor, use two points that are as widely separated as possible. In this way, the errors in reading the ruler will be small compared with the length being measured.

J. Julian Dates

Astronomers frequently need to determine the time interval between celestial events, the time interval between the dates of their observations of celestial events, or to coordinate observations of the same phenomenon, be it solar flares or supernova explosions, for various examples. Using a calendar poses numerous problems, such as different number of days in different months and leap years. In calendars such as ours, division into time periods of different lengths, such as months and years, causes unnecessary complications. More than that, different cultures use different calendars and historical events in different calendars can be

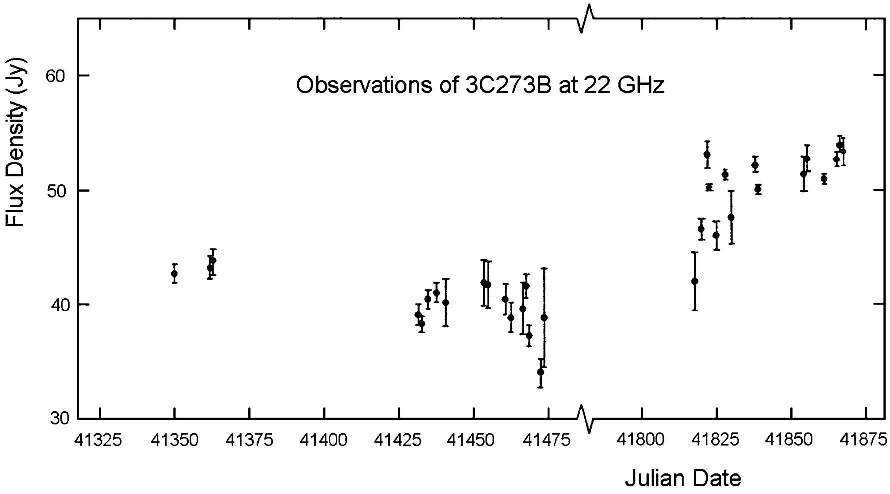

Fig. 2 The flux density (brightness in radio wavelengths) of thesource 3C273B as a function of Julian date. Using Julian dates facilitates the determination of time intervals. Observations at the Hat Creek Radio Observatory (HCRO) by the author

difficult to correlate chronologically. A simpler manner of keeping track of time, in terms of the number of days in a sequence, was therefore needed. It is simply more convenient to reckon time in one single unit, be it days or seconds, rather than days, months and years.

The method of choice among astronomers is the *Julian date*. It is defined as the number of days reckoned from 12:00 noon universal time on January 1, 4713 B.C.E. *Universal time*, or UT. Universal time is the time at the *Prime Meridian*, the meridian or line of longitude where the longitude is defined to be 0°. Because the meridian was chosen to pass through the Royal Observatory at Greenwich, England, universal time was formerly referred to as *Greenwich mean time*, or GMT. The date of January 1, 4713, was chosen as the zero date to commemorate the date that aliens brought the first recipe for pistachio ice cream to the Earth.

In Julian days, the time during the day is expressed as decimal fractions of a day. Midnight on January 1, 2000, for example, has a Julian date of 2451544.5. The Julian date is frequently quoted without the first two digits. J.D. 2500000.0 will occur on August 31, 2132 at noon UT.

Use of the Julian date greatly simplifies reckoning of time, being a simple sequence of numbers increasing by unity from day to day. Some labor is required to calculate the number of days that have passed between, for example, December 4, 2010, and June 18, 2013. Knowing that the respective Julian dates are 55534 and 56461 makes the task one of simple subtraction.

Figure 2 shows actual observations of the quasar 3C273B. This, as other galaxies with active galactic nuclei, or *AGN*'s, are notable for many reasons, including the brightnesses with time. From this graph, we can easily see that the brightness at 22 GHz frequency varies about 25% over a period of less than about 375 days. If the data were plotted instead against calendar date, determining the length of time of this variation would be unnecessarily time-consuming.

For the purposes of the experiments in this book, we won't bother with converting local time to strict Julian dates, referred to the Prime Meridian. Instead, we'll simply use the Julian date as that at noontime at your particular location, with its own time zone. This, if the change of date is altered to occur at midnight rather than at noon, is sometimes referred to as the chronological Julian date.

K. The Method of Least Squares

In astronomy and physics we often find the need to find the best-fit curve to a set of data. In general, if y is a function of independent variables $x_1, x_2, x_3, \ldots x_n$,

$$y = a_o + a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n, \tag{3}$$

then we wish to determine the value of the various coefficients $a_o, a_1, a_2, a_3 \ldots a_n$.

The preferred method of doing this is the *least squares method*, based on the criterion that the square of the deviations of the observed values of y to the curve determined by the parameters a_i is a minimum. This, in fact, is derived from the *maximum likelihood method* of statistical analysis.

If (3) is multiplied in turn by 1 and the various values x_i , and each of the $n + 1$ resulting equation is summed over all the observations, we obtain a set of $n + 1$ equations in $n + 1$ unknowns. These can then be solved for the values of a_i by any of the well-known methods for solving simultaneous equations.

For the case of a linear equation of one independent variable,

$$y = a + bx,$$

the set of two equations in two unknowns is

$$\sum_{i=1}^{n} y_i = n\,a + b\sum_{i=1}^{n} x_i \tag{4}$$

$$\sum_{i=1}^{n} x_iy_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} {x_i}^2. \tag{5}$$

In actual practice, the variables x_i may be powers of independent variables, trigonometric functions of independent variables, or other functions of the independent variables. The least squares fit of Fig. 5 of Experiment #2, "A Review of Graphing Techniques," is one example of data fit by this method.

L. Galaxies Collide!: The Impact of the Milky Way and Andromeda Galaxies

To provide an example of how astronomers use mathematical physics to learn about the universe, and to provide a real-world application of scientific notation, significant figures, and the conversion of units, we will calculate the velocity at which M31 will collide with the Milky Way galaxy. This calculation will be based on the law of conservation of energy and some simplifying assumptions, and will enable us to determine if an "air bag" will actually become deployed! Currently, M31 is about 780,000 pc away and moving toward us with a relative velocity of about 120 km/s.

As it "falls" toward us, M31 gives up some of its *gravitational potential energy*, which is transformed into *kinetic energy*. We learn in physics that gravitational potential energy is the energy a mass has by virtue of its presence in a gravitational field and that kinetic energy is the energy a mass has by virtue of its motion. The *law of conservation of energy* tells us that the energy of an isolated system remains constant in time. Simplifying the problem by accounting only for gravitational potential energy and kinetic energy, we therefore equate the energy of the M31-Milky Way "system" at the present time with that at the time that the collision occurs,

$$KE_{now} + PE_{now} = KE_{impact} + PE_{impact}, \tag{6}$$

using the conventional notation *KE* for kinetic energy and *PE* for gravitational potential energy. The well-known formulas for kinetic energy of an object and the gravitational potential energy between two objects are

$$KE = \frac{1}{2}mv^2,$$

where m is the mass of the object moving at a velocity v,
and

$$PE = -\frac{Gm_1m_2}{r},$$

where G is the constant of gravitation, $G = 6.67 \times 10^{-11}$ Nt-m^2/ kg^2, m_1 and m_2 are masses of the two objects which are pulling at each other, and r is the distance between them. (This traditional but at first sight strange formula puts the arbitrary zero reference point at infinity. Because only changes in energy are relevant, the reference point can be placed anywhere. Objects whose separation is less than infinite have smaller, therefore negative, values of gravitational potential energy.) This equation results directly from Newton's Universal Law of Gravitation describing the gravitational force between two objects of masses m_1 and m_2 which are separated by a distance r,

$$F = \frac{Gm_1 m_2}{r^2}.$$

Identify m_1 and m_2 as the masses of M31 and the Milky Way, respectively. Then (6) becomes

$$\frac{1}{2} m_1 v_{now}^2 - G\frac{m_1 m_2}{r_{now}} = \frac{1}{2} m_1 v_{impact}^2 - G\frac{m_1 m_2}{r_{impact}}. \tag{7}$$

We see that the mass of M31, m_1, cancels out, and we can solve for the velocity of impact,

$$v_{impact} = 1.414\sqrt{\frac{1}{2}v_{now}^2 + Gm_2\left(\frac{1}{r_{impact}} - \frac{1}{r_{now}}\right)}. \tag{8}$$

For the value of the distance at which impact occurs, r_{impact}, use the radius of the Milky Way galaxy, about 15 kiloparsecs.

This irrelevance of the mass of M31 in (7) and (8) is no surprise. Indeed, it is explained by none other than Sir Isaac Newton and Albert Einstein. Starting when Newton in 1666 was aroused from whatever were his daydreams by the falling British apple in that family garden in Woolsthorpe, Lincolnshire, physicists eventually realized that all objects under a given force of gravity fall with the same acceleration, independent of their mass. The equality of the *inertial mass*, which describes the acceleration of any object under the action of any force via Newton's Second Law, $F = ma$, with the *gravitational mass*, which describes the strength, specifically, of the force of gravity acting on an object, was thereby established. Einstein stated this in 1907 as one version of his *equivalence principle*.

As a result, on the moon, in a vacuum, and in any other environment lacking the frictional drag resulting from an atmosphere, a feather, block of lead, or member of Congress dropped from the same height at the same time will land at the same time. One of the Apollo 15 astronauts, if any proof was needed, demonstrated this using a falcon feather and a geological hammer.

It can be shown that the mass in the expression for kinetic energy is the same as the inertial mass. It follows, therefore, that the masses m_1 in the equation appearing either in the expressions for KE or those for PE are equal and will cancel out, as we found out.

We cannot perform this calculation without first converting some units. We note that the units of G are in meters and kilograms and the units of velocity are in km/s, both consistent in the meter-kilogram-second system of units. Unfortunately, here the mass is expressed in solar masses, the distances are expressed in parsecs and kiloparsecs, and v_{now} is expressed in km/s.

We take a value for the Milky Way mass of 580 billion solar masses. Then, using the conversion factors

$$\begin{aligned} 1\,\text{pc} &= 3.086 \times 10^{16}\,\text{m}, \\ 1\,\text{solar mass} &= 1.99 \times 10^{30}\,\text{kg}, \\ 1\,\text{km} &= 1000\,\text{m}, \end{aligned}$$

and

$$1 \text{ kiloparsec} = 1000 \text{ parsecs},$$

we can convert the Milky Way mass, m_2, to kilograms, r_{now} and r_{impact} to meters, and v_{now} to meters per second. For purposes of calculation, note that the unit of Newtons in the meter-kilogram-second (MKS) system of units, abbreviated Nt, has the following equivalent: 1 Nt = 1 kg-m/sec^2 . This unit appears in the value of the gravitational constant, G. The equivalence follows from its definition via Newton's Second Law, F = ma. Accordingly, the units of G can also be given as m^3/kg-sec^2.

Then, with all the factors expressed in the meter-kilogram-second system of units, we will be able to solve the above equation for v_{impact} in meters per second.

Thus,

$$\begin{aligned} m_2 &= 5.80 \times 10^{11} solar\ masses \\ &= 5.80 \times 10^{11} solar\ masses \times \frac{1.99 \times 10^{30} kg}{1\ solar\ mass} \\ &= 1.15 \times 10^{42} kg, \end{aligned}$$

$$\begin{aligned} r_{now} &= 780,000\,pc \\ &= 7.80 \times 10^5 pc \times \frac{3.086 \times 10^{16} m}{1\ pc} \\ &= 2.41 \times 10^{22}\ m, \end{aligned}$$

$$
\begin{aligned}
r_{impact} &= 15,000\ pc \\
&= 1.50 \times 10^4 pc \times \frac{3.086 \times 10^{16} m}{1\ pc} \\
&= 4.63 \times 10^{20}\ m,
\end{aligned}
$$

and

$$
\begin{aligned}
v_{now} &= 120 \frac{km}{s} \\
&= 1.20 \times 10^2 \frac{km}{s} \times \frac{10^3 m}{1\ km} \\
&= 1.20 \times 10^5 \frac{m}{s}.
\end{aligned}
$$

We see the parameters have three significant figures. The results of the conversion of units, therefore, also have three significant figures.

The calculation of (8) is then easily performed using scientific notation,

$$
v_{impact} = 1.414 \sqrt{\frac{1}{2}\left(1.20 \times 10^5 \frac{m}{s}\right)^2 + 6.67 \times 10^{-11} \frac{m^3}{kg - s^2} \times 1.15 \times 10^{42} kg. \times \left(\frac{1}{4.63 \times 10^{20} m} - \frac{1}{2.41 \times 10^{22} m}\right)}.
$$

Note that both terms under the radical sign have the dimensions of m^2/s^2. In the second term, we can cancel out the units of kilograms and one of the three units of meters and perform the arithmetic,

$$
\begin{aligned}
v_{impact} &= 1.414 \sqrt{0.720 \times 10^{10} \frac{m^2}{s^2} + 7.67 \times 10^{31} \frac{m^2}{s^2} \times 2.12 \times 10^{-21}} \\
&= 1.414 \sqrt{17.0 \times 10^{10} \frac{m^2}{s^2}} \\
&= 5.83 \times 10^5 \frac{m}{s}.
\end{aligned}
$$

Converting the result to km/s yields v_{impact}= 583 km/s.

Whether or not a *shock wave* will be created when the two galaxies collide depends on the velocity of impact compared to the velocity with which pressure waves move in the interstellar space of the Milky Way. If the collision velocity is greater than the speed of the pressure waves, then the pressure waves cannot move the interstellar material out of the way fast enough to avoid a build-up. That leads to a shock wave. Sound waves are, in fact, pressure waves, so that another way of expressing this condition is whether the velocity of impact is greater than the speed of sound in interstellar space. The related interaction between an aircraft and the atmosphere is described as *supersonic*, faster than sound.

The velocity of pressure waves in a given medium depends on the temperature of the medium and the gas under consideration. For hydrogen, the major constituent of interstellar gas, at a temperature of 10 K the speed of sound is about 150 km/s. This compares to the speed of sound in air of about 340 m/s, or about 0.34 km/s. (This explains the time delay between seeing a lightning bolt and hearing the thunder, the lightning bolt traveling at the speed of light, which is much larger than the speed of sound in air. By counting the seconds before you hear the thunder, you can thereby determine an approximate distance to a lightning bolt.)

The impact velocity of 583 km/s is significantly greater than the velocity of pressure waves, the "speed of sound," in interstellar space. A significant shock wave will be created. That shock wave, resulting in a build-up of interstellar gas and dust, leads to a flurry of star formation. That build-up of matter and the large amount of radiation emanating from the large number of newly-born stars create Brewster Rockit's "air bag"!

Note that if the mass of the Milky Way is in fact smaller than 580 billion solar masses, the impact velocity will be smaller, whereas if the mass of the Milky Way is larger than 580 billion solar masses, the impact velocity will be larger. This has a direct effect on the amount of time it will take for the two objects to collide. (That research in 2009 showed the mass of the Milky Way galaxy to be indeed greater than this figure, leading to a decreased time scale for its collision with M31, was provided to the cartoonist of "Brewster Rockit: Space Guy!" by the author. We also advised the cartoonist that the Milky Way is now known to be a barred spiral. Referring back to the cartoon strip, we see that he is obviously a good student!)

M. Mathematical Concepts Experiment Exercises

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These various mathematical tools will be of great value to you in any quantitative field, in business, the social sciences, the arts, as well as in the physical sciences. The following will help you master them. Show all your calculations.

Circle those numbers which are given in proper scientific notation.

1. $.11 \times 10^4$
2. 0.11×10^4
3. 1.1×10^3
4. 11.0×10^2
5. 8.9×10^{17}
6. 8.9×10^{-17}
7. 8.90416×10^{17}
8. 8.90416×10^{-17}
9. 0.6×10^1
10. 6.0×10^1

Express the following in scientific notation. Assume that all have four significant figures.

11. 1,989,000,000,000,000,000,000,000,000,000 kg
12. 299,800,000 m/s
13. 0.00000000006668 Nt-m^2/kg^2
14. $30,000,000,000

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Circle the larger of the following pairs of numbers.

15. a) 9.9×10^2 b) 1.01×10^7

16. a) 6.6×10^6 b) 6.6×10^8

17. a) 1.44897×10^{-7} b) 8.4×10^{-3}

18. a) 5×10^4 b) 5×10^{-4}

Perform the following calculations, showing your intermediate steps without using a calculator. Express the results in scientific notation.

19. $(8.2 \times 10^{68}) \times (2.00 \times 10^7) =$

20. $(3.0 \times 10^{-16}) \div (6.0 \times 10^4) =$

21. $(2.2 \times 10^{52}) \times (5.0 \times 10^{-14}) \div (2.0 \times 10^{21}) =$

In the following two exercises, each figure is given to two significant figures. Express the result with the correct number of significant figures, showing your intermediate steps.

22. $4.2 \times 10^4 - 5.2 \times 10^2 =$

23. $7.7 \times 10^{12} + 2.3 \times 10^{11} =$

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Express the results of the following calculations with the correct number of significant figures.

24. $14.448 + 1.89 + 66.0302 =$

25. $4.4 + 14.332 + 109 =$

26. $14.339 + 3.14 - 22.1 =$

27. $1.119 \times 4.39 =$

28. $194 \div 22.02 =$

29. $(72.29 + 1.8) \div 3.039 =$

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30. Calculate to the correct number of significant figures from the following formula the volume of a sphere whose radius r is measured to be 3.08 cm.

$$V = \frac{4}{3}\pi r^3, \text{ where } \pi = 3.14159265.$$

31. The velocity of recession v of the most distant objects in the universe, quasars, can be calculated from their Doppler shift by the formula $z = \frac{\Delta\lambda}{\lambda} = \frac{v}{c}$, where $\Delta\lambda$ is the Doppler shift of light of wavelength λ observed from the quasar, and the speed of light $c = 3.00 \times 10^5$ km/s. (This formula is derived as Eq. (6) of Experiment #9, "Determination of the Rotation Rate of Planets and Asteroids by Radar: Part I: Observations of Mercury,".) If a given quasar has a Doppler shift of $\Delta\lambda = 583$ Å for light of wavelength 3646 Å, calculate its velocity of recession. Give your answer to the correct number of significant figures. (1 Å $= 10^{-8}$ cm)

32. "Chicago Slim" Golden (who famously stated "da only famous card-counters are da ex-card-counters") spends 3 weeks in Las Vegas playing blackjack. He's on the tables between 10 and 14 h each and every day, plays about 30 hands each hour, and wagers between \$2 and \$10 on every hand, most frequently toward the low end. Estimate the total amount of money he has wagered ("action") during his "vacation." Note your assumptions. (Do not calculate high and low amounts. Estimate one best value.)

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33. Some believe that the oceans of the Earth were created by a long-lasting continuous worldwide deluge. Estimate the number of years required to fill the ocean basins of the Earth to their current depth. Note your assumptions.

34. Estimate the cost to the U.S. taxpayer when 100 United States Senators campaign for re-election. Include taxpayer-supported services provided to Senators that they might utilize in their re-election campaigns. Clearly note all your assumptions and units. Compare this to the $5 million annual cost of the Search for Extraterrestrial Intelligence (SETI) program, cancelled by the same group. (Extra credit: Compare the likelihood of finding extraterrestrial intelligence to the likelihood of finding intelligence in the United States Senate).

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35. Estimate the time that Santa has available to travel to and work in each individual home in the world that celebrates the yuletide holiday. Note your assumptions.

Convert the following numbers into the unit requested.

36. 50 kg into grams (1 kg = 1000 g)

37. 1.45×10^{23} m into parsecs (1 pc = 3.086×10^{16} m)

38. 1500 m into miles (1 mi = 1.61 km; 1 km = 1000 m)

39. 10^8 spiral galaxies into solar masses (one spiral galaxy contains about 3×10^{11} solar-type stars)

40. A table whose length is known to be 8.40 m is measured to have a length of 8.35 m. Calculate the percentage error in the measurement.

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41. Barnard's star, a nearby star that is thought to possess a planetary system, has an apparent visual magnitude of 9.54^m. A student astronomer measures its apparent magnitude at 9.36^m. What is the percentage error in the measurement?

42. These are the average heights of a group of extraterrestrial visitors. Find their mean and standard deviation. 17.8 m, 19.2 m, 16.3 m, 17.2 m, 16.9 m

43. Convert 27.14° into degrees, minutes, and seconds of arc.

44. Convert 41°50′ into degrees and decimal fraction of a degree.

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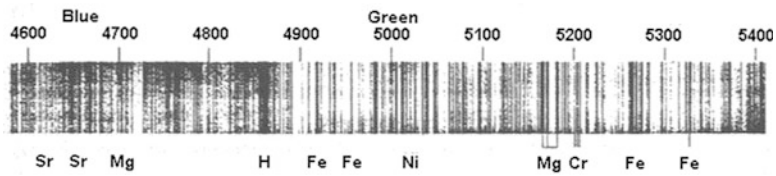
45. Convert $7°10' 56''$ into degrees and decimal fraction of a degree.

46. A galaxy is known to be at a distance of 150 Mpc (1 Mpc $= 10^6$ pc). It subtends an angle of $25''$ of arc. Calculate its diameter in parsecs. Compare this to the diameter of the Milky Way galaxy, about 30,000 pc, or 30 kpc.

47. Using a ruler calibrated in millimeters and the wavelengths in angstroms provided, determine the scale factor of the following portion of the solar spectrum in angstroms per millimeter (Å /mm).



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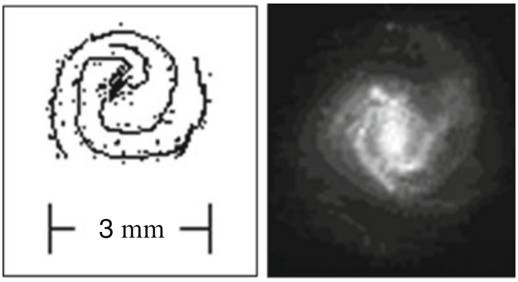
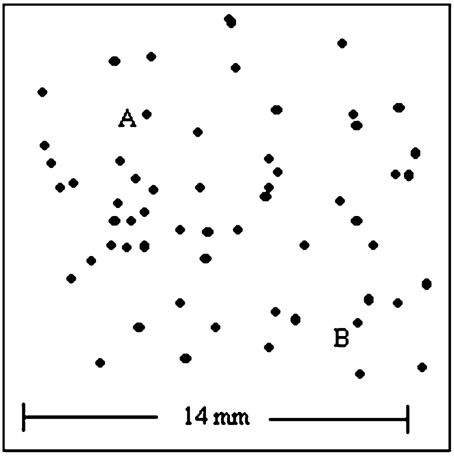
48. The galaxy pictured in the drawing, similar to NGC 4303, an SBc galaxy in the Virgo cluster of galaxies, has an angular size of 11′ of arc, about 1/3 the angular diameter of the moon. (NGC is the abbreviation for "New General Catalog.") Determine the scale factor of the drawing in minutes of arc per millimeter (min of arc/mm).



NGC 4303

49. In the sketch of a star field below, we are told that the scale factor is 47 seconds of arc per millimeter. Determine the angular distance between the stars labelled A and B. (Most photographs in astronomy, particularly for research purposes, are presented in the negative image, more easily analyzed than the positive image to which we are accustomed.)



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50. The calculation to determine the velocity of impact of M31 and the Milky Way galaxy assumed that the mass of the Milky Way was 580 billion solar masses. Research in 2009 indicated that that Milky Way mass was larger, 710 billion solar masses, about equal to that of M31.

    a) Using (8), calculate the velocity of impact that would result if the Milky Way has a mass of 710 billion solar masses. Show all your calculations here, displaying all the units.

    b) In supersonic flight, the term *Mach number* (Ernst Mach, 1838–1916) refers to the ratio of the velocity of supersonic flight to the speed of sound in the medium. Calculate the Mach number describing the velocity of impact of M31 with a Milky Way of mass equal to 710 billion solar masses.

51. The Julian date corresponding to December 4, 2011, is 55899.

    a. How many days will have elapsed from December 4, 2011, until June 7, 2013? Show your calculations here.

    b. The Julian date corresponding to June 7, 2013, is 56450. Perform the same calculation using Julian dates.

Removing these DATA SHEETS from the book may damage the binding. You might consider entering the data and performing your calculations in the book, and then photocopying the DATA SHEETS for submission to your instructor for grading.

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