Chapter 2
Mechanics of Materials

2.1 Introduction

This text is concerned with the mechanics of solids. Mechanics is the study of the motion of bodies. A solid is defined to be an object that retains its shape when it is unloaded and unconfined. A body that deforms without being loaded is called a fluid. Necessarily, fluids do not undergo fracture, as they are for all intents and purposes fractured by the absence (or paucity) of molecular bonds. Solids, on the other hand, can and do undergo fracture when subjected to loading conditions sufficient to induce and/or propagate cracks. Sometimes these cracks can accumulate and/or grow in such a way as to cause the object to fail in the sense that it no longer is capable of performing its intended function. ***This then, is a major objective of this text—to develop models that can be used to design solid objects that are capable of withstanding fracture (as well as other modes of failure) when subjected to mechanical loading.***

In the previous chapter, we learned that people have been studying mechanics for at least several millennia, both as it pertains to the motions of the heavenly bodies and as it pertains to building construction. A body deforms when the distance between any two points in the body changes. Commencing with Galileo Galilei's text *Two New Sciences,* significant effort has been devoted by scientists and engineers to the development of models for predicting the response of deformable bodies (Timoshenko 1953).

Solid objects utilized for structural purposes can fail due to a variety of causes such as excessive deformations, buckling, excessive cost, and fracture. While all of these need to be considered in the design process, the failure mechanism that will be considered in detail in this text is fracture. Failure of structures due to fracture is by no means a new subject. Indeed, of the seven ancient wonders of the world, only one of these wonders remains more or less intact today—the great pyramids. The others failed long ago, usually due to fracture.

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Fig. 2.1 Photo of Stonehenge on the Salisbury Plain in Southern England

Fig. 2.2 Street scene in Pompeii

Early engineers undoubtedly had some rudimentary understanding of the cause of fracture, as evidenced by the beam–arch–column structures shown in Figs. 2.1, 2.2, and 2.3.

Stonehenge, on the Salisbury plain in southern England, is apparently a very old site, dating back possibly as far as 8,000 BC. Carbon dating of wooden fragments

Fig. 2.3 Photo of the Canopus at Hadrian's Villa

found inside the dirt berm suggests that the structure may have originally been wooden. It is likely that early engineers realized that in order to build structures that would stand the test of time, a more durable material was needed—stone. And indeed, almost all structures remaining today from antiquity are stone. Unfortunately, as demonstrated by the large horizontal stones in Fig. 2.1, the low tensile strength of stone limits its use as a beam of substantial length.

Excavations at Pompeii have revealed roofless structures virtually everywhere within the city destroyed by the eruption of Vesuvius in 79 AD, as shown in Fig. 2.2. As stone construction was extremely expensive even then, most structures used wooden beams for roof structures in Pompeii.

The Romans invented the arch in order to provide larger spans for stone structures, and this invention allowed the Romans to create many of the most famous structures still standing today from that time period. A telling example is the portico from the Canopus at Hadrian's villa in Tivoli, built in the second century AD (Fig. 2.3). The portico has both flat and curved stone members between the arches, and the discerning reader will recognize that the span between the arches is slightly larger than that between the beams, attesting to the fact that arches can span larger dimensions than beams made of stone because they carry loads strictly in compression, whereas beams necessarily undergo tensile loading on one side or the other, a circumstance that precludes the use of stone for large spans.

The Romans expanded this understanding of compression to build perhaps their most amazing structure—a dome—in the second century, again during the reign of Hadrian. The Pantheon, the last completely intact Roman structure in Rome, stands today as a monument to the ingenuity of the Romans, as shown in Fig. 2.4.

It is not known today exactly who undertook the reconstruction of the old temple built by Augustus Caesar's close friend Marcus Agrippa, but what is certain is that it

Fig. 2.4 Photos of the Pantheon, exterior photo on *left*, and interior photo on *right* showing the oculus

Fig. 2.5 Photo of the Pont du Gard (note people standing on the lower deck)

was revolutionary. The dome is made of concrete, a technology that was lost after the fall of the Roman Empire in the fifth century until the nineteenth century, when the French reinvented concrete technology. A careful study of this structure will lend credence to the enormous impact that the Romans had on western civilization.

Despite their proven ability to construct both massive and impressive structures, ancient engineers did not possess rigorous design methodologies. Theirs was an experimental and necessarily expensive discipline. For example, it is known that the Pont du Gard, built in the first century AD (Fig. 2.5), was constructed at a cost that would have bankrupted a small nation today. It is noteworthy that this massive aqueduct in the south of France still stands today, so that the cost may not sound so impressive if amortized over two millennia.

2.2 Modern Models

The models we use today to design structures are robust, in the sense that essentially all of the controllable inputs can be manipulated analytically (meaning—*without actually having to build the structure!)* in order to produce an acceptable design. More importantly, today's models have been shown repeatedly through careful experimentation to be accurate. All of the models that will be developed in this text are built on three important but distinct types of variables: ***independent variables, input variables, and output variables.***

Independent variables are composed of time and spatial coordinates. In this course it will be assumed that structural response is time independent, so that the first of these independent variables will not appear in our models. Spatial coordinates will normally appear via an assigned coordinate system, such as Cartesian coordinates (x, y, z), after the French mathematician Descartes. It should be pointed out that choosing both the origin and the orientation of a coordinate system is completely arbitrary and is therefore at the discretion of the modeler. However, in this course the following convention will be employed throughout:

1. ***The coordinate origin will always be placed at the left end of the long axis of the object.***
2. ***The coordinate system will always be right handed.***
3. ***The y coordinate direction will (almost) always be placed normal to the x-axis and in the plane of the page.***

Input variables in all mechanics problems are of three distinct types: ***loads, geometry, and material properties.*** These compose the complete set of necessary information that must be known before a structural analysis can be carried out. For example, the shape of the structure must be known a priori, before the analysis can be performed, and this is termed the geometry of the structure.

Output variables are the set of items that result from the model development. They generally consist of kinetic (such as stress, to be defined below) and kinematic quantities (such as displacements). Once a cogent model has been constructed, the output variables will appear as explicit functions of the independent variables and the input variables. Thus, for example, in a beam, the displacement field will be modeled as a function of the input loads, the material properties of the beam, and the shape of the beam.

As a consequence of the continuous nature of structural components, the resulting models in this course will employ differential calculus, so that the models will be at least in part in the form of differential equations (Malvern 1969; Glover and Jones 1992). Thus, it will be necessary to solve these equations for specific sets of loads, geometry, and material properties so as to describe them via simpler algebraic equations. This type of robust model can then be inverted in such a way that the input loads, geometry, and material properties can be ***designed*** so as to create a structure that will satisfy any and all design constraints. The subject of structural design will be addressed in Chap. 7.

In this chapter, we develop (and in some cases review) the fundamental mechanics that are required in order to develop models capable of predicting the response of solids to mechanical loadings. These fundamentals fall into three general classes: kinetics, kinematics, and constitution. Insofar as they relate to the current subject matter, these are discussed in some detail below.

2.2.1 Kinetics

Kinetics is the study of mechanical loads acting on objects. There are two fundamentally different ways that mechanical loadings can be imparted to bodies: via body forces (expressed in force per unit volume) or via surface tractions (expressed in force per unit area). For example, when a person sits still in a chair, the mass of the person results in a force per unit volume that acts on every mass point (in the interior of the person) in the direction of the center of mass of the Earth. This force, F, is called a gravitational force, and it is directly proportional to the mass, m_1, of the person, the mass, m_2, of the earth, and inversely proportional to the square of the distance between the mass point and the center of gravity of the earth, r. The law describing ***body forces*** was first espoused rigorously by Isaac Newton in his book *The Principia*, and for that reason it is called Newton's gravitational law, given by

$$F = G\frac{m_1 m_2}{r^2} \tag{2.1}$$

where F is the magnitude of the force, G is the gravitational constant, and the direction of the force is through the straight-line connecting the mass point in question to the center of mass of the earth.

Unlike body forces***, surface tractions*** (expressed in units of force per unit area and to be defined below) act on the surface of objects when they come in contact with other objects. From Newton's third law, we know that two objects that are in contact with one another exert equal and opposite forces on one another. It has also been proven (by Augustin Cauchy) that two objects in contact with one another exert equal and opposite surface tractions on one another, meaning that not only are the forces identical, but the distribution of those forces is also identical between the two bodies in contact.

When a person sits in a chair, the surface tractions act on the part of the person's body that comes in contact with the chair, and those acting on the chair are of identical magnitude and opposite sign to those acting on the person. So why doesn't the person in the chair fly upwards due to the force being applied to him or her by the chair? The answer is that the resultant of the surface tractions upwards (caused by the chair pushing on the person) is exactly equilibrated by the resultant of the body force downwards (equal to the weight of the person), and this is described by Newton's first law (sometimes called conservation of momentum):

$$\sum \vec{F} = 0, \; \sum \vec{M} = 0 \tag{2.2}$$

If the body is in motion, then the right hand side of the above two equations is not zero, but is expressed as the rate of change of the momentum, called Newton's second law. In this course, we will consider only objects that are in equilibrium, so that (2.2) is sufficient to guarantee that momentum is conserved. The above laws describe the kinetics of all bodies in the universe that are at rest.

There are several other conservation laws that have been espoused over the last three centuries. These include: (1) conservation of mass, (2) conservation of energy, (3) conservation of charge, and (4) the entropy production inequality. These laws are for most practical circumstances demonstrated to be true everywhere on our planet. Therefore, the discerning reader may ask why these laws are not utilized in the current text. The answer is that they are not needed because in all of the circumstances that will be considered herein they are all trivially satisfied, meaning that they provide no additional useful information, while still being true. Therefore, they will not be considered further herein.

The concept of load intensity has long been recognized as a means of estimating the load carrying ability of a solid. A rough approximation of load intensity can be constructed by dividing the total load acting on a surface by the area of the surface. For example, the load intensity of the great pyramid of Cheops is equal to the total force caused by the mass of the pyramid in earth's gravitational field divided by the footprint of the base of the pyramid. This force is equal to the mass of the pyramid multiplied by Earth's gravitational constant, g. A further example of this concept can be seen in Fig. 2.6, wherein a truncated pyramid of weight (force), W, is shown resting on a plane both upright and inverted.

It is clear from Fig. 2.6 that the average pressure on the base of the inverted pyramid, p_I, is greater than the pressure on the upright pyramid, p_U, because the weight of the pyramid is the same whether it is upright or inverted, and $b>a$. Thus, if the pyramid is resting on a soft base, such as sand (in the Egyptian desert!), it is much more likely to cause failure of the base material if it is inverted than if it is upright. Thus, it is clear *that load **intensity** is more important for predicting failure than is load itself.*

A terminology has been developed with respect to this load intensity and it is called traction. The traction acting on a surface, as shown in Fig. 2.7, is defined as follows.

$$\vec{t}(\vec{n}) \equiv \lim_{\Delta A \to 0} \frac{\overrightarrow{\Delta F}}{\Delta A} \tag{2.3}$$

Fig. 2.6 Average pressure on the base of a truncated pyramid

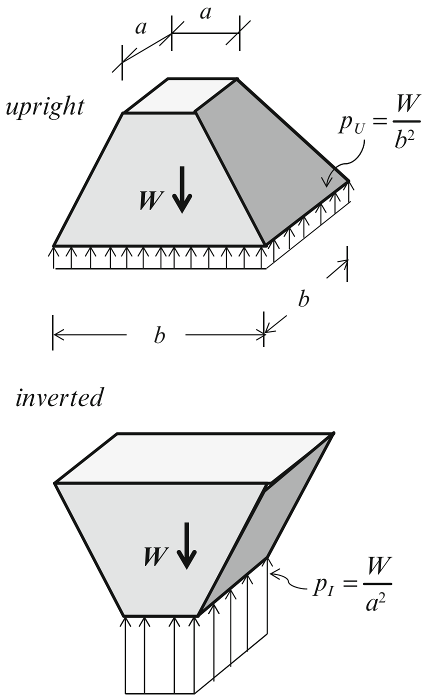
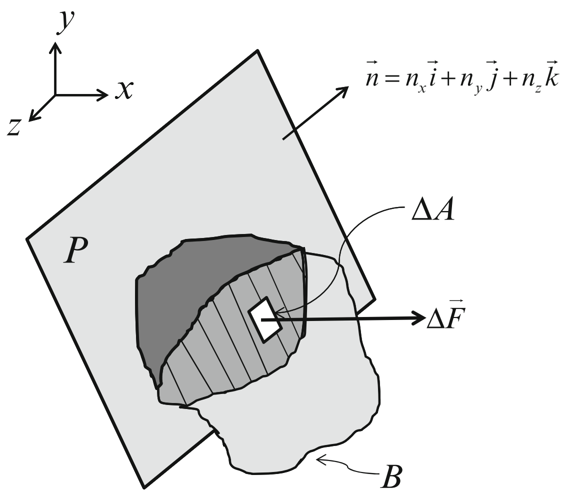


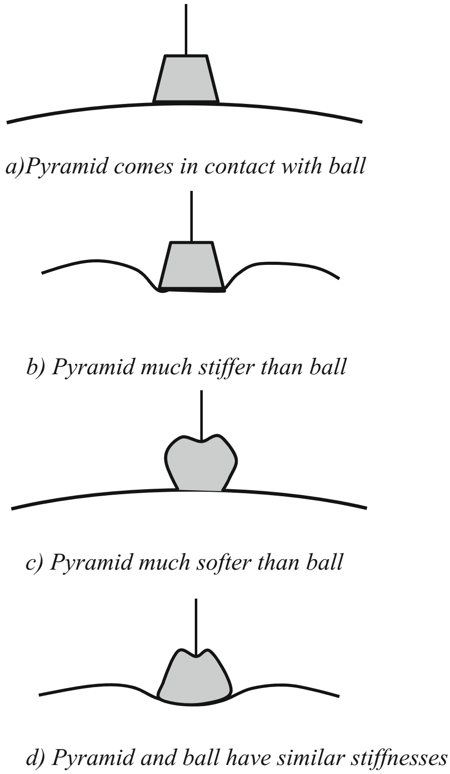
Fig. 2.7 Traction, $\vec{t}$, on plane *P*, with unit outer normal vector, $\vec{n}$, in body *B*



Where, as shown in the figure, $\vec{n}$ is the unit outer normal vector to the plane, ΔA is the area of the plane, and $\Delta\vec{F}$ is the force acting on the plane. Note that $\vec{t}(\vec{n})$ may be written in its component form as follows:

$$\vec{t}(\vec{n}) = t_x\vec{i} + t_y\vec{j} + t_z\vec{k} \tag{2.4}$$

Fig. 2.8 Physics of different types of mechanical boundary conditions



where $\vec{i}, \vec{j}$, and $\vec{k}$ are unit base vectors in the x, y, and z coordinate directions, respectively.

It can be seen from the above definition that the traction vector is described in units of force per unit area. Thus, if the traction vector is normal to the surface of interest, it is equivalent to a pressure, but it can also have components parallel to the surface of interest. Furthermore, the traction vector may act on any plane (defined by the unit outer normal vector, $\vec{n}$) and at any point inside or on the surface of an object.

2.2.2 Boundary Conditions

Suppose that the plane of interest is chosen to coincide with the boundary of the object in question. Due to the physics of the problem, it is necessary to develop a mathematical model describing what is occurring at the boundary, termed the so-called boundary conditions. In order to describe the physics, consider a block that is lowered slowly onto a curved surface until the two objects touch at a single point, as shown in Fig. 2.8a. As the block continues to be lowered, there are three possible results (assuming that the block is held in place over the curved surface):

CASE 1: The surface conforms to the shape of the block (Fig. 2.8b)

The block is obviously much stiffer than the surface, such as the case of a steel block resting on a foam rubber surface. In this case, the foam rubber surface conforms to the shape of the block, resulting in what is termed traction boundary conditions applied to the boundary of the block where it is in contact with the surface.

$$\textbf{Traction boundary conditions}: \vec{t}(\vec{n}) = known\ on\ the\ boundary\ of\ the\ block$$

CASE 2: The block conforms to the shape of the surface (Fig. 2.8c)

The surface is clearly much stiffer than the block, such as the case of a foam rubber block resting on a steel surface. In this case, the block conforms to the shape of the surface, resulting in what is termed displacement boundary conditions applied to the block where it is in contact with the surface.

$$\textbf{Displacement boundary conditions}: \vec{u} = known\ on\ the\ boundary\ of\ the\ block$$

CASE 3: Both the block and the surface deform significantly (Fig. 2.8d)

Both the block and the surface are made of the same (or similar) materials. This case, called a structural interaction problem, requires that both objects be analyzed simultaneously. Where the two come in contact with one another, the boundary conditions are not known. All that is known is that both the shapes and tractions must match, called matching conditions.

We will not consider this last possibility in this course, as it is an advanced subject that is beyond the scope of this text.

The reader will perhaps understand the above discussion best if the example of his/her own body is considered. If a person sits on pavement, his/her body will conform to the shape of the pavement where there is direct contact, and this part of the person's exterior is subjected to displacement boundary conditions. Note that sitting on the pavement is generally discomforting, and this is due to locally large stresses within the person near the points of contact. Note also that the part of the person's exterior that is not in contact with the pavement is subjected to air pressure. Since air is relatively compliant compared to the person, this part of the person's exterior is subjected to traction boundary conditions. Conversely, if a person is placed on a relatively compliant object, such as a water bed, then the bed will conform to the shape of the person's exterior, and the person is therefore subjected to traction boundary conditions on the portion of their exterior that is in contact with the bed. This, of course, explains in large measure why beds are more comfortable than pavement—stress concentrations are largely mitigated by traction type boundary conditions.

As we will see later in the text, both traction and displacement boundary conditions occur often in the analysis of solids. However, due to the physics described above, only one type of boundary condition is possible in each coordinate direction at a point on the surface of an object.

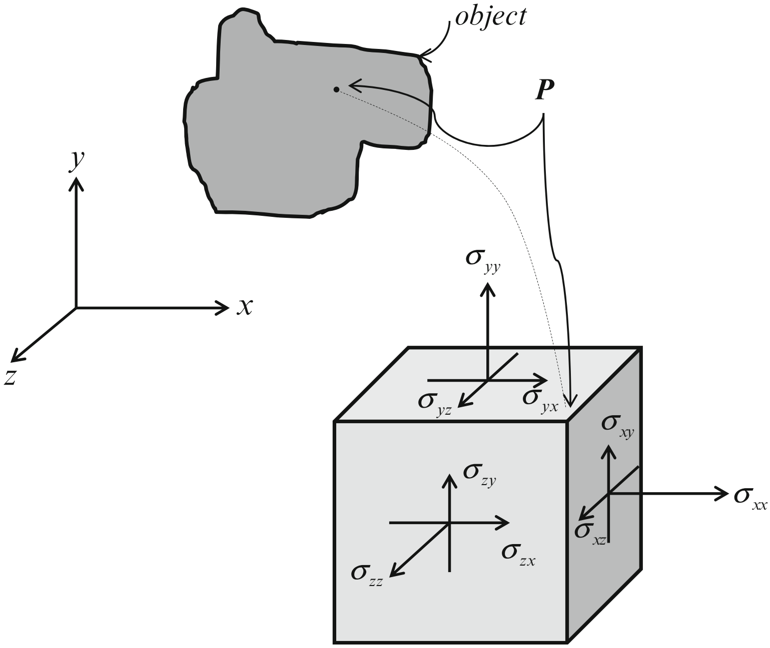


Fig. 2.9 Components of the stress tensor at a point *P* in an object

2.2.3 *The Stress Tensor*

Now let us examine the interior of the object of interest. Suppose that three planes are chosen so as to pass through an arbitrary point *P* in the interior of the object and perpendicular to the three coordinate axes. Then the traction vector on these three planes will take on a particular significance, as defined by Augustin Cauchy. In this case, the nine resulting components are called the Cauchy stress tensor, as shown in Fig. 2.9, and are given by

$$\begin{aligned}\vec{t}(\vec{i}) &\equiv \sigma_{xx}\vec{i} + \sigma_{xy}\vec{j} + \sigma_{xz}\vec{k}\\ \vec{t}(\vec{j}) &\equiv \sigma_{yx}\vec{i} + \sigma_{yy}\vec{j} + \sigma_{yz}\vec{k}\\ \vec{t}(\vec{k}) &\equiv \sigma_{zx}\vec{i} + \sigma_{zy}\vec{j} + \sigma_{zz}\vec{k}\end{aligned} \tag{2.5}$$

The sign convention for subscripts on the nine components of the stress tensor described above can be seen to be as follows***: the first subscript is associated with the unit normal vector for that plane, and the second subscript is the direction that the stress component is oriented.*** Note also that if the subscripts are the same for any component of stress, that component is perpendicular to the plane of interest and is therefore termed a ***normal stress***. If the subscripts are not the same for any component of stress, that component is parallel to the plane of interest and is therefore termed a ***shear stress***. Finally, the stress is called a tensor because the orientation of the components of the stress transform in a very complicated way (unlike the components of a vector), as we will see in Chap. 6.

Equation (2.5) is a very important definition in the history of mechanics. Monsieur Cauchy is responsible for many important developments—this is one of his very best. As we will see, the ability to predict the components of stress at all points in a body of arbitrary shape is at the very heart of structural design. Cauchy went a step further with the above definition of stress by using Newton's first law (summing forces) to prove *that the nine components of stress at a point in an object are sufficient to uniquely define the state of loading at that point in the object*. Therefore, it has been established that the stress tensor is the key kinetic quantity necessary to determine the state of loading at every point in a body. While the significance of this was not immediately apparent to the engineering community when Cauchy first reported it in 1822, it was clear by the middle of the nineteenth century that if the stress tensor could be predicted at every point in an object, it could be utilized as a means of predicting whether or not the body would be capable of withstanding the loads applied to it. Thus, it may be said that ***stress is the most important concept underlying all of modern structural and solid mechanics.***

The discerning reader may ask the question—how can there be stress at the molecular or atomistic scale? Of course, the answer is that there really is no such thing as a continuum, from whence the concept of stress emanates. In fact, at the scale that we normally employ it stress is nothing more than an ensemble average of molecular and atomistic forces per unit area. As such, it does not exist in reality. And furthermore, it cannot be measured. It can only be inferred by measuring displacements and employing constitutive equations, to be described below. Nevertheless, this ingenious concept has been shown through experimental observation to be a powerful means of predicting failure due to yielding, buckling, creep, excessive deformations, and fracture. Interestingly, Sophie Germain and Augustin Cauchy appear to be among the first to develop their models for media by starting from the assumption that the media can be idealized as continua. Prior to their attempts, such as the work of Navier, models proceeded from the molecular scale. For cases where the object of interest is large compared to the molecular scale, these latter approaches have given way to the continuum approach employed by Germain and Cauchy.

By summing moments at an arbitrary point in an object it can also be shown that the stress tensor is symmetric, i.e.,

$$\sigma_{xy} = \sigma_{yx}, \ \sigma_{xz} = \sigma_{zx}, \ \sigma_{yz} = \sigma_{zy} \tag{2.6}$$

Thus, there are only six unique components of stress at a given point in an object.

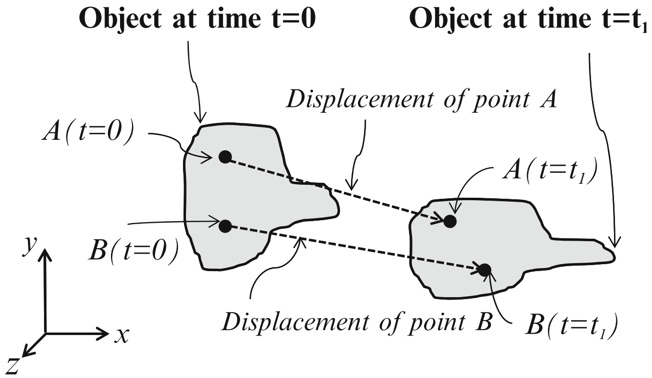


Fig. 2.10 Depiction of two points in a deformable body

2.2.4 Kinematics

Kinematics is the study of the motions of objects without reference to the forces involved. Three-dimensional objects sometimes move in such a way that the object in question may be considered for practical circumstances to be rigid. The term rigid implies that the object does not deform, so that all materials points in the object retain their relative distances from one another throughout the motion of the object. This is clearly an approximation to reality, and this approximation is not sufficient for determining whether the body will fail due to deformations. The current course is focused on deformable body motions, so that the case of rigid body motions will not be considered herein.

When a body deforms, each material point may undergo a distinct path of motion in time, as demonstrated in Fig. 2.10 for two points in a typical deformable body.

It is clear from Fig. 2.10 that the displacement vector, $\vec{u}$, is a function of coordinate location in the body, that is, $\vec{u} = \vec{u}(x, y, z, t)$, where the components of the displacement vector are given by

$$\vec{u}(x, y, z, t) = u(x, y, z, t)\vec{i} + v(x, y, z, t)\vec{j} + w(x, y, z, t)\vec{k} \tag{2.7}$$

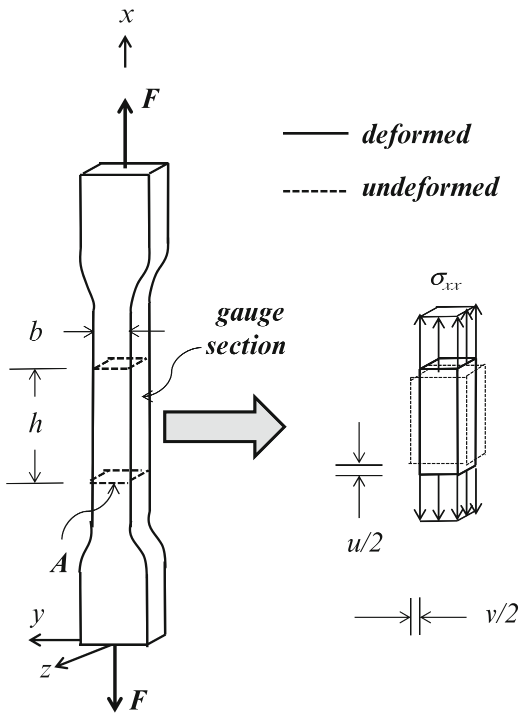
Because the displacement vector is a function of position, the spatial gradient of the displacement will not necessarily be zero in a deformable body. Therefore, suppose that a new variable is introduced, called the strain tensor, with the following components:

$$\varepsilon_{xx} \equiv \frac{\partial u}{\partial x}, \varepsilon_{yy} \equiv \frac{\partial v}{\partial y}, \varepsilon_{zz} \equiv \frac{\partial w}{\partial z}$$

$$\varepsilon_{xy} \equiv \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \quad \varepsilon_{xz} \equiv \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \quad \varepsilon_{yz} \equiv \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{2.8}$$

where we have employed the symbol $\equiv$ to mean "is defined to be."

Fig. 2.11 A uniaxial test



As has been discussed above, when a body translates as a rigid body, the displacements at all points in the body are equivalent, so that all of the components of the strain tensor defined in (2.8) will be identically zero. Thus, the definition of strain given above can be viewed as a means of filtering out rigid body translations. The importance of this revelation will become important in the next section.

2.2.5 Material Behavior

In order to complete the development of a rigorous model for predicting the mechanical response of structural components, it is necessary to develop a model describing the material behavior of the component to be modeled. Such a model requires the construction of a well-designed experiment called a constitutive test, to be defined below. Toward this end, the experiment originally proposed by Leonardo Da Vinci in Chap. 1 is useful for supplying a wealth of information that is relevant to this text. A modern version of Da Vinci's experiment consists of applying a load to the end of a prismatic bar composed of the material in question and measuring the deformation as a function of the applied load, as shown in Fig. 2.11. ***This type of test is termed a uniaxial test*** because the load is applied parallel to the long axis of the bar (Allen and Haisler 1985). Of course, it is not possible to measure load directly in the laboratory (or anywhere else, for that matter). Thus, this is accomplished indirectly by placing a device in series with the specimen called a load cell, which is nothing more than a fancy name for a spring that obeys Hooke's law. The load cell can then be used to deduce the load, and a variety of techniques (such as strain gauges, linear voltage differential transducers (LVDTs), or optical measuring devices) may be used to measure the displacement (and resulting axial strain) in the bar. This test then results in what is called an inverse problem.

It can be shown that the stresses and strains in this experiment are spatially homogeneous in the so-called gauge section, as shown in Fig. 2.11 due to something called St. Venant's principle and Newton's laws so long as the displacements are measured at least as far as the dimension b from the shoulder in the specimen. Thus, using the definitions of stress and strain given earlier in this chapter, it can be deduced that

$$\sigma_{xx} = \frac{F}{A} \tag{2.9}$$

$$\varepsilon_{xx} = \frac{u}{h} \tag{2.10}$$

and

$$\varepsilon_{yy} = \frac{v}{b} \tag{2.11}$$

where the quantities in the above equations are as shown in Fig. 2.11.

Therefore, this is a rather unique situation in which the unknowns in the problem are deducible from the loads and geometry of the specimen, and for that reason this test is termed ***a constitutive test***. Thus, a plot of the axial stress versus the axial strain is easy to construct, and from this it is possible to deduce the relation between the axial stress and the axial strain.

There are other tests that may be used in this way to deduce material properties, but this test is generally the simplest (and least expensive) to perform in the laboratory for many structural materials. There are some solids for which this type of test is impractical due to the makeup of the material. Examples include certain types of soils that cannot carry significant tensile load. Concrete is another example. For materials of this type, more complicated tests are necessary. Due to their complexity, they will not be covered in this text. However, the principle of all ***constitutive tests*** is identical to that of a uniaxial test—to devise a means of deducing the stresses and strains inside the object directly from experimentally determined information on the boundary of the specimen.

An example of the observed relation between the stress and strain for the case of constant loading rate is shown in Fig. 2.12. Metals tested at temperatures below about 30 % of their melting temperature will typically behave in a way that is termed elasto-plastic due to the fact that they will display a relationship between stress and strain that is linear so long as the stress does not exceed a critical value called the yield point, and denoted in this text by, σ^{T}, which is a material constant.

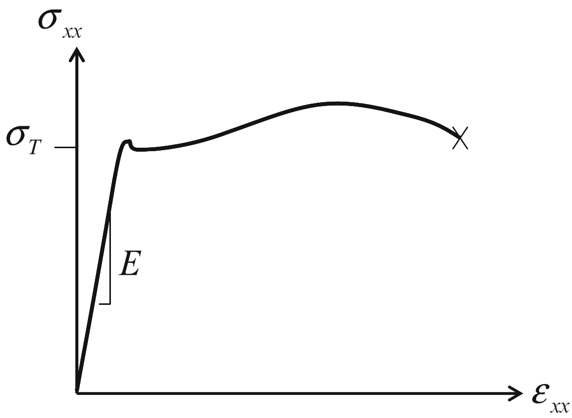


Fig. 2.12 Typical uniaxial stress–strain curve for a metal

If the test is performed in compression, then the yield strength is denoted by σ^{C}. The yield strengths in tension and compression are often equivalent in metals. However, in other materials, such as soils and concrete, the compressive yield strength is normally much larger than it is in tension.

Up to this value, it is known that essentially all of the deformation is caused by aggregated molecular bond stretching, which is approximately linear and recoverable. Beyond this point, the behavior is not linear, and the material will undergo permanent deformation. Other responses of materials are possible. It would be nice if we could make materials behave the way we want them to, but that simply is not the case. Try as we may, they will behave as they wish. Thus, there are numerous different material models that may be required to model the stress–strain behavior, even in a simple uniaxial test. Human tissue, for example, behaves in a highly nonlinear and rate-dependent way when loaded uniaxially.

Such materials are beyond the scope of this text. Herein, we will consider only materials wherein the stress–strain behavior is linear, called Hookean, after the first person to notice this effect, Robert Hooke.

When materials behave linearly and the relation between the stress and strain is unique, the material is termed "linear elastic." In this case, the uniaxial stress–strain relation is described by

$$\varepsilon_{xx} = \frac{\sigma_{xx}}{E} \Leftrightarrow \sigma_{xx} = E\varepsilon_{xx} \tag{2.12}$$

where the x-axis is aligned with the loading direction, as shown in Fig. 2.11. The symbol E is clearly a measurable quantity that results directly from (2.9) and (2.10) and is represented by the slope of the uniaxial stress–strain curve shown in Fig. 2.12. E is called Young's modulus, after Thomas Young. The lateral strain may also be measured in the same test, as described by (2.11), and this measurement, together with that obtained from (2.10), may be used to obtain the following material constant, called Poisson's ratio, after Siméon Denis Poisson:

$$\nu \equiv -\frac{\varepsilon_{yy}}{\varepsilon_{xx}} \tag{2.13}$$

A generalized three-dimensional representation of (2.12) for a generally anisotropic linear elastic material is given by the following:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.14}$$

where the C matrix can be shown to be symmetric using thermodynamic constraints and thus contains 21 unique constants. Therefore, it might be necessary to perform quite a few experiments in order to obtain all of the coefficients in the C matrix. Fortunately, there are no practical circumstances where materials are generally anisotropic. The most general case of material anisotropy commonly found is called orthotropic material behavior. Such is the case for continuous fiber composites, such as those used in the airframes of the Boeing 787 Dreamliner and the Airbus A380, as well as downhill skies, golf clubs, fishing rods, and tennis rackets, to name a few. For materials such as this, it can be shown that (2.14) simplify to the following form:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.15}$$

When a material is tested, it is often observed that the test may be performed along any orientation and the results are the same. This type of response is called isotropic. Many structural materials are isotropic, and we will confine our models to isotropic materials in the current text. In this case, (2.15) simplifies to the following:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.16}$$

Thus, for isotropic linear elastic materials, there are at most three unique material constants to be measured experimentally—C_{11}, C_{12}, and C_{44}. These can be related to Young's modulus and Poisson's ratio as follows. First note that in the uniaxial test performed in Fig. 2.11 the stress state is uniaxial, i.e., $\sigma_{xx} \neq 0, \sigma_{yy} = \sigma_{zz} = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$. Substituting this into (2.16) results in the following:

$$\varepsilon_{xx} = C_{11}\sigma_{xx} \tag{2.17}$$

$$\varepsilon_{yy} = \varepsilon_{zz} = C_{12}\sigma_{xx} \tag{2.18}$$

Comparing (2.12) and (2.17) reveals the following equivalence

$$C_{11} = \frac{1}{E} \tag{2.19}$$

Comparing (2.13) and (2.18) and using (2.17) and (2.19) results in

$$C_{12} = -\frac{v}{E} \tag{2.20}$$

Thus, two of the three unknown coefficients in the C matrix for an isotropic linear elastic material can be obtained from a uniaxial test. It will now be shown that the third coefficient, C_{44}, may also be obtained from these same coefficients. This can be accomplished by introducing a new experiment, called a ***pure shear test***, as shown in Fig. 2.12. In this test, similar to the uniaxial bar test described above, the stress may be plotted versus the applied strain, with the result that for a linear elastic material:

$$\sigma_{xy} = G\varepsilon_{xy} \tag{2.21}$$

where the slope of the curve, G, is called the shear modulus. The yield strength in this test is denoted by σ^{S}. By equating (2.16) and (2.21) it is apparent that for an isotropic material

$$C_{44} = C_{55} = C_{66} = \frac{1}{G} \tag{2.22}$$

It will now be shown that G is redundant for isotropic media. To do this, consider once again the shear test in Fig. 2.13. Given the shear stress $\sigma_{xy} = k$, where k is an arbitrary constant value of stress applied in the shear test, a cutting plane passed through the stress block in the shear test will result in ***free body diagram A***, as shown in the lower left portion of the figure. Summing forces in the x' coordinate direction on this diagram will result in

$$\sum F_{x'} = 0 = \sigma_{y'x'}A + \sigma_{xy}A\sin 45^\circ \cos 45^\circ - \sigma_{xy}A\sin 45^\circ \cos 45^\circ \Rightarrow \sigma_{y'x'} = 0 \tag{2.23}$$

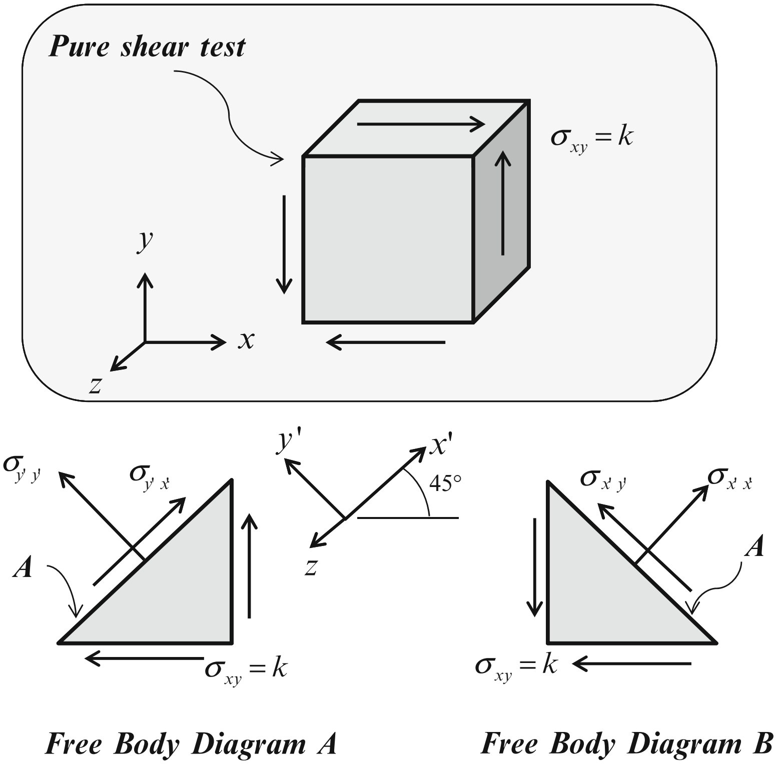


Fig. 2.13 Pure shear test

Summing forces in the

$$\sum F_{y'} = 0 = \sigma_{y'y'}A + \sigma_{xy}\cos^2 45^\circ + \sigma_{xy}\sin^2 45^\circ \Rightarrow \sigma_{y'y'} = -\sigma_{xy} = -k \quad (2.24)$$

Similarly, summing forces in the x' and y' coordinate directions in ***free body diagram B*** will result in the following:

$$\sigma_{x'x'} = k \quad (2.25)$$

and

$$\sigma_{x'y'} = 0 \quad (2.26)$$

Thus, it can be seen that the stress states shown in Fig. 2.14 are mechanically equivalent.

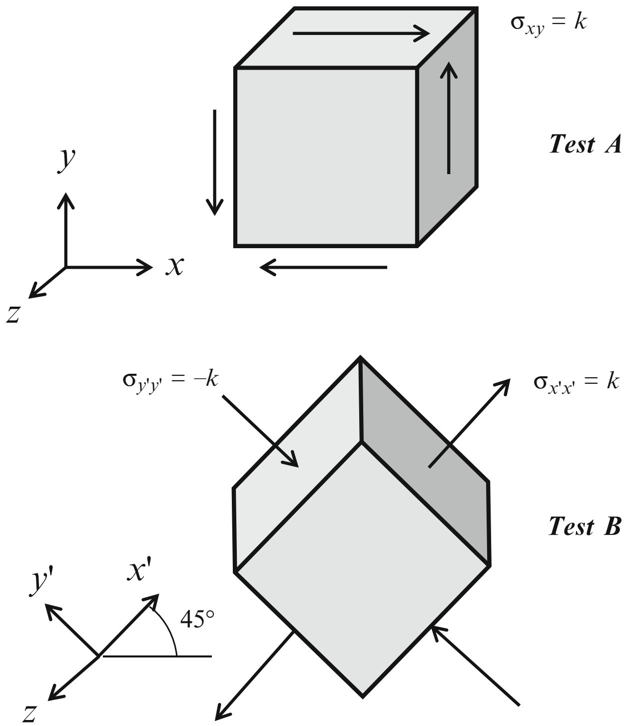


Fig. 2.14 Two mechanically equivalent tests

It is not actually necessary to perform these two tests. Rather, they are performed as a thought experiment in order to relate G to E and v. In order to accomplish this, recall that the material is assumed to be isotropic, so that the stress–strain equations described in (2.16) apply both in the primed and in the unprimed coordinate systems. Applying Test B in Fig. 2.14 to (2.16) and employing (2.19) and (2.20) thus results in the following:

$$\varepsilon_{x'x'} = \frac{(1+v)}{E}k \tag{2.27}$$

But since Tests A and B are mechanically equivalent, it follows that $\varepsilon_{x'x'} = \varepsilon_{xy}$, and (2.21) and (2.27) can be equated, with the result that

$$G = \frac{E}{(1+v)} \tag{2.28}$$

Thus, there are only two unique material constants for linear elastic materials, and both constants may be obtained from a uniaxial test: E and v. Therefore, substituting (2.19), (2.20), and (2.28) into (2.16) results in the following:

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{yz} \\ \varepsilon_{xz} \\ \varepsilon_{xy} \end{Bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{bmatrix} \begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{Bmatrix} \tag{2.29}$$

As a small footnote, the reader may be interested to know that when the three-dimensional theory of isotropic linear elastic media was first reported in 1822 by Augustin Cauchy, it was thought that there was only a single material constant necessary to describe the constitution. A small but spirited controversy erupted over whether there was one or two material constants for isotropic media, as a fair proportion of the scientific community thought there to be only one. Those supporting the position that there was but a single constant held the view that Poisson's ratio was the same for all materials and should therefore not be included. The controversy seems to have raged almost all the way to the end of the nineteenth century, finally settling on the now well-known correct value of two material constants with the publication of A.E.H. Love's two volume text in 1892–1893 (Love 1892). Perhaps this controversy can be attributed to the fact that in the nineteenth century the state of experimental laboratories was such that it was not possible to measure Poisson's ratio to even one significant digit accuracy. Fortunately, a broad range of experimental techniques were developed in the twentieth century, and today we have accurate values for Poisson's ratio for essentially all materials.

This completes the description of the constitutive behavior of isotropic linear elastic media. A table of material properties for typical engineering materials is given in the appendix.

2.3 Units of Measure

The choice of a system of units for the purpose of this course is not a trivial one. On the one hand, we have the US (or English) system that is commonly used in this country. On the other hand, we have the SI (or metric) system, short for Système International, which is commonly used everywhere else in the world today (except Myanmar!).

In order to explain this strange divergence of systems, it is perhaps necessary to go all the way back to the year 1066, when the French nobleman William of Normandy defeated the English king Harold at Hastings and became the King of England. As a result of this famous battle and the complex system of choosing nobility in Europe, there ensued major differences of opinion between the French and the English that have in some ways continued down to the present. Chief among these were the Hundred Years War and the War of the Roses, but smaller and yet still significant differences of opinion have pervaded Western European culture as well.

For example, when in 1582 Pope Gregory decided to correct the Julian calendar, which had by that time become badly out of time with the sun, the French adopted the Gregorian calendar, but the English refused to accept it until 1752, with the dismaying result that the English Channel divided two countries not only by water, but also by at least 10 days on the calendar.

Such differences may seem overwrought today, but 300 years ago they were all too common. In 1707 a British fleet was destroyed, along with 1,400 sailors, off the Isles of Scilly because it was not possible to measure longitude accurately at the time. Thus, a prize was set by the English monarchy for the first person who could measure longitude accurately. This prize was eventually claimed by John Harrison (in 1767), who invented the ship's chronometer, a clock that is capable of measuring time accurately on a ship at sea. His four clocks, developed over a 31 year span, are still displayed today at the Royal Observatory in Greenwich, east of London (see Fig. 2.15). It is for this reason that the English are credited with establishing the units of time, and the Prime (0°) Meridian is universally accepted to be at the Greenwich Observatory.

The French were not satisfied with this development by the English. Thus, they set out to construct a universally accepted measure of distance. This was accomplished by two French surveyors, Monsieurs Jean Baptiste Joseph Delambre and Pierre Méchain during the 1790s. They were commissioned by the French Academy of Sciences to survey from Barcelona to the Pas de Calais so that the distance from the North Pole to the equator could be accurately determined. The meter was subsequently defined to be on ten-millionth of that distance, and a platinum bar was constructed as a reference, as shown in Fig. 2.16.

Today, the meter is accepted almost everywhere on Earth as the standard unit of distance. Therefore, in this textbook this unit of distance will be adopted, along with the SI system in its entirety. Accordingly, a unit of mass, called a gram, as it was originally conceived was defined to be the mass of 1 dm^3 of water at 0°C. This has now been altered (without destroying the original spirit of the definition) to the mass of a physical prototype preserved by the International Bureau of Weights and Measures.

An important feature of the SI system of units is that there is a clear difference between force and mass, which is often not the case in the US system. In the SI system, the distinction is given by the following formula:

$$F = mg^{\mathrm{E}} \tag{2.30}$$

where force, F, is measured in Newtons (N), m is mass, measured in grams (g), and g^{E} is Earth's gravitational constant, given by

$$g^{\mathrm{E}} = \frac{9.80665\,\mathrm{N}}{10^3\,\mathrm{g}} \tag{2.31}$$

Fig. 2.15 Photos of John Harrison's chronometers, numbers one through four clockwise from *top left*

Table 2.1 shows a table demonstrating the important units of measure in the SI system.

Table 2.2 shows a table demonstrating the important units of measure in the US system.

Table 2.3 is useful for converting from one system to the other. In keeping with the necessity for three significant digits accuracy, conversions are shown to four significant digits accuracy.

Fig. 2.16 Photo of platinum bar representing the meter in the Musée des Artes et Metiers, Paris

Table 2.1 SI units of measure in mechanics

Unit of measure	SI units	Other SI units	Other SI units	Other SI units
Mass	Gram (g)	Kilogram = 10^3 g	mg = 10^{-3} g	
Force	Newton (N)	kN = 10^3 N		
Length	Meter (m)	km = 10^3 m	mm = 10^{-3} m	micron (μm) = 10^{-6}m
Stress	Pascal (Pa = N/M^2)	KPa = 10^3 Pa	MPa = 10^6 Pa	GPa = 10^9 Pa
Moment	N m			
Density	g/m^3	kg/m^3		

Table 2.2 US units of measure in mechanics

Unit of measure	US units	Other US units	Other US units	Other US units
Mass	Pound (lb)	Ton = 2 × 10^3 lb	Ounce(oz) = lb/16	
Force	Pound force (lbf)			
Length	Inch (in.)	Foot (ft) = 12 in.	Yard (yd) = 3 ft	Mile (mi) = 5,280 ft
Stress	lbf/in.^{2} (psi)	lbf/ft^2(psf)	ksi = 10^3 psi	Msi = 10^6 psi
Moment	ft-lbf			
Density	lb/in.^{3}	lb/ft^3		

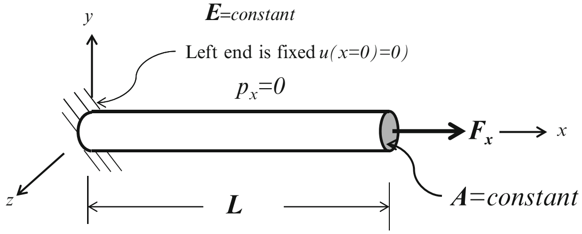
Table 2.3 Table for converting from SI to US system (to convert to the units at top multiply the value in the column on the left by the factor in the table)

To	lb	lbf	in.	ft	yd	mi	psi	psf	ksi	ft-lbf	lb/in.^{2}	lb/ft^2
From												
g	2.205×10^{-3}											
kg	2.205											
mg	2.205×10^{-4}											
N		0.2248										
kN		0.2248×10^{1}										
m			39.37	3.281	1.094	0.6214×10^{-2}						
km			39.37×10^{1}	3.281×10^{2}	1.094×10^{-2}	0.6214						
mm			39.37×10^{-2}	3.281×10^{-3}	1.094×10^{-3}	0.6214×10^{-6}						
μm			39.37×10^{-4}	3.281×10^{-4}	1.094×10^{-4}	0.6214×10^{-5}						
Pa							0.1450×10^{-3}	0.02089	0.1450×10^{-4}			
kPa							0.145	20.89	0.1450×10^{-3}			
MPa							0.145×10^{3}	20.89×10^{2}	0.145			
Gpa							0.145×10^{6}	20.89×10^{4}	0.145×10^{6}			
N-m										0.7376		
g/m^2											0.036	62.43

2.4 Assignments

PROBLEM 2.1

GIVEN: A uniaxial bar of length, L, is prismatic (meaning the cross-section in the y–z plane does not vary in the x direction), with cross-sectional area, A, and is subjected to an axial load, F_x, at the free end.



REQUIRED: Consider the axial displacement, $u(x = L)$, of the free end

(a) Propose based on your intuition whether $u(x = L) \propto F$ or $u(x = L) \propto 1/F$ (where $\propto$ means "proportional to"), assuming L and A are held fixed. Plot a graph of $u(x = L)$ vs. F.
(b) Propose based on your intuition whether $u(x = L) \propto L$ or $u(x = L) \propto 1/L$, assuming F and A are held fixed. Plot a graph of $u(x = L)$ vs. F.
(c) Propose based on your intuition whether $u(x = L) \propto A$ or $u(x = L) \propto 1/A$, assuming F and L are held fixed. Plot a graph of $u(x = L)$ vs. A.
(d) Based on your results obtained in parts (a)–(c), propose a general formula for $u(x = L)$ as a function of F, L, and A.

PROBLEM 2.2

GIVEN: The Hoover damn shown below is subjected to a loading that produces stress in the object at all points.

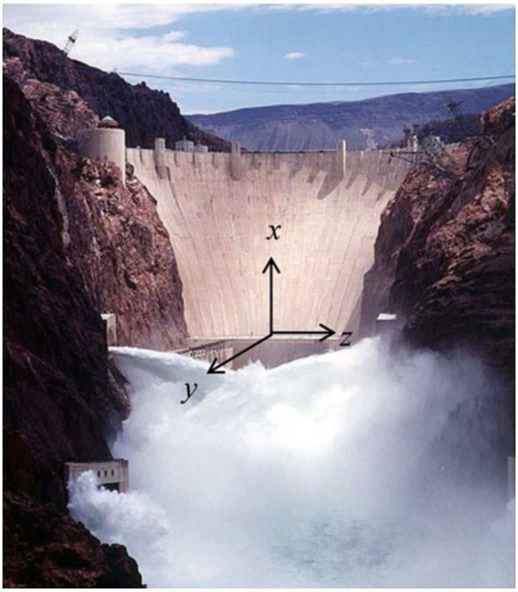


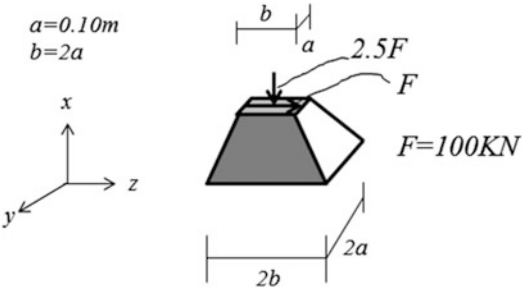
Photo of Hoover Dam Releasing Water (courtesy Bureau of Reclamation PD-USGOV)

The stress state at the midpoint bottom of the damn is given by

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} = \begin{bmatrix} -200 & 20 & 10 \\ 20 & -50 & -25 \\ 10 & -25 & -10 \end{bmatrix}$$

REQUIRED: Draw a depiction of the material point (including coordinate axes) with the stress components labeled on the faces of the resulting cube.

PROBLEM 2.3
GIVEN: The object shown below is subjected to the loads shown. Assume that the object is in equilibrium and that the stresses on the bottom surface of the object are evenly distributed.



REQUIRED: Determine all nine components of the stress on the bottom surface of the object and draw these components on the stress cube, showing the coordinate axes.

PROBLEM 2.4

GIVEN: The definition of a creep and recovery test is to take a prismatic bar of homogeneous material and apply a constant load in the axial direction, measuring the deformation as a function of time. After a period of time, the load is removed, and the deformation is measured for a length of time equivalent to that over which the load was applied.

REQUIRED

(1) Select a person from this class and work in a team of two, turn in your report together, and try to make it clear, concise, and professional.
(2) Go on the web and find a recipe for making either silly putty or gak; use this recipe to make your own batch of either.
(3) Use your batch to make a cylindrical prismatic bar; measure the length and cross-sectional area of the specimen and report them.
(4) Select a gage length within the specimen and draw dots on the bar at the upper and lower points of the gage length.
(5) Devise a means of holding up the specimen with the long axis aligned vertically, and apply a load axially at the bottom of the specimen in tension in accordance with the given above; report the weight of the applied load; take a photo and include in your report.
(6) Measure the deformation within the gage length at selected time intervals and record them in an excel spreadsheet.
(7) After a selected interval of time (during which the bar does not fracture), remove the load and continue to record the deformation as a function of time in accordance with the given above.
(8) Using the results of (3)–(7), calculate the applied stress as a function of time, as well as the strain as a function of time, and record them in your spreadsheet; turn in your spreadsheet with your report.
(9) Plot graphs of the stress and strain versus time and include in your report.

PROBLEM 2.5

GIVEN: Two identical uniaxial bars with cross-sectional area 0.01 m^2 are subjected to loadings in the x direction at two different loading rates $(\mathrm{d}F/\mathrm{d}t)_1 = 50$ kN/s (Test #1), $(\mathrm{d}F/\mathrm{d}t)_2 = 100$ kN/s (Test #2)

Time (s)	ε_{xx} (Test #1)	ε_{xx} (Test #2)
0	0	0
5	0.0008	0.0012
10	0.0012	0.0020
15	0.0017	0.0030
20	0.0020	0.0040
25	0.0025	0.0065
30	0.0030	0.0200

(continued)

continued

Time (s)	ε_{xx} (Test #1)	ε_{xx} (Test #2)
35	0.0035	0.0630
40	0.0040	X
45	0.0045	
50	0.0095	
55	0.0270	
60	0.0580	
65	X	

X means "fracture"

REQUIRED

(1) Plot a single input diagram showing σ_{xx} versus time for both tests.
(2) Plot a single output diagram showing ε_{xx} versus time for both tests.
(3) Plot a single crossplot diagram of σ_{xx} versus ε_{xx} for both tests.

PROBLEM 2.6
GIVEN: Stress may be expressed in either the metric system (Pascals, meaning Newtons per square meter) or the English system (*psi*, meaning *lbf* per square inch)
REQUIRED

1. Derive the relation between *Pa* and *psi*
2. Derive the relation between *MPa* and *ksi*
3. Derive the relation between *ksi* and *GPa*

References

Allen D, Haisler W (1985) Introduction to aerospace structural analysis. Wiley, New York
Glover C, Jones H (1992) Conservation principles for continuous media. McGraw-Hill, New York
Love AEH (1892) Treatise on the mathematical theory of elasticity. Dover, New York
Malvern L (1969) Introduction to the mechanics of a continuous medium. Prentice-Hall, Englewood Cliffs
Timoshenko S (1953) History of strength of materials. McGraw-Hill, New York

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