Chapter 2
Dithering

Analog dithering is an elegant way to modify hard nonlinearities. In servomechanism study in control theory, dither has been defined as a high frequency signal, compared to system dynamics, that will reduce the sector size of the nonlinear characteristic [1, 2]. This is in turn related to system characteristics such as stability and linearity. However this definition does not cover all of the possible applications of the concept for any kind of dynamics and input frequency content to the system.

In this chapter, first a general and broader definition of the dithering concept is introduced. Then the concept of 'equivalent nonlinearity' is defined. This functional equivalence explains the linearization process in terms of dithering and simplifies the analysis of the dithered system. The insight obtained from equivalent nonlinearity can be utilized to find the distortion of amplifiers.

Based on the dithering frequency, the dithering process can be divided to high and low frequency. If the dither frequency is much higher than the maximum input frequency, it is called high frequency dithering; reversely, if it is lower than the zero crossing rate of the input band-pass signal, it is called low frequency dithering.

Two approaches can be followed in parallel, to obtain the equivalent nonlinearity function, which are called Fourier series method and the statistical method. The first approach is based on Fourier expansion, the second one relies on the statistical averaging concept. The advantages and disadvantages of each method are discussed in detail.

The frequency content of the high and low frequency dithering are studied based on the generic output spectral form. A different bandwidth constraint formula is derived for each. Afterwards the statistical conceptual model is applied to high and low frequency to derive the equivalent nonlinearity.

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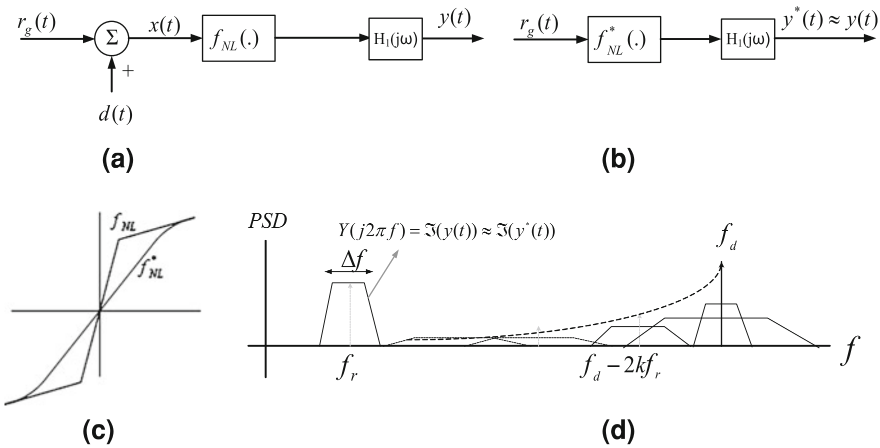

Fig. 2.1 **a** Continuous time dithering applied to a nonlinear apparatus; **b** equivalent nonlinearity concept model; **c** the shape of a hard nonlinearity f_{NL} is effectively transformed to softer one f^*_{NL}; **d** the output frequency domain content of the linear operator f_{NL} f^*_{NL} will generate the same baseband part as f_{NL} at the output of the filter

2.1 Dithering Concept

Dithering is injection of an external signal to a linear or nonlinear system to obtain several possible objectives. Among them are augmenting the linearity of the open or closed loop system, robustness, and asymptotic stability, for a general class of nonlinear apparatus, reduction of quantization noise in data convertors; and adaptive enhancement of the closed loop linearity. The dithering can also be utilized to affect the dynamic behavior of the system and stop undesired chaotic behavior or undesired limit cycle oscillations in the system, or alleviate the resonance jump phenomenon in sliding mode control applications [3].

The dithering signal may be analog or digital, and can have any statistical and spectral characteristics. It can be a random or a deterministic signal and it can be correlated or uncorrelated to the input signal. Being a random signal, its value at each moment can be dependent or independent of its previous values in time, or in the other words it can be with memory or memory less.

In case of having a nonlinear system, described with a general input output nonlinear operator $y = f_{NL}(x)$, the concept of the dithering is illustrated in Fig. 2.1a, wherein signal $d(t)$ is the continuous time dithering, $x(t)$ is the input signal, and $y(t)$ is the output signal of the nonlinear system. The output of the system is related to the total input as:

$$y = h_1(t) * f_{NL}(x(t)) = h_1(t) * f_{NL}(r_g(t) + d(t) \tag{2.1}$$

where $h_1(t)$ is the impulse response of the filter $H_1(j\omega)$. Having a mathematical description of the single-input single-output $f_{NL}(.)$ we can find the output response to any desired input. The only assumption is that the f_{NL} function is assumed to be memory-less, i.e. its output at each moment depends only on its input at the same instant.

2.2 Equivalent Nonlinearity

The most important effect of dithering is *linearization* of hard nonlinear characteristics, i.e. the original nonlinearity is transformed to another one which has a smaller 'sector size',[1] as depicted in Fig. 2.1c. This is due to the fact that the *dither effectively acts as a moving average filter, and applying this kind of process to a hard nonlinearity, will always make it softer*, provided that the proceeding filter H_1 is able to omit the undesirable spurious components at the output. This property is illustrated in Fig. 2.1c, where the f^*_{NL} is *the equivalent (linearized) nonlinearity,* after applying dither signal $d(t)$ to f_{NL} and proceeding filtering of its output. From now on, this new (transformed) mathematical operator is called *equivalent nonlinearity*.

In other terms, equivalent nonlinearity is a new function, which includes the linearization effect of the dithering signal. If the input $r_g(t)$ is a base-band signal (extending from dc to maximum frequency of f_r), and the dithering frequency is a higher frequency, the output of the original nonlinearity will include a replication of the signal (with in-band distortion) and sidebands around the dithering signal frequency component and its higher order harmonics, as shown in Fig. 2.1d.

The mathematical mechanism behind this phenomenon will be discussed more in the upcoming sections, and in Chap. 3. So the definition of the equivalent nonlinearity implies that the output filter $H_1(j\omega)$ has enough out of band rejection to omit the sidebands around the dithering frequency (and its harmonics) resulting in an output signal processed by the $f^*(.)$ equivalent operator. The main advantages of the equivalent nonlinearity are:

1. It is a single input single output system: the effect of the dithering signal is embedded within the new function; therefore we won't need to analyze the system for combination of two input signals.
2. The equivalent function is a softer nonlinearity. Hence, it is much simpler to analyze. The extreme case is having an ideal limiter, which is very memory-intensive to analyze with numerical methods (due to discontinuity of the output). The 'softened' or linearized version of it is of a lower polynomial order and is easier to analyze with commercial simulators.
3. The formulation of the equivalent nonlinearity is insightful to understand the dithering effect intuitively.

[1] The sector size is defined as the maximum angle between the line crossing origin, and the horizontal axis. Generally a system with less sector size is assumed to be smoother.

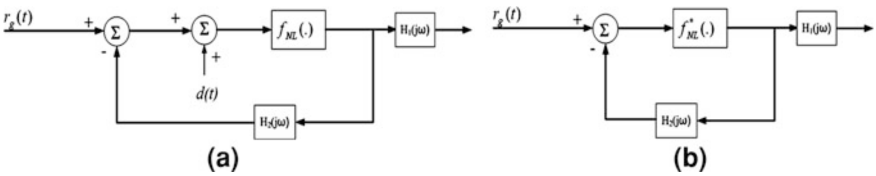

Fig. 2.2 **a** Continuous time dithering applied to a nonlinear apparatus with negative feedback; **b** equivalent system, using the concept of equivalent nonlinearity

The downside of the equivalent nonlinearity model is that it does not take into account the spurious side-bands of the dither frequency that penetrate into the signal band-width. When the frequency of the dither is chosen close to the bandwidth edge, it is impossible to filter out the spurs and they affect and degrade the linearity of the system as well. This issue is illustrated in Fig. 2.1d, where the modulated side-bands penetrate into the input signal band when we reduce the f_d.

The system in which the dithering is applied to may be placed in an environment that provides sort of feedback from output to the input, or is intentionally designed to have a feedback, as illustrated in block diagram of Fig. 2.2. The same concept of equivalent nonlinearity can also be used for the analysis of dithering in a negative feedback loop, provided that the feedback filter (H_2 in Fig. 2.2) is sharp enough to effectively omit the dither component and its modulated side band spurious.

The complete analysis of the systems in Figs. 2.1 and 2.2 means determination of signals in time and frequency domain in different points of the block diagram. For a linear systems analysis, everything is determined by the impulse response of the system. The output of the system is related to the input through the convolution integral.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau).h(t,\tau)d\tau \tag{2.2}$$

or equivalently in frequency domain, $Y(j\omega) = X(j\omega).H(j\omega)$.There are also several properties attributed to the linear systems and they are considered equivalent to each other: existence and uniqueness of the solution for a given known input, continuous dependence of the output on the input and boundedness of the output in case of boundedness of the input.

These properties are not fulfilled even in the simplest cases of a memory-less nonlinear system. Some examples and reviews are given in [4]. Any system for which the superposition principle does not hold is defined to be nonlinear. *In this case there is no possibility of generalizing from the responses for any class of inputs to the response for any other input*. This constitutes a fundamental and important difficulty which necessarily requires any study of nonlinear systems to be quite specific. One can attempt to calculate the response for a specific case of initial conditions and input, but make very little inference based on this result regarding response characteristics in other cases.

In order to understand and be able to mathematically describe the behavior, the nonlinear system has to be described. There are several approaches to define and describe the nonlinear block performance. The analysis will be focused on the system block diagram of Fig. 2.1. The methods and necessary tools will be developed to find the response of the open loop system to some classes of input signal and the results obtained will be used for the nonlinear feedback system of Fig. 2.2. From now on, for the sake of simplicity, we limit the scope of discussion to memory-less nonlinearity, i.e. the output of the nonlinear system can only depend on its input at the same time and f_{NL} is a static single valued function. The memory effect model and analysis will be presented in Sect. 5.3.

2.3 High and Low Frequency Dithering

The dithering signal frequency can be classified to two different modes: dithering frequencies which are higher than the input signal frequencies, which we call high frequency dithering (HFD) and those which are below the input signal frequencies and are called low frequency dithering (LFD).

The low frequency dithering can be used for linearization of a nonlinear system, that process band-pass input signals. The band pass signals are defined as those signals which have no spectral components below a certain frequency. High frequency can be used for any kind of input signal including DC, below the dithering frequency. The main difference in formulation of equivalent nonlinearity for high and low frequency dithering is that for high frequency, the rate of variations of the dither is usually much higher than the input signal, therefore the equivalent nonlinearity is obviously time-invariant, and high frequency fluctuations are averaged out simply by a low pass filter placed at the output. On the other hand, for low frequency dithering, the dither signal dynamics is slower with regards to the signal value, but should be much higher than the envelope bandwidth. In case of low frequency dithering, the averaging process is performed on the complex envelope of the band-pass input signal. Hence in terms of instantaneous value, the system is time variant and therefore the equivalent nonlinearity versus value is meaningless. Due to this basic difference, the equivalent nonlinearity is defined versus value in high frequency and versus envelope value in low frequency dithering, as will be explained and formulated mathematically in 2.5. The spectral distribution for the high and low frequency cases imposes two different estimations of the 'linear bandwidth',[2] and those estimations are given in Sect. 2.4.

There are two basic methods to calculate the equivalent nonlinearity, which are namely Fourier series expansion approach (assuming a periodic high or low frequency dither) or time domain averaging and statistical averaging approach.

[2] Linear bandwidth is the frequency range in which the spurious are low enough and the overall linearity is good.

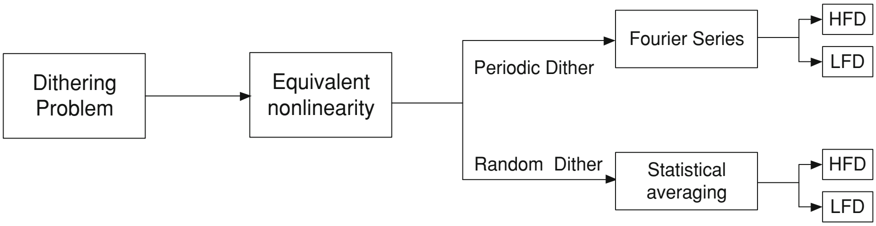

Fig. 2.3 Different approaches for random, high and low frequency dithers

The first approach is based on multi-dimensional Fourier expansion of the output versus input, for periodic input, while the second one is applicable to any kind of dither signal, under certain mathematical assumptions. The assumption made about the dither in the latter is weaker than the periodicity and the dither should only be 'repetitive', i.e. its value probability distribution function should repeat itself versus time. A good example of repetitive signal is a two tone signal, when the two tones are not harmonically related. By this definition all of the periodic signals are repetitive but *not* vice versa. The advantage of statistical view is that it is generic in terms of input signal, and it is applicable to a broad family of signals, including periodic signals.

The drawback of the statistical view compared to the Fourier series expansion is that it overlooks the spurious sidebands around the dithering frequency and its harmonics. These components can be harmful to the signal integrity, but sometimes are unavoidable, especially when the dither frequency is close to the signal, as illustrated in Fig. 2.1d. When the modulated spurious sidebands penetrate into the signal band, the linearity is always compromised. On the other hand, calculation of the Fourier coefficients for a general input signal and a general periodic dither can be very time consuming, tedious and memory-intensive, especially when the number of the frequency components grows large. An alternative approach to find the equivalent nonlinearity is developed in Chap. 6

In the following parts, first an intuitive understanding of the smoothing mechanism, based on rotating vector representation, for sinusoidal dither, is introduced. Then the Fourier series based approach for memory-less nonlinearities is explained, for high and low frequency. Afterwards the statistical point of view is developed for these two modes and different formulas for equivalent nonlinearity are derived, based on the value for high and envelope for low frequency dithering. Figure 2.3 sums up the order and distinction of the two approaches to dithering problem. The memory effect can be super-imposed to the analysis on both of the approaches, and is discussed detail in Chap. 5.

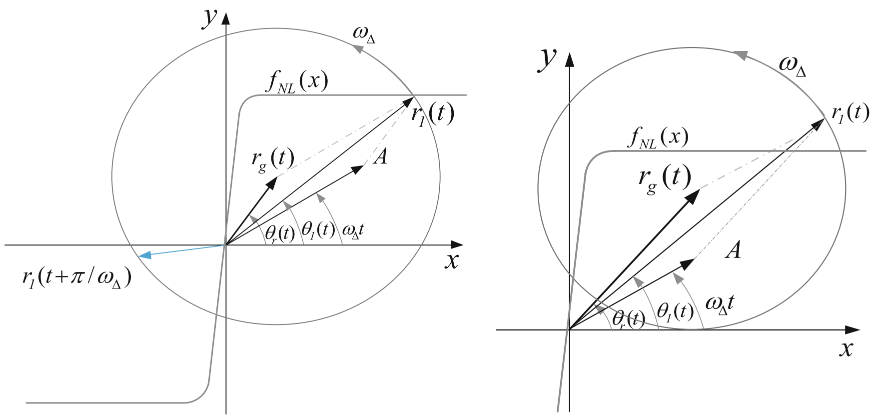

Fig. 2.4 Illustration of sinusoidal low frequency dithering effect on envelope r. The effective complex envelope vector rotates around a circle with a frequency much higher than the signal bandwidth (**a**) Linear averaging as the superposition vector sees the negative and positive values with proportion to the signal; (**b**) saturated averaging, in which the time duration experienced by summation is limited to either positive or negative y axis

2.4 Rotating Vector Representation

An intuitive understanding of the dithering effect for sinusoidal dither can be obtained through the following expansion of the entire complex envelope around carrier frequency ω_r:

$$\begin{aligned} x(t) &= \left|r_g(t)\right| \sin(\omega_r t + \theta_r(t)) + Asin(\omega_d t) \\ &= r_1(t) \sin(\omega_r t + \theta_1(t)) \end{aligned} \tag{2.3a}$$

while r_1(t) is the vector superposition of the phasors of the input complex envelope $r(t)$ and the dithering phasor $Aexp(j\omega_\Delta t)$, as follows:

$$r_1(t) \triangleq \left|\left|r_g(t)\right| \exp(j\theta_r(t)) + Aexp(j\omega_\Delta t)\right| \tag{2.3b}$$

and $\omega_\Delta = \omega_r - \omega_d$.

Provided that the rotation speed of the dithering is much higher than that of the input signal complex envelope bandwidth (i.e. $\omega_d \gg \Delta\omega$), we can assume that the vector $\bar{r}_g(t)$ is not moving in time and the dither vector is rotating around it with an angular speed of ω_Δ. Then after the averaging process (through a band-pass filter around ω_r or a low pass filter from zero to frequency ω_r) the system would be effectively time invariant with respect to the input envelope $r_g(t)$.

The averaging process is depicted in the diagram of Fig. 2.4a, b in the x–y plane, in which x and y represent the input and output signals of the nonlinear block respectively ($y = f_{NL}(x)$), and the imaginary part of the output complex envelope. The tip of the superposition vector $r_1(t)$ (sum of rotating A, and $r_g(t)$)

moves along a circle. This circle is centered at point $r_g(t)$ and its radius is the dither amplitude A.

Depending on the relative amplitudes of the dither and signal vectors, two different situations can be distinguished as follows:

Linear operation: For $|r_g(t)| < A$, as illustrated in Fig. 2.4a, the dither component acts like a moving average filter and the average of the rotating vector will be proportional to the input vector $r_g(t)$, as long as the mapping of $r_1(t)$ on the y axis passes through both positive and negative y. Intuitively, the average of the projection on the y axis will be proportional to the amplitude of the input signal $r_g(t)$ and thus the dithering makes the hard saturation function linear.

Saturation: When the value of $r(t)$ becomes larger than A ($|r_g(t)| \geq A$), the superposition vector $r_1(t)$ will be entirely projected on the positive (or entirely negative depending on the input sign) side of y, as shown in Fig. 2.4b, which drives the nonlinear system into positive or negative saturation.

Hence the saturation threshold is a function of r_g/A ratio. The linearization effect will also depend on the average time spent in positive and negative sides of the x axis, which is not uniform for sinusoidal dither.

It should be noted that the same interpretation is true for the real part of the output envelope, either if the superposition signal is projected on the x axis and x is a saturation function of y instead, or using the same chart as Fig. 2.4, for another vector orthogonal to the original one. This intuitive demonstration is valid for both HFD and LFD sinusoidal ditherings.

2.5 Fourier Series Approach

As long as the dithering signal is periodic, two-dimensional Fourier series expansion in combination with time domain averaging is a viable approach to extract the equivalent nonlinearity. We need to assume that the dither and signals frequency are non-commensurable, to have two degrees of freedom in Fourier series. Another way to express that condition in practical situations is to assume that the greatest common divisor of the two frequencies is much smaller than both of them gcd $(f_d, f_r) \ll f_d, f_r$ provided that both of the frequencies are integers. This should be true for both HFD and LFD.

2.5.1 High Frequency Dithering

In high frequency case, the dither signal, denoted as $d_H(t)$ is assumed to have a frequency much higher than the maximum frequency component of the input signal $r_g(t)$. Usually the fundamental frequency of $d_H(t)$, or the minimum

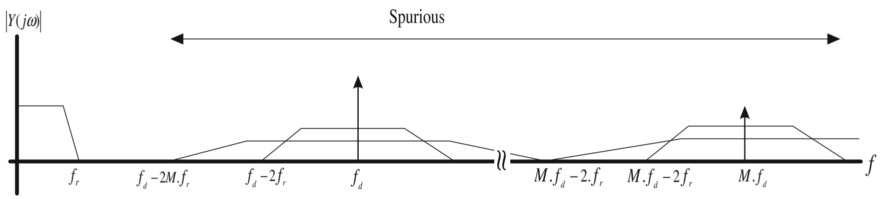

Fig. 2.5 Frequency content, the output of the nonlinear component after a high frequency dithering, with a single frequency f_d.

frequency component of its Fourier transform $D_H(\omega)$ is much higher than the input signal frequency.

The dithering signal $d_H(t)$ can be expressed as:

$$d_H(t) = \sum_{m=1}^{p} D_m.\sin(m\omega_d t + \varphi_m),\ while\ \omega_d \gg \omega_r \tag{2.4}$$

in which subscript H denotes high frequency as expressed by the condition. The output of the nonlinear operator is a function of the signal $x(t)$, which has a combination of the input signal frequency and dithering frequency. f_{NL} can be expanded in a two dimensional Fourier series as:

$$\begin{aligned} y(t) &= f_{NL}\left(r_g(t) + d_H(t)\right) = \sum_{m=0}^{\infty} g_m\left(r_g\right).\sin(m\omega_d t + \varphi_m) \\ &= g_0\left(r_g, D_1, D_2, \ldots, D_P\right) + \sum_{m=1}^{\infty} g_m\left(r_g, D_1, D_2, \ldots, D_P\right).\sin(m\omega_d t + \varphi_m) \end{aligned} \tag{2.5}$$

where $g_0(r)$ is the transformed nonlinearity characteristic, shaped (linearized) by the high frequency dither d_H or the equivalent nonlinearity defined in Sect. 2.2. There are averaging methods to achieve this term of the Fourier series which will be explained in the coming sections.

Under assumption of odd nonlinear characteristics for f_{NL}, Fig. 2.5 shows the output spectral content for a general nonlinear function f_{NL} in which the right hand term of (2.5) is truncated to 2M + 1. As we see from the figure, the closest modulated sideband of the output of the nonlinearity had a widened and up converted version of the input, with intermodulation order of 2M.

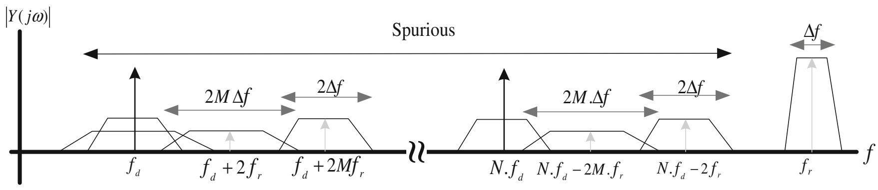

Fig. 2.6 Low frequency dithered frequency content, the output of the nonlinear component, with a single frequencyf_d.

2.5.2 *Low Frequency Dithering*

The injection of a dither component with the main frequency lower than the input band pass signal (f_r) can again modify and linearize the nonlinear characteristics. The dither is called d_L as:

$$d_L(t) = \sum_{m=1}^{p} D_m.\sin(m\omega_d t + \varphi_m),\ while\ \omega_d < \omega_r$$

and the input is a band pass modulated signal around a central frequency f_r:

$$r_g(t) = Re\{\tilde{r}_g(t).\exp(j\omega_r t + \varphi_r(t))\} = |\tilde{r}_g(t)|.\cos(\omega_r t + \varphi_r(t)) \tag{2.6}$$

As $\omega_r = 2\pi f_r$ and $\tilde{r}_g(t)$ is the complex envelope of the signal. The output Fourier series expansion can be written as:

$$\begin{aligned} y(t) &= f_{NL}\left(r_g(t) + d_L(t)\right) \\ &= \sum_{m,n=0}^{\infty} g_{mn}\left(\tilde{r}_g(t), D_1, D_2, \ldots, D_P\right).\sin(m\omega_r t + \varphi_r(t) + n\omega_d t) \\ &= g_{10}\left(\tilde{r}_g(t), D_1, D_2, \ldots, D_P\right) \\ &\quad + \sum_{m,n=1}^{\infty} g_{mn}\left(\tilde{r}_g(t), D_1, D_2, \ldots, D_P\right).\sin(m\omega_r t + \varphi_r(t) + n\omega_d t) \end{aligned} \tag{2.7}$$

while again $g_{10}(r, D_1, D_2, \ldots, D_p)$ is the equivalent nonlinearity f_{NL}^{*} defined in Sect. 2.2 and will be derived again in part 2–6. The generic spectral domain is according to Eq. (2.7) is depicted in Fig. 2.6, where again like high frequency case, the odd nonlinearity order is truncated to 2 M + 1 instead of ∞.

If the input signal is selected '*properly*', so that the spurs don't interfere with the desired signal band, g_{10} is capable of linear amplification-processing of the input signal. This will be explained in detail in Chap. 5.

When the dither frequency is selected to be much higher than the input signal, according to Fig. 2.5, the spurious sidebands won't affect the original linearized signal. This provides a way to estimate the bandwidth f_{rH}, as to avoid overlap between the M'th order sideband and original signal at the output:

$$f_d - 2Mf_{rH} > f_{rH} \tag{2.8}$$

which yields to the following equation:

$$f_{rH} < \frac{f_d}{2M+1} \tag{2.9}$$

where again the subscript H denotes high frequency dithering. By setting the M, the upper bound is derived for the bandwidth f_{rH}. The order M will depend on the order of the hard nonlinearity. Setting M = 1, results in $\frac{f_d}{3}$ as the fundamental limit, which agrees with the time domain analysis in [5].

For low frequency dithering, the fundamental dithering frequency f_d has to be at least half the band-width of the input signal and the input and dithering frequencies should be incommensurable. This can be explained based on the condition of not overlapping modulated dither around input carrier at $f_r - f_d$ with the signal itself, as:

$$f_r - f_d < f_r - \frac{\Delta f_{max}}{2} \quad \therefore \quad f_d > \frac{\Delta f_{max}}{2} \tag{2.10}$$

The linearization process is illustrated in Fig. 2.5. The dither and its higher order harmonics will absorb the spurious cross modulations and the input signal is processed without distortion.

Based on the notation used in Fig. 2.5, a limit can be derived for the maximum bandwidth of the input band pass signal

$$\frac{m\Delta f_0}{2} < mf_d + \Delta f_{max} < f_r - \frac{\Delta f_{max}}{2} \tag{2.11}$$

while m is the integer floor function of the ratio of the two frequencies f_r and f_d, and Δf_{max} and Δf_0 are the bandwidths of the system and the output band pass filter respectively. Based on (2.10), we can write:

$$\Delta f_L = 2f_r - m\Delta f_0 \tag{2.12}$$

where L denotes the low frequency and Δf_L indicates the maximum frequency span of the input band-pass signal. Equations (2.9) and (2.12) say nothing about the optimum points for the dithering frequency. This will be dealt with in Chap. 6.

2.6 Statistical Approach

As explained in the previous sections, the nonlinear system characteristic effectively changes in presence of the dithering signal. The method to study and describe this phenomenon will depend on the type of the input and dithering signal and of course the nonlinear apparatus itself.

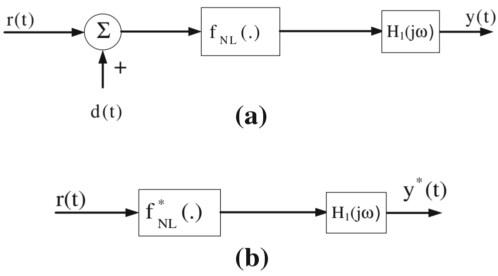

Fig. 2.7 **a** Open loop dither applied to static nonlinearity; **b** equivalent nonlinearity

The concept of 'equivalent nonlinearity' was chosen to explain and achieve insight to the dithering effect. The terms left out of the summation, i.e. g_0 in (2.5) and g_{10} in (2.7), define the equivalent nonlinearity respectively for high and low frequency dithering. The describing function concept defines a quasi-linear gain for different cases of static (time invariant and memory-less) nonlinear blocks, but this representation will depend on the stochastic behavior of the input signal. On the other hand there are approaches that do not depend on the deterministic mathematical description of the signal that drives the equivalent nonlinearity; these approaches are based on a so called probabilistic view.

Probabilistic view aims to find the equivalent nonlinearity based on the statistical description of the dithering signal, without imposing any limitation on the mathematical expression of the input signal. The set of conditions imposed on the dithering signal and the message signal does not include anymore the periodicity of the dither signal. This periodicity is replaced with a more relaxed condition of repetitive function [1]. The concept of equivalent nonlinearity is illustrated in Fig. 2.7a, b.

The two systems generate the same outputs as certain conditions on frequency of the dither and filter dynamics are met, and the modified f^* function is convolution of the probability distribution of the dithering signal with the original nonlinear function as follows:

$$f_{NL}^*(r_g) = \int_{-\infty}^{\infty} f_{NL}\left(r_g + \xi\right) P_d(\xi) d\xi \tag{2.13}$$

in which P_d represents the value distribution function of the dither signal and ξ is the instantaneous value of the dither signal. It can be achieved by taking the derivative of the cumulative distribution function of the dither signal, noted with capital letter P, defined as follows:

$$P_d(\xi) = \frac{\mu(t \in \{t_1, t_2\}, d(t) < \xi)}{t_2 - t_1} \tag{2.14}$$

in which μ is the measure function [6], or the duration in which the function is less than a certain value ξ, in the observation interval t_1, t_2. P_d is a monotonic increasing function of its argument. It is equal to 1 for infinite argument and 0 for minus infinite argument. The value probability distribution is:

$$p_d(\xi) = \frac{d(P_d(\xi))}{d\xi} \tag{2.15}$$

The condition in which the two functions f^* (and f) generate the same output depends on the frequency content of the input and the dithering signals, and the transfer function of the successive filter $(H_1(j\omega))$. Two different cases have to be distinguished for high and low frequency dithering.

From now on the dither signal should be repetitive. Assuming ΔT_d is the period of the dither signal, or the supermom[3] of the repetition time of the $p_d(t)$, the two cases are established below.

2.6.1 Equivalent Nonlinearity for High Frequency Dithering

If the high frequency attenuation criterion:

$$\lim_{\omega \to \infty} (\omega H_1(j\omega)) = 0$$

is met, the outputs of the two systems in Fig. 2.6 are identical, if the dither period tends to zero [7]:

$$\lim_{\Delta T_d \to 0} (y^*(.) - y(.)) = 0$$

while modified f* function is the convolution of the probability distribution of the dithering signal with the original nonlinear function as follows:

$$y^*(.) = f^*_{NL}(r_g) = \int_{-\infty}^{\infty} f_{NL}\big(r_g + \xi\big) P_d(\xi) d\xi \tag{2.16}$$

A detailed mathematical proof of this theorem is given in [7].

[3] In mathematics, given a subset S of a totally or partially ordered set T, the supremum (sup) of S, if it exists, is the least element of T that is greater than or equal to every element of S. Consequently, the supremum is also referred to as the least upper bound (lub or LUB). If the supremum exists, it is unique.

2.6.2 Equivalent Nonlinearity for Low Frequency Dithering

Assuming ω_r is the central (carrier) frequency of the band pass signal, and $\omega_r/2 < \omega_d < \omega_r - \Delta\omega$, where $\Delta\omega$ is the bandwidth of the band pass input signal, if the low frequency attenuation criteria are met, as stated below:

$$\lim_{|\omega-\omega_d|\to 0}((\omega - \omega_r).H_1(j\omega)) = 0$$

Then the outputs of the two systems are identical, if the ratio of the dither period to the signal period tends to zero:

$$\lim_{\Delta T_d/\Delta T_0 \to 0}(y^*(.) - y(.)) = 0$$

where $\Delta T_0 = 2\pi/\Delta\omega$. The equivalent nonlinearity as a function of the real envelope of the modulated signal can be formulated as:

$$f_{NL}^*(r_g) \triangleq \int_0^{2\pi}\int_{-\infty}^{\infty}(\frac{1}{\pi})p_d(\xi_2).f_{NL}(r_g cos\xi_1 + \xi_2)cos\xi_1.d\xi_2.d\xi_1 \quad (2.17)$$

while $p_d(.)$ is the probability distribution function of the value of the dithering signal $d(t)$, ξ_1 represents the phase of the input sinusoid and ξ_2 represents the dither.

This theorem is similar to the correlation equation presented in [1] for low pass signals, but it formulates the envelope transfer function for low frequency dithering. It shows the smoothed shape of the nonlinearity for any kind of periodic dithering signal, under the following conditions:

1. The frequency of the dithering signal is high enough compared to the complex envelope bandwidth.
2. It is high enough compared to the filter bandwidth.
3. It is low enough compared to the carrier frequency.
4. Spurious cross modulation components of dithering and signal are negligible.

Then the two systems of Fig. 2.7a, b become equivalent.

It is easy to apply equations 2.16 and 2.17, to various nonlinear functions and see the result of application of different dithering signals on equivalent nonlinearity.

2.7 Conclusion

In this chapter, the basic concept of the dithering was introduced. A distinction was made between low and high frequency dithering schemes, and the usable linearization bandwidth was derived for each case separately. The concept of equivalent nonlinearity was introduced which simplifies the analysis of the dither phenomena. The frequency domain Fourier approach and statistical approaches were introduced, which are capable of giving the equivalent nonlinearity for periodic and random signals. The equivalent nonlinearity was derived through Fourier expansion and also related to the statistical properties of the dithering signal and shape of the original memory-less nonlinearity.

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