

Chapter 2

Problem of Minimum Resistance to Translational Motion of Bodies

Newton's aerodynamic problem consists in minimizing the resistance to the *translational motion* of a three-dimensional body moving in a homogeneous medium of resting particles. The particles do not interact with each other and reflect off elastically in collisions with the body. This problem has been considered for various classes of admissible bodies. In Newton's initial setting [40], the class of admissible bodies consisted of convex axisymmetric bodies of fixed length and width, that is, bodies inscribed in a fixed right circular cylinder. The problem was later considered for various classes of (convex and axisymmetric) bodies, for example, for bodies whose front generator has a fixed length (and whose width is also fixed) [4, 32], for bodies of fixed volume [5], and so on. A major step forward was made in the 1990s, when unexpected and striking results were obtained for some classes of *nonaxisymmetric* bodies and, later, for *nonconvex* bodies [8, 11, 12, 16–18, 29–31]). However, the authors kept the initial assumption that the body must have a fixed length and width, that is, can be inscribed in a fixed right circular cylinder.

A further constraint imposed on all classes of bodies was as follows. A particle cannot hit a body more than once. Here we do not impose this constraint. In Sect. 2.1, we consider two classes of (generally speaking, nonconvex and nonsymmetric) bodies inscribed in a circular cylinder. These classes differ in accordance with how one understands the expression “inscribed in a cylinder.” We show that in each class the infimum of the resistance is zero, that is, there exist “almost perfectly streamlined” bodies. In Sect. 2.2, this result is generalized to right cylinders with an arbitrary (not only circular) section. In Sect. 2.3, we demonstrate that any convex body can be modified within a small neighborhood of its boundary so that the resulting body displays a resistance less than an arbitrarily small $\varepsilon > 0$. In fact, any body can be made “almost perfectly streamlined” by making microscopic longitudinal “grooves” in its surface. In Sect. 2.4, we consider an analog of Newton's problem in the two-dimensional case. Here the minimum resistance is always positive but smaller than that of convex bodies. In Sect. 2.5, we consider the problem of minimum *specific resistance* for *unbounded bodies* in a parallel flow of particles.

Most of the results presented in this chapter were first published in [43, 44, 47, 52, 55].

2.1 Bodies Inscribed in a Circular Cylinder

We consider the problem of minimizing the functional $R_{\delta_{v_0}}$ in (1.7) with $v_0 = (0, 0, -1)$ and continuous function c satisfying $c(v, v) = 0$, over various classes of bodies. In the important case $c(v, v^+) = (v - v^+) \cdot v$, integral (1.7) denotes the resistance to the translational motion of a body with velocity $-v_0$ in a medium of resting particles.

Throughout this chapter, we assume without further mention that the measurable function $v_B^+(\cdot, v_0)$ is defined almost everywhere on $\mathbb{R}^3$ (in view of the translational invariance, this means that it is defined almost everywhere on $\mathbb{R}^2 \times \{0\}$ and is measurable there). This ensures the existence of the integral $R_{\delta_{v_0}}(B)$ in (1.7). In essence, the condition means the regularity of the scattering of particles falling in the direction of v_0 . In Figs. 2.1 and 2.2, we present two examples of bodies for which the condition of regularity fails.

A set B not satisfying the regularity condition is obtained by revolution of a plane set S through 360° about the vertical axis AB (Fig. 2.1). The set S is obtained from the rectangle $ABCD$ by removing a subset part of whose boundary is an arc of a parabola with a vertical axis and with a focus at a singular point F of the boundary of this subset. The velocity v_0 of the flow of particles is directed downward: $v_0 = (0, 0, -1)$. The particles reflecting from the parabolic part of the boundary go to the singular point F , and after hitting this point their further motion is not defined. Thus, the function $v_B^+(\cdot, v_0)$ is not defined on a positive-measure subset corresponding to these particles.

Another mechanism to break regularity is due to the fact that a positive-measure set of particles can remain within a bounded set forever, that is, be trapped, and in that case the function $v_B^+(\cdot, v_0)$ is not defined on the set corresponding to these particles. An example of a trap made from arcs of ellipse and parabola is given in Sect. 2.2 of [68]; a similar construction is reconstructed here in Fig. 2.2.

Below we consider the minimization problem in classes of (generally) nonconvex bodies inscribed in a given cylinder. The notion of a body inscribed in a cylinder can

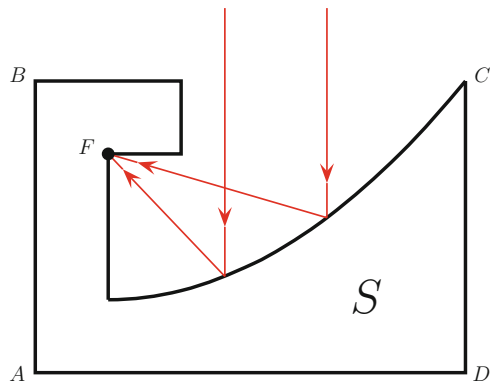


Fig. 2.1 Example of a body with irregular scattering. A positive-measure set of particles hit a singular point of the body's boundary

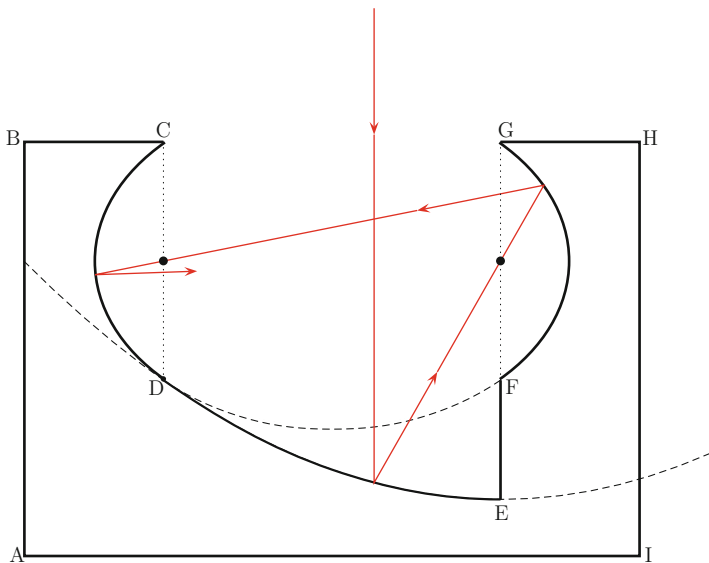


Fig. 2.2 The body ABCDEFGHI is a trap. Curves CD and GF are arcs of an ellipse, and DE is an arc of the parabola with a vertical axis and focus at one of the foci of the ellipse. The set of particles falling downward on arc DE will remain forever in the hollow formed by curve $CDEFG$

be defined in different ways. We introduce two distinct classes of bodies inscribed in a right circular cylinder of radius 1 and height h : the class $\mathcal{P}(h)$ of bodies whose projection on the horizontal plane is a disc and the class $\mathcal{S}(h)$ of bodies containing at least one horizontal section of the cylinder. Note that the class $\mathcal{P}(h)$ is wider than $\mathcal{S}(h)$: $\mathcal{S}(h) \subset \mathcal{P}(h)$. The infima over both classes are equal to 0. We first prove this for $\mathcal{P}(h)$ and then give a sketch of the proof for $\mathcal{S}(h)$.

2.1.1 The Class of Bodies with Fixed Horizontal Projection

Definition 2.1. We denote by $\mathcal{P}(h)$ the class of connected bodies B lying in the cylinder $C_h = \text{Ball}_1(0) \times [-h, 0]$ and such that the orthogonal projection of B on the plane Ox_1x_2 is the unit disc $\text{Ball}_1(0)$.

Proposition 2.1. $\inf_{B \in \mathcal{P}(h)} R_{\delta_{v_0}}(B) = 0$.

Proof. We shall construct a one-parameter family $B_\varepsilon \in \mathcal{P}(h)$, $\varepsilon > 0$, of bodies such that $R_{\delta_{v_0}}(B_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. First we carry out the construction for $h \geq 1/2$ and then for $h < 1/2$.

- (i) For $h \geq 1/2$ we fix ε and consider in the plane Ox_1x_3 the two curvilinear triangles in the rectangle $A OCD = [0, 1] \times [-1/2, 0]$ obtained by cutting

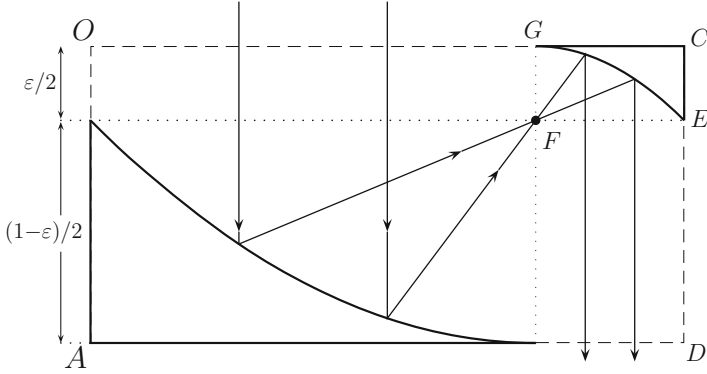


Fig. 2.3 The case $h \geq 1/2$: an auxiliary construction with two parabolas

off the lower left and upper right corners of the rectangle by arcs of parabolas (Fig. 2.3). These parabolas have the common focus $F = (1 - \varepsilon, -\varepsilon/2)$ and a common vertical axis. The arc of the first parabola has endpoints $(1 - \varepsilon, -1/2)$ and $(0, -\varepsilon/2)$, and it cuts off the larger triangle. The arc of the second parabola has endpoints $G = (1 - \varepsilon, 0)$ and $E = (1, -\varepsilon/2)$, and it cuts off the smaller triangle CEG with size of order ε .

Let $B_\varepsilon = B_\varepsilon(h) = B'_\varepsilon \cup B''_\varepsilon$, where the set B'_ε is obtained by revolution of both triangles about the vertical axis Ox_3 (containing side AO). We add the set B''_ε to make the resulting set B_ε connected. The scattering of particles is determined by the set B'_ε , whereas B''_ε perturbs the scattering slightly and can be selected in various ways. For instance, we can take B''_ε to be a cylindrical sector with a small opening: the intersection of the cylinder C_h with the set $|x_1| \leq \varepsilon|x_2|$.

It is easy to calculate $v_{B_\varepsilon}^+(\xi, v_0)$, where $\xi = (x_1, x_2, 0)$, $x_1^2 + x_2^2 \leq 1$. If $|x_1| < \varepsilon|x_2|$ or $1 - \varepsilon < \sqrt{x_1^2 + x_2^2} < 1$, then a particle is reflected vertically upward, so that $v_{B_\varepsilon}^+(\xi, v_0) = -v_0$. On the other hand, if $|x_1| > \varepsilon|x_2|$ and $\sqrt{x_1^2 + x_2^2} < 1 - \varepsilon$, then a particle is reflected from the larger parabolic arc, passes through the common focus, is reflected from the smaller parabolic arc, and then moves vertically downward. Thus, on leaving cylinder C the particle has velocity $v_{B_\varepsilon}^+(\xi, v_0) = v_0$. The quantity $R_{\delta_{v_0}}(B_\varepsilon)$ is the product of $c(v_0, -v_0)$ and the area of the upper cross section of the set B_ε , which is the union of the annulus $1 - \varepsilon \leq \sqrt{x_1^2 + x_2^2} \leq 1$ and the two sectors $|x_1| < \varepsilon|x_2|$. Hence $R_{\delta_{v_0}}(B_\varepsilon) = O(\varepsilon)$.

- (ii) For $h < 1/2$ the construction is slightly more complicated. We take an integer n such that $1/n < 2h$ [for instance, $n = n(h) = \lfloor 1/(2h) \rfloor + 1$], and let $b = 1/n - 3\varepsilon/2$. Below we define a set in the plane Ox_1x_3 that is the union of n pairs of curvilinear triangles and one right isosceles triangle with side $n\varepsilon$ (Fig. 2.4). First we consider the sequence of n rectangles with horizontal side

2.1.2 The Class of Sets Containing a Section of the Cylinder

Definition 2.2. We denote by $\mathcal{S}(h)$ the class of connected bodies lying in the cylinder C_h and containing at least one horizontal section, $\text{Ball}_1(0) \times \{c\}$, $-h \leq c \leq 0$, of the cylinder.

Proposition 2.2. $\inf_{B \in \mathcal{S}(h)} R_{\delta_{v_0}}(B) = 0$.

Proof. As was done previously, we construct a family of bodies B_ε such that $R_{\delta_{v_0}}(B_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. We obtain this family by a slight modification of the construction used in the proof of Proposition 2.1, namely, we add a set containing the bottom base $\text{Ball}_1(0) \times \{-h\}$ of the cylinder. In addition, for $h > 1/2$ we rotate slightly the axis of the upper (smaller) parabolic arc about the fixed focus, and in the case $h \leq 1/2$, we make the slope of the hypotenuse of the right triangle slightly less than 45° . All these modifications are required to make a particle reflected from the upper parabolic arc or from the hypotenuse of the right triangle move along a slightly inclined (not vertical) line and to go past the lower base of the cylinder without further collisions.

- (i) For $h > 1/2$ we consider two sets in place of the two curvilinear triangles in the previous construction (Fig. 2.3). First, the arc of the lower parabola is extended to the right until it intersects the right lateral side of the rectangle $\square = [0, 1] \times [-h, 0]$, and we consider the set M_1 cut off of the rectangle $\square$ by this arc and lying under the arc. (Note that $\square$ contains the rectangle $OADC$ and generates the cylinder C_h when revolved about the axis OA .)

Next, the line joining this intersection point with the focus F is the axis of the upper parabola. Thus, the upper parabola has the same axis $F = (1 - \varepsilon, -\varepsilon/2)$ and passes through the same point $E = (1, -\varepsilon/2)$ as in the previous construction [part (i) of the proof of Proposition 2.1], but now the axis of the parabola will form an angle of order ε with the vertical direction. This parabola intersects the boundary of the smaller rectangle $CEFG = [1 - \varepsilon, 1] \times [-\varepsilon/2, 0]$ at two points: one is point E and the other lies on side FG . The second set M_2 is the part of rectangle $CEFG$ cut off by the parabola and lying over it.

The set B'_ε is the result of the revolution of these two sets $M_1 \cup M_2$ about the axis Ox_3 , and the set B''_ε is as above. The set $B_\varepsilon = B'_\varepsilon \cup B''_\varepsilon$ contains the section $\text{Ball}_1(0) \times \{-1/2\}$ of the cylinder. A particle reflected from the arc of the lower parabola passes through the common focus F , is reflected from the arc of the upper parabola, and then moves freely at an angle of magnitude $O(\varepsilon)$ with the vertical direction. The contribution of such particles to the functional is $o(1)$, whereas the contribution of the particles reflecting back vertically upward is $O(\varepsilon)$, as before. Thus, $R_{\delta_{v_0}}(B_\varepsilon) = o(1)$.

- (ii) For $h \leq 1/2$ we add the rectangle $[0, 1] \times [-h, -h + \varepsilon]$ to the system of triangles ($2n$ curvilinear triangles and one right triangle). The right triangle from the system should now be modified as follows. It should have the same two vertices $(1, 0)$ and $(1, -n\varepsilon)$ as before, whereas the third vertex now has

coordinates $(1 - n\varepsilon', 0)$. Here we choose ε' such that each particle moving horizontally to the right, after reflection from the hypotenuse of the triangle, passes to the right of $(1, -h + \varepsilon)$ and so does not hit the indicated (added) rectangle. For this we can take $\varepsilon' = \varepsilon / \sqrt{1 - 2n\varepsilon/(h - \varepsilon)} = \varepsilon + O(\varepsilon^2)$. Finally, we set $b = 1/n - \varepsilon' - \varepsilon/2$. Taking account of this modification, we construct n pairs of curvilinear triangles as before.

A particle falling on one of the larger arcs of parabolas in this construction will be successfully reflected by a larger and a smaller arc of parabolas, and then it is reflected from the hypotenuse of the right triangle and moves freely at an angle $\arctan \frac{\varepsilon'^2 - \varepsilon^2}{2\varepsilon\varepsilon'} = O(\varepsilon)$ with the vertical direction. Thus, the difference between the initial and final velocities of the particle is $O(\varepsilon^2)$.

The set B'_ε is obtained by the revolution of n pairs of triangles, the right triangle, and the additional rectangle about the axis AO . The set B''_ε is as above. The set $B_\varepsilon = B'_\varepsilon \cup B''_\varepsilon$ contains the section $\text{Ball}_1(0) \times \{-h\}$ of the cylinder. As before (see the proof of Proposition 2.1), we define the set $\mathcal{N}(h, \varepsilon)$ and show that $\text{Area}(\mathcal{N}(h, \varepsilon)) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Furthermore, for $\xi \in \mathcal{N}(h, \varepsilon)$ the particle reflects back vertically upward, so that $v_{B_\varepsilon}^+(\xi, v_0) = -v_0$, while if $\xi \notin \mathcal{N}(h, \varepsilon)$, then $v_{B_\varepsilon}^+(\xi, v_0) = v_0 + o(1)$. Thus, $R_{\delta_{v_0}}(B_\varepsilon) = c(v_0, -v_0) \cdot \text{Area}(\mathcal{N}(h, \varepsilon)) + o(1) = o(1)$ as $\varepsilon \rightarrow 0$. The proof of Proposition 2.2 is complete. $\square$

Remark 2.1. Note that in case (i) each particle hits the body B_ε at most twice, and in case (ii) at most three times. An open question is: for which h are two hits enough, that is, what is the minimal h_0 such that for $h > h_0$ there exists a sequence of sets with resistance tending to zero such that particles in the flow have only one or two collisions with them? This question can be posed for the classes of sets $\mathcal{P}(h)$ and $\mathcal{S}(h)$. It follows from the proofs of Propositions 2.1 and 2.2 that $0 \leq h_0 \leq 1/2$ in both cases.

Remark 2.2. The body in Fig. 2.4 is actually a (disconnected) *two-dimensional* body of arbitrarily small resistance. It is contained in the rectangle $[0, 1] \times [-h, 0] \subset \mathbb{R}_{x_1 x_3}^2$, and its projection on the x_1 -axis is $[0, 1]$.

Substituting the segment $[0, 1]$ with a generic set $I \subset \mathbb{R}_{x_1}$, we come to the following more general statement, which will be used in the next section.

Proposition 2.3. *Let $I = \cup_i I_i \subset \mathbb{R}_{x_1}$ be the union of a finite number of disjoint compact segments. Set $x_0 = (0, -1)$. Consider a family of segments J_δ , $\delta > 0$ and assume that $|J_\delta| = \delta$, $I \cap J_\delta = \emptyset$ and $\text{dist}(\text{Conv}(I), J_\delta) \rightarrow 0$ as $\delta \rightarrow 0$. Then the following holds.*

- (a) *For any ε there exists a two-dimensional body $B \subset I \times [-h, 0] \subset \mathbb{R}_{\{x_1, x_2\}}^2$ such that its projection on the x_1 -axis coincides with I and its resistance is smaller than ε , that is, $R_{\delta_{x_0}}(B) < \varepsilon$.*
- (b) *For δ sufficiently small there exists a body $B_\delta \subset I \times [-h, 0]$ such that the projection of B_δ on the x_1 -axis coincides with I , $\lim_{\delta \rightarrow 0} R_{\delta_{x_0}}(B_\delta) = 0$, and no particle of the vertical flow intersects $(\mathbb{R} \setminus J_\delta) \times [-h, -h + \delta]$.*

The proof of this proposition is obtained by a slight modification of the construction of Fig. 2.4 and is omitted here.

2.2 Bodies Inscribed in an Arbitrary Cylinder

Here we state a generalization of the results of the previous section. Consider a connected body $\Omega \subset \mathbb{R}^2$.

- Definition 2.3.** (a) We denote by $\mathcal{P}(\Omega, h)$ the class of connected sets B contained in the cylinder $\Omega \times [-h, 0]$ and such that the orthogonal projection of B on the plane Ox_1x_2 is Ω .
 (b) We denote by $\mathcal{S}(\Omega, h)$ the class of connected sets B contained in the cylinder $\Omega \times [-h, 0]$ and containing a section of the cylinder $\Omega \times \{c\}$, $-h \leq c \leq 0$.

Theorem 2.1. (a) $\inf_{B \in \mathcal{P}(\Omega, h)} |R_{\delta_{v_0}}(B)| = 0$.

(b) $\inf_{B \in \mathcal{S}(\Omega, h)} |R_{\delta_{v_0}}(B)| = 0$.

Proof. Clearly, $\mathcal{S}(\Omega, h) \subset \mathcal{P}(\Omega, h)$; therefore (b) implies (a). Nevertheless, we will first provide a proof for (a) that makes clearer the reasoning in the proof of (b).

Generally speaking, the boundary $\partial\Omega$ is the union of a finite number of closed non-self-intersecting (and not intersecting each other) piecewise smooth curves of finite length. We shall assume that $\partial\Omega$ is a single curve; the general case is obtained by a slight modification of the argument. Let l be the length of this curve, $|\partial\Omega| = l$. Consider the square lattice formed by the lines $x_1 = n_1\delta$, $x_2 = n_2\delta$, $n_1, n_2 \in \mathbb{Z}$ in the plane Ox_1x_2 containing Ω . The squares of this grid have size $\delta \times \delta$. The squares that have a nonempty intersection with $\partial\Omega$ will be called *boundary squares*. Let us show that for δ sufficiently small the number of boundary squares does not exceed $5l/\delta$.

Indeed, divide the curve $\partial\Omega$ into $n = \lfloor l/(2\delta) \rfloor + 1$ pieces of equal length (clearly, this length does not exceed 2δ) and mark the midpoint on each piece; this point divides the piece into two parts of length δ . Take the square containing this point and eight squares surrounding it. We get a large square composed of these nine squares, which will be called a *tile*. The distance from the midpoint to each point of the piece does not exceed δ , and the distance from the midpoint to the boundary of the tile is at least δ ; this implies that the piece lies in the tile. Hence the curve is contained in the union of at most n tiles and, thus, in the union of at most $9n$ squares. Taking into account that $n \leq l/(2\delta) + 1$ and choosing δ sufficiently small, we get that the number of boundary squares does not exceed $5l/\delta$. Thus the area of the union of boundary squares does not exceed $5l\delta$.

- (a) Let Ω_δ be the union of the squares contained in Ω , and let Ω_δ^n be the intersection of Ω with the strip $n\delta \leq x_1 \leq (n+1)\delta$. Thus, Ω_δ^n is composed of squares of size $\delta \times \delta$ lying in a strip of width δ . Note that the sets Ω_δ^n are nonempty only for a finite number of values n and have the form $\Omega_\delta^n = [n\delta, (n+1)\delta] \times I_\delta^n$,

where I_δ^n is the union of a finite number of segments whose length is a multiple of δ . One has $\Omega_\delta = \cup_n \Omega_\delta^n$. Let Ω'_δ be the intersection of Ω with the union of boundary squares; then one has $\Omega = \Omega_\delta \cup \Omega'_\delta$.

The required set of small resistance is $B_\delta = \tilde{B} \cup \hat{B}$, where $\tilde{B} = \tilde{B}_\delta$ and $\hat{B} = \hat{B}_\delta$ are constructed as follows. First we define the intersection of $\tilde{B}$ with $\Omega'_\delta \times \mathbb{R}$ and, second, the intersection of $\tilde{B}$ with each set $\Omega_\delta^n \times \mathbb{R}$, $n \in \mathbb{Z}$. By this, $\tilde{B}$ will be uniquely defined. Further, $\hat{B}$ is the intersection of the union of “walls” $W_n = W_n^\delta = \{x = (x_1, x_2, x_3) : |x_1 - n\delta| < \delta^2\}$ with the cylinder $\Omega \times [-h, 0]$, $\hat{B} = (\cup_{n \in \mathbb{Z}} W_n) \cap (\Omega \times [-h, 0])$. The “sandwich” B_δ is the union of these sets; adding the union of “layers” $\hat{B}$ makes it connected and changes the resistance only by $O(\delta)$.

Set $\tilde{B} \cap (\Omega_\delta^n \times \mathbb{R}) = \Omega_\delta^n \times [-h, -h + \delta]$; this is a cylinder of height δ . This is a part of the body; it makes a small contribution (of order δ) to the resistance.

Further, using the statement of Proposition 2.3(a), for each value n such that Ω_δ^n is nonempty we construct a body $D_\delta^n \subset I_\delta^n \times [-h, 0]$. The resistance of this body is smaller than δ , and its projection on the x_1 -axis is I_δ^n . Then we set

$$\tilde{B} \cap (\Omega_\delta^n \times \mathbb{R}) = [n\delta, (n+1)\delta] \times D_\delta^n.$$

Thus we have $R_{\delta_{x_0}}(\tilde{B} \cap (\Omega_\delta^n \times \mathbb{R})) < \delta^2$. As a result one obtains

$$R_{\delta_{x_0}}(B_\delta) = R_{\delta_{x_0}}(\tilde{B} \cap (\Omega'_\delta \times \mathbb{R})) + \sum_n R_{\delta_{x_0}}(\tilde{B} \cap (\Omega_\delta^n \times \mathbb{R})) + R_{\delta_{x_0}}(\hat{B}) = O(\delta), \quad \delta \rightarrow 0.$$

Finally, it is clear from the construction that $B \subset \mathcal{P}(\Omega, h)$.

- (b) A row of squares is called *special* if it contains more than $\sqrt{5l/\delta}$ boundary squares. Clearly, the number of special rows does not exceed $\sqrt{5l/\delta}$; otherwise we would have a number of boundary squares larger than $5l/\delta$.

The intersection of Ω with the union of all boundary squares and all special rows is denoted by Ω''_δ . Clearly, the area of Ω''_δ is $O(\sqrt{\delta})$.

For each n corresponding to a nonspecial row we select a square of the lattice $[n\delta, (n+1)\delta] \times J_\delta^n$ that does not intersect Ω and has the distance at most $\sqrt{5l\delta}$ from $\text{Conv}(\Omega''_\delta)$. Note that this selection can be made due to the property of a nonspecial row: there are at most $\sqrt{5l/\delta}$ boundary squares between the square $[n\delta, (n+1)\delta] \times J_\delta^n$ and the rectangle $\text{Conv}(\Omega''_\delta)$. Then, using Proposition 2.3(b), one constructs a set $\tilde{D}_\delta^n$ lying in $I_\delta^n \times [-h, 0]$, with the resistance $o(1)$ as $\delta \rightarrow 0$, and such that no particle reflected from $\tilde{D}_\delta^n$ intersects $(\mathbb{R} \setminus J_\delta^n) \times [-h, -h + \delta]$.

Let

$$B_\delta = \tilde{B} \cup \hat{B} \cup B',$$

where $\hat{B} = \hat{B}_\delta$ is as in (a),

$$B' = B'_\delta = \Omega \times [-h, -h + \delta]$$

and

$$\tilde{B} = \tilde{B}_\delta = \cup'_n([n\delta, (n+1)\delta] \times \tilde{D}_\delta^n),$$

where the union is taken over nonspecial n .

By adding $\tilde{B}$, as in (a), we ensure that the resulting body B_δ is connected, and adding B' ensures that the body belongs to the class $\mathcal{S}(\Omega, h)$. Finally, the set of small resistance $\tilde{B}$ is constructed in such a way that particles reflected from it do not hit B' . Thus, we have $R_{\delta_{x_0}}(B_\delta) = o(1)$, $\delta \rightarrow 0$. $\square$

The topic of this section will be continued in Chap. 8 (Theorem 8.2).

2.3 Bodies Modified in a Neighborhood of Their Boundary

Here we show that a convex body can be “modified” in a small neighborhood of its boundary so that the resulting body displays an arbitrarily small resistance to the parallel flow of particles falling on it.

Let us first provide an “astronautical” interpretation of the problem. Suppose we are traveling in a spaceship $C_2 \subset \mathbb{R}^3$, which is a bounded convex set. The inner space of the spaceship coincides with another convex set $C_1 \subset C_2$ (Fig. 2.5). The spaceship body is then $C_2 \setminus C_1$; it is natural to require that $\partial C_1 \cap \partial C_2 = \emptyset$ (meaning that the thickness of the spacecraft body is everywhere positive).

We are going to process the metallic body of the spaceship aiming to minimize the velocity slowdown when going through space clouds. The processing may result in dimples, hollows, grooves, etc. on the spaceship surface. In general, we assume that any body B satisfying the inclusions $C_1 \subset B \subset C_2$ can be obtained by such a processing. We pose the following question.

Question: Given the convex bodies C_1 and C_2 , the spaceship velocity v , and the cloud density, what is the minimum resistance of the resulting body B ?

Note in passing that the resistance of the original body C_2 can be very easily decreased just by making dimples. Indeed, let the direction of v be vertical and consider a region $\mathcal{U} \subset \partial C_2$ in the upper part of the surface ∂C_2 whose inclination

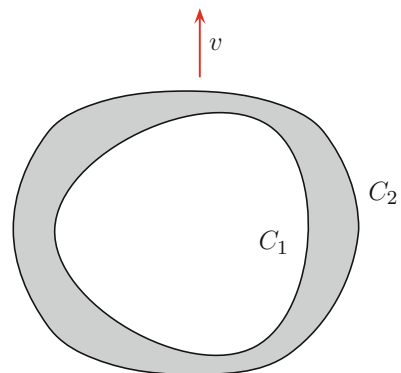


Fig. 2.5 Spaceship. The direction of motion is indicated by v

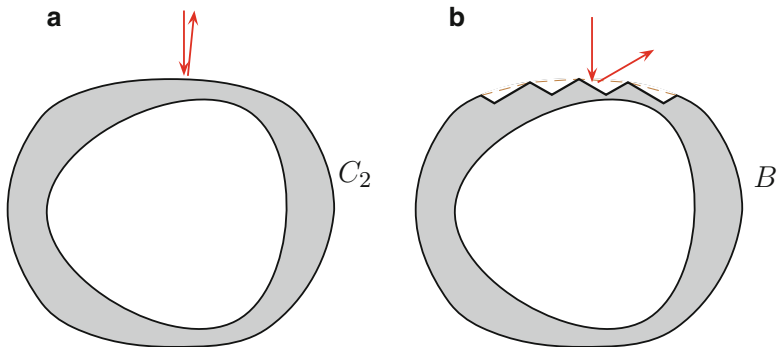


Fig. 2.6 Original spaceship C_2 (a) and spaceship B with dimples (b)

relative to the horizontal plane is less than 30° . (We assume that $\mathcal{U}$ is not empty.) Then make several conical dimples in this region, with the inclination of the cone surface being exactly 30° (Fig. 2.6b).

The resistance of the resulting body B is smaller than that of the original body C_2 . Indeed, in a reference system attached to the body we observe a parallel flow of particles falling vertically downward. A particle hitting C_2 in the region $\mathcal{U}$ is reflected at an angle smaller than 60° relative to the vertical (Fig. 2.6a). In contrast, a particle hitting B in a conical dimple will be reflected exactly at a 60° angle (Fig. 2.6b). Therefore, the momentum transmitted by the particle to the body is smaller in the latter case than in the former one, and summing up all the transmitted momenta, we get that the resistance of B is smaller than that of C_2 .

In a similar way, the resistance of Newton's optimal body can be decreased by making dimples on its front (flat) surface. This observation was first made by Buttazzo and Kawohl in [12]. The techniques of making dimples and grooves were further developed by Comte and Lachand-Robert in [16–18] in their study of generalizations of Newton's problem in classes of nonconvex bodies.

The answer to our question is surprising: the resistance of the resulting body can be made arbitrarily small. That is, by processing the surface of our spaceship, one can make it almost perfectly streamlined!

Theorem 2.2. *In the three-dimensional case we have*

$$\inf\{|R_v(B)| : C_1 \subset B \subset C_2, B \text{ connected}\} = 0.$$

The proof of this theorem is based on the following auxiliary two-dimensional result. Let C_1 and C_2 be bounded convex bodies, $C_1 \subset C_2 \subset \mathbb{R}^2$, $\partial C_1 \cap \partial C_2 = \emptyset$, and $v \in S^1$.

Theorem 2.3. *In the two-dimensional case we have*

$$\inf\{|R_v(B)| : C_1 \subset B \subset C_2\} = 0.$$

These results were announced, with a brief outline of the proof, in [52], and the authors' detailed exposition is given in [55].

Remark 2.3. The statement of the two-dimensional theorem is weaker than that of the three-dimensional one since the infimum is taken over the wider class of (generally) *disconnected* bodies. In contrast, the infimum over *connected* bodies is always positive in two dimensions.

The following plausible conjecture is intended to further elucidate the difference between the two-dimensional and three-dimensional cases.

Definition 2.4. Let $C \subset \mathbb{R}^d$ be a bounded convex body and $v \in S^{d-1}$. The value

$$\mathcal{R}_v(C) := \sup_{C_1} \inf \{ |R_v(B)| : C_1 \subset B \subset C, B \text{ connected} \},$$

where the supremum is taken over all convex bodies C_1 such that $C_1 \subset C$ and $\partial C_1 \cap \partial C_2 = \emptyset$, is called the *minimal resistance of bodies obtained by roughening C*.

Conjecture 2.1. (a) For any $v \in S^2$ and any convex $C \subset \mathbb{R}^3$,

$$\mathcal{R}_v(C) = 0$$

holds.

(b) For any $v \in S^1$ and any convex $C \subset \mathbb{R}^2$,

$$1/4 < \frac{\mathcal{R}_v(C)}{|R_v(C)|} \leq 1/2 \quad (2.1)$$

holds.

Statement (a) of this conjecture is a direct consequence of Theorem 2.2, but statement (b) has not been proved yet. Note, however, that in the particular case where C is symmetric with respect to an axis parallel to v , the double inequality (2.1) can be easily derived from the proof of Theorem 2 in [52].

2.3.1 Preliminary Constructions

Here we explain the basic two-dimensional construction that will be used in the proofs of Theorems 2.3 and 2.2. The main idea of these proofs consists in making a large number of “diversion channels” penetrating the body near its boundary. Each channel is the union of three sets: a *front funnel*, a *tube*, and a *rear funnel*. The front funnel is turned to the flow, and the rear one, to the opposite direction. Each particle of the flow goes into the front funnel of a channel, then moves through the channel along the body boundary, and finally leaves through the rear funnel, and its final velocity only slightly differs from the velocity of incidence.

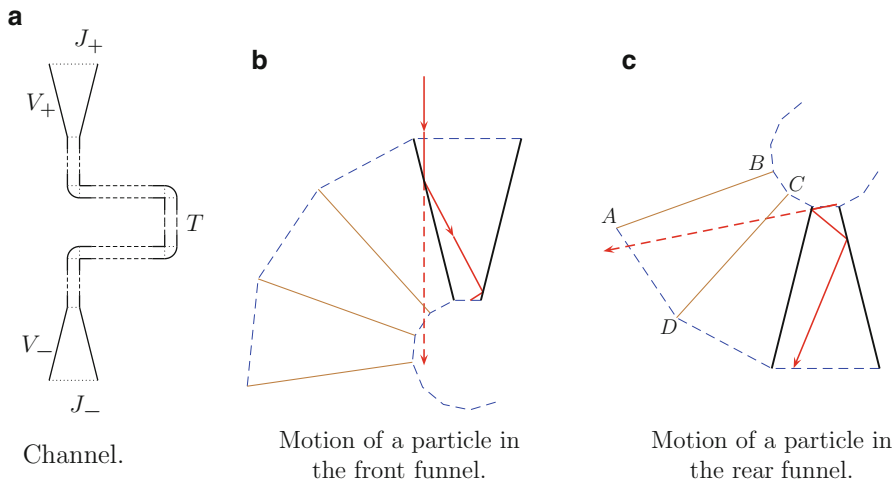


Fig. 2.7 Dynamics of a particle in a channel

Introduce the coordinates x_1, x_2 on a plane such that $v = (0, -1)$, the x_1 -axis being considered horizontal, and the x_2 -axis, vertical. Fix the parameter $0 < \varepsilon < 1$. The front and rear ε -funnels $V_{\pm}$ are the trapezoids $|x_1| \leq \varepsilon|x_2|$, $\varepsilon^2 \leq \pm x_2 \leq \varepsilon$, respectively. The point $(0, \pm\varepsilon^2)$ is called the vertex of the corresponding funnel. The front and rear sides of the front funnel are, respectively, its larger and smaller bases, that is, the segments $\{|x_1| \leq \varepsilon^2, x_2 = \varepsilon\}$ and $\{|x_1| \leq \varepsilon^3, x_2 = \varepsilon^2\}$. In contrast, the front and rear sides of the rear funnel are its smaller and larger bases, that is, the segments $\{|x_1| \leq \varepsilon^3, x_2 = -\varepsilon^2\}$ and $\{|x_1| \leq \varepsilon^2, x_2 = -\varepsilon\}$. A parallel translation of the front (rear) funnel is also called a front (rear) funnel (Fig. 2.7a).

An ε -tube is a finite sequence of figures: rectangles and circle sectors. These figures are called *elements* of the tube. The rectangles are vertically or horizontally oriented; they are called *v*- and *h*-rectangles, respectively. In a *v*-rectangle, one of the horizontal (upper and lower) sides is considered the *front* side, and the other horizontal side is the *rear* one. Their length equals $2\varepsilon^3$. In an *h*-rectangle, the length of the vertical (left and right) sides equals $2\varepsilon^3$; one of these sides is the front one, and the other side is the rear one. Each circle sector has an angular size of 90° ; it is a quarter of a circle of radius $2\varepsilon^3$. One of the radii bounding the sector is vertical, and the other one is horizontal; one of these radii is called the front one, and the other the rear one. In the sequence of figures forming the tube, rectangles and circle sectors alternate; the first and last figures are *v*-rectangles, the upper side of the first rectangle is the front one, and the lower side of the last rectangle is the rear one (Fig. 2.7a). Further, in the subsequence composed of rectangles, the *v*- and *h*-rectangles alternate. Finally, in the sequence of figures (rectangles and circle sectors) forming the tube, the rear side of the preceding figure coincides with the front side of the subsequent figure, and there are no other points of pairwise intersection of the figures.

It may happen, in particular, that the tube is a single v -rectangle; in this case its upper side is the front one, the lower side is the rear one, and the length of these sides is $2\varepsilon^3$.

Definition 2.5. An ε -channel is the union

$$K = V_+ \cup T \cup V_- \subset \mathbb{R}_{\{x_1, x_2\}}^2$$

of a front ε -funnel V_+ , an ε -tube T , and a rear ε -funnel V_- satisfying the following conditions: the rear side of the front funnel coincides with the front side of the tube, the rear side of the tube coincides with the front side of the rear funnel, and there are no other points of pairwise intersection for these figures. The front part of the front funnel is called the *front side* of the channel (denoted by J_+ in Fig. 2.7a), and the rear side of the rear funnel is called the *rear side* of the channel (denoted by J_- in Fig. 2.7a). The rest of the channel boundary is called the *lateral boundary* of the channel.

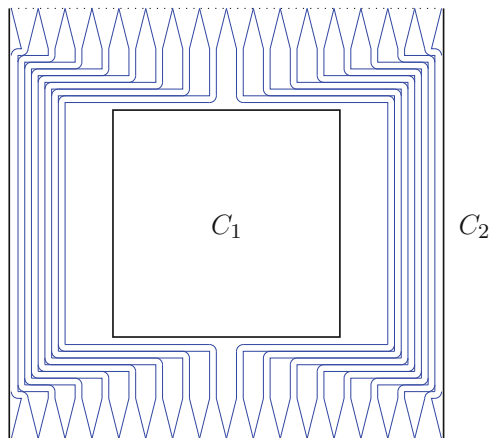
Lemma 2.1. Consider a billiard in an ε -channel. If a particle starts its motion with velocity $v = (0, -1)$ at a point of the front side of the channel, then after making a finite number of reflections from the lateral boundary it finally comes into a point of the rear side of the channel with velocity $v + O(\varepsilon)$, $\varepsilon \rightarrow 0$. Here $O(\varepsilon)$ is uniform with respect to all ε -channels and all initial positions.

Proof. First we prove that the particle, after a finite number of reflections from the lateral boundary, crosses the rear side of the channel (and not its front side). The proof of this statement is inductive. That is, for each figure forming the channel (trapezoid, rectangle, circle sector) we will prove the following: if the particle enters the figure through its front side, then after a while it will leave the figure through its rear side.

The motion in a rectangle is unidirectional, from the front to the rear side; this is obvious. Further, notice that when moving in a circle, the angular coordinate of the particle changes monotonically. This implies that if the particle intersects the front radius of a sector, then after several (maybe none) reflections from the arc it will intersect the rear radius.

It remains to consider the motion in the funnels. The particle starts moving vertically down from the front side of the front funnel (that is, from the larger side of the trapezoid). Apply the method of unfolding of the trajectory; (Fig. 2.7b). In a convenient reference system, the trapezoid takes the form $|x_1| \leq \varepsilon x_2$, $\varepsilon^2 \leq x_2 \leq \varepsilon$. The unfolded trajectory is a vertical line at a distance less than ε^2 from the origin; therefore, it intersects the circle $\text{Ball}_{\varepsilon^2}(0)$. On the other hand, the sequence of images of the smaller side of the trapezoid under the unfolding forms a broken line winding around the origin and touching the same circle. (Notice that this broken line is contained in the larger circle $\text{Ball}_{\varepsilon^2\sqrt{1+\varepsilon^2}}(0)$; we will use it later.) Hence the unfolded trajectory intersects the broken line; this means that the original trajectory, after several reflections from the lateral sides of the trapezoid, will intersect its smaller side.

Fig. 2.8 Two concentric squares with a built-in channel system



Finally, when considering the motion in the rear funnel we again use the unfolding method. This time we unfold the final part of the trajectory starting from the point of intersection with the front side of the funnel (that is, the smaller base of the trapezoid; see Fig. 2.7c). The unfolded trajectory intersects one of the images, under the unfolding, of the larger base of the trapezoid; this image is AD in the figure. This means that the particle, after several reflections from the lateral sides of the trapezoid, finally reaches the rear side of the channel. Using Fig. 2.7c one gets an estimate for the particle velocity at the point of intersection with the rear side of the funnel. The angle the velocity vector forms with the vertical is obviously smaller than the largest angle formed by the symmetry axis of $ABCD$ with the tangent lines from A to the circle $\text{Ball}_{\varepsilon^2\sqrt{1+\varepsilon^2}}(0)$. The latter quantity equals $\arctan \varepsilon + \arcsin \varepsilon$. Thus, the difference between the initial and final velocities, v and v^+ , of a particle in an ε -channel can be estimated from above as follows:

$$|v - v^+| \leq 4 \sin^2((\arctan \varepsilon + \arcsin \varepsilon)/2) = O(\varepsilon).$$

□

Figure 2.8 shows what the channel system might look like in the case where C_1 and C_2 are concentric squares. A body of small resistance is obtained by removing the channels from the larger square C_2 .

The construction of the channel system in the general case is more complicated. We will start with a method of constructing a special ε -channel that will be used later in this section. Consider two rectangles Π^+ and Π^- with horizontal sides of length $2\varepsilon^2$ and vertical sides of length $\varepsilon - \varepsilon^2$, let A^+ be the midpoint of the lower side of Π^+ , and let A^- be the midpoint of the upper side of Π^- . A broken line joining points A^+ and A^- and satisfying the conditions stated later in this paragraph will be called an ε -axis, and these points will be called the front and rear endpoints of the axis. The broken line consists of a finite number of vertical and horizontal segments. The initial and final segments are vertical ones of lengths greater than or equal to ε^3 ,

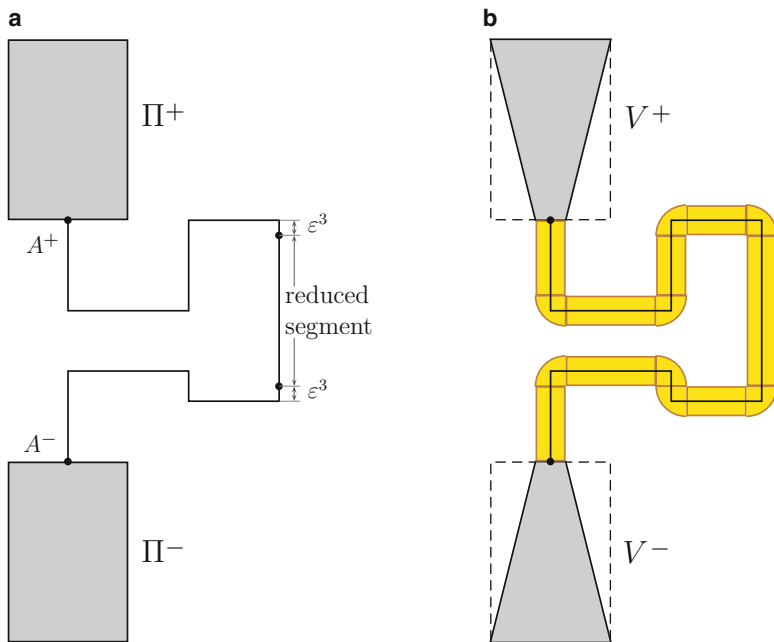


Fig. 2.9 An ε -contour (a) and the channel generated by this contour (b)

and the lengths of the other segments are greater than or equal to $2\varepsilon^3$. The broken line does not have points of self-intersection and or points of intersection with Π^+ and Π^- other than endpoints A^+ and A^- . The endpoints of the segments, except for A^+ and A^- , are called *vertices*. Thus, both the initial and final segments have only one vertex, and the other segments have two vertices. A shortened segment of the broken line, with (one or two) segments of length ε^3 adjacent to its vertices removed, is called a *reduced segment*. A reduced segment may be a true segment, or it may be a single point. The union of the rectangles Π^+ and Π^- and the ε -axis is called an ε -contour (Fig. 2.9a).

Now suppose that we have an ε -contour. To each reduced segment of the ε -axis we assign a rectangle of width $2\varepsilon^3$ such that the segment is a midline of the rectangle and divides it into two rectangles of width ε^3 (Fig. 2.9b). In the degenerated case, where the reduced segment is a point, the assigned rectangle is a segment of length $2\varepsilon^3$. To each vertex we assign a circle sector of radius $2\varepsilon^3$ such that the two radii bounding the sector coincide with sides of the rectangles assigned to the adjacent reduced segments. Finally, to the rectangles Π^+ and Π^- we assign the inscribed trapezoids V^+ and V^- such that the midpoints of two sides of these rectangles, A^+ and A^- , are also midpoints of smaller bases of length $2\varepsilon^3$ of the trapezoids, and the opposite sides of the rectangles coincide with the larger bases of the trapezoids. If the obtained figures (rectangles, sectors, and trapezoids) do not mutually intersect, then their union is an ε -channel. It will be called a *channel generated by the given ε -contour*.

2.3.2 Proof of Theorem 2.3

Consider two plane convex bodies C_1 and C_2 . Without loss of generality we assume that $v = (0, -1)$. The proof of Theorem 2.3 amounts to constructing a family of bodies B_ε , $C_1 \subset B_\varepsilon \subset C_2$ with resistance going to zero, $\lim_{\varepsilon \rightarrow 0} R_v(B_\varepsilon) = 0$. In what follows, we will write R instead of R_v , omitting the subscript v .

It suffices to provide a family B_ε in the special case where

$$\text{dist}(\partial C_1, \partial C_2) > 4\sqrt{2}. \quad (2.2)$$

Indeed, in the general case, take $k > 0$ large enough so that $\text{dist}(\partial(kC_1), \partial(kC_2)) > 4\sqrt{2}$ and find a family $\tilde{B}_\varepsilon$, $kC_1 \subset \tilde{B}_\varepsilon \subset kC_2$ such that $\lim_{\varepsilon \rightarrow 0} R(\tilde{B}_\varepsilon) = 0$. Then the family $B_\varepsilon = k^{-1}\tilde{B}_\varepsilon$ satisfies the required relations $C_1 \subset B_\varepsilon \subset C_2$ and $R(B_\varepsilon) = k^{-1}R(\tilde{B}_\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Thus, the general case is reduced to the special case (2.2).

Consider the partition of $\mathbb{R}^2$ into (closed) squares of size 2×2 with vertices in $2\mathbb{Z} \times 2\mathbb{Z}$ and denote by D the union of squares contained in the interior of C_2 . One easily sees that $C_1 \subset D$. The squares of the partition that are contained in D and have nonempty intersection with ∂D will be called *boundary squares*. The boundary squares do not intersect C_1 .

Indeed, each boundary square (let it be Q) has a nonempty intersection with another square of a partition (say, Q') that does not belong to D . By definition of D , Q' contains a point from ∂C_2 . The diameter of the union $Q \cup Q'$ does not exceed $4\sqrt{2}$, and therefore by (2.2) $Q \cup Q'$ does not contain points of ∂C_1 . This implies that $Q \cap C_1 = \emptyset$.

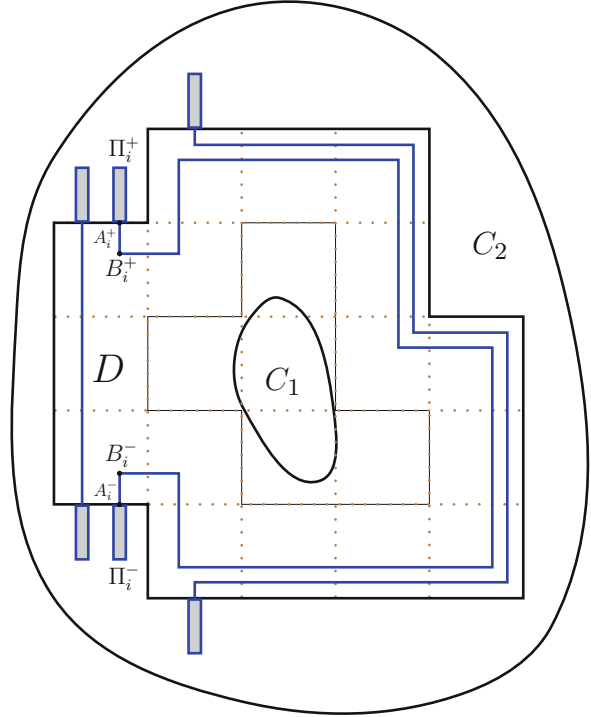
In Fig. 2.10, C_1 and C_2 are bounded by black closed curves, and D is bounded by a thick polygonal line. The boundary squares are situated between the thick and thin black solid polygonal lines.

For future convenience we will use the small parameter ε of the form $\varepsilon = 1/(2n + 1)$, where n is a positive integer, and impose the restriction $\varepsilon < \text{dist}(D, \partial C_2)$. Denote by $l = l(D) = \max\{|x_1 - y_1| : (x_1, x_2), (y_1, y_2) \in D\}$ the width of D and impose one more restriction $\varepsilon < 1/l$.

Denote by $\partial_+ D$ and $\partial_- D$ the upper and lower parts of the boundary ∂D , that is, the intersection of ∂D with the union of upper (lower) sides of the squares forming D . Introduce the metric $\tilde{d}$ in $\mathbb{R}^2$ by $\tilde{d}(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. In this metric a ball of radius r is a square of size $2r \times 2r$ with vertical and horizontal sides.

Take $d_i = (2i - 1)\varepsilon^3$, $N = l/(2\varepsilon^2)$, and denote by L_i the set of points $x \in D$ such that $\tilde{d}(x, \partial D) = d_i$, $i = 1, \dots, N$. The curve L_i will be called the *i*th-level line. Due to the choice of ε one always has $d_i < 1$, so each curve L_i is contained in the union of boundary squares and, therefore, does not intersect C_1 . The curves L_i are closed, do not have self-intersections, and are composed of vertical and horizontal segments. Let us divide each level line L_i into two curves by two points with maximal and minimal x_1 -coordinates; then the x_1 -coordinate will change monotonically along each of these curves. Finally, the x_1 -coordinate

Fig. 2.10 The construction of a built-in channel system: the ε -contours for three values of i are shown. The rectangles Π_i^+ , Π_i^- and the points A_i^+ , A_i^- , B_i^+ , B_i^- are indicated for a single value of i



of each vertical segment forming L_i differs by $(2i - 1)\varepsilon^3$ from a multiple of 2; hence the difference of x_1 -coordinates of any two vertical segments belonging to any two level lines is a multiple of $2\varepsilon^3$. The same is valid for the x_2 -coordinate. These observations imply that the length of each segment in each level line is greater than or equal to $2\varepsilon^3$.

Divide the upper boundary, $\partial_+ D$, into segments of length $2\varepsilon^2$. The number of these segments is N , and the upper side of each square forming $\partial_+ D$ contains exactly ε^{-2} segments (recall that this number is an integer). Denote the segments from right to left (that is, from the larger to the smaller x_1 -coordinate) by $I_1^+, \dots, I_N^+$, and construct the rectangles $\Pi_1^+, \dots, \Pi_N^+$ of height $\varepsilon - \varepsilon^2$ resting on these segments; that is, I_i^+ is the lower side of Π_i^+ . Similarly, divide the lower boundary, $\partial_- D$, into segments of the same length, enumerate them from right to left, $I_1^-, \dots, I_N^-$, and take the rectangles $\Pi_1^-, \dots, \Pi_N^-$ of the same height $\varepsilon - \varepsilon^2$ resting on these resting on these segments; that is, each segment I_i^- is the upper side of Π_i^- . The rectangles Π_i^+ and Π_i^- corresponding to three different values of i are shown in Fig. 2.10.

All the rectangles $\Pi_i^\pm$ are contained in $C_2 \setminus D$. Denote by A_i^+ the midpoint of the segment I_i^+ and by A_i^- the midpoint of I_i^- . The x_1 -coordinate of both A_i^+ and A_i^- equals

$$x_1(A_i^+) = \varepsilon^2(1 + 2N - 2i). \quad (2.3)$$

The set

$$\{x : x_1 = x_1(A_i^+), \bar{d}(x, \partial D) \leq d_i\}$$

is either (i) the vertical segment $A_i^+ A_i^-$ or (ii) a union of two segments $A_i^+ B_i^+ \cup A_i^- B_i^-$ (Fig. 2.10). In case (ii), one has $\bar{d}(B_i^+, \partial D) = \bar{d}(B_i^-, \partial D) = d_i$.

Define the broken line Γ_i as follows. In case (i), take $\Gamma_i = A_i^+ A_i^-$. In case (ii), Γ_i is the union of the segments $A_i^+ B_i^+$ and $A_i^- B_i^-$ and the part of the curve L_i contained in the half-plane $x_1 > x_1(A_i^+)$. In this case, the length of each of the segments $A_i^+ B_i^+$ and $A_i^- B_i^-$ is greater than or equal to ε^3 .

By formula (2.3) and by the choice of ε , the value $x_1(A_i^+) - \varepsilon^3$ is a multiple of $2\varepsilon^3$. In addition, it has already been established that each vertical segment of the broken line L_i also has this property: denoting by x_1 the first coordinate of the segment, we have that $x_1 - \varepsilon^3$ is a multiple of $2\varepsilon^3$. Thus, the distance from each of the vertical segments $A_i^+ B_i^+$, $A_i^- B_i^-$ to the nearest vertical segment of L_i is a multiple of $2\varepsilon^3$. This implies that the lengths of the first and last horizontal segments of Γ_i are greater than or equal to $2\varepsilon^3$. The other intermediate segments of Γ_i are at the same time segments of L_i and, therefore, also have lengths greater than or equal to $2\varepsilon^3$.

Thus, the broken line Γ_i is composed of vertical and horizontal segments, the lengths of the initial and final segments are greater than or equal to ε^3 , and the lengths of the other segments are greater than or equal to $2\varepsilon^3$. In particular, this broken line may coincide with a single vertical segment. It starts at point A_i^+ and finishes at point A_i^- and has no points of self-intersection. Therefore, Γ_i is an ε -contour joining the rectangles Π_i^+ and Π_i^- . The $\bar{d}$ -distance between different curves Γ_i and Γ_j is at least $2\varepsilon^3$:

$$\bar{d}(\Gamma_i, \Gamma_j) \geq 2\varepsilon^3 \quad \text{for } i \neq j.$$

Fix i and consider the rectangles and sectors generated by the reduced segments and vertices of the broken line Γ_i . Notice that any triple of consecutive elements—a v -rectangle, a sector, and an h -rectangle—contains elements that do not intersect pairwise. On the other hand, if a pair of elements does not belong to such a triple, then the minimal union of the squares of the partition containing one element does not intersect the minimal union of the squares containing the other element. Therefore, the elements do not mutually intersect; hence these elements, jointly with the trapezoids $V_i^+ \subset \Pi_i^+$ and $V_i^- \subset \Pi_i^-$ generated by the rectangles Π_i^+ and Π_i^- , form a channel; let it be denoted by $\tilde{K}_i$. We assume that the channel is open, that is, does not intersect its boundary.

Now let us show that the channels defined previously do not mutually intersect. Let K_i' be the union of the rectangles (or the rectangle) generated by the segments $A_i^+ B_i^+$, $A_i^- B_i^-$ and the adjacent sectors. Let K_i'' be the union of the other sets generated by the broken line Γ_i and forming the channel. One obviously has

$$\tilde{K}_i = K_i' \cup K_i'' \cup V_i^+ \cup V_i^- \subset K_i' \cup K_i'' \cup \Pi_i^+ \cup \Pi_i^-. \quad (2.4)$$

We have $K'_i \subset \{x : \bar{d}(x, \partial D) \leq 2i\varepsilon^3\}$ and $K''_i \subset \{x : (2i-2)\varepsilon^3 < \bar{d}(x, \partial D) < 2i\varepsilon^3\}$; hence

$$K''_i \cap K'_j = \emptyset \text{ for } i \neq j \quad \text{and} \quad K'_i \cap K''_j = \emptyset \text{ for } i < j. \quad (2.5)$$

Further, $K'_i \subset \{x : x_1(A_i^+) - \varepsilon^3 < x_1 < x_1(A_i^+) + \varepsilon^3\}$ and $K''_i \subset \{x : x_1 > x_1(A_i^+) + \varepsilon^3\}$, and hence

$$K'_i \cap K'_j = \emptyset \quad \text{and} \quad K''_i \cap K'_j = \emptyset \text{ for } i < j. \quad (2.6)$$

Finally, neither the set $\Pi_i^+ \cup \Pi_i^-$ nor the set $\Pi_j^+ \cup \Pi_j^-$, $i \neq j$ intersects D and their projections on the horizontal axis do not intersect; therefore,

$$(\Pi_i^+ \cup \Pi_i^-) \cap (\Pi_j^+ \cup \Pi_j^-) = \emptyset \quad \text{and} \quad (\Pi_i^+ \cup \Pi_i^-) \cap (K'_j \cup K''_j) = \emptyset \text{ for } i \neq j. \quad (2.7)$$

Relations (2.4)–(2.7) imply that the channels $\tilde{K}_i$, $i = 1, \dots, N$ do not mutually intersect. Moreover, the union of the front sides of these channels is a horizontal segment of length l shielding the vertical flow of particles incident on D .

Denote

$$\tilde{B}_\varepsilon = D \cup (\cup_i \Pi_i^+) \cup (\cup_i \Pi_i^-) \setminus (\cup_i \tilde{K}_i). \quad (2.8)$$

Thus, the set $\tilde{B}_\varepsilon$ is obtained by adding all the rectangles $\Pi_i^\pm$ to the set D and then subtracting all the channels $\tilde{K}_i$. A particle incident on D with initial velocity $v = (0, -1)$ intersects the front side of a channel, passes through the channel in the positive direction, then intersects its rear side and continues to move with a velocity $v^+ = v + O(\varepsilon)$. Therefore, the resistance of $\tilde{B}_\varepsilon$ equals

$$R(\tilde{B}_\varepsilon) = O(\varepsilon), \quad \varepsilon \rightarrow 0. \quad (2.9)$$

Notice that, roughly speaking, the set $\tilde{B}_\varepsilon$ is not a body since it is locally one-dimensional. Indeed, the intersection of $\tilde{B}_\varepsilon$ with a neighborhood of a point on the common boundary of neighbor v - or h -rectangles is a rectilinear interval. To improve the construction, replace the ε -channels $\tilde{K}_i$ in formula (2.8) with ε' -channels K_i contained in $\tilde{K}_i$, with $\varepsilon' = \varepsilon + O(\varepsilon^2) < \varepsilon$. (We additionally require that the lateral boundaries of $\tilde{K}_i$ and K_i be disjoint.) The resulting set $B_\varepsilon = D \cup (\cup \Pi_i^+) \cup (\cup \Pi_i^-) \setminus (\cup K_i)$ is a true body, and it also satisfies the relation $R(B_\varepsilon) = O(\varepsilon)$, $\varepsilon \rightarrow 0$ and the inclusion $C_1 \subset B_\varepsilon \subset C_2$. The proof of Theorem 2.3 is complete.

2.3.3 Proof of Theorem 2.2

Fix bodies C_1 and C_2 and assume without loss of generality that $v = (0, 0, -1)$. As in the previous section, we construct here a family of connected bodies B_ε , $C_1 \subset B_\varepsilon \subset C_2$ with vanishing resistance, $\lim_{\varepsilon \rightarrow 0} R(B_\varepsilon) = 0$.

A typical body of the family is sandwich-shaped: it is the union of several thin sheets of two kinds: “sheets of small resistance” and “solid sheets.” These two kinds of sheets alternate in the sandwich. The plane of the sheets is parallel to v . The sheets of small resistance are constructed with the use of Theorem 2.3 proved in the previous section. The solid sheets are much thinner than the sheets of small resistance and “glue them together,” so that the resulting body is connected.

Let us proceed to a description of the construction. For a convex body $C \subset \mathbb{R}^3$ denote

$$C^t = \{(x_2, x_3) : (t, x_2, x_3) \in C\}.$$

In other words, C^t is the projection of the cross section $C \cap \{x_1 = t\}$ on the plane $\mathbb{R}_{\{x_2, x_3\}}^2$. Further, for a set $A \subset \mathbb{R}$ define the sets

$$\underline{C}^A = \bigcap_{t \in A} C^t \quad \text{and} \quad \overline{C}^A = \text{Conv}(\bigcup_{t \in A} C^t).$$

One easily sees that for any $t \in A$

$$\overline{C}^A \subset C^t \subset \underline{C}^A$$

holds.

Define the set

$$I = \{t : C_1^t \neq \emptyset\};$$

that is, I is the projection of C_1 on $\mathbb{R}_{\{x_1\}}$. Without loss of generality assume that the sets C_1 and C_2 are open. Then I is a bounded open interval, $I = (a, b)$.

Further, one has $C_1^t \subset C_2^t$. Moreover, for any $t \in I$ there exists an open interval $\mathcal{U}_t$ containing t such that

$$\overline{C}_1^{\mathcal{U}_t} \subset \underline{C}_2^{\mathcal{U}_t} \quad \text{and} \quad \partial(\overline{C}_1^{\mathcal{U}_t}) \cap \partial(\underline{C}_2^{\mathcal{U}_t}) = \emptyset.$$

Fix $\varepsilon > 0$ and choose a finite subset of intervals $\mathcal{U}_{t_i}$ covering the segment $[a + \varepsilon, b - \varepsilon]$. Next choose disjoint intervals $I_i^\varepsilon \subset \mathcal{U}_{t_i}$ such that

$$\bigcup_i I_i^\varepsilon = [a + \varepsilon, b - \varepsilon].$$

Due to the choice of these intervals, for each i one has

$$\overline{C}_1^{I_i^\varepsilon} \subset \underline{C}_2^{I_i^\varepsilon} \quad \text{and} \quad \partial(\overline{C}_1^{I_i^\varepsilon}) \cap \partial(\underline{C}_2^{I_i^\varepsilon}) = \emptyset.$$

Define

$$C(\varepsilon) = C_2 \cap \{x_1 \in (a, a + \varepsilon) \cup (b - \varepsilon, b)\},$$

and denote by $C_{x_1, x_2}(\varepsilon)$ the projection of $C(\varepsilon)$ on the plane $\mathbb{R}_{\{x_1, x_2\}}^2$. More precisely, one has

$$C_{x_1, x_2}(\varepsilon) = \{(t, \tau) : t \in (a, a + \varepsilon) \cup (b - \varepsilon, b) \text{ and } C_2^t \cap \{x_2 = \tau\} \neq \emptyset\}.$$

Let a_ε be the area of $C_{x_1, x_2}(\varepsilon)$; one has $\lim_{\varepsilon \rightarrow 0} a_\varepsilon = 0$. The resistance of $C(\varepsilon)$ can be estimated as

$$|R(C(\varepsilon))| \leq 2a_\varepsilon. \quad (2.10)$$

Using Theorem 2.3 we select plane bodies B_i^ε such that

$$\overline{C}_1^{I_i} \subset B_i^\varepsilon \subset \underline{C}_2^{I_i} \quad \text{and} \quad |R(B_i^\varepsilon)| < \varepsilon. \quad (2.11)$$

Then the resistance of the three-dimensional set $I_i \times B_i^\varepsilon$ can be estimated as

$$|R(I_i \times B_i^\varepsilon)| = |I_i| \cdot |R(B_i^\varepsilon)| < \varepsilon |I_i|, \quad (2.12)$$

where $|I|$ means the length of interval I . Define the body

$$\tilde{B}_\varepsilon = C(\varepsilon) \cup \left(\bigcup_i I_i \times B_i^\varepsilon \right).$$

Using the first relation in (2.11) and the definition of $C(\varepsilon)$ one concludes that

$$C_1 \subset \tilde{B}_\varepsilon \subset C_2.$$

Consider a particle incident on $\tilde{B}_\varepsilon$. If its x_1 -coordinate belongs to $(a, a + \varepsilon) \cup (b - \varepsilon, b)$, then it makes a single reflection at a point of $\partial C(\varepsilon)$. If the x_1 -coordinate belongs to an interval I_i , then the particle makes several reflections at points of $I_i \times B_i^\varepsilon$ and never hits any other subset constituting the body $\tilde{B}_\varepsilon$. Therefore, the resistance of $\tilde{B}_\varepsilon$ is the sum of resistances of its subsets:

$$R(\tilde{B}_\varepsilon) = R(C(\varepsilon)) + \sum_i R(I_i \times B_i^\varepsilon).$$

Using (2.10) and (2.12) one obtains the estimate

$$|R(\tilde{B}_\varepsilon)| \leq 2a_\varepsilon + \varepsilon(b - a),$$

that is, $\lim_{\varepsilon \rightarrow 0} R(\tilde{B}_\varepsilon) = 0$.

However, the body $\tilde{B}_\varepsilon$ is not connected. Let us therefore modify it in the following way. Take an open set $J_\varepsilon \subset \mathbb{R}$ and require that it be the disjoint union of open intervals of total length less than ε and contains the endpoints of all intervals I_i , that is,

$$\bigcup_i \partial I_i \subset J_\varepsilon \quad \text{and} \quad |J_\varepsilon| < \varepsilon.$$

Define

$$D(\varepsilon) = C_2 \cap \{x_1 \in J_\varepsilon\}.$$

Then the body

$$B_\varepsilon = \tilde{B}_\varepsilon \cup D(\varepsilon)$$

is connected and satisfies the relations

$$C_1 \subset B_\varepsilon \subset C_2 \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} R(B_\varepsilon) = 0.$$

The proof of Theorem 2.2 is complete.

2.4 Two-Dimensional Problem

The results obtained in the three-dimensional case can be easily generalized to higher dimensions $d > 3$. The infimum of the functional $R_{\delta_{v_0}}$ over the corresponding classes of nonconvex bodies is also equal to 0. By contrast, in the two-dimensional case, the infimum (whether taken over the class of convex or nonconvex bodies) is positive.

2.4.1 Minimum Resistance of Convex Bodies

We discuss briefly the known results for the two-dimensional analog of Newton's problem with $c(v, v^+) = (v - v^+) \cdot v$. Here, as above, we consider a flow of particles falling vertically downward with velocity $v_0 = (0, -1)$. Let $C = C_h = [-1, 1] \times [-h, 0]$ be a rectangle in a plane with variables x_1, x_2 , and let B be a convex body inscribed in C . The upper part of the boundary of B is the graph of a concave function $-f_B$, and the resistance can be represented by the functional $R_{\delta_{v_0}}(B) = 2 \int_{-1}^1 (1 + f_B'^2(x))^{-1} dx$. Thus, the problem of minimizing the resistance assumes the following form:

$$\text{Find } \inf_{f \in \text{Convex}_2(h)} \mathcal{R}_2(f), \quad \text{where } \mathcal{R}_2(f) = \int_{-1}^1 \frac{dx}{1 + f'^2(x)}. \quad (2.13)$$

Here $\text{Convex}_2(h)$ is the class of convex functions $f : [-1, 1] \rightarrow [0, h]$.

The solution to this problem is well known (see, e.g., [12]): the body of minimum resistance is an isosceles triangle if $h \geq 1$ and a trapezoid if $h < 1$ (Fig. 2.11). The smallest value of the resistance is

$$2 \inf_{f \in \text{Convex}_2(h)} \mathcal{R}_2(f) = \begin{cases} 4 - 2h, & \text{if } h < 1, \\ 4/(1 + h^2), & \text{if } h \geq 1. \end{cases} \quad (2.14)$$

Finally, in the two-dimensional case (in contrast to the three-dimensional case), the solution in the class of arbitrary convex bodies inscribed in C is the same as in the narrower class of convex bodies symmetric relative to a vertical axis.

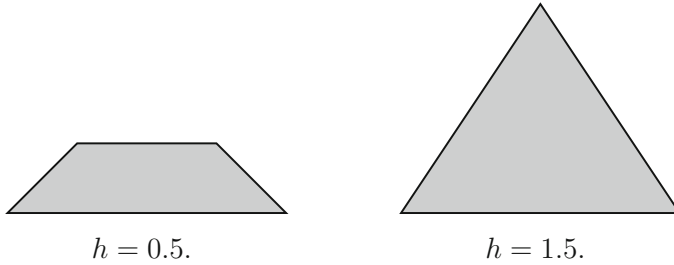


Fig. 2.11 Solution to two-dimensional problem for convex bodies

2.4.2 Minimum Resistance of Nonconvex Bodies

Let $\mathcal{P}_2(h)$ denote the class of connected bodies lying in the rectangle C_h and intersecting both its left and right sides $\{-1\} \times [-h, 0]$ and $\{1\} \times [-h, 0]$. For the resistance of such bodies we have the formula

$$R_{\delta_{v_0}}(B) = \int_{[-1,1] \times \{0\}} \langle v_0 - v_B^+(\xi, v_0), v_0 \rangle d\xi. \quad (2.15)$$

It is clear that if $B \in \mathcal{P}_2(h)$, then also $\text{Conv} B \in \mathcal{P}_2(h)$.

Theorem 2.4. $\inf_{B \in \mathcal{P}_2(h)} R_{\delta_{v_0}}(B) = 2(1 - h/\sqrt{1 + h^2})$.

Remark 2.4. The minimum value is attained on the subclass of $\mathcal{P}_2(h)$ of bodies that particles in the flow hit at most twice. This is clear from the proof below.

Remark 2.5. It follows from this theorem that the least resistance over the class of *nonconvex* bodies is positive but still much smaller than over the class of *convex* bodies. Comparing the statement of the theorem with (2.14) we conclude that the ratio

$$\frac{\text{the least resistance of } \mathbf{nonconvex} \text{ bodies}}{\text{the least resistance of } \mathbf{convex} \text{ bodies}}$$

decreases monotonically from 1/2 to 1/4 as h increases from 0 to $+\infty$.

Proof. Recall that $\text{Convex}_2(h)$ is the set of convex functions $f : [-1, 1] \rightarrow [0, h]$. Consider the functional $\hat{R}$ on this set defined by the formula

$$\hat{R}(f) = \int_{-1}^1 \left(1 - \frac{|f'(x)|}{\sqrt{1 + f'^2(x)}} \right) dx. \quad (2.16)$$

We have the following lemmas, which we prove subsequently.

Lemma 2.2. *The functional $\hat{R}$ takes its minimum value on the function $f_h(x) = h|x|$. This value is $2(1 - h/\sqrt{1 + h^2})$. In other words,*

$$\inf_{f \in \text{Convex}_2(h)} \hat{R}(f) = \hat{R}(f_h) = 2 \left(1 - \frac{h}{\sqrt{1 + h^2}} \right).$$

Lemma 2.3. *For each $B \in \mathcal{P}_2(h)$,*

$$R_{\delta_{v_0}}(B) \geq \hat{R}(f_{\text{Conv} B}).$$

Let Δ_h be the triangle $-h \leq x_2 \leq -f_h(x_1)$. This is the isosceles triangle in the rectangle C_h with vertices at $(1, -h)$, $(-1, -h)$, and $(0, 0)$.

Recall that the variable $\xi = (x, 0)$ ranges over $[-1, 1] \times \{0\}$.

Lemma 2.4. *There exists a family of bodies $B_\varepsilon \in \mathcal{P}_2(h)$ such that*

$$\text{Conv } B_\varepsilon = \Delta_h,$$

and the corresponding family of functions $v_{B_\varepsilon}^+(\xi, v_0)$ converges in measure to the function $(\text{sgn } x, -h)/\sqrt{1 + h^2}$ as $\varepsilon \rightarrow 0^+$.

These three lemmas immediately yield the required result. Indeed, by Lemmas 2.2 and 2.3,

$$\inf_{B \in \mathcal{P}_2(h)} R_{\delta_{v_0}}(B) \geq 2 \left(1 - \frac{h}{\sqrt{1 + h^2}} \right),$$

whereas by Lemma 2.4 and (2.15),

$$\lim_{\varepsilon \rightarrow 0^+} R_{\delta_{v_0}}(B_\varepsilon) = 2 \left(1 - \frac{h}{\sqrt{1 + h^2}} \right),$$

so that the reverse inequality holds:

$$\inf_{B \in \mathcal{P}_2(h)} R_{\delta_{v_0}}(B) \leq 2 \left(1 - \frac{h}{\sqrt{1 + h^2}} \right).$$

□

Proof of Lemma 2.2. Let $p(u) = 1 - u/\sqrt{1 + u^2}$. In what follows we use the observations that for $u > 0$ the functions $p(u)$ and $u \cdot p(1/u)$ are convex and $p(u)$ is decreasing.

The result of the lemma is an immediate consequence of the relations

$$\int_{-1}^1 p(|f'(x)|) dx \geq 2p(h) \quad \text{for each } f \in \text{Convex}_2(h), \quad (2.17)$$

$$\int_{-1}^1 p(|f'_h(x)|) dx = 2p(h). \quad (2.18)$$

Relation (2.18) follows from the identity $|f'_h(x)| \equiv h$. To prove (2.17), we consider two cases.

- (i) The function f is (not strictly) monotone. Assuming without loss of generality that f is nondecreasing, and taking into account the convexity of p , we obtain from Jensen's integral inequality that

$$\frac{1}{2} \int_{-1}^1 p(f'(x)) dx \geq p\left(\frac{1}{2} \int_{-1}^1 f'(x) dx\right) = p\left(\frac{f(1) - f(-1)}{2}\right) \geq p(h/2) > p(h).$$

This proves (2.17) in this case.

- (ii) The function f is not monotone, so there exists x_0 , $-1 < x_0 < 1$ such that f is monotonically nonincreasing for $x \leq x_0$ and nondecreasing for $x \geq x_0$. We have

$$\int_{-1}^1 p(|f'(x)|) dx = \int_{-1}^{x_0} p(-f'(x)) dx + \int_{x_0}^1 p(f'(x)) dx.$$

Applying Jensen's inequality to both terms on the right hand side of this equation, we obtain

$$\begin{aligned} \frac{1}{1+x_0} \int_{-1}^{x_0} p(-f'(x)) dx &\geq p\left(\frac{1}{1+x_0} \int_{-1}^{x_0} (-f'(x)) dx\right), \\ \frac{1}{1-x_0} \int_{x_0}^1 p(f'(x)) dx &\geq p\left(\frac{1}{1-x_0} \int_{x_0}^1 f'(x) dx\right). \end{aligned}$$

Since $\int_{-1}^{x_0} (-f'(x)) dx \leq h$ and $\int_{x_0}^1 f'(x) dx \leq h$, it follows that

$$\int_{-1}^1 p(|f'(x)|) dx \geq (1+x_0) p\left(\frac{h}{1+x_0}\right) + (1-x_0) p\left(\frac{h}{1-x_0}\right). \quad (2.19)$$

The function $u \cdot p(1/u)$ is convex; therefore, the right-hand side of (2.19) is a convex function of x_0 , which is also even. Hence at $x_0 = 0$ it takes a minimum value, which is $2p(h)$. The proof of (2.17) is complete. $\square$

Proof of Lemma 2.3. We introduce the shorter notation $f_{\text{Conv}B} =: f$ and $v_B^+(\xi, v_0) =: v^+(x) = (v_1^+(x), v_2^+(x))$, where $\xi = (x, 0)$. Then formula (2.15) assumes the form

$$R_{\delta v_0}(B) = \int_{-1}^1 (1 + v_2^+(x)) dx. \quad (2.20)$$

We define the following modified law of reflection from the convex body $\text{Conv}B$: the velocity of the particle after reflection is the unit vector with a nonpositive second component that is tangent to $\partial(\text{Conv}B)$ at the point of reflection (Fig. 2.12). If the tangent is horizontal, then we agree to choose, for instance, the vector pointing to the right, that is, $(1, 0)$. In other words, the velocity after reflection makes the

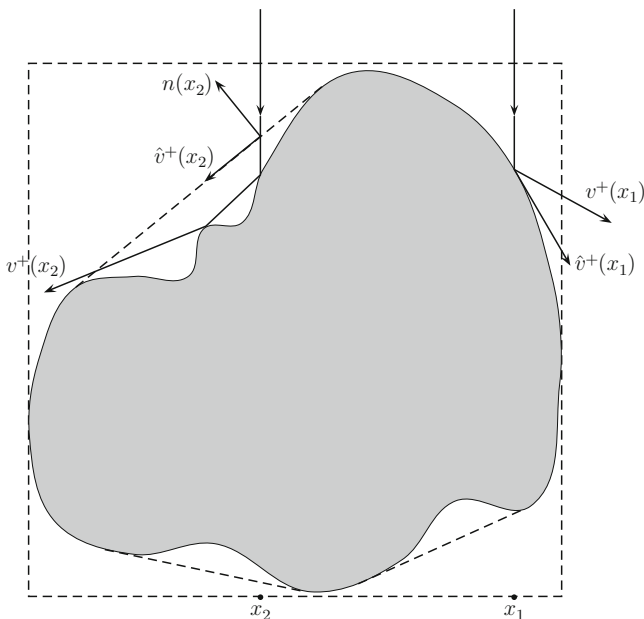


Fig. 2.12 Ordinary (billiard) and modified laws of reflection

smallest possible angle with the initial velocity $v_0 = (0, -1)$. Thus, if the reflection point is $(x, -f(x))$, then in accordance with the modified law the velocity of the reflected particle is

$$\hat{v}^+(x) = (\hat{v}_1^+(x), \hat{v}_2^+(x)) = \frac{(\operatorname{sgn} f'(x), -|f'(x)|)}{\sqrt{1 + f'^2(x)}}.$$

Here $\operatorname{sgn} z = 1$ if $z \geq 0$ and -1 otherwise. Using this notation, we can represent the functional $\hat{R}(f)$ as

$$\hat{R}(f) = \int_{-1}^1 (1 + \hat{v}_2^+(x)) dx. \quad (2.21)$$

This is the resistance of the body $\operatorname{Conv} B$ if the modified law of reflection holds.

We assert that $v_2^+(x) \geq \hat{v}_2^+(x)$. Then from (2.20) and (2.21) we immediately obtain the required result. We consider two cases.

- (i) $(x, -f(x)) \in \partial(\operatorname{Conv} B) \cap \partial B$. In this case, the billiard particle makes a unique elastic collision with ∂B , after which the second component of the velocity becomes

$$v_2^+(x) = \frac{1 - f'^2(x)}{1 + f'^2(x)} > -\frac{|f'(x)|}{\sqrt{1 + f'^2(x)}} = \hat{v}_2^+(x)$$

(see Fig. 2.12 for $x = x_1$).

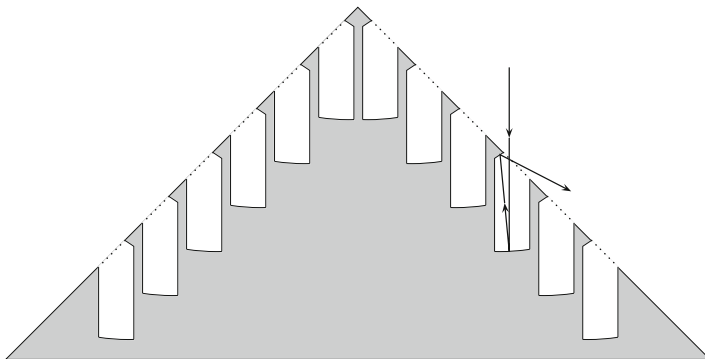


Fig. 2.13 Solution to two-dimensional nonconvex problem

- (ii) $(x, -f(x)) \in \partial(\text{Conv} B) \setminus \partial B$. In this case, the billiard particle crosses a line interval that is a connected component of $\partial(\text{Conv} B) \setminus \partial B$ and goes into a “hollow,” which is a connected component of $\text{Conv} B \setminus B$. After several reflections it intersects the same interval and leaves the hollow at some velocity $v^+(x)$ that makes an acute angle with the normal $n(x)$ to $\partial(\text{Conv} B)$ at $(x, -f(x))$, that is, $\langle v^+(x), n(x) \rangle > 0$, where $n(x) = (f'(x), 1)/\sqrt{1 + f'^2(x)}$ (see Fig. 2.12 for $x = x_2$). This easily yields the required inequality. $\square$

Proof of Lemma 2.4. We set $B_\varepsilon = \Delta_h \setminus (\cup_{-n_0 \leq n(\neq 0) \leq n_0} \Omega_\varepsilon^n)$, where the value $n_0 = n_0(\varepsilon)$ will be defined below, the sets Ω_ε^n , $n = 1, \dots, n_0$ are translations of a set Ω_ε also defined below, and each set Ω_ε^{-n} is symmetric to Ω_ε^n relative to the vertical axis Ox_2 (Fig. 2.13). The projections of $\Omega_\varepsilon^{-n_0}, \dots, \Omega_\varepsilon^{-1}, \Omega_\varepsilon^1, \dots, \Omega_\varepsilon^{n_0}$ on the horizontal axis Ox_1 are disjoint line segments of length ε^2 ; each successive segment lies at a distance of ε^3 from the preceding one. The part of the boundary of Ω_ε^n indicated by a dotted line lies on the corresponding lateral side of the triangle Δ_h . The quantity n_0 is the largest n such that $\Omega_\varepsilon^n \subset \Delta_h \cap \{x_2 > -h\}$, so that the sets Ω_ε^n lie in the triangle but are disjoint from its base.

The set Ω_ε is the part of the vertical strip $0 \leq x_1 \leq \varepsilon^2$ bounded by the segments GC and EG and the arc AD of a parabola (Fig. 2.14). (Note that the positions of points $G, C, \dots$ depend on ε , so strictly speaking we should write $G_\varepsilon, C_\varepsilon$, and so on.) We set $O = (0, 0)$, $C = (\varepsilon^2, -h\varepsilon^2)$, $D = (\varepsilon^2, -\varepsilon)$, $F = (\varepsilon^3/2, -h\varepsilon^3)$. Thus, segment OC lies on a lateral side of the triangle Δ_h . Arc AD is a piece of a parabola with upward branches, focus at F , and a vertical axis; EG is a segment containing F such that a “billiard particle” coming from D and reflected from EG at F goes to C (we indicate the corresponding trajectory by a dotted line). Side GC is indicated by a dashed line.

Consider the billiard in Ω_ε represented in Fig. 2.14. The billiard particle moves vertically downward starting from some point in the segment GC . It is reflected from the arc AD , then from EG at point F , and again goes to some point in GC , with velocity before reflection making an angle of magnitude $O(\varepsilon)$ with FC . Since

three-dimensional body $z \leq u(x)$ bounded from above by the graph of a piecewise smooth function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$. We assume that the axis Oz is directed upward and, thus, the flow is falling vertically downward. The function u is invariant with respect to a group G of motions of the plane, that is, $u \circ g = u$ for any $g \in G$. In addition, it is assumed that each particle impinging on the body vertically downward and making a reflection at a nonsingular point of the body boundary will move freely afterward, that is, without any further reflection from the body. This condition on u is called the *single-impact assumption* (s.i.a.). It can also be written analytically; that is, for any nonsingular point $x \in \mathbb{R}^2$ and any $t > 0$

$$\frac{u(x - t\nabla u(x)) - u(x)}{t} \leq \frac{1}{2} (1 - |\nabla u(x)|^2) \quad (2.22)$$

holds.

Let Ω be a fundamental region of G . We assume that Ω is bounded; therefore, the function u is also bounded. By $|\Omega|$ we denote the area of Ω . By definition, the quantity

$$F(u) = \frac{1}{|\Omega|} \int_{\Omega} \frac{dx}{1 + |\nabla u(x)|^2} \quad (2.23)$$

is called the *specific resistance* of the body. Clearly, it does not depend on the choice of the fundamental region. This definition has a simple physical interpretation: $F(u)$ is the force of pressure of a flow of large cross section divided by the area of this section.

The greatest value of specific resistance is $F(u) = 1$, and the maximizers are constants $u \equiv \text{const}$; in this case the body is a half-space. On the other hand, for any body satisfying the s.i.a. we have $|\nabla u| \leq 1$. Indeed, assuming that $|\nabla u(x_0)| > 1$ at a point x_0 and taking into account (2.22), we get that u tends to $-\infty$ along the ray $x - t\nabla u(x)$, $t > 0$, which is impossible since it is bounded. It follows from (2.23) that $F(u) \geq 0.5$, so we have

$$0.5 \leq F(u) \leq 1.$$

This formula is in agreement with physical intuition. Indeed, consider a particle with mass m that falls with velocity $(0, 0, -1)$ on a body and is reflected by it. The vertical component of velocity after the reflection is nonnegative; therefore, the vertical component of the momentum transmitted to the body by the particle is at least m . On the other hand, the maximal value of this component is $2m$ and is achieved when the gradient of u at the point of reflection is zero. Thus, the vertical component of the transmitted momentum is at least one half its maximum value; therefore, the specific resistance of a body is at least one half its maximum value.

The problem consists in finding the minimal value of specific resistance under the single-impact condition. It has not been solved until now. Note that the least known value is approximately $F(u_*) \approx 0.580778$; the corresponding function u_* was constructed in [53]. Therefore, one has

$$0.5 \leq \inf_u F(u) \leq 0.581.$$

Now consider a slightly modified problem. That is, we shall consider bodies whose boundary is not necessarily the graph of a function; in addition, we shall allow multiple reflections of the flow particles from the body. Note that in this broader class of bodies the explicit formula (2.23) for the specific resistance is no longer valid. Nevertheless, the problem in this form becomes easier and admits an explicit solution.

Let us pass to rigorous mathematical definitions. Following [47], we consider the problem in arbitrary dimension $d \geq 2$. Fix a unit vector $n = (n_1, \dots, n_d)$ such that $n_d > 0$ in Euclidean space $\mathbb{R}^d$ with orthogonal coordinates $x = (x_1, \dots, x_d)$, and consider a set B with a piecewise smooth boundary and containing the half-space $x \cdot n < 0$. Further, we consider a flow of particles with velocity $v = (0, \dots, 0, -1)$ impinging on the body. Initially (prior to reflections from the body) the coordinate of the particle is $x(t) = x + vt$; then it is reflected finitely many times from B at regular points of the boundary ∂B , and it finally moves freely. Let $v_B^+(x)$ denote its final velocity. The momentum transmitted by the particle to the body is proportional to $v - v_B^+(x)$, where the proportionality ratio is equal to the mass of the particle. Thus, we have the mapping $x \mapsto v_B^+(x)$ defined on a subset of $\mathbb{R}^d$ and translationally invariant in direction v . We also impose the regularity condition on the scattering of particles by the body.

Let $\mathcal{B}_n$ be the set of bodies $B \subset \mathbb{R}^d$ such that

- (a) B contains the half-space $\{x \cdot n < 0\}$ and
- (b) The scattering of particle flow by B is regular.

For a set $A \subset \{x_d = 0\}$ with a finite $(d-1)$ -dimensional Lebesgue measure denote

$$R(B, A) = \frac{1}{|A|} \int_A (v - v_B^+(x)) dx.$$

The quantity $R(B, A)$ admits a natural interpretation: it is the resistance of B to the flow with cross section A divided by the area of this section or, in other words, the flow pressure averaged over A .

Notice that $v_B^+(x) \cdot n \geq 0$, hence $(v - v_B^+(x)) \cdot n \leq -n_d$. On the other hand, v and $v_B^+(x)$ are unit vectors; therefore, $|v - v_B^+(x)| \leq 2$. This implies that

$$n_d \leq |R(B, A)| \leq 2 \tag{2.24}$$

for any $B \in \mathcal{B}_n$ and any Borel set A of a finite Lebesgue measure.

The upper estimate in (2.24) is exact; let, for instance, $\mathbb{R}^{d-1} = \cup_i A_i$, where A_i are pairwise disjoint bounded Borel sets with boundaries of measure zero, and let $\tilde{n} = (n_1, \dots, n_{d-1})$ and

$$B^* = \cup_i (A_i \times (-\infty, c_i)),$$

where $c_i \geq -\frac{1}{n_d} \inf\{\tilde{x} \cdot \tilde{n} : \tilde{x} \in A_i\}$. Then $B^* \in \mathcal{B}_n$ and

$$|R(B, A)| = 2$$

for any Borel set $A \subset \{x_d = 0\}$ of finite measure.

Let us show that the lower estimate in (2.24) is also exact.

Theorem 2.5. *There exists a sequence of sets $B_k \in \mathcal{B}_n$ such that*

$$\lim_{k \rightarrow \infty} |R(B_k, A)| = n_d$$

for any Borel set A with a finite Lebesgue measure. Moreover, each particle of the flow collides with B_k at most twice, and the sequence B_k approximates the half-space $x \cdot n < 0$ or, in more precise terms,

$$\{x : x \cdot n < 0\} \subset B_k \subset \{x : x \cdot n < \alpha_k\}$$

where $\lim_{k \rightarrow \infty} \alpha_k = 0$.

The following corollary is a direct consequence of the theorem.

Corollary 2.1. *For any set A of finite Lebesgue measure*

$$\inf_{B \in \mathcal{B}_n} |R(B, A)| = n_d$$

holds.

Proof. The proof of Theorem 2.5 is based on a direct construction of bodies B_k , and the case of higher dimensions reduces to the two-dimensional case $d = 2$. Indeed, suppose that for any $n^{(2)} = (n_1, n_2) \in S^1$ there exists a sequence of bodies $B_k^{(2)}$ providing the solution in two dimensions. Then the sequence $B_k^{(d)} = \mathbb{R}^{d-2} \times B_k^{(2)}$ is a solution for the vector $n^{(d)} = (0, \dots, 0, n_1, n_2) \in S^{d-1}$ in the d -dimensional case, $d \geq 3$. Now, to obtain a solution for arbitrary unit vector $n = (n_1, \dots, n_d)$, one first constructs a sequence B'_k that provides a solution for $n' = (0, \dots, 0, \sqrt{\sum_1^{d-1} n_j^2}, n_d)$, and then applies to each B'_k an isometry sending n' to n and v to v . Let B_k be the image of B'_k under this isometry; then the sequence B_k is a solution for n .

It remains to consider the two-dimensional case. First we briefly describe the idea of construction. Consider the plane with coordinates x, y , and assume that the axis Oy is directed upward. Take the half-plane $x \cdot n < \alpha_k$ and make a series of identical dimples on its boundary; as a result, we obtain the body B_k . A part of the boundary of the dimple is formed by two pairs of arcs of confocal parabolas, where the length of one arc in each pair is much smaller than that of the other one. A parallel beam of incident particles is reflected by the larger arc, passes through the common focus, is reflected by the second arc, and transforms again into a parallel beam (of smaller width), with the direction almost parallel to the boundary of the half-plane.

Let us now state a rigorous proof. Choose $\varphi \in (-\pi/2, \pi/2)$ such that $n = (\sin \varphi, \cos \varphi)$. Fix $a > 0$ and $\varepsilon > 0$ and set

$$x_0 = \frac{1 - \sin \varphi}{2} a. \quad (2.25)$$

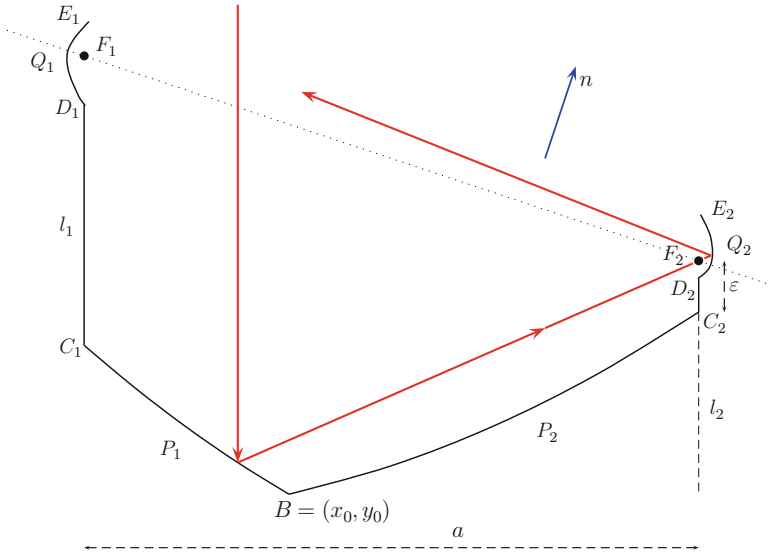


Fig. 2.15 A hollow

Consider the point $B = (x_0, y_0)$, with the value y_0 to be specified below. Draw two parabolas through B with upward branches, namely, parabola P_1 with focus at $F_2 = (a, -a \tan \varphi)$ and parabola P_2 with focus $F_1 = (0, 0)$. The axis of P_1 is denoted by l_2 , and the axis of P_2 by l_1 . The axes are vertical and pass through F_2 and F_1 , respectively. Choose y_0 such that point C_1 of the intersection of P_1 with l_1 would lie below F_1 , and point C_2 of intersection of P_2 with l_2 would lie below F_2 , and additionally $\min_{i=1,2} |C_i F_i| = \varepsilon$. Take two points F'_i , $i = 1, 2$ on the line l_i above F_i such that $|F_i F'_i| = \varepsilon$, and take two parabolas Q_1 and Q_2 such that

- (i) Q_1 has its focus at F_1 and the axis $F_1 F'_1$, and Q_2 has its focus at F_2 and the axis $F_2 F'_2$;
- (ii) $\max \{|D_i F_i|, |E_i F_i|, i = 1, 2\} = \varepsilon$, where D_i and E_i are, respectively, the lower and upper points of intersection of Q_i with l_i (Fig. 2.15).

Now consider the curve $\Gamma_{a,\varepsilon} = E_1 D_1 C_1 B C_2 D_2 E_2$ composed of the arcs of parabolas BC_i , $D_i E_i$ and the line segments $C_i D_i$, $i = 1, 2$. A particle falling vertically downward in accordance with $x(t) = (x, -t)$, $0 < x < x_0$, first reflects from the arc BC_1 , passes through F_2 , reflects from the arc $D_2 E_2$, and finally moves freely with velocity

$$v_{a,\varepsilon}^+(x) = (-\cos \varphi, \sin \varphi) + \beta_{\varepsilon/a}^1, \quad (2.26)$$

where $\lim_{\omega \rightarrow 0} \beta_{\omega}^1 = 0$. If $x_0 < x < a$, then the particle first reflects from the arc BC_2 , then passes through F_1 , reflects from $D_1 E_1$, and finally moves freely with velocity

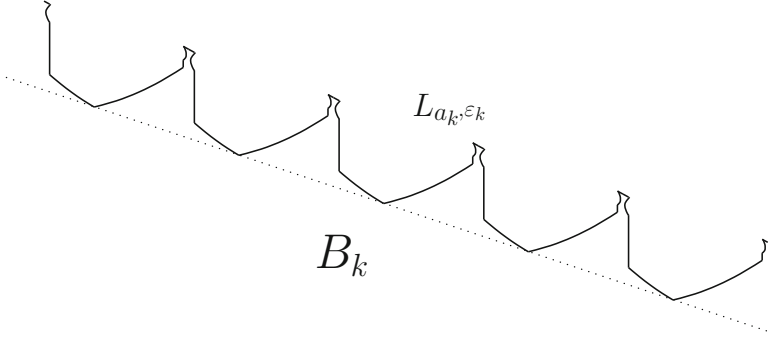


Fig. 2.16 A body B_k from the minimizing sequence is bounded above by the periodic curve L_{a_k, ε_k}

$$v_{a, \varepsilon}^+(x) = (\cos \varphi, -\sin \varphi) + \beta_{\varepsilon/a}^2, \quad (2.27)$$

where $\lim_{\omega \rightarrow 0} \beta_{\omega}^2 = 0$.

Taking into account (2.25)–(2.27), one obtains

$$\int_0^a (v - v_{\varepsilon/a}^+(x)) dx = -a \cos \varphi \varepsilon n + o(1), \quad \varepsilon/a \rightarrow 0.$$

Let $\Gamma_{a, \varepsilon}^{(m)}$ be the curve obtained by shifting $\Gamma_{a, \varepsilon}$ by the vector $-(x_0, y_0) + m(a + \varepsilon, -(a + \varepsilon) \tan \varphi)$, and consider the continuous line $L_{a, \varepsilon}$ composed of the curves $\Gamma_{a, \varepsilon}^{(m)}$, $m \in \mathbb{Z}$ and of line segments joining endpoints of neighboring curves. The line $L_{a, \varepsilon}$ is situated between the parallel straight lines $x \cdot n = 0$ and $x \cdot n = \alpha$, where $\alpha = x_0 \sin \varphi + y_0 \cos \varphi$. The parameters x_0 and y_0 , and therefore the parameter α , depend on a and ε and approach 0 when a and ε go to 0.

Choose sequences a_k and ε_k such that $\lim_{k \rightarrow \infty} a_k = 0$, $\lim_{k \rightarrow \infty} (\varepsilon_k/a_k) = 0$, and take the sets B_k bounded above by L_{a_k, ε_k} (Fig. 2.16). Clearly, $B_k \in \mathcal{B}_n$ and $\{x \cdot n < 0\} \subset B_k \subset \{x \cdot n < \alpha_k\}$ for the sequence of corresponding values α_k approaching zero as $k \rightarrow \infty$. For any line segment $A \subset \Pi = \{y = 0\}$, and therefore, for any Borel set $A \subset \Pi$

$$R(B_k, A) = -\cos \varphi n + o(1), \quad k \rightarrow \infty$$

holds. The proof of Theorem 2.5 is complete. $\square$

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