

Spectral Theory for Forward Nonautonomous Parabolic Equations and Applications

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Dedicated to Professor George Sell on the occasion of his 70th birthday

Abstract We introduce the concept of the principal spectrum for linear forward nonautonomous parabolic partial differential equations. The principal spectrum is a nonempty compact interval. Fundamental properties of the principal spectrum for forward nonautonomous equations are investigated. The paper concludes with applications of the principal spectrum theory to the problem of uniform persistence in some population growth models.

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1 Introduction

The current paper is devoted to the study of principal spectrum of the following linear nonautonomous parabolic equation:

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$$\begin{aligned}
u_t = & \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a_{ij}(t, x) \frac{\partial u}{\partial x_j} + a_i(t, x) u \right) \\
& + \sum_{i=1}^N b_i(t, x) \frac{\partial u}{\partial x_i} + c_0(t, x) u, \quad t > s, \quad x \in D,
\end{aligned} \tag{1}$$

endowed with the boundary condition

$$\mathcal{B}(t)u = 0, \quad t > s, \quad x \in \partial D, \tag{2}$$

where $D \subset \mathbb{R}^N$, $s \geq 0$, a_{ij} , a_i , b_i , and c_0 are appropriate functions on $[0, \infty) \times D$, and \mathcal{B} is a boundary operator of either the Dirichlet or Neumann or Robin type, that is,

$$\mathcal{B}(t)u = \begin{cases} u & \text{(Dirichlet)} \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(t, x) \partial_j u + a_i(t, x) u \right) \nu_i & \text{(Neumann)} \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(t, x) \partial_j u + a_i(t, x) u \right) \nu_i + d_0(t, x) u, & \text{(Robin)} \end{cases} \tag{3}$$

where d_0 is an appropriate function on $[0, \infty) \times \partial D$. Let $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ with $d_0 \equiv 0$ in the Dirichlet or Neumann boundary condition case. To indicate the dependence of (1)+(2) on a , we may write (1)+(2) as $(1)_a + (2)_a$.

Among others, (1)+(2) arise from linearization of nonautonomous nonlinear parabolic equations at a global solution (i.e., a solution which exists for all $t \geq 0$) as well as from linearization of autonomous nonlinear parabolic equations at a global time dependent solution.

Concerning the linearization of a nonlinear problem at a global solution, it is of great importance to study the dynamical behavior of solutions of (1) + (2) as $s \rightarrow \infty$ and $t - s \rightarrow \infty$, where s represents the initial time. This paper is focused on the study of the least upper bound of exponential growth rates of solutions of (1) + (2) as $s \rightarrow \infty$ and $t - s \rightarrow \infty$, which is equivalent to the study of so called principal spectrum of (1) + (2) introduced in this paper.

Observe that (1)+(2) is called forward nonautonomous because, first, we are mainly interested in the properties of solutions as $s \rightarrow \infty$, $t - s \rightarrow \infty$, and a_{ij} , a_i , b_i , c_0 , and d_0 are not necessarily defined for $t < 0$, and, second, the set of forward limiting equations can contain elements depending on time.

Principal spectrum for nonautonomous parabolic equations defined for all $t \in \mathbb{R}$ is well studied in several works (see [9–12, 14–18, 21], and references therein) and has also found great applications (see [8, 13, 19, 22, 27], etc.). Principal spectrum for such nonautonomous parabolic equations reflects the growth rates of solutions as $t - s \rightarrow \infty$, where s represents the initial time.

As the focus for forward nonautonomous parabolic equations is on the study of the behavior of solutions as $s \rightarrow \infty$ and $t - s \rightarrow \infty$, the principal spectral theory developed for nonautonomous parabolic equations defined for all $t \in \mathbb{R}$ cannot be applied to forward nonautonomous ones directly. The objective of this paper is to establish some principal spectral theory for forward nonautonomous parabolic equations, and discuss its applications to nonlinear parabolic equations of Kolmogorov type.

In order to do so, we first in Sect. 2 introduce the assumptions and the notion of weak solutions of (1)+(2) and present some basic properties of weak solutions.

In Sect. 3, we give the definition of principal spectrum of (1)+(2) and establish some fundamental properties. Let $U_a(t, s)u_0$ denote the weak solution of (1)+(2) with initial condition $u(s) = U_a(s, s)u_0 = u_0 \in L_2(D)$ ($s \geq 0$). Roughly speaking, the *principal spectrum* of (1)+(2) is the complement in \mathbb{R} of all the $\lambda \in \mathbb{R}$ satisfying either of the following conditions:

- There are $\eta > 0$, $M \geq 1$, and $T > 0$ such that

$$\|U_a(t, s)\| \leq Me^{(\lambda - \eta)(t - s)} \quad \text{for } t > s \geq T;$$

- There are $\eta > 0$, $M \in (0, 1]$, and $T > 0$ such that

$$\|U_a(t, s)\| \geq Me^{(\lambda + \eta)(t - s)} \quad \text{for } t > s \geq T$$

(see Definition 3.2). Among others, it is proved in Sect. 3 that

- The principal spectrum of (1)+(2) is a compact interval $[\lambda_{\min}(a), \lambda_{\max}(a)]$ (see Theorem 3.1).
- $\lambda_{\min}(a) = \liminf_{\substack{s \rightarrow \infty \\ t - s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t - s}$ and $\lambda_{\max}(a) = \limsup_{\substack{s \rightarrow \infty \\ t - s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t - s}$ (see Theorem 3.2).

In Sect. 4, we investigate the relation of the principal spectrum of (1)+(2) and the principal spectrum, denoted by $[\tilde{\lambda}_{\min}(a), \tilde{\lambda}_{\max}(a)]$, of its forward limiting equations, and show that if some extension of (1)+(2) together with its limiting equations admits a so-called exponential separation, then

- $[\lambda_{\min}(a), \lambda_{\max}(a)] = [\tilde{\lambda}_{\min}(a), \tilde{\lambda}_{\max}(a)]$ (see Theorem 4.3);
- $\lambda_{\min}(a) = \liminf_{\substack{s \rightarrow \infty \\ t - s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t - s}$ and $\lambda_{\max}(a) = \limsup_{\substack{s \rightarrow \infty \\ t - s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t - s}$ for any nonzero nonnegative $u_0 \in L_2(D)$ (see Theorem 4.3);
- If, moreover, (1)+(2) is asymptotically uniquely ergodic, which includes the asymptotically periodic as a special case, then $[\lambda_{\min}(a), \lambda_{\max}(a)]$ is a singleton, i.e., $\lambda_{\min}(a) = \lambda_{\max}(a)$, and in the asymptotically periodic case, $\lambda_{\max}(a)$ equals the principal eigenvalue of the forward limiting periodic parabolic equation (see Corollary 4.5).

In Sect. 5, we establish more properties of the principal spectrum $[\lambda_{\min}(a), \lambda_{\max}(a)]$ of (1)+(2), including

- $\lambda_{\min}(a)$ and $\lambda_{\max}(a)$ continuously depend on a in the norm topology (see Theorem 5.1);
- When a_{ij} , a_i and b_i depend only on x , the principal spectrum of (1)+(2) is greater than or equal to that of its time-averaged equations (see Theorem 5.3).

The properties mentioned above provide some important tools for the principal spectrum analysis as well as its computation.

We remark that the theories and techniques developed in this paper would have applications to the study of long time behavior in various forward nonautonomous nonlinear equations arising from biology and chemistry. In particular, they would have applications to the extensions of the existing dynamical theories for asymptotically periodic systems (see [28–31], etc.) to asymptotically uniquely ergodic ones, which include asymptotically periodic and almost periodic systems as special cases. In the last section (i.e., Sect. 6), we discuss applications of the principal spectrum theory for forward nonautonomous parabolic equations to the asymptotic dynamics of nonlinear parabolic equations of Kolmogorov type. In particular, we provide sufficient conditions for the uniform persistence (see Theorem 6.1).

Throughout the paper $D \subset \mathbb{R}^N$ is a bounded domain (an open and connected subset).

The norm in $L_2(D)$ is denoted by $\|\cdot\|$. Also, the norm in the Banach space $\mathcal{L}(L_2(D), L_2(D))$ of bounded linear operators from $L_2(D)$ into $L_2(D)$ is denoted by $\|\cdot\|$.

For the meaning of some symbols, like $C^{k+\alpha, l+\beta}(E_1 \times E_2)$, or $\mathcal{D}(E)$, etc., the reader is referred to the authors' monograph [18].

2 Assumptions and Weak Solutions

In this section, we state the assumptions, introduce the definition of weak solutions, and present some basic properties of weak solutions.

2.1 Assumptions

Consider (1)+(2). Our first assumption is on the regularity of the domain D .

(A1) (Boundary regularity) *For Dirichlet boundary conditions, D is a bounded domain. For Neumann or Robin boundary conditions, D is a bounded domain with Lipschitz boundary.*

If (A1) holds, D is always considered with the N -dimensional Lebesgue measure, whereas, in the case of Robin boundary conditions, ∂D is considered with the $(N-1)$ -dimensional Hausdorff measure, which is equivalent to the surface measure.

The second assumption regards boundedness of the coefficients of the equations (and of the boundary conditions):

(A2) (Boundedness) $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ belongs to $L_\infty([0, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0, \infty) \times \partial D, \mathbb{R})$ (in the Dirichlet or Neumann case d_0 is set to be constantly equal to zero).

We may write $a = (a_{ij}, a_i, b_i, c_0, d_0)$ for $a = ((a_{ij})_{i,j=1}^N, (a_i)_{i=1}^N, (b_i)_{i=1}^N, c_0, d_0)$ if no confusion occurs.

The next assumption is about the uniform ellipticity.

(A3) (Uniform ellipticity) *There exists $\alpha_0 > 0$ such that there holds*

$$\sum_{i,j=1}^N a_{ij}(t, x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^N \xi_i^2 \quad \text{for a.e. } (t, x) \in [0, \infty) \times D \text{ and all } \xi \in \mathbb{R}^N,$$

$$a_{ij}(t, x) = a_{ji}(t, x) \quad \text{for a.e. } (t, x) \in [0, \infty) \times D, \quad i, j = 1, 2, \dots, N. \quad (4)$$

Sometimes we will use the forward limit equations to study the principal spectrum of (1)+(2). For any $t \geq 0$ we define the *time-translate* $a \cdot t$ of a by

$$a \cdot t := ((a_{ij} \cdot t)_{i,j=1}^N, (a_i \cdot t)_{i=1}^N, (b_i \cdot t)_{i=1}^N, c_0 \cdot t, d_0 \cdot t),$$

where $a_{ij} \cdot t(\tau, x) := a_{ij}(\tau + t, x)$ for $\tau \in [-t, \infty)$, $x \in D$, etc.

For a given sequence $(t_n) \subset [0, \infty)$ with $t_n \rightarrow T^*$ ($T^* \leq \infty$) and $\tilde{a} = ((\tilde{a}_{ij})_{i,j=1}^N, (\tilde{a}_i)_{i=1}^N, (\tilde{b}_i)_{i=1}^N, \tilde{c}_0, \tilde{d}_0) \in L_\infty((-T^*, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty((-T^*, \infty) \times \partial D, \mathbb{R})$, we say that $a \cdot t_n$ converges to \tilde{a} in the weak-* topology if for any $T > -T^*$, $a \cdot t_n \rightarrow \tilde{a}$ in the weak-* topology of $L_\infty([T, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([T, \infty) \times \partial D, \mathbb{R})$.

Recall that the Banach space $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ is the dual of $L_1(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_1(\mathbb{R} \times \partial D, \mathbb{R})$. We denote the duality pairing by $\langle \cdot, \cdot \rangle_{L_1, L_\infty}$.

We fix a countable dense subset $\{g_1, g_2, \dots\}$ of the unit ball in $L_1(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_1(\mathbb{R} \times \partial D, \mathbb{R})$ such that for each $k \in \mathbb{N}$ there exists $K = K(k) > 0$ with the property that $g_k(t, \cdot) = 0$ for a.e. $t \in \mathbb{R} \setminus [-K, K]$.

For any $\tilde{a}^{(1)}, \tilde{a}^{(2)} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ put

$$d(\tilde{a}^{(1)}, \tilde{a}^{(2)}) := \sum_{k=1}^{\infty} \frac{1}{2^k} |\langle g_k, (\tilde{a}^{(1)} - \tilde{a}^{(2)}) \rangle_{L_1, L_\infty}|. \quad (5)$$

For any $\tilde{a} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$, $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0)$, and any $t \in \mathbb{R}$ we define the *time-translate* $\tilde{a} \cdot t$ of \tilde{a} by

$$\tilde{a} \cdot t := ((\tilde{a}_{ij} \cdot t)_{i,j=1}^N, (\tilde{a}_i \cdot t)_{i=1}^N, (\tilde{b}_i \cdot t)_{i=1}^N, \tilde{c}_0 \cdot t, \tilde{d}_0 \cdot t),$$

where $\tilde{a}_{ij} \cdot t(\tau, x) := \tilde{a}_{ij}(\tau + t, x)$ for $\tau \in \mathbb{R}, x \in D$, etc.

We may extend a to functions belonging to $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ to study the forward limits of a . A function $\bar{a} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$, $\bar{a} = ((\bar{a}_{ij})_{i,j=1}^N, (\bar{a}_i)_{i=1}^N, (\bar{b}_i)_{i=1}^N, \bar{c}_0, \bar{d}_0)$, is called an *extension of a* if $\bar{a}_{ij}(t, x) = a_{ij}(t, x)$, $\bar{a}_i(t, x) = a_i(t, x)$, $\bar{b}_i(t, x) = b_i(t, x)$, and $\bar{c}_0(t, x) = c_0(t, x)$ for a.e. $(t, x) \in [0, \infty) \times D$, and $\bar{d}_0(t, x) = d_0(t, x)$ for a.e. $(t, x) \in [0, \infty) \times \partial D$.

The lemma below will be instrumental in showing that the forward limits of a do not depend on the extension of a to a function in $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$.

Lemma 2.1. *Let $\bar{a}^{(1)} = (\bar{a}_{ij}^{(1)}, \bar{a}_i^{(1)}, \bar{b}_i^{(1)}, \bar{c}_0^{(1)}, \bar{d}_0^{(1)})$ and $\bar{a}^{(2)} = (\bar{a}_{ij}^{(2)}, \bar{a}_i^{(2)}, \bar{b}_i^{(2)}, \bar{c}_0^{(2)}, \bar{d}_0^{(2)})$ be extensions of $a \in L_\infty([0, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0, \infty) \times \partial D, \mathbb{R})$. Then, for any $t_n \rightarrow \infty$, one has $d(\bar{a}^{(1)} \cdot t_n, \bar{a}^{(2)} \cdot t_n) \rightarrow 0$. In particular, $\bar{a}^{(1)} \cdot t_n$ converges in the weak-* topology to $\tilde{a} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ if and only if $\bar{a}^{(2)} \cdot t_n$ converges in the weak-* topology to \tilde{a} .*

Proof. For $\varepsilon > 0$, take $k_0 \in \mathbb{N}$ such that

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} < \frac{\varepsilon}{2M},$$

where M denotes the maximum of the $(L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R}))$ -norms of $\bar{a}^{(1)}$ and $\bar{a}^{(2)}$. Then we have

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} |\langle g_k, (\bar{a}^{(1)} \cdot \tau - \bar{a}^{(2)} \cdot \tau) \rangle_{L_1, L_\infty}| < \varepsilon$$

for each $\tau \in \mathbb{R}$. Let $K > 0$ be such that $g_k(t, x) = 0$ for a.e. t outside $[-K, K]$, for all $k = 1, 2, \dots, k_0 - 1$. We have

$$\sum_{k=1}^{k_0-1} \frac{1}{2^k} |\langle g_k, (\bar{a}^{(1)} \cdot t_n - \bar{a}^{(2)} \cdot t_n) \rangle_{L_1, L_\infty}| = 0$$

for $n \in \mathbb{N}$ so large that $t_n > K$. As a result, $d(\bar{a}^{(1)} \cdot t_n, \bar{a}^{(2)} \cdot t_n) < \varepsilon$ for such n . Therefore

$$d(\bar{a}^{(1)} \cdot t_n, \bar{a}^{(2)} \cdot t_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and then $\bar{a}^{(1)} \cdot t_n$ converges in the weak-* topology to $\tilde{a} \in L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ if and only if $\bar{a}^{(2)} \cdot t_n$ converges in the weak-* topology to \tilde{a} . \square

For an extension \bar{a} of a , the set $\{\bar{a} \cdot t : t \in \mathbb{R}\}$ is (norm-)bounded, hence has compact closure in the weak-* topology. We define

$$Y(\bar{a}) := \text{cl}\{\bar{a} \cdot t : t \in \mathbb{R}\}, \quad (6)$$

where the closure is taken in the weak-* topology. When not remarked to the contrary, $Y(\bar{a})$ is considered with the weak-* topology. $Y(\bar{a})$ is a compact metrizable space, with a metric given by $d(\cdot, \cdot)$.

For $\bar{a} \in Y(\bar{a})$ and $t \in \mathbb{R}$ we write $\sigma_t \bar{a} := \bar{a} \cdot t$. $(Y(\bar{a}), \{\sigma_t\}_{t \in \mathbb{R}})$ is a compact flow (i.e., $\sigma_t \bar{a}$ is continuous in $t \in \mathbb{R}$ and $\bar{a} \in Y(\bar{a})$, and $\sigma_0 = \text{Id}$, $\sigma_{t+s} = \sigma_t \circ \sigma_s$ for any $t, s \in \mathbb{R}$).

Let \bar{a} be an extension of a . Put

$$Y_0(\bar{a}) := \bigcap_{s \geq 0} \text{cl}\{\bar{a} \cdot t : t \in [s, \infty)\}. \quad (7)$$

In other words, $Y_0(\bar{a})$ equals the ω -limit set of \bar{a} for the compact flow $(Y(\bar{a}), \{\sigma_t\}_{t \in \mathbb{R}})$. By standard results in the theory of topological dynamical systems, $Y_0(\bar{a})$ is invariant, nonempty, compact and connected. Also, $\bar{a} \in Y_0(\bar{a})$ if and only if there is a sequence $t_n \rightarrow \infty$ such that $\bar{a} \cdot t_n \rightarrow \bar{a}$ as $n \rightarrow \infty$.

In view of Lemma 2.1, $Y_0(\bar{a})$ does not depend on the choice of extension \bar{a} of a . We can (and will) thus write $Y_0(a)$. Further, $\tilde{a} \in Y_0(a)$ if and only if there is a sequence $t_n \rightarrow \infty$ such that $a \cdot t_n \rightarrow \tilde{a}$ as $n \rightarrow \infty$.

The next assumption will be instrumental in proving the continuous dependence of solutions on parameters.

(A4) (Convergence almost everywhere)

In the Dirichlet or Neumann case:

(A4a) For any sequence $(t_n) \subset [0, \infty)$ with $t_n \rightarrow T^*$ ($T^* \leq \infty$) such that $a \cdot t_n$ converges to \tilde{a} in the weak-* topology we have that $a_{ij} \cdot t_n \rightarrow \tilde{a}_{ij}$, $a_i \cdot t_n \rightarrow \tilde{a}_i$, $b_i \cdot t_n \rightarrow \tilde{b}_i$ pointwise a.e. on $[T, \infty) \times D$, for any $T > -T^*$,

and

(A4b) for any sequence $(\tilde{a}^{(n)}) \subset Y_0(a)$ converging to \tilde{a} in the weak-* topology we have that $\tilde{a}_{ij}^{(n)} \rightarrow \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \rightarrow \tilde{a}_i$, $\tilde{b}_i^{(n)} \rightarrow \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$.

In the Robin case:

(A4a) For any sequence $(t_n) \subset [0, \infty)$ with $t_n \rightarrow T^*$ ($T^* \leq \infty$) such that $a \cdot t_n$ converges to \tilde{a} in the weak-* topology we have that $a_{ij} \cdot t_n \rightarrow \tilde{a}_{ij}$, $a_i \cdot t_n \rightarrow \tilde{a}_i$, $b_i \cdot t_n \rightarrow \tilde{b}_i$ pointwise a.e. on $[T, \infty) \times D$, and $d_0 \cdot t_n \rightarrow \tilde{d}_0$ pointwise a.e. on $[T, \infty) \times \partial D$, for any $T > -T^*$,

and

(A4b) for any sequence $(\tilde{a}^{(n)}) \subset Y_0(a)$ converging to \tilde{a} in the weak-* topology we have that $\tilde{a}_{ij}^{(n)} \rightarrow \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \rightarrow \tilde{a}_i$, $\tilde{b}_i^{(n)} \rightarrow \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$, and $\tilde{d}_0^{(n)} \rightarrow \tilde{d}_0$ pointwise a.e. on $\mathbb{R} \times \partial D$.

To study the continuous dependence of the weak solutions and principal spectrum of (1)+(2) with respect to its coefficients, we may imbed the extensions of a into a subset Y of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying

(A2)' (Boundedness and invariance) Y is a bounded subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ and is closed (hence, compact) in the weak-* topology of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$. Moreover, Y is invariant: For any $\tilde{a} \in Y$ and any $t \in \mathbb{R}$ there holds $\tilde{a} \cdot t \in Y$.

(It should be remarked here that, under Assumption (A2)', $(Y, \{\sigma_t\}_{t \in \mathbb{R}})$, where $\sigma_t \tilde{a} := \tilde{a} \cdot t$, is a compact flow.)

(A3)' (Uniform ellipticity) There exists $\alpha_0 > 0$ such that for any $\tilde{a} \in Y$ there holds

$$\sum_{i,j=1}^N \tilde{a}_{ij}(t,x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^N \xi_i^2 \quad \text{for a.e. } (t,x) \in \mathbb{R} \times D \text{ and all } \xi \in \mathbb{R}^N,$$

$$\tilde{a}_{ij}(t,x) = \tilde{a}_{ji}(t,x) \quad \text{for a.e. } (t,x) \in \mathbb{R} \times D, \quad i, j = 1, 2, \dots, N. \quad (8)$$

At some places, we may assume

(A4)' (Convergence almost everywhere)

In the Dirichlet or Neumann case:

For any sequence $(\tilde{a}^{(n)}) \subset Y$ converging to \tilde{a} in the weak-* topology we have that $\tilde{a}_{ij}^{(n)} \rightarrow \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \rightarrow \tilde{a}_i$, $\tilde{b}_i^{(n)} \rightarrow \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$.

In the Robin case:

For any sequence $(\tilde{a}^{(n)}) \subset Y$ converging to \tilde{a} in the weak-* topology we have that $\tilde{a}_{ij}^{(n)} \rightarrow \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \rightarrow \tilde{a}_i$, $\tilde{b}_i^{(n)} \rightarrow \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$, and $\tilde{d}_0^{(n)} \rightarrow \tilde{d}_0$ pointwise a.e. on $\mathbb{R} \times \partial D$.

Observe that for a given a satisfying (A2) and (A3), $Y = Y_0(a)$ satisfies (A2)' and (A3)'.

For $a \in L_\infty([0, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0, \infty) \times \partial D, \mathbb{R})$ satisfying (A2) and (A3) we denote by $\bar{a} = (\bar{a}_{ij}, \bar{a}_i, \bar{b}_i, \bar{c}_0, \bar{d}_0)$ the extension of a given by

$$\begin{aligned} \bar{a}_{ij}(t,x) &:= \alpha_0 \delta_{ij} & \text{for } t < 0, x \in D, \\ \bar{a}_i(t,x) &:= 0 & \text{for } t < 0, x \in D, \\ \bar{b}_i(t,x) &:= 0 & \text{for } t < 0, x \in D, \\ \bar{c}_0(t,x) &:= 0 & \text{for } t < 0, x \in D, \\ \bar{d}_0(t,x) &:= 0 & \text{for } t < 0, x \in \partial D \end{aligned} \quad (9)$$

(δ_{ij} denotes the Kronecker delta).

Sometimes, for a fulfilling (A2) and (A3), we pick up some extension \bar{a} of a so that $Y = Y(\bar{a})$ satisfies (A2)' and (A3)'. We may say that such \bar{a} satisfies (A2)' and (A3)'. If $Y = Y(\bar{a})$ satisfies (A4)', we say \bar{a} satisfies (A4)'.

Clearly \bar{a} defined by (9) satisfies $(A2)'$ and $(A3)'$.

Lemma 2.2. *For a satisfying $(A2)$ – $(A4)$ the extension \bar{a} given by (9) satisfies $(A4)'$.*

Proof. In the following, the expression “ $\tilde{a}^{(n)}$ converges pointwise a.e. to \tilde{a} ” means that $\tilde{a}_{ij}^{(n)} \rightarrow \tilde{a}_{ij}$, $\tilde{a}_i^{(n)} \rightarrow \tilde{a}_i$, $\tilde{b}_i^{(n)} \rightarrow \tilde{b}_i$ pointwise a.e. on $\mathbb{R} \times D$, and $\tilde{d}_0^{(n)} \rightarrow \tilde{d}_0$ pointwise a.e. on $\mathbb{R} \times \partial D$.

Note that the proof reduces to proving the following subcases:

- (i) *For any real sequence (t_n) with $\lim_{n \rightarrow \infty} t_n = -\infty$ we have that $\bar{a} \cdot t_n$ converges pointwise a.e. to $(\alpha_0 \delta_{ij}, 0, 0, 0, 0)$.*

This is straightforward.

- (ii) *For any real sequence (t_n) with $\lim_{n \rightarrow \infty} t_n = T \in \mathbb{R}$ we have that $\bar{a} \cdot t_n$ converges pointwise a.e. to $\bar{a} \cdot T$.*

The fact that the corresponding coefficients converge pointwise a.e. on $[T, \infty) \times D$ (resp. pointwise a.e. on $[T, \infty) \times \partial D$) is a consequence of $(A4a)$. The pointwise convergence a.e. on $(-\infty, T) \times D$ (resp. on $(-\infty, T) \times \partial D$) follows by the construction of \bar{a} .

- (iii) *For any real sequence (t_n) with $\lim_{n \rightarrow \infty} t_n = \infty$ such that $\bar{a} \cdot t_n$ converges to $\tilde{a} \in Y_0(a)$ in the weak-* topology we have that $\bar{a} \cdot t_n$ converges pointwise a.e. to \tilde{a} .*

This is a consequence of $(A4a)$.

- (iv) *For any sequence $(a^{(n)}) \subset Y_0(a)$ convergent to $\tilde{a} \in Y_0(a)$ in the weak-* topology we have that $a^{(n)}$ converges pointwise a.e. to \tilde{a} .*

This is just $(A4b)$. □

The next result is a consequence of the Ascoli–Arzelà theorem.

Lemma 2.3. *Assume that the boundary ∂D of D is of class C^β , for some $\beta > 0$.*

- (1) *If $a_{ij}, a_i, b_i, c_0 \in C^{\beta_1, \beta_2}([0, \infty) \times D)$, and $d_0 \in C^{\beta_1, \beta_2}([0, \infty) \times \partial D)$, where $0 < \beta_2 \leq \beta$, then $a = (a_{ij}, a_i, b_i, c_0, d_0)$ satisfies $(A4)$.*
- (2) *Assume that Y satisfies $(A2)'$. If for each $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y$ there holds $\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0 \in C^{\beta_1, \beta_2}(\mathbb{R} \times D)$, and $\tilde{d}_0 \in C^{\beta_1, \beta_2}(\mathbb{R} \times \partial D)$, where $0 < \beta_2 \leq \beta$, and the $C^{\beta_1, \beta_2}(\mathbb{R} \times D)$ -norms of $\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0$ are bounded uniformly in $\tilde{a} \in Y$ and the $C^{\beta_1, \beta_2}(\mathbb{R} \times \partial D)$ -norms of \tilde{d}_0 are bounded uniformly in $\tilde{a} \in Y$, then Y satisfies $(A4)'$.*

2.2 Weak Solutions: Definition

Throughout this subsection, D satisfies $(A1)$ and Y is a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying $(A2)'$ – $(A3)'$.

Let a satisfy $(A2)$, $(A3)$, and let \bar{a} be an extension of a such that $Y(\bar{a}) \subset Y$. In particular, \bar{a} satisfies $(A2)'$ and $(A3)'$.

We define V as follows:

$$V := \begin{cases} \mathring{W}_2^1(D) & \text{(Dirichlet)} \\ W_2^1(D) & \text{(Neumann)} \\ W_{2,2}^1(D, \partial D), & \text{(Robin)} \end{cases} \quad (10)$$

where $\mathring{W}_2^1(D)$ is the closure of $\mathcal{D}(D)$ in $W_2^1(D)$ and $W_{2,2}^1(D, \partial D)$ is the completion of

$$V_0 := \{v \in W_2^1(D) \cap C(\bar{D}) : v \text{ is } C^\infty \text{ on } D \text{ and } \|v\|_V < \infty\}$$

with respect to the norm $\|v\|_V := (\|\nabla v\|_2^2 + \|v\|_{2,\partial D}^2)^{1/2}$, where $\mathcal{D}(D)$ is the space of smooth real functions having compact support in D .

If no confusion occurs, we will write $\langle u, u^* \rangle$ for the duality between V and V^* , where $u \in V$ and $u^* \in V^*$.

For $s \leq t$, let

$$W = W(s, t; V, V^*) := \{v \in L_2((s, t), V) : \dot{v} \in L_2((s, t), V^*)\} \quad (11)$$

equipped with the norm

$$\|v\|_W := \left(\int_s^t \|v(t)\|_V^2 dt + \int_s^t \|\dot{v}(t)\|_{V^*}^2 dt \right)^{\frac{1}{2}},$$

where $\dot{v} := dv/dt$ is the time derivative in the sense of distributions taking values in V^* (see [5, Chap. XVIII] for definitions).

For a given $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y$, consider

$$u_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N \tilde{a}_{ij}(t, x) \frac{\partial u}{\partial x_j} + \tilde{a}_i(t, x) u \right) + \sum_{i=1}^N \tilde{b}_i(t, x) \frac{\partial u}{\partial x_i} + \tilde{c}_0(t, x) u, \quad x \in D, \quad (12)$$

endowed with the boundary condition

$$\tilde{\mathcal{B}}(t)u = 0, \quad x \in \partial D, \quad (13)$$

where $\tilde{\mathcal{B}}$ is a boundary operator of either the Dirichlet or Neumann or Robin type, that is, $\tilde{\mathcal{B}}(t)u = \mathcal{B}(t)u$, where $\mathcal{B}(t)u$ is as in (3) with a being replaced by \tilde{a} . Sometimes we write the nonautonomous problem (12)+(13) as (12) $_{\tilde{a}}$ +(13) $_{\tilde{a}}$.

Denote by $B_{\tilde{a}} = B_{\tilde{a}}(t, \cdot, \cdot)$ the bilinear form on V associated with $\tilde{a} \in Y$,

$$B_{\tilde{a}}(t, u, v) := \int_D ((\tilde{a}_{ij}(t, x) \partial_j u + \tilde{a}_i(t, x) u) \partial_i v - (\tilde{b}_i(t, x) \partial_i u + \tilde{c}_0(t, x) u) v) dx, \quad (14)$$

$(u, v \in V)$ in the Dirichlet and Neumann boundary condition cases, and

$$\begin{aligned} B_{\bar{a}}(t, u, v) := & \int_D ((\tilde{a}_{ij}(t, x) \partial_j u + \tilde{a}_i(t, x) u) \partial_i v - (\tilde{b}_i(t, x) \partial_i u + \tilde{c}_0(t, x) u) v) dx \\ & + \int_{\partial D} \tilde{d}_0(t, x) uv dH_{N-1}, \end{aligned} \quad (15)$$

$(u, v \in V)$ in the Robin boundary condition case, where H_{N-1} stands for the $(N-1)$ -dimensional Hausdorff measure, which is, by (A1), equivalent to the surface measure (we used the summation convention in the above).

Definition 2.1 (Weak solution). (1) Let $\tilde{a} \in Y$. A function $u \in L_2((s, t), V)$ is a *weak solution of (12) $_{\bar{a}}$ +(13) $_{\bar{a}}$ on $[s, t] \times D$, $s < t$, with initial condition $u(s) = u_0$* if

$$- \int_s^t \langle u(\tau), v \rangle \dot{\phi}(\tau) d\tau + \int_s^t B_{\bar{a}}(\tau, u(\tau), v) \phi(\tau) d\tau - \langle u_0, v \rangle \phi(s) = 0 \quad (16)$$

for all $v \in V$ and $\phi \in \mathcal{D}([s, t])$, where $\mathcal{D}([s, t])$ is the space of all smooth real functions having compact support in $[s, t]$.

(2) If \bar{a} is an extension of a and $s \geq 0$, a weak solution $u \in L_2((s, t), V)$ of (12) $_{\bar{a}}$ +(13) $_{\bar{a}}$ on $[s, t] \times D$ with initial condition $u(s) = u_0$ is called a *weak solution of (1)+(2) on $[s, t] \times D$ with initial condition $u(s) = u_0$* .

Definition 2.2 (Global weak solution). (1) Let $\tilde{a} \in Y$. A function $u \in L_{2, \text{loc}}((s, \infty), V)$ is a *global weak solution of (12) $_{\bar{a}}$ +(13) $_{\bar{a}}$ with initial condition $u(s) = u_0$, $s \in \mathbb{R}$* , if for each $t > s$ its restriction $u|_{[s, t]}$ is a weak solution of (12)+(13) on $[s, t] \times D$ with initial condition $u(s) = u_0$.

(2) If \bar{a} is an extension of a and $s \geq 0$, a global solution of (12) $_{\bar{a}}$ +(13) $_{\bar{a}}$ on $[s, \infty)$ is called a *global solution of (1)+(2) on $[s, \infty)$* .

We remark that the (global) weak solutions of (1)+(2) are independent of the choices of the extensions of a . Sometimes we will write of (global) weak solutions of (1) $_a$ +(2) $_a$.

2.3 Weak Solutions: Basic Properties

Throughout this subsection, D satisfies (A1) and Y is a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying (A2)'–(A4)'.

Let a satisfy (A2)–(A4), and let \bar{a} be an extension of a such that $Y(\bar{a}) \subset Y$.

We recall some basic properties of weak solutions of (12)+(13) ((1)+(2)) from [18] to be used in later sections. The reader is referred to [18] for various other important properties.

Proposition 2.4 (Existence of global solution). *For any $\tilde{a} \in Y$, $s \in \mathbb{R}$, and any $u_0 \in L_2(D)$ there exists a unique global weak solution $u(t; s, \tilde{a}, u_0)$ of (12) $_{\tilde{a}}$ +(13) $_{\tilde{a}}$ with initial condition $u(s; s, \tilde{a}, u_0) = u_0$.*

Proof. See [3, Theorem 2.4]. □

As, for $s \leq t$ and $\tilde{a} \in Y$ fixed, the assignment $[L_2(D) \ni u_0 \mapsto u(t; s, \tilde{a}, u_0) \in L_2(D)]$ is linear, we write $U_{\tilde{a}}(t, s)u_0$ for $u(t; s, \tilde{a}, u_0)$.

Proposition 2.5. (i) *For any $s \leq t$ and any $\tilde{a} \in Y$ there holds*

$$U_{\tilde{a}}(t, s) = U_{\tilde{a}, s}(t - s, 0).$$

(ii) *For any $s \leq t_1 \leq t_2$ and any $\tilde{a} \in Y$ there holds*

$$U_{\tilde{a}}(t_2, s) = U_{\tilde{a}}(t_2, t_1) \circ U_{\tilde{a}}(t_1, s).$$

As a consequence, for any $s \leq t$ and any $\tilde{a} \in Y$ there holds

$$U_{\tilde{a}}(s + t, 0) = U_{\tilde{a}, s}(t, 0) \circ U_{\tilde{a}}(s, 0).$$

Proof. See [18, Propositions 2.1.6, 2.1.7 and 2.1.8]. □

We may write $U_{\tilde{a}}(t, s)$ as $U_a(t, s) = U_{a, s}(t - s, 0)$ if $t \geq s \geq 0$.

Proposition 2.6 (L_2 - L_2 estimates). *There are constants $M > 0$ and $\gamma > 0$ such that*

$$\|U_{\tilde{a}}(t, 0)\| \leq Me^{\gamma t} \tag{17}$$

for $\tilde{a} \in Y$ and $t > 0$.

Proof. See [18, Proposition 2.2.2]. □

Proposition 2.7 (Compactness). *For any given $0 < t_1 \leq t_2$, if E is a bounded subset of $L_2(D)$ then $\{U_{\tilde{a}}(t, 0)u_0 : \tilde{a} \in Y, t \in [t_1, t_2], u_0 \in E\}$ is relatively compact in $L_2(D)$.*

Proof. See [18, Proposition 2.2.5]. □

For $u, v \in L_2(D)$ we write $u \leq v$ (or $v \geq u$) if $u(x) \leq v(x)$ for a.e. $x \in D$. We denote $L_2(D)^+ := \{u \in L_2(D) : u \geq 0\}$.

Proposition 2.8 (Monotonicity on initial data). *Let $\tilde{a} \in Y$, $t > 0$ and $u_1, u_2 \in L_2(D)$.*

(1) *If $u_1 \leq u_2$ then $U_{\tilde{a}}(t, 0)u_1 \leq U_{\tilde{a}}(t, 0)u_2$.*

(2) *If $u_1 \leq u_2$, $u_1 \neq u_2$, then $(U_{\tilde{a}}(t, 0)u_1)(x) < (U_{\tilde{a}}(t, 0)u_2)(x)$ for $x \in D$.*

Proof. See [18, Proposition 2.2.9]. □

Lemma 2.9. *Let $\tilde{a} \in Y$ and $t > 0$. Then $\|U_{\tilde{a}}(t, 0)\| = \sup\{U_{\tilde{a}}(t, 0)u_0 : u_0 \in L_2(D)^+, \|u_0\| = 1\}$.*

Proof. See [18, Lemma 3.1.1]. □

Proposition 2.10 (Monotonicity on coefficients). *Assume that $a^{(1)}$ and $a^{(2)}$ satisfy (A2)–(A4).*

- (1) *Assume the Dirichlet boundary condition. Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, but $c_0^{(1)} \leq c_0^{(2)}$, where equalities and inequalities are to be understood a.e. on $[T, \infty) \times D$. Then*

$$U_{a^{(1)}}(t, s)u_0 \leq U_{a^{(2)}}(t, s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$.

- (2) *Assume the Neumann or Robin boundary condition. Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, but $c_0^{(1)} \leq c_0^{(2)}$, $d_0^{(1)} \geq d_0^{(2)}$, where equalities and inequalities are to be understood a.e. on $[T, \infty) \times D$ or a.e. on $[T, \infty) \times \partial D$. Then*

$$U_{a^{(1)}}(t, s)u_0 \leq U_{a^{(2)}}(t, s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$.

- (3) *Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, $c_0^{(1)} = c_0^{(2)}$, but $d_0^{(1)} \geq 0$, $d_0^{(2)} = 0$, where equalities and inequalities are to be understood a.e. on $[T, \infty) \times D$ or a.e. on $[T, \infty) \times \partial D$. Then*

$$U_{a^{(1)}}^R(t, s)u_0 \leq U_{a^{(2)}}^N(t, s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$, where $U_a^R(t, s)u_0$ and $U_a^N(t, s)u_0$ denote the solutions of (1)_a+(2)_a with Robin and Neumann boundary conditions, respectively.

- (4) *Let, for some $T \geq 0$, $a_{ij}^{(1)} = a_{ij}^{(2)}$, $a_i^{(1)} = a_i^{(2)}$, $b_i^{(1)} = b_i^{(2)}$, $c_0^{(1)} = c_0^{(2)}$, but $d_0^{(2)} \geq 0$, where equalities and inequalities are to be understood a.e. on $[T, \infty) \times D$ or a.e. on $[T, \infty) \times \partial D$. Then*

$$U_{a^{(1)}}^D(t, s)u_0 \leq U_{a^{(2)}}^R(t, s)u_0$$

for any $t > s \geq T$ and any $u_0 \in L_2(D)^+$, where $U_a^D(t, s)u_0$ and $U_a^R(t, s)u_0$ denote the solutions of (1)_a+(2)_a with Dirichlet and Robin boundary conditions, respectively.

Proof. Compare [18, Proposition 2.2.10]. □

Proposition 2.11 (Continuous dependence). *For any real sequences $(s_n)_{n=1}^\infty$ with $s_n \rightarrow \infty$ and $(t_n)_{n=1}^\infty$ with $t_n \rightarrow t \in (0, \infty)$, if*

$$\lim_{n \rightarrow \infty} a \cdot s_n = \tilde{a},$$

then for any $u_0 \in L_2(D)$, $U_a(s_n + t_n, s_n)u_0 = U_{a \cdot s_n}(t_n, 0)u_0$ converges to $U_{\tilde{a}}(t, 0)u_0$ in $L_2(D)$.

Proof. It follows from the arguments of [18, Theorem 2.4.1]. \square

Proposition 2.12 (Continuous dependence). *For any sequence $(\tilde{a}^{(n)})_{n=1}^\infty \subset Y$, any real sequence $(t_n)_{n=1}^\infty$ and any sequence $(u_n)_{n=1}^\infty \subset L_2(D)$, if $\lim_{n \rightarrow \infty} \tilde{a}^{(n)} = \tilde{a}$, $\lim_{n \rightarrow \infty} t_n = t$, where $t \in (0, \infty)$, and $\lim_{n \rightarrow \infty} u_n = u_0$ in $L_2(D)$, then $U_{\tilde{a}^{(n)}}(t_n, 0)u_n$ converges in $L_2(D)$ to $U_{\tilde{a}}(t, 0)u_0$.*

Proof. It follows from [18, Theorem 2.4.1]. \square

We denote by $\Pi(Y) = \{\Pi(Y)_t\}_{t \geq 0}$ the *topological linear skew-product semiflow* generated by the family (12) $_{\tilde{a}}$ +(13) $_{\tilde{a}}$, $\tilde{a} \in Y$, on the product bundle $L_2(D) \times Y$:

$$\Pi(Y)(t; u_0, \tilde{a}) = \Pi(Y)_t(u_0, \tilde{a}) := (U_{\tilde{a}}(t, 0)u_0, \sigma_t \tilde{a}) \quad (t \geq 0, \tilde{a} \in Y, u_0 \in L_2(D)).$$

For $Y = Y(\tilde{a})$, we will denote $\Pi(Y)$ by $\Pi(\tilde{a})$.

3 Principal Spectrum

In this section, we introduce the definition of the principal spectrum of (1)+(2) and establish some fundamental properties of it. Throughout the present section, we assume that D and a satisfy (A1)–(A4). Let \tilde{a} be an extension of a such that it satisfies (A2)'–(A4)'.

3.1 Definition

Definition 3.1 (Principal resolvent). A real number λ belongs to the *principal resolvent* of (1) $_a$ +(2) $_a$ or $\{U_a(t, s)\}_{t \geq s \geq 0}$, denoted by $\rho(a)$, if either of the following conditions holds:

- There are $\eta > 0$, $M \geq 1$, and $T > 0$ such that

$$\|U_a(t, s)\| \leq M e^{(\lambda - \eta)(t - s)} \quad \text{for } t > s \geq T$$

(such λ are said to belong to the *upper principal resolvent*, denoted by $\rho_+(a)$),

- There are $\eta > 0$, $M \in (0, 1]$, and $T > 0$ such that

$$\|U_a(t, s)\| \geq M e^{(\lambda + \eta)(t - s)} \quad \text{for } t > s \geq T$$

(such λ are said to belong to the *lower principal resolvent*, denoted by $\rho_-(a)$).

Definition 3.2 (Principal spectrum). The *principal spectrum* of $(1)_a + (2)_a$ or $\{U_a(t, s)\}_{t \geq s \geq 0}$, denoted by $\Sigma(a)$, equals the complement in \mathbb{R} of the principal resolvent $\rho(a)$.

3.2 Fundamental Properties

Theorem 3.1. The principal spectrum $\Sigma(a)$ of $(1)_a + (2)_a$ is a compact nonempty interval $[\lambda_{\min}(a), \lambda_{\max}(a)]$.

In the following, $[\lambda_{\min}(a), \lambda_{\max}(a)]$ denotes the principal spectrum $\Sigma(a)$ of $(1)_a + (2)_a$ unless otherwise specified.

Theorem 3.2.

$$\lambda_{\min}(a) = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} \leq \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} = \lambda_{\max}(a).$$

Theorem 3.3. Assume that there is $T \geq 0$ such that there holds: $a_i(t, x) = b_i(t, x) = 0$ for a.e. $(t, x) \in [T, \infty) \times D$, and $c_0(t, x) \leq 0$ for a.e. $(t, x) \in [T, \infty) \times D$. Then $[\lambda_{\min}(a), \lambda_{\max}(a)] \subset (-\infty, 0]$.

Theorem 3.4. In the case of the Dirichlet boundary condition, assume that there is $T \geq 0$ such that there holds: $a_i(t, x) = b_i(t, x) = 0$ for a.e. $(t, x) \in [T, \infty) \times D$, and $c_0(t, x) \leq 0$ for a.e. $(t, x) \in [T, \infty) \times D$. Then $\lambda_{\max}(a) < 0$.

To prove the above theorems, we first prove some lemmas.

- Lemma 3.5.** (1) For any $t_0 > 0$ there is $K_1 = K_1(t_0) \geq 1$ such that $\|U_a(s+t, s)\| \leq K_1$ for all $s \geq 0$ and all $t \in [0, t_0]$.
 (2) For any $t_0 > 0$ there is $K_2 = K_2(t_0) > 0$ such that $\|U_a(s+t, s)\| \geq K_2$ for all $s \geq 0$ and all $t \in [0, t_0]$.

Proof. See [18, Lemma 3.1.2]. □

Lemma 3.6. (1) A real number λ belongs to the upper principal resolvent if and only if there are $\delta_0 > 0$, $T > 0$, $\eta > 0$ and $\tilde{M} > 0$ such that

$$\|U_a(t, s)\| \leq \tilde{M} e^{(\lambda - \eta)(t-s)} \quad \text{for } t-s \geq \delta_0 \text{ and } s \geq T.$$

- (2) A real number λ belongs to the lower principal resolvent if and only if there are $\delta_0 > 0$, $T > 0$, $\eta > 0$ and $\tilde{M} > 0$ such that

$$\|U_a(t, s)\| \geq \tilde{M} e^{(\lambda + \eta)(t-s)} \quad \text{for } t-s \geq \delta_0 \text{ and } s \geq T.$$

Proof. The “only if” parts follow from Definition 3.1 in a straightforward way.

To prove the “if” part in (1), it suffices to notice that, by Lemma 3.5(1), there is $K_1 = K_1(\delta_0) > 0$ such that $\|U_a(t, s)\| \leq K_1 \leq (K_1 \max\{1, e^{-\delta_0(\lambda-\eta)}\})e^{(\lambda-\eta)(t-s)}$ for all $t > s \geq T$ with $t - s \leq \delta_0$.

To prove the “if” part in (2), it suffices to notice that, by Lemma 3.5(2), there is $K_2 = K_2(\delta_0) > 0$ such that $\|U_a(t, s)\| \geq K_2 \geq (K_2 \min\{1, e^{-\delta_0(\lambda+\eta)}\})e^{(\lambda+\eta)(t-s)}$ for all $t > s \geq T$ with $t - s \leq \delta_0$. \square

Lemma 3.7. *There exist $\delta_1 > 0$, $M_1 > 0$ and a real $\underline{\lambda}$ such that $\|U_a(t, s)\| \geq M_1 e^{\underline{\lambda}(t-s)}$ for all $s \geq 0$ and all $t - s \geq \delta_1$.*

Proof. See [18, Lemma 3.1.4]. \square

Proof of Theorem 3.1. We prove first that the upper principal resolvent $\rho_+(a)$ is nonempty. Indeed, by the L_2 - L_2 estimates (Proposition 2.6), there are $M > 0$ and $\gamma > 0$ such that $\|U_a(t, s)\| \leq M e^{\gamma(t-s)}$ for all $t \geq s \geq 0$. Consequently, $\gamma + 1 \in \rho_+(a)$. Further, it follows from the definition that $\rho_+(a)$ is a right-unbounded open interval $(\lambda_{\max}(a), \infty)$.

The lower principal resolvent $\rho_-(a)$ is nonempty, too, since it contains, by Lemma 3.7 combined with Lemma 3.6(2), the real number $\underline{\lambda} - 1$. Further, it follows from the definition that $\rho_-(a)$ is a left-unbounded open interval $(-\infty, \lambda_{\min}(a))$.

As $\rho_-(a) \cap \rho_+(a) = \emptyset$, one has $\Sigma(a) = \mathbb{R} \setminus \rho(a) = [\lambda_{\min}(a), \lambda_{\max}(a)]$. \square

Proof of Theorem 3.2. First, by Definition 3.2, for any sequences $(t_n)_{n=1}^\infty \subset (0, \infty)$, $(s_n)_{n=1}^\infty \subset (0, \infty)$, such that $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ as $n \rightarrow \infty$ there holds

$$\lambda_{\min}(a) \leq \liminf_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)\|}{t_n - s_n} \leq \limsup_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)\|}{t_n - s_n} \leq \lambda_{\max}(a). \quad (18)$$

Notice that, since $\lambda_{\min}(a) \notin \rho_-(a)$, it follows from Definition 3.1 that for each $n \in \mathbb{N}$ there are $n \leq s_{n,1} < t_{n,1}$ with the property that

$$\|U_a(t_{n,1}, s_{n,1})\| < \frac{1}{n} \exp((\lambda_{\min}(a) + \frac{1}{n})(t_{n,1} - s_{n,1})).$$

We claim that $\lim_{n \rightarrow \infty} (t_{n,1} - s_{n,1}) = \infty$ as $n \rightarrow \infty$. Indeed, if not then there is a bounded subsequence $(t_{n_k,1} - s_{n_k,1})_{k=1}^\infty$, $n_k \rightarrow \infty$ as $k \rightarrow \infty$. It follows that $\|U_a(t_{n_k,1}, s_{n_k,1})\| \rightarrow 0$ as $k \rightarrow \infty$, which contradicts Lemma 3.5(2). Thus we have

$$\limsup_{n \rightarrow \infty} \frac{\ln \|U_a(t_{n,1}, s_{n,1})\|}{t_{n,1} - s_{n,1}} \leq \lambda_{\min}(a). \quad (19)$$

Notice also that, since $\lambda_{\max}(a) \notin \rho_+(a)$, it follows from Definition 3.1 that for each $n \in \mathbb{N}$ there are $n \leq s_{n,2} < t_{n,2}$ with the property that

$$\|U_a(t_{n,2}, s_{n,2})\| > n \exp((\lambda_{\max}(a) - \frac{1}{n})(t_{n,2} - s_{n,2})).$$

We claim that $\lim_{n \rightarrow \infty} (t_{n,2} - s_{n,2}) = \infty$ as $n \rightarrow \infty$. Indeed, if not then there is a bounded subsequence $(t_{n_k,2} - s_{n_k,2})_{k=1}^\infty$, $n_k \rightarrow \infty$ as $k \rightarrow \infty$. It follows that $\|U_a(t_{n_k,2}, s_{n_k,2})\| \rightarrow \infty$ as $k \rightarrow \infty$, which contradicts Lemma 3.5(1). Thus we have

$$\liminf_{n \rightarrow \infty} \frac{\ln \|U_a(t_{n,2}, s_{n_k,2})\|}{t_{n,2} - t_{n_k,2}} \geq \lambda_{\max}(a). \quad (20)$$

The theorem then follows from (18)–(20). \square

Proof of Theorem 3.3. Fix $u_0 \in L_2(D)^+$ with $\|u_0\| = 1$, and put $u(t, x) := (U_a(t, T)u_0)(x)$, $t \geq T$, $x \in D$. It follows from [18, Proposition 2.1.4] that

$$\begin{aligned} \|u(t, \cdot)\|^2 - \|u(s, \cdot)\|^2 &= -2 \int_s^t B_a(\tau, u(\tau, \cdot), u(\tau, \cdot)) d\tau \\ &\leq -2 \int_s^t \int_D \left(\sum_{i,j=1}^N a_{ij}(\tau, x) \partial_{x_i} u(\tau, x) \partial_{x_j} u(\tau, x) \right) dx d\tau \leq 0 \end{aligned}$$

for any $T \leq s < t$. Consequently, with the help of Lemma 2.9 we have $\|U_a(t, s)\| \leq 1$ for any $T \leq s < t$. Therefore $(0, \infty) \subset \rho_+(Y_0)$. \square

Proof of Theorem 3.4. It follows by the Poincaré inequality (see [6, Theorem 3 in Sect. 5.6]) that there is $\alpha_1 > 0$ such that $\|u\| \leq \alpha_1 \|\nabla u\|$ for any $u \in \dot{W}_2^1(D)$.

Starting as in the proof of Theorem 3.3 we estimate

$$\begin{aligned} \|u(t, \cdot)\|^2 - \|u(s, \cdot)\|^2 &= -2 \int_s^t B_a(\tau, u(\tau, \cdot), u(\tau, \cdot)) d\tau \\ &\leq -2 \int_s^t \int_D \left(\sum_{i,j=1}^N a_{ij}(\tau, x) \partial_{x_i} u(\tau, x) \partial_{x_j} u(\tau, x) \right) dx d\tau \\ &\stackrel{\text{by (A2)}}{\leq} -2\alpha_0 \int_s^t \|\nabla u(\tau, \cdot)\|^2 d\tau \leq \frac{-2\alpha_0}{(\alpha_1)^2} \int_s^t \|u(\tau, \cdot)\|^2 d\tau \end{aligned}$$

for $T \leq s < t$. An application of the regular Gronwall inequality and Lemma 2.9 gives that

$$\|U_a(t, s)\| \leq e^{-\lambda_0(t-s)}$$

for any $T \leq s < t$, where $\lambda_0 := \alpha_0/\alpha_1^2 > 0$. Consequently, $[-\lambda_0, \infty) \subset \rho_+(a)$ and $\lambda_{\max}(a) < -\lambda_0$. \square

3.3 Monotonicity and Continuity with Respect to Zero Order Terms

In this subsection, we explore the monotonicity and continuity of the principal spectrum of (1)+(2) with respect to the zero order terms.

Let $a^{(1)}, a^{(2)}$ be such that they satisfy properties (A1)–(A4).

We assume that there is $T \geq 0$ such that the following assumptions are satisfied:

(MC1) $a_{ij}^{(1)}(\cdot, \cdot) = a_{ij}^{(2)}(\cdot, \cdot)$, $a_i^{(1)}(\cdot, \cdot) = a_i^{(2)}(\cdot, \cdot)$, $b_i^{(1)}(\cdot, \cdot) = b_i^{(2)}(\cdot, \cdot)$, for a.e. $(t, x) \in [T, \infty) \times D$.

(MC2) $d_0^{(2)}(\cdot, \cdot) = d_0^{(1)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times \partial D$.

(MC3) One of the following conditions, (a), (b), (c), (d), or (e) holds:

(a) Both $(1)_{a(1)}$ and $(1)_{a(2)}$ are endowed with the Dirichlet boundary conditions, and

- $c_0^{(1)}(\cdot, \cdot) \leq c_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times D$,

(b) Both $(1)_{a(1)}$ and $(1)_{a(2)}$ are endowed with the Robin boundary conditions, and

- $c_0^{(1)}(\cdot, \cdot) \leq c_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times D$,
- $d_0^{(1)}(\cdot, \cdot) \geq d_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times \partial D$.

(c) Both $(1)_{a(1)}$ and $(1)_{a(2)}$ are endowed with the Neumann boundary conditions, and

- $c_0^{(1)}(\cdot, \cdot) \leq c_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times D$,

(d) $(1)_{a(1)}$ is endowed with the Dirichlet boundary condition and $(1)_{a(2)}$ is endowed with the Robin boundary condition, and $d_0^{(2)}(\cdot, \cdot) \geq 0$

- $c_0^{(1)}(\cdot, \cdot) = c_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times D$.

(e) $(1)_{a(1)}$ is endowed with the Robin boundary condition and $(1)_{a(2)}$ is endowed with the Neumann boundary condition, and $d_0^{(2)}(\cdot, \cdot) \geq 0$

- $c_0^{(1)}(\cdot, \cdot) = c_0^{(2)}(\cdot, \cdot)$ for a.e. $(t, x) \in [T, \infty) \times D$.

Theorem 3.8. Assume that (MC1) and (MC3) hold. Then $\lambda_{\min}(a^{(1)}) \leq \lambda_{\min}(a^{(2)})$ and $\lambda_{\max}(a^{(1)}) \leq \lambda_{\max}(a^{(2)})$.

Proof. We prove only the first inequality, the proof of the other being similar.

By Theorem 3.2, there are sequences $(s_n)_{n=1}^\infty$, $(t_n)_{n=1}^\infty$, with $0 < s_n < t_n$, $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ as $n \rightarrow \infty$, such that

$$\lim_{n \rightarrow \infty} \frac{\ln \|U_{a^{(2)}}(t_n, s_n)\|}{t_n - s_n} = \lambda_{\min}(a^{(2)}).$$

Proposition 2.10 implies that for each $u_0 \in L_2(D)^+$ there holds

$$\|U_{a^{(1)}}(t_n, s_n)u_0\| \leq \|U_{a^{(2)}}(t_n, s_n)u_0\|$$

for $T \leq s_n < t_n$, which implies, via Lemma 2.9, that $\|U_{a^{(1)}}(t_n, s_n)\| \leq \|U_{a^{(2)}}(t_n, s_n)\|$ for sufficiently large n . By Theorem 3.2,

$$\lambda_{\min}(a^{(1)}) \leq \liminf_{n \rightarrow \infty} \frac{\ln \|U_{a^{(1)}}(t_n, s_n)\|}{t_n - s_n} \leq \lim_{n \rightarrow \infty} \frac{\ln \|U_{a^{(2)}}(t_n, s_n)\|}{t_n - s_n} = \lambda_{\min}(a^{(2)}). \quad \square$$

Theorem 3.9. *Assume that (MC1) and (MC2) hold. Then $|\lambda_{\min}(a^{(1)}) - \lambda_{\min}(a^{(2)})| \leq r$ and $|\lambda_{\max}(a^{(1)}) - \lambda_{\max}(a^{(2)})| \leq r$, where $r := \lim_{\tau \rightarrow \infty} \text{ess sup} \{ |c_0^{(1)}(t, x) - c_0^{(2)}(t, x)| : t \in (\tau, \infty), x \in D \}$.*

Proof. For $m \in \mathbb{N}$, put $a^{(1)} \pm (r + \frac{1}{m})$ to be $a^{(1)}$ with $c_0^{(1)}$ replaced by $c_0^{(1)} \pm (r + \frac{1}{m})$.

By using arguments as in the proof of [18, Lemma 4.3.1] we see that

$$U_{a^{(k)} \pm (r + \frac{1}{m})}(t, s) = e^{\pm (r + \frac{1}{m})(t-s)} U_{a^{(k)}}(t, s) \quad (0 \leq s < t)$$

for $k = 1, 2$. Consequently, by Theorem 3.2,

$$\lambda_{\text{ext}}(a^{(1)} \pm (r + \frac{1}{m})) = \lambda_{\text{ext}}(a^{(1)}) \pm (r + \frac{1}{m}),$$

where λ_{ext} stands for λ_{\min} or λ_{\max} .

Observe that for any $m \in \mathbb{N}$ there is $T_m > 0$ such that

$$c_0^{(1)}(t, x) - (r + \frac{1}{m}) \leq c_0^{(2)}(t, x) \leq c_0^{(1)}(t, x) + (r + \frac{1}{m})$$

for a.e. $(t, x) \in (T_m, \infty) \times D$.

It then follows from Theorem 3.8 that

$$\lambda_{\text{ext}}(a^{(1)} - (r + \frac{1}{m})) \leq \lambda_{\text{ext}}(a^{(2)}) \leq \lambda_{\text{ext}}(a^{(1)} + (r + \frac{1}{m})),$$

hence

$$\lambda_{\text{ext}}(a^{(1)}) - (r + \frac{1}{m}) \leq \lambda_{\text{ext}}(a^{(2)}) \leq \lambda_{\text{ext}}(a^{(1)}) + (r + \frac{1}{m}).$$

As $m \in \mathbb{N}$ is arbitrary, this gives the desired result. \square

4 Exponential Separation and Equivalent Definition

In this section, we investigate the relation between the principal spectrum of (1)+(2) and that of the forward limit equations of (1)+(2). To do so, we employ the so-called exponential separation theory for general time dependent linear parabolic equations, which together with principal spectrum theory extends principal eigenvalue and principal eigenfunction theory for time periodic parabolic equations.

4.1 Definitions and Characterizations

We first introduce the principal spectrum of $(12)_a + (13)_a$ over $Y_0(a)$ and the exponential separation of $\Pi(Y)$ over Y . We then show that the principal spectrum of $(1)_a + (2)_a$ equals that of $(12)_a + (13)_a$ over $Y_0(a)$ provided that $\Pi(\bar{a})$ admits an exponential separation on $Y(\bar{a})$.

Throughout the present subsection, Y is a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying $(A2)' - (A4)'$.

Let D and a satisfy $(A1) - (A4)$, and let \bar{a} be an extension of a such that $Y(\bar{a}) \subset Y$. In particular, \bar{a} satisfies $(A2)' - (A4)'$.

Definition 4.1. $\lambda \in \mathbb{R}$ belongs to the *principal resolvent* of $Y_0(a)$ or the *principal resolvent* of $(12)_a + (13)_a$ over $Y_0(a)$, denoted by $\tilde{\rho}(a)$, if either of the following conditions is satisfied:

- There are $\eta > 0$ and $M \geq 1$ such that

$$\|U_{\bar{a}}(t, 0)\| \leq M e^{(\lambda - \eta)t} \quad \text{for } t > 0, \quad \bar{a} \in Y_0(a)$$

(such λ are said to belong to the *upper principal resolvent* of $Y_0(a)$, denoted by $\tilde{\rho}_+(a)$),

- There are $\eta > 0$ and $M \in (0, 1]$ such that

$$\|U_{\bar{a}}(t, 0)\| \geq M e^{(\lambda + \eta)t} \quad \text{for } t > 0, \quad \bar{a} \in Y_0(a)$$

(such λ are said to belong to the *lower principal resolvent* of $Y_0(a)$, denoted by $\tilde{\rho}_-(a)$).

Definition 4.2. The *principal spectrum* of $(12)_a + (13)_a$ over $Y_0(a)$, denoted by $\tilde{\Sigma}(a)$, equals the complement in \mathbb{R} of the principal resolvent of $(12)_a + (13)_a$ over $Y_0(a)$.

Remark 4.1. In the terminology of the monograph [18], the principal resolvent of $Y_0(a)$ (resp. the principal spectrum of $Y_0(a)$) is called the *principal resolvent* of $\Pi(\bar{a})$ over $Y_0(a)$ (resp. the *principal spectrum* of $\Pi(\bar{a})$ over $Y_0(a)$).

Theorem 4.2. $\tilde{\Sigma}(a)$ is a nonempty interval $[\tilde{\lambda}_{\min}(a), \tilde{\lambda}_{\max}(a)]$.

Proof. See [18, Theorem 3.1.1]. □

Definition 4.3. Let Y' be a closed invariant subset of Y . We say that $\Pi(Y)$ admits an *exponential separation with separating exponent* $\gamma_0 > 0$ over Y' if there are an invariant one-dimensional subbundle X_1 of $L_2(D) \times Y'$ with fibers $X_1(\bar{a}) = \text{span}\{w(\bar{a})\}$, $\|w(\bar{a})\| = 1$, and an invariant complementary one-codimensional subbundle X_2 of $L_2(D) \times Y'$ with fibers $X_2(\bar{a}) = \{v \in L_2(D) : \langle v, w^*(\bar{a}) \rangle = 0\}$ having the following properties:

- (i) $w(\bar{a}) \in L_2(D)^+$ for all $\bar{a} \in Y'$,
- (ii) $X_2(\bar{a}) \cap L_2(D)^+ = \{0\}$ for all $\bar{a} \in Y'$,

(iii) There is $M \geq 1$ such that for any $\tilde{a} \in Y'$ and any $v \in X_2(\tilde{a})$ with $\|v\| = 1$,

$$\|U_{\tilde{a}}(t, 0)v\| \leq Me^{-\gamma_0 t} \|U_{\tilde{a}}(t, 0)w(\tilde{a})\| \quad (t > 0).$$

For more on bundles, etc., see [18, Sect. 3.2].

Let (A5) stand for the following assumption.

(A5) $\Pi(\tilde{a})$ admits an exponential separation over $Y(\tilde{a})$, for some extension \tilde{a} of a .

In the next subsection, we will show that if both D and a are sufficiently smooth, (A5) is satisfied.

Theorem 4.3. Assume (A5). Then

(i)

$$\begin{aligned} \lambda_{\min}(a) &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\tilde{a} \cdot s)\|}{t-s} = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t-s} \\ &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} \end{aligned}$$

for each nonzero $u_0 \in L_2(D)^+$,

(ii)

$$\begin{aligned} \lambda_{\max}(a) &= \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\tilde{a} \cdot s)\|}{t-s} = \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t-s} \\ &= \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} \end{aligned}$$

for each nonzero $u_0 \in L_2(D)^+$.

(iii) $\Sigma(a) = \tilde{\Sigma}(a)$, i.e., $\lambda_{\min}(a) = \tilde{\lambda}_{\min}(a)$ and $\lambda_{\max}(a) = \tilde{\lambda}_{\max}(a)$.

Before proving the above theorem, we first recall a lemma from [18].

Lemma 4.4. Assume (A5). Then

(1) $\lambda \in \mathbb{R}$ belongs to $\tilde{p}_+(a)$ if and only if there are $\eta > 0$ and $M \geq 1$ such that

$$\|U_{\tilde{a}}(t, 0)w(\tilde{a})\| \leq Me^{(\lambda-\eta)t} \quad \text{for } t > 0 \text{ and } \tilde{a} \in Y_0(a),$$

(2) $\lambda \in \mathbb{R}$ belongs to $\tilde{p}_-(a)$ if and only if there are $\eta > 0$ and $M \in (0, 1)$ such that

$$\|U_{\tilde{a}}(t, 0)w(\tilde{a})\| \geq Me^{(\lambda+\eta)t} \quad \text{for } t > 0 \text{ and } \tilde{a} \in Y_0(a).$$

Proof. See [18, Lemma 3.2.6]. □

We remark that the complement of the set of those $\lambda \in \mathbb{R}$ for which either of the conditions in Lemma 4.4 holds is called the *dynamical spectrum* or the *Sacker–Sell spectrum* of $\Pi|_{X_1 \cap (L_2(D) \times Y_0(a))}$. The reader is referred to [23–26] for the fundamental spectral theory for nonautonomous linear evolution equations.

Proof of Theorem 4.3 First of all, by [18, Lemma 3.2.5], we have

$$\begin{aligned}
 & \liminf_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)u_0\|}{t_n - s_n} \\
 &= \liminf_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)\|}{t_n - s_n} \\
 &= \liminf_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)w(\bar{a} \cdot s_n)\|}{t_n - s_n} \\
 &\leq \limsup_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)w(\bar{a} \cdot s_n)\|}{t_n - s_n} \\
 &= \limsup_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)\|}{t_n - s_n} = \limsup_{n \rightarrow \infty} \frac{\ln \|U_a(t_n, s_n)u_0\|}{t_n - s_n} \quad (21)
 \end{aligned}$$

for any $(s_n)_{n=1}^\infty, (t_n)_{n=1}^\infty$ such that $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$, and any nonzero $u_0 \in L_2(D)^+$. By Theorem 3.2, there holds

$$\lambda_{\min}(a) = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} \leq \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} = \lambda_{\max}(a). \quad (22)$$

(i) and (ii) then follow from (21) and (22).

Next, we prove (iii). We first prove

$$\tilde{\lambda}_{\min}(a) \leq \lambda_{\min}(a) \leq \lambda_{\max}(a) \leq \tilde{\lambda}_{\max}(a). \quad (23)$$

Fix, for the moment, $\varepsilon > 0$. As $\tilde{\lambda}_{\min}(a) - \varepsilon \in \tilde{\rho}_-(a)$, it follows from Lemma 4.4(2) that there is $T > 0$ such that for any $t \geq T$ and $\bar{a} \in Y_0(a)$ there holds

$$\ln \|U_{\bar{a}}(t, 0)w(\bar{a})\| > (\tilde{\lambda}_{\min}(a) - \varepsilon)t. \quad (24)$$

By Proposition 2.12, there is $\delta > 0$ such that for any $\tilde{a}^{(1)}, \tilde{a}^{(2)} \in Y(\bar{a})$ with $d(\tilde{a}^{(1)}, \tilde{a}^{(2)}) < \delta$ there holds

$$-\varepsilon T \leq \ln \|U_{\tilde{a}^{(1)}}(T, 0)w(\tilde{a}^{(1)})\| - \ln \|U_{\tilde{a}^{(2)}}(T, 0)w(\tilde{a}^{(2)})\| \leq \varepsilon T. \quad (25)$$

For the above $\delta > 0$ there is $T_1 > 0$ such that for any $s \geq T_1$ there is $\bar{a} \in Y_0(a)$ such that

$$d(\bar{a} \cdot s, \bar{a}) < \delta.$$

It then follows from (24) and (25) that

$$\ln \|U_a(T+s, s)w(\bar{a} \cdot s)\| \geq (\tilde{\lambda}_{\min}(a) - 2\varepsilon)T,$$

and hence

$$\|U_a(T+s, s)w(\bar{a} \cdot s)\| \geq e^{(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T}, \quad (26)$$

for any $s \geq T_1$.

We then have, applying Proposition 2.5,

$$\begin{aligned} & \|U_a(nT+s, s)w(\bar{a} \cdot s)\| \\ &= \|U_a((n-1)T+s, T+s)U_a(T+s, s)w(\bar{a} \cdot s)\| \\ &\geq \|U_a((n-1)T+s, T+s)w(\bar{a} \cdot (T+s))\| \cdot e^{(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T} \\ &\geq \|U_a((n-2)T+s, 2T+s)w(\bar{a} \cdot (2T+s))\| \cdot e^{2(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T} \\ &\geq \dots \\ &\geq e^{n(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T} \end{aligned} \quad (27)$$

for any $s \geq T_1$ and $n \in \mathbb{N}$.

Therefore for any $s \geq T_1$ and $t > 0$ with $t-s = nT + \tau$ for some $n \in \{0, 1, 2, \dots\}$ and $0 \leq \tau < T$ there holds

$$\begin{aligned} \|U_a(t, s)w(\bar{a} \cdot s)\| &= \|U_a(t, nT+s)U_a(nT+s, s)w(\bar{a} \cdot s)\| \\ &\geq \|U_a(t, nT+s)w(\bar{a} \cdot (nT+s))\| \cdot e^{n(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T} \\ &\geq Me^{n(\tilde{\lambda}_{\min}(a) - 2\varepsilon)T}, \end{aligned} \quad (28)$$

where $M := \inf \{ \|U_{\bar{a}}(\tau, 0)w(\bar{a})\| : 0 \leq \tau \leq T, \bar{a} \in Y(\bar{a}) \} > 0$. This implies that

$$\liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\bar{a} \cdot s)\|}{t-s} \geq \tilde{\lambda}_{\min}(a) - 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we get

$$\liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\bar{a} \cdot s)\|}{t-s} \geq \tilde{\lambda}_{\min}(a). \quad (29)$$

Similarly we prove that

$$\limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\bar{a} \cdot s)\|}{t-s} \leq \tilde{\lambda}_{\max}(a). \quad (30)$$

(23) then follows from (22), (29), and (30).

Next, we prove

$$\lambda_{\min}(a) \leq \tilde{\lambda}_{\min}(a) \leq \tilde{\lambda}_{\max}(a) \leq \lambda_{\max}(a). \quad (31)$$

Let $\lambda \in \rho_+(a)$. By Definition 3.1, there are $\eta > 0$, $M \geq 1$ and $T > 0$ such that

$$\|U_a(t, s)\| \leq M e^{(\lambda - \eta)(t-s)} \quad \text{for } t > s \geq T.$$

In particular, $\|U_a(t, s)w(\tilde{a} \cdot s)\| \leq M e^{(\lambda - \eta)(t-s)}$ for any $t > s \geq T$.

For each $\tilde{a} \in Y_0(a)$ there is $(s_n)_{n=1}^\infty \subset (0, \infty)$ with $s_n \rightarrow \infty$ such that $\tilde{a} \cdot s_n \rightarrow \tilde{a}$. Then by Proposition 2.12, for any $t > 0$

$$U_a(t + s_n, s_n)w(\tilde{a} \cdot s_n) \rightarrow U_{\tilde{a}}(t, 0)w(\tilde{a})$$

as $n \rightarrow \infty$. Hence

$$\|U_{\tilde{a}}(t, 0)w(\tilde{a})\| \leq M e^{(\lambda - \eta)t}$$

for any $t > 0$. It then follows via Lemma 4.4(1) that $\lambda \in \tilde{\rho}_+(a)$. Consequently,

$$\tilde{\lambda}_{\max}(a) \leq \lambda_{\max}(a). \quad (32)$$

Let $\lambda \in \rho_-(a)$. By Definition 3.1, there are $\eta > 0$, $M \in (0, 1)$ and $T > 0$ such that

$$\|U_a(t, s)\| \geq M e^{(\lambda + \eta)(t-s)} \quad \text{for } t > s \geq T.$$

By [18, Lemma 3.2.3], there is $M_2 \geq 1$ such that $\|U_a(t, s)\| \leq M_2 \|U_a(t, s)w(\tilde{a} \cdot s)\|$ for all $t > s$. Therefore, $\|U_a(t, s)w(\tilde{a} \cdot s)\| \geq \tilde{M} e^{(\lambda + \eta)(t-s)}$ for any $t > s \geq T$, where $\tilde{M} := M/M_2 \in (0, 1)$.

For each $\tilde{a} \in Y_0(a)$ there is $(s_n)_{n=1}^\infty \subset (0, \infty)$ with $s_n \rightarrow \infty$ such that $\tilde{a} \cdot s_n \rightarrow \tilde{a}$. Then by Proposition 2.12, for any $t > 0$

$$U_a(t + s_n, s_n)w(\tilde{a} \cdot s_n) \rightarrow U_{\tilde{a}}(t, 0)w(\tilde{a})$$

as $n \rightarrow \infty$. Hence

$$\|U_{\tilde{a}}(t, 0)w(\tilde{a})\| \geq \tilde{M} e^{(\lambda + \eta)t}$$

for any $t > 0$. It then follows via Lemma 4.4(2) that $\lambda \in \tilde{\rho}_-(a)$. Consequently,

$$\tilde{\lambda}_{\min}(a) \geq \lambda_{\min}(a). \quad (33)$$

(31) follows from (32) and (33).

By (23) and (31), $\Sigma(a) = \tilde{\Sigma}(a)$, i.e., (iii) holds. \square

Corollary 4.5. *Assume (A5). If a is asymptotically uniquely ergodic (i.e., $Y_0(a)$ is uniquely ergodic), then $\lambda_{\min}(a) = \lambda_{\max}(a)$. If, furthermore, a is asymptotically*

periodic with period T (i.e., $Y_0(a) = \{\tilde{a} \cdot t : t \in [0, T]\}$ for some \tilde{a}), then $\lambda := \lambda_{\min}(a) (= \lambda_{\max}(a))$ is the principal eigenvalue of the following periodic eigenvalue problem,

$$\begin{cases} -u_t + \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N \tilde{a}_{ij}(t, x) \frac{\partial u}{\partial x_j} + \tilde{a}_i(t, x) u \right) \\ \quad + \sum_{i=1}^N \tilde{b}_i(t, x) \frac{\partial u}{\partial x_i} + \tilde{c}_0(t, x) u = \lambda u, & x \in D, \\ \tilde{B}(t) u = 0, & x \in \partial D, \\ u(t+T, x) = u(t, x). \end{cases} \quad (34)$$

Proof. By [18, Corollary 3.2.2], we have $\tilde{\lambda}_{\max}(a) = \tilde{\lambda}_{\min}(a)$. It then follows from Theorem 4.3 that $\lambda_{\max}(a) = \lambda_{\min}(a)$. \square

4.2 The Classical Case: An Example

In this subsection, we consider the so-called classical case, i.e., both D and the coefficients of (1)+(2) are sufficiently smooth (see (SM1) and (SM2) in the following) and show that for such a case (A5) is satisfied.

- (SM1) (Boundary regularity) $D \subset \mathbb{R}^N$ is a bounded domain, where its boundary ∂D is an $(N-1)$ -dimensional manifold of class $C^{3+\alpha}$ for some $\alpha > 0$.
- (SM2) (Smoothness) There is $\alpha > 0$ such that the functions a_{ij} ($= a_{ji}$) and a_i belong to $C^{2+\alpha, 3+\alpha}([0, \infty) \times \bar{D})$, the functions b_i and c_0 belong to $C^{2+\alpha, 1+\alpha}([0, \infty) \times \bar{D})$, and the function d_0 belongs to $C^{2+\alpha, 3+\alpha}([0, \infty) \times \partial D)$.
- (SM3) (Ellipticity) There exists $\alpha_0 > 0$ such that

$$\sum_{i,j=1}^N a_{ij}(t, x) \xi_i \xi_j \geq \alpha_0 \sum_{i=1}^N \xi_i^2 \text{ for all } x \in \bar{D}, \xi \in \mathbb{R}^N \text{ and } t \geq 0.$$

We extend a to \bar{a} by putting $\bar{a}_{ij}(t, x) := a_{ij}(0, x)$ ($i, j = 1, 2, \dots, N$), $\bar{a}_i(t, x) := a_i(0, x)$ ($i = 1, 2, \dots, N$), $\bar{b}_i(t, x) := b_i(0, x)$ ($i = 1, 2, \dots, N$), $\bar{c}_0(t, x) := c_0(0, x)$, for all $t < 0$ and $x \in \bar{D}$, and $\bar{d}_0(t, x) := d_0(0, x)$ for all $t < 0$ and $x \in \partial D$.

(SM1) implies the fulfillment of (A1), (SM2) implies the fulfillment of (A2), and (SM3) is just (A3). By Lemma 2.3(1), (SM1) and (SM2) imply (A4). (SM2) and (SM3) together with the construction of \bar{a} give (A2)'–(A3)' with $Y = Y(\bar{a})$. The satisfaction of (A4)' with $Y = Y(\bar{a})$ follows now from (SM2) via Lemma 2.3(2).

We claim that the problem (1)+(2) satisfies (A5). We have

$$Y(\bar{a}) = \alpha(\bar{a}) \cup \{\bar{a} \cdot t : t \in \mathbb{R}\} \cup Y_0(\bar{a}),$$

where $\alpha(\bar{a}) = \{(a_{ij}(0, \cdot), a_i(0, \cdot), b_i(0, \cdot), c_0(0, \cdot), d_0(0, \cdot))\}$.

Let $\tilde{a} = (\tilde{a}_{ij}, \tilde{a}_i, \tilde{b}_i, \tilde{c}_0, \tilde{d}_0) \in Y(\bar{a})$.

- Assume $\tilde{a} \in Y_0(a)$, or $\tilde{a} \in \alpha(\bar{a})$. It follows by the Ascoli–Arzelà theorem that the functions \tilde{a}_{ij} and \tilde{a}_i belong to $C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \bar{D})$, the functions \tilde{b}_i and \tilde{c}_0 belong to $C^{2+\alpha, 1+\alpha}(\mathbb{R} \times \bar{D})$ and the function \tilde{d}_0 belongs to $C^{2+\alpha, 3+\alpha}(\mathbb{R} \times \partial D)$. Applying the theory in [1] (see [1, Corollary 15.3]), we have that $U_{\tilde{a}}(\cdot, 0)u_0$ is a classical solution on $[t_0, \infty)$, for any $t_0 > 0$ and $u_0 \in L_2(D)$.
- Assume $\tilde{a} = \bar{a} \cdot \tau$ for some $\tau \geq 0$. Then the functions \tilde{a}_{ij} and \tilde{a}_i belong to $C^{2+\alpha, 3+\alpha}([0, \infty) \times \bar{D})$, the functions \tilde{b}_i and \tilde{c}_0 belong to $C^{2+\alpha, 1+\alpha}([0, \infty) \times \bar{D})$ and the function \tilde{d}_0 belongs to $C^{2+\alpha, 3+\alpha}([0, \infty) \times \partial D)$. Again applying the theory in [1], we have that $U_{\tilde{a}}(\cdot, 0)u_0$ is a classical solution on $[t_0, \infty)$, for any $t_0 > 0$ and $u_0 \in L_2(D)$.
- Assume $\tilde{a} = \bar{a} \cdot \tau$ for some $\tau < 0$. Applying the theory in [1] and the theory in [18], we have that $[(0, T) \times D \ni (t, x) \mapsto (U_{\tilde{a}}(t, 0)u_0)(x)] \in W_p^{1,2}((0, T) \times D)$ for any $T > 0$ and $p > 1$, and $U_{\tilde{a}}(t, 0)u_0$ is a strong solution on (t_0, T) , for any $0 < t_0 < T$ and $u_0 \in L_2(D)$.

Then in the Dirichlet case, by [11, Theorem 2.1 and Lemma 3.9], there hold

(HI1) (Harnack type inequality for quotients) *For each $\delta_1 > 0$ there is $C_1 = C_1(\delta_1) > 1$ with the property that*

$$\sup_{x \in D} \frac{(U_{\tilde{a}}(t, 0)u_0^{(1)})(x)}{(U_{\tilde{a}}(t, 0)u_0^{(2)})(x)} \leq C_1 \inf_{x \in D} \frac{(U_{\tilde{a}}(t, 0)u_0^{(1)})(x)}{(U_{\tilde{a}}(t, 0)u_0^{(2)})(x)}$$

for any $\tilde{a} \in Y(\bar{a})$, $t \geq \delta_1$ and any $u_0^{(1)}, u_0^{(2)} \in L_2(D)^+$ with $u_0^{(2)} \neq 0$.

(HI2) (Pointwise Harnack inequality) *There is $\varsigma \geq 0$ such that for each $\delta_2 > 0$ there is $C_2 = C_2(\delta_2) > 0$ with the property that*

$$(U_{\tilde{a}}(t, 0)u_0)(x) \geq C_2(d(x))^{\varsigma} \|U_{\tilde{a}}(t, 0)u_0\|_{\infty} \quad (35)$$

for any $\tilde{a} \in Y(\bar{a})$, $t \geq \delta_2$, $u_0 \in L_2(D)^+$ and $x \in D$, where $d(x)$ denotes the distance of $x \in D$ from the boundary ∂D of D .

In the Neumann or Robin cases, [9, Theorem 2.5] states that (HI2) is satisfied with $\varsigma = 0$, which implies, via [18, Lemma 3.3.1], the fulfillment of (HI1). The above reasoning can be repeated for the adjoint equation, hence, by [18, Theorem 3.3.3], the topological linear skew-product semiflow $\Pi(\bar{a})$ admits an exponential separation over $Y(\bar{a})$.

For $t \geq 0$ we define

$$\kappa(t) := -B_{\bar{a} \cdot t}(0, w(\bar{a} \cdot t), w(\bar{a} \cdot t)),$$

that is,

$$\begin{aligned}\kappa(t) = & - \sum_{i=1}^N \int_D \left(\sum_{j=1}^N a_{ij}(t, x) \partial_j w(\bar{a} \cdot t) + a_i(t, x) w(\bar{a} \cdot t) \right) \partial_i w(\bar{a} \cdot t) dx \\ & + \int_D \left(\sum_{i=1}^N b_i(t, x) \partial_i w(\bar{a} \cdot t) + c_0(t, x) w(\bar{a} \cdot t) \right) w(\bar{a} \cdot t) dx\end{aligned}$$

in the Dirichlet and Neumann boundary condition cases, and

$$\begin{aligned}\kappa(t) = & - \sum_{i=1}^N \int_D \left(\sum_{j=1}^N a_{ij}(t, x) \partial_j w(\bar{a} \cdot t) + a_i(t, x) w(\bar{a} \cdot t) \right) \partial_i w(\bar{a} \cdot t) dx \\ & + \int_D \left(\sum_{i=1}^N b_i(t, x) \partial_i w(\bar{a} \cdot t) + c_0(t, x) w(\bar{a} \cdot t) \right) w(\bar{a} \cdot t) dx \\ & - \int_{\partial D} d_0(t, x) (w(\bar{a} \cdot t))^2 dH_{N-1}\end{aligned}$$

in the Robin boundary condition case, where H_{N-1} stands for the $(N-1)$ -dimensional Hausdorff measure (which is, under our assumption (SM1), equivalent to the $(N-1)$ -dimensional Lebesgue measure).

Observe that the function $\kappa: [0, \infty) \rightarrow \mathbb{R}$ is well defined and continuous (see [18] for detail).

Lemma 4.6. Assume (SM1)–(SM3). For $0 \leq s < t$ put $\eta(t; s) := \|U_a(t, s)w(\bar{a} \cdot s)\|$. Then

$$\eta_t(t; s) = \kappa(t)\eta(t; s)$$

for any $0 \leq s < t$.

Proof. See the proof of [18, Lemma 3.5.3]. □

In view of Lemma 4.6 we have the following extension of Theorem 4.3.

Theorem 4.7. Assume (SM1)–(SM3). For any nonzero $u_0 \in L_2(D)^+$ there holds

$$\begin{aligned}\lambda_{\min}(a) &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t-s} \\ &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\bar{a} \cdot s)\|}{t-s} \\ &= \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \kappa(\tau) d\tau \leq \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \kappa(\tau) d\tau\end{aligned}$$

$$\begin{aligned}
&= \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)w(\bar{a} \cdot s)\|}{t-s} = \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)\|}{t-s} \\
&= \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t, s)u_0\|}{t-s} = \lambda_{\max}(a).
\end{aligned}$$

5 More Properties of Principal Spectrum

5.1 Continuity with Respect to the Coefficients

In the present subsection we investigate continuous dependence of the principal spectrum on the whole of the coefficients.

Assume (A1). We let Y be a subset of $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ satisfying (A2)'–(A4)'.

Throughout the present subsection we make also the following assumption.

(A5)' $\Pi(Y)$ admits an exponential separation over Y .

Let $d_{\text{norm}}(\cdot, \cdot)$ denote the metric on $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ generated by the norm, and let $d(\cdot, \cdot)$ be given by (5).

For $a^{(1)}, a^{(2)} \in L_\infty([0, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0, \infty) \times \partial D, \mathbb{R})$ and $s \geq 0$, by $d_{\text{norm}}^s(a^{(1)}, a^{(2)})$ we denote the $L_\infty([s, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([s, \infty) \times \partial D, \mathbb{R})$ -norm of the difference of the restrictions of $a^{(1)}, a^{(2)}$ to $[s, \infty) \times D ([s, \infty) \times \partial D)$.

Definition 5.1. We say that $a \in L_\infty([0, \infty) \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty([0, \infty) \times \partial D, \mathbb{R})$ is Y -admissible if a satisfies (A2)–(A4) and, moreover, $Y_0(a) \subset Y$.

We remark here that, for a Y -admissible a , it follows from [18, Theorem 3.2.3] (the uniqueness of exponential separation) that the restrictions to $Y_0(a)$ of the one-dimensional subbundles (resp. one-codimensional subbundles) appearing in the definition of an exponential separation over $Y(\bar{a})$ and over Y are the same.

For the rest of the subsection we fix a Y -admissible $a^{(0)}$.

Theorem 5.1. For each $\varepsilon > 0$ there is $\eta > 0$ such that for any Y -admissible a , if $\limsup_{s \rightarrow \infty} d_{\text{norm}}^s(a, a^{(0)}) < \eta$ then

$$|\lambda_{\min}(a) - \lambda_{\min}(a^{(0)})| \leq \varepsilon \quad \text{and} \quad |\lambda_{\max}(a) - \lambda_{\max}(a^{(0)})| \leq \varepsilon.$$

Lemma 5.2. For each $\varepsilon > 0$ there is $\eta > 0$ with the following property. Let $\hat{a}, \check{a} \in Y$ be such that $d(\hat{a} \cdot t, \check{a} \cdot t) < \eta$ for all $t \in \mathbb{R}$. Then, for any integer sequences $(k_n)_{n=1}^\infty, (l_n)_{n=1}^\infty$, such that $l_n - k_n \rightarrow \infty$ as $n \rightarrow \infty$ and

$$\lim_{n \rightarrow \infty} \frac{\ln \|U_{\hat{a}}(l_n, k_n)w(\hat{a} \cdot k_n)\|}{l_n - k_n} = \lambda,$$

one has

$$\lambda - \varepsilon \leq \liminf_{n \rightarrow \infty} \frac{\ln \|U_{\tilde{a}}(l_n, k_n) w(\tilde{a} \cdot k_n)\|}{l_n - k_n} \leq \limsup_{n \rightarrow \infty} \frac{\ln \|U_{\tilde{a}}(l_n, k_n) w(\tilde{a} \cdot k_n)\|}{l_n - k_n} \leq \lambda + \varepsilon.$$

Proof. It follows from [18, Lemma 4.4.2]. \square

Proof of Theorem 5.1. Fix $\varepsilon > 0$, and take a Y -admissible a such that $\limsup_{s \rightarrow \infty} d_{\text{norm}}^s(a, a^{(0)}) < \eta$, where $\eta > 0$ is as in Lemma 5.2.

By Theorem 4.3 and [18, Theorems 3.2.5 and 3.2.6], there exist an ergodic invariant measure μ_{\min} for the compact flow $(Y_0(a^{(0)}), \{\sigma_t\})$ and a Borel set $Y_1 \subset Y_0(a^{(0)})$ with $\mu_{\min}(Y_1) = 1$ such that

$$\lim_{t \rightarrow \infty} \frac{\ln \|U_{\tilde{a}}(t, 0) w(\tilde{a})\|}{t} = \lambda_{\min}(a^{(0)})$$

for any $\tilde{a} \in Y_1$. Fix some $\tilde{a} \in Y_1$. Let $(t_n)_{n=1}^\infty$ be a sequence with $\lim_{n \rightarrow \infty} t_n = \infty$ such that $\tilde{a}^{(0)} \cdot t_n$ converges to \tilde{a} . We can extract a subsequence (t_{n_k}) such that $\tilde{a} \cdot t_{n_k}$ converges, as $k \rightarrow \infty$, to some \tilde{a} .

We claim that $d(\tilde{a} \cdot t, \tilde{a} \cdot t) < \eta$ for all $t \in \mathbb{R}$. Denote $\eta_1 := \limsup_{s \rightarrow \infty} d_{\text{norm}}^s(a, a^{(0)})$ ($< \eta$), and let M_1 stand for the maximum of the $L_\infty(\mathbb{R} \times D, \mathbb{R}^{N^2+2N+1}) \times L_\infty(\mathbb{R} \times \partial D, \mathbb{R})$ -norms of \tilde{a} and $\tilde{a}^{(0)}$. Fix $t \in \mathbb{R}$. Take $k_0 \in \mathbb{N}$ so large that $1/2^{k_0-1} < M_1(\eta - \eta_1)/6$. Then we have

$$\sum_{k=k_0}^{\infty} \frac{1}{2^k} |\langle g_k, \tilde{a} \cdot \tau - \tilde{a}^{(0)} \cdot \tau \rangle_{L_1, L_\infty}| < \frac{\eta - \eta_1}{3} \quad (36)$$

for all $\tau \in \mathbb{R}$. Take $M > 0$ such that $g_k(\tau, \cdot) = 0$ for all $\tau \leq -M$ and all $k = 1, 2, \dots, k_0 - 1$. Further, take $s_0 > 0$ such that $d_{\text{norm}}^s(a, a^{(0)}) < (\eta + 2\eta_1)/3$ for all $s > s_0 - M$. Finally, let $l_0 \in \mathbb{N}$ be such that $t + t_{n_l} > s_0 - M$ for all $l > l_0$.

Then we have

$$|\langle g_k, \tilde{a} \cdot (t + t_{n_l}) - \tilde{a}^{(0)} \cdot (t + t_{n_l}) \rangle_{L_1, L_\infty}| \leq (\eta + 2\eta_1)/3$$

for $k = 1, 2, \dots, k_0 - 1$ and all $l > l_0$, hence

$$\sum_{k=1}^{k_0-1} \frac{1}{2^k} |\langle g_k, \tilde{a} \cdot (t + t_{n_l}) - \tilde{a}^{(0)} \cdot (t + t_{n_l}) \rangle_{L_1, L_\infty}| < \frac{\eta + 2\eta_1}{3} \quad (37)$$

for all $l > l_0$.

Taking (36) and (37) into account we see that $d(\tilde{a} \cdot (t + t_{n_l}), \tilde{a}^{(0)} \cdot (t + t_{n_l})) < (2\eta + \eta_1)/3$ for sufficiently large l . By letting l go to infinity we have $d(\tilde{a} \cdot t, \tilde{a} \cdot t) \leq (2\eta + \eta_1)/3 < \eta$.

By Lemma 5.2,

$$\begin{aligned} \lambda_{\min}(a^{(0)}) - \varepsilon &\leq \liminf_{n \rightarrow \infty} \frac{\ln \|U_{\check{a}}(n, 0)w(\check{a} \cdot n)\|}{n} \\ &\leq \limsup_{n \rightarrow \infty} \frac{\ln \|U_{\check{a}}(n, 0)w(\check{a} \cdot n)\|}{n} \leq \lambda_{\min}(a^{(0)}) + \varepsilon. \end{aligned}$$

As a consequence of Theorem 4.3 and [18, Theorem 3.1.2 and Lemma 3.2.5], both $\liminf_{n \rightarrow \infty} \frac{1}{n} \ln \|U_{\check{a}}(n, 0)w(\check{a} \cdot n)\|$ and $\limsup_{n \rightarrow \infty} \frac{1}{n} \ln \|U_{\check{a}}(n, 0)w(\check{a} \cdot n)\|$ are in $\Sigma(a)$. Hence we have found $\lambda \in [\lambda_{\min}(a), \lambda_{\max}(a)]$ with $|\lambda - \lambda_{\min}(a^{(0)})| \leq \varepsilon$. By interchanging the rôles of $a^{(0)}$ and a we obtain that there is $\lambda \in [\lambda_{\min}(a^{(0)}), \lambda_{\max}(a^{(0)})]$ with $|\lambda - \lambda_{\min}(a)| \leq \varepsilon$.

We proceed in the same way with λ_{\max} , obtaining that the Hausdorff distance between $[\lambda_{\min}(a), \lambda_{\max}(a)]$ and $[\lambda_{\min}(a^{(0)}), \lambda_{\max}(a^{(0)})]$ is not bigger than ε , which is equivalent to the statement of Theorem 5.1. \square

5.2 Time Averaging

In the present subsection we assume that $a_{ij}(t, x) \equiv a_{ij}(x)$, $a_i(t, x) \equiv a_i(x)$, $b_i(t, x) \equiv b_i(x)$, and D and a satisfy (SM1)–(SM3).

Let \bar{a} be the extension of a as in Sect. 4.2. $\Pi(\bar{a})$ admits an exponential separation over $Y(\bar{a})$.

We call $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$ a *time-averaged function* of a if

$$\hat{c}_0(x) = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} c_0(t, x) dt \quad \text{for all } x \in \bar{D},$$

and

$$\hat{d}_0(x) = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} d_0(t, x) dt \quad \text{for all } x \in \partial D,$$

for some real sequences $(s_n)_{n=1}^\infty, (t_n)_{n=1}^\infty$ with $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ as $n \rightarrow \infty$.

The time independent equation

$$\begin{cases} u_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_j} + a_i(x)u \right) + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + \hat{c}_0(x)u, & x \in D, \\ \mathcal{B}_{\hat{a}} u = 0, & x \in \partial D, \end{cases} \quad (38)$$

where

$$\mathcal{B}_{\hat{a}}u = \begin{cases} u & \text{(Dirichlet)} \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(x) \partial_j u + a_i(x)u \right) v_i & \text{(Neumann)} \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(x) \partial_j u + a_i(x)u \right) v_i + \hat{d}_0(x)u, & \text{(Robin),} \end{cases}$$

is called a *time-averaged equation* of (1)+(2) if $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$ is a time-averaged function of a .

The eigenvalue problem associated to (38) reads as

$$\begin{cases} \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_j} + a_i(x)u \right) + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + \hat{c}_0(x)u = \lambda u, & x \in D, \\ \mathcal{B}_{\hat{a}}u = 0, & x \in \partial D. \end{cases} \quad (39)$$

It is well known that (39) has a unique eigenvalue, denoted by $\lambda_{\text{princ}}(\hat{a})$, which is real, simple, has an eigenfunction $\phi_{\text{princ}}(\hat{a}) \in L_2(D)^+$ associated to it, and for any other eigenvalue λ of (39), $\text{Re } \lambda < \lambda_{\text{princ}}(\hat{a})$ (see [2, 4]). We call $\lambda_{\text{princ}}(\hat{a})$ the *principal eigenvalue* of (38) and $\phi_{\text{princ}}(\hat{a})$ a *principal eigenfunction* (in the literature, sometimes, $-\lambda_{\text{princ}}(\hat{a})$ is called the principal eigenvalue of (38)).

Let

$$\hat{Y}(a) := \left\{ \hat{a} : \exists 0 \leq s_n < t_n \text{ with } s_n \rightarrow \infty \text{ and } t_n - s_n \rightarrow \infty \text{ such that} \right.$$

$$\left. \begin{aligned} \hat{c}_0(x) &= \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} c_0(t, x) dt \quad \text{for all } x \in \bar{D}, \\ \hat{d}_0(x) &= \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} d_0(t, x) dt \quad \text{for all } x \in \partial D \end{aligned} \right\}.$$

It follows from our assumptions, via the Ascoli–Arzelà theorem, that $\hat{Y}(a)$ is nonempty, and consists of functions belonging to $C^{3+\alpha}(\bar{D}, \mathbb{R}^{N^2+N}) \times C^{1+\alpha}(\bar{D}, \mathbb{R}^{N+1}) \times C^{3+\alpha}(\partial D)$, with their $C^{3+\alpha}(\bar{D}, \mathbb{R}^{N^2+N}) \times C^{1+\alpha}(\bar{D}, \mathbb{R}^{N+1}) \times C^{3+\alpha}(\partial D)$ -norms uniformly bounded. Moreover, the convergence in the definition of $\hat{Y}(a)$ is uniform in $x \in \bar{D}$ (resp. uniform in $x \in \partial D$).

Theorem 5.3. (1) *There is $\hat{a} \in \hat{Y}(a)$ such that $\lambda_{\min}(a) \geq \lambda_{\text{princ}}(\hat{a})$.*

(2) *$\lambda_{\max}(a) \geq \lambda_{\text{princ}}(\hat{a})$ for any $\hat{a} \in \hat{Y}(a)$.*

(3) *Assume moreover that a is asymptotically uniquely ergodic. Then $\hat{Y}(a)$ is a singleton $\{\hat{a}\}$, $\lambda_{\max}(a) = \lambda_{\min}(a) \geq \lambda_{\text{princ}}(\hat{a})$, and $\lambda_{\min}(a) = \lambda_{\text{princ}}(\hat{a})$ if and*

only if there is a sequence $(s_n)_{n=1}^\infty \subset [0, \infty)$ with $\lim_{n \rightarrow \infty} s_n = \infty$ with the property that the following two conditions are satisfied:

- There are a continuous function $c_{01}: \bar{D} \rightarrow \mathbb{R}$ and a bounded continuous function $c_{02}: (-\infty, \infty) \rightarrow \mathbb{R}$ such that $c_0(t + s_n, x)$ converges, as $n \rightarrow \infty$, to $c_{01}(x) + c_{02}(t)$, uniformly on compact subsets of $\mathbb{R} \times \bar{D}$,
- There is a continuous function $d_{01}: \partial D \rightarrow \mathbb{R}$ such that $d_0(t + s_n, x)$ converges, as $n \rightarrow \infty$, to $d_{01}(x)$, uniformly on compact subsets of $\mathbb{R} \times \partial D$.

To prove the above theorem, we first recall a lemma from [18].

Lemma 5.4. Let $\tilde{v}(t, x) := w(\bar{a} \cdot t)(x)$ ($t \geq 0, x \in \bar{D}$) and

$$\hat{w}(x; s, t) := \exp \left(\frac{1}{t-s} \int_s^t \ln w(\bar{a} \cdot \tau)(x) d\tau \right)$$

($0 \leq s < t, x \in D$). Then $\hat{w}(x; s, t)$ satisfies

$$\begin{aligned} & \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a_{ij}(x) \frac{\partial \hat{w}}{\partial x_j} + a_i(x) \hat{w} \right) + \sum_{i=1}^N b_i(x) \frac{\partial \hat{w}}{\partial x_i} \\ & \leq \left(\frac{1}{t-s} \int_s^t \frac{1}{\tilde{v}} \frac{\partial \tilde{v}}{\partial \tau}(\tau, x) d\tau \right) \hat{w} + \left(\frac{1}{t-s} \int_s^t \kappa(\tau) d\tau - \frac{1}{t-s} \int_s^t c_0(\tau, x) d\tau \right) \hat{w} \end{aligned} \quad (40)$$

for $x \in D$ and

$$\hat{B}(s, t) \hat{w} = 0$$

for $x \in \partial D$, where

$$\hat{B}(s, t) \hat{w} := \begin{cases} \hat{w} & (\text{Dirichlet}) \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(x) \partial_{x_j} \hat{w} + a_i(x) \hat{w} \right) \nu_i & (\text{Neumann}) \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(x) \partial_{x_j} \hat{w} + a_i(x) \hat{w} \right) \nu_i \\ \quad + \left(\frac{1}{t-s} \int_s^t d_0(\tau, x) d\tau \right) \hat{w} & (\text{Robin}). \end{cases} \quad (41)$$

Proof. See the proof of [18, Lemma 5.2.1]. □

Proof of Theorem 5.3(1) and (2). (1) For given $0 \leq s < t$ put

$$\eta(t; s) := \|U_a(t, s) w(\bar{a} \cdot s)\|$$

and

$$\hat{w}(x; s, t) := \exp \left(\frac{1}{t-s} \int_s^t \ln w(\bar{a} \cdot \tau)(x) d\tau \right), \quad x \in \bar{D}.$$

By Theorem 4.7, there are sequences $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ with $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ such that

$$\frac{\ln \eta(t_n; s_n)}{t_n - s_n} = \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \kappa(t) dt \rightarrow \lambda_{\min}(a).$$

It follows from (SM2) with the help of the Ascoli–Arzelà theorem that (after possibly taking a subsequence and relabeling) $\lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} c_0(t, x) dt$ and $\lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} d_0(t, x) dt$ exist, and the limits are uniform in $x \in \bar{D}$ and in $x \in \partial D$, respectively. Denote these limits by $\hat{c}_0(x)$ and $\hat{d}_0(x)$. Let $\hat{a} := (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)$. It then follows by arguments similar to those in [18, Theorem 5.2.2(1)] that $\lambda_{\min}(a) \geq \lambda_{\text{princ}}(\hat{a})$.

- (2) For any $\hat{a} = (a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0) \in \hat{Y}(a)$ there are $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$ with $s_n \rightarrow \infty$ and $t_n - s_n \rightarrow \infty$ such that

$$\frac{1}{t_n - s_n} \int_{s_n}^{t_n} c_0(t, x) dt \rightarrow \hat{c}_0(x) \quad \text{and} \quad \frac{1}{t_n - s_n} \int_{s_n}^{t_n} d_0(t, x) dt \rightarrow \hat{d}_0(x)$$

uniformly in $x \in \bar{D}$ and in $x \in \partial D$, respectively. By passing (if necessary) to subsequences and relabeling we can assume that there is λ_0 such that

$$\lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \kappa(t) dt = \lambda_0.$$

By arguments similar to those in the proof of (1), $\lambda_0 \geq \lambda_{\text{princ}}(\hat{a})$. It follows from Theorem 4.7 that $\lambda_{\max}(a) \geq \lambda_0$. Then we have $\lambda_{\max}(a) \geq \lambda_{\text{princ}}(\hat{a})$. \square

Before proving Theorem 5.3(3) we formulate and prove the following auxiliary result.

Lemma 5.5. *Assume that a is asymptotically uniquely ergodic. Then*

- (i) *For each $x \in \bar{D}$ and each $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0) \in Y_0(a)$ the limits*

$$\lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t c_0(\tau, x) d\tau \quad \text{and} \quad \lim_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \tilde{c}_0(\tau, x) d\tau \quad (42)$$

exist and are equal, and

- (ii) *For each $x \in \bar{D}$ and each $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0) \in Y_0(a)$ the limits*

$$\lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t d_0(\tau, x) d\tau \quad \text{and} \quad \lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \tilde{d}_0(\tau, x) d\tau \quad (43)$$

exist and are equal.

In particular, it follows that $\hat{Y}(a) = \{(a_{ij}, a_i, b_i, \hat{c}_0, \hat{d}_0)\}$, where \hat{c}_0 is the common limit in (42) and \hat{d}_0 is the common limit in (43).

Proof. Let \mathbb{P} be the unique ergodic invariant measure for the compact flow $(Y_0(a), \{\sigma_t\}_{t \in \mathbb{R}})$. \mathbb{P} is also the unique ergodic invariant measure for the compact semiflow $(\tilde{Y}_0(\bar{a}), \{\sigma_t\}_{t \geq 0})$, where $\tilde{Y}_0(\bar{a}) := \{\bar{a} \cdot t : t \geq 0\} \cup Y_0(a) = \text{cl}\{\bar{a} \cdot t : t \geq 0\}$.

For each $x \in \bar{D}$ (resp. for each $x \in \partial D$) we define a function $\tilde{c}_0[x] : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$ (resp. a function $\tilde{d}_0[x] : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$) as:

$$\tilde{c}_0[x](\tilde{a}) := \tilde{c}_0(0, x), \quad \tilde{a} \in \tilde{Y}_0(\bar{a}),$$

$$\tilde{d}_0[x](\tilde{a}) := \tilde{d}_0(0, x), \quad \tilde{a} \in \tilde{Y}_0(\bar{a}).$$

where $\tilde{a} = (a_{ij}, a_i, b_i, \tilde{c}_0, \tilde{d}_0)$. The functions $\tilde{c}_0[x] : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$ (resp. $\tilde{d}_0[x] : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$) are, for each $x \in \bar{D}$ (resp. for each $x \in \partial D$), continuous.

As $(\tilde{Y}_0(\bar{a}), \{\sigma_t\}_{t \geq 0})$ is uniquely ergodic, it follows from the results in [20] that for any continuous $g : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$ and any $\varepsilon > 0$ there is $T_0 = T_0(g, \varepsilon) > 0$ such that

$$\left| \frac{1}{t} \int_0^t g(\tilde{a} \cdot \tau) d\tau - \int_{\tilde{Y}_0(\bar{a})} g(\cdot) d\mathbb{P}(\cdot) \right| < \varepsilon$$

for each $t > T_0$ and each $\tilde{a} \in \tilde{Y}_0(a)$. In particular, for any continuous $g : \tilde{Y}_0(\bar{a}) \rightarrow \mathbb{R}$ there holds

$$\lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t g(\bar{a} \cdot \tau) d\tau = \lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t g(\tilde{a} \cdot \tau) d\tau = \int_{\tilde{Y}_0(\bar{a})} g(\cdot) d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. By substituting in the above, for a fixed $x \in \bar{D}$, the function $\tilde{c}_0[x]$ for g we have

$$\lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t c_0(\tau, x) d\tau = \lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \tilde{c}_0(\tau, x) d\tau = \int_{\tilde{Y}_0(\bar{a})} \tilde{c}_0[x](\cdot) d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. Similarly, by substituting, for a fixed $x \in \partial D$, the function $\tilde{d}_0[x]$ for g we have

$$\lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t d_0(\tau, x) d\tau = \lim_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \tilde{d}_0(\tau, x) d\tau = \int_{\tilde{Y}_0(\bar{a})} \tilde{d}_0[x](\cdot) d\mathbb{P}(\cdot),$$

for each $\tilde{a} \in \tilde{Y}_0(a)$. □

Proof of Theorem 5.3(3). By Theorem [18, Theorem 5.2.2(3)] and Theorem 4.3, we have

$$\lambda_{\max}(a) = \tilde{\lambda}_{\max}(a) = \tilde{\lambda}_{\min}(a) = \lambda_{\min}(a).$$

Let $Y_1 \subset Y_0(a)$ be a minimal invariant set. By the unique ergodicity of $(Y_0(a), \{\sigma_t\})$, the compact flow $(Y_1, \{\sigma_t\})$ is both minimal and uniquely ergodic. Let $\tilde{a} \in Y_1$. In view of Lemma 5.5 we can apply [18, Theorem 5.2.2(3)] to have that $\tilde{\lambda}_{\max}(a) = \lambda(\tilde{a})$ if and only if there are c_{01} , c_{02} , and d_{01} such that

$$\tilde{c}_0(t, x) = c_{01}(t) + c_{02}(x) \quad \text{and} \quad \tilde{d}_0(t, x) = d_{01}(x).$$

Note that there is $s_n \rightarrow \infty$ such that $a \cdot s_n \rightarrow \tilde{a}$. Therefore, $\tilde{\lambda}_{\max}(a) = \lambda(\tilde{a})$ if and only if there is $s_n \rightarrow \infty$ such that

$$\lim_{n \rightarrow \infty} c_0(t + s_n, x) = c_{01}(t) + c_{02}(x) \quad \text{and} \quad \lim_{n \rightarrow \infty} d_0(t, x) = d_{01}(x),$$

where the convergence is uniform on compact subsets of $\mathbb{R} \times \bar{D}$ (resp. on compact subsets of $\mathbb{R} \times \partial D$). \square

5.3 Space-Averaging

In the present subsection we assume that $a_{ij}(t, x) \equiv a_{ij}(t)$, $a_i(t, x) \equiv 0$, $b_i(t, x) \equiv 0$, and the boundary condition is Neumann. We also assume that D and a satisfy (SM1)–(SM3).

Let $\check{c}_0(t) := \frac{1}{|D|} \int_D c_0(t, x) dx$, $t \geq 0$. We call $\check{a} := (a_{ij}, 0, 0, \check{c}_0, 0)$ the *space-averaged* of a , and call the problem

$$\begin{cases} u_t = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(\sum_{j=1}^N a_{ij}(t) \frac{\partial u}{\partial x_j} \right) + \check{c}_0(t)u, & t > s \geq 0, x \in D, \\ \sum_{i=1}^N \left(\sum_{j=1}^N a_{ij}(t) \partial_j u \right) \nu_i = 0, & t > s \geq 0, x \in \partial D \end{cases} \quad (44)$$

the *space-averaged equation* of (1)+(2).

The theory presented in Sect. 4.2 applies to (44).

Denote by $[\lambda_{\min}(\check{a}), \lambda_{\max}(\check{a})]$ the principal spectrum interval of (44).

Theorem 5.6. (1) $[\lambda_{\min}(\check{a}), \lambda_{\max}(\check{a})] = \{ \lambda : \exists s_n < t_n \text{ with } s_n \rightarrow \infty \text{ and } t_n - s_n \rightarrow \infty \text{ such that } \lambda = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \check{c}_0(t) dt \}$.

(2) $\lambda_{\min}(a) \geq \lambda_{\min}(\check{a})$ and $\lambda_{\max}(a) \geq \lambda_{\max}(\check{a})$.

Proof. (1) Observe that the function $u: [0, \infty) \times \bar{D} \rightarrow \mathbb{R}$ defined as $u(t, x) := \exp\left(\int_0^t \check{c}_0(\tau) d\tau\right)$, $t \geq 0$, $x \in \bar{D}$, is a solution of (44) satisfying $u(0, \cdot) \in L_2(D)^+ \setminus \{0\}$, and apply Theorem 4.7 to obtain (1).

To prove (2), we use the following inequality, which was proved as a part of the proof of [18, Theorem 5.3.1(2)]:

$$\frac{1}{t-s} \int_s^t \check{c}_0(\tau) d\tau \leq \frac{\ln \|U_a(t,s)w(\bar{a} \cdot s)\|}{t-s} + \frac{1}{|D|} \frac{1}{t-s} \int_D \ln \frac{w(\bar{a} \cdot t)(x)}{w(\bar{a} \cdot s)(x)} dx, \quad 0 \leq s < t. \quad (45)$$

It follows from [18, Lemma 5.2.3(2)] that the set $\{w(\bar{a}^{(1)})(x)/w(\bar{a}^{(2)})(x) : \bar{a}^{(1)}, \bar{a}^{(2)} \in Y(\bar{a}), x \in D\}$ is bounded and bounded away from zero. Therefore the limit, as $s \rightarrow \infty$ and $t-s \rightarrow \infty$, of the second term on the right-hand side of (45) equals zero. Consequently,

$$\lambda_{\min}(\check{a}) = \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \check{c}_0(\tau) d\tau \leq \liminf_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t,s)w(\bar{a} \cdot s)\|}{t-s} = \lambda_{\min}(a)$$

and

$$\lambda_{\max}(\check{a}) = \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{1}{t-s} \int_s^t \check{c}_0(\tau) d\tau \leq \limsup_{\substack{s \rightarrow \infty \\ t-s \rightarrow \infty}} \frac{\ln \|U_a(t,s)w(\bar{a} \cdot s)\|}{t-s} = \lambda_{\max}(a).$$

This concludes the proof of (2). \square

6 Applications to Nonlinear Equations of Kolmogorov Type

In this section we study the asymptotic dynamics of nonlinear parabolic equations of Kolmogorov type. In particular, we provide conditions for (forward) uniform persistence of the nonlinear Kolmogorov equations by utilizing the principal spectrum associated to proper forward nonautonomous linear parabolic equations.

Throughout the present section we make the following assumption.

(NA1) $D \subset \mathbb{R}^N$ is a bounded domain, where its boundary ∂D is an $(N-1)$ -dimensional manifold of class $C^{3+\alpha}$, for some $\alpha > 0$.

Further, \mathcal{B} will stand for the boundary operator either of the Dirichlet type

$$\mathcal{B}u = u \quad \text{on } \partial D,$$

or of the Neumann type

$$\mathcal{B}u = \frac{\partial u}{\partial \mathbf{v}} \quad \text{on } \partial D,$$

where \mathbf{v} denotes the unit normal vector pointing out of D .

Let φ_{princ} be the unique (nonnegative) principal eigenfunction of the elliptic boundary value problem

$$\begin{cases} \Delta u = \lambda u & \text{on } D, \\ \mathcal{B}u = 0 & \text{on } \partial D, \end{cases} \quad (46)$$

normalized so that $\sup\{\varphi_{\text{princ}}(x) : x \in \bar{D}\} = 1$. By the elliptic strong maximum principle and the Hopf boundary point principle, in the Dirichlet case $\varphi_{\text{princ}}(x) > 0$ for each $x \in D$ and $(\partial\varphi_{\text{princ}}/\partial\mathbf{v})(x) < 0$ for each $x \in \partial D$. In the Neumann case $\varphi_{\text{princ}} \equiv 1$.

Let X be a fractional power space of the Laplacian operator Δ in $L_p(D)$ with the boundary condition $\mathcal{B}u = 0$ such that X is compactly imbedded into $C^1(\bar{D})$. We denote the norm in X by $\|\cdot\|_X$.

Denote $X^+ := \{u \in X : u(x) \geq 0 \text{ for all } x \in \bar{D}\}$. The interior X^{++} of X^+ is nonempty, and is characterized in the following way: In the case of Dirichlet boundary conditions, $X^{++} = \{u \in X^+ : u(x) > 0 \text{ for all } x \in D \text{ and } (\partial u/\partial\mathbf{v})(x) < 0 \text{ for all } x \in \partial D\}$, and in the case of Neumann boundary conditions, $X^{++} = \{u \in X^+ : u(x) > 0 \text{ for all } x \in \bar{D}\}$ (see [18, Lemma 7.1.8]). In particular, observe that $\varphi_{\text{princ}} \in X^{++}$.

For $u_1, u_2 \in X$ we write $u_1 \ll u_2$ (or $u_2 \gg u_1$) if $u_2 - u_1 \in X^{++}$.

Consider the following nonautonomous partial differential equation of Kolmogorov type:

$$u_t = \Delta u + f(t, x, u), \quad x \in D, \quad (47)$$

with $f: [0, \infty) \times \bar{D} \times [0, \infty) \rightarrow \mathbb{R}$, endowed with the boundary conditions

$$\mathcal{B}u = 0, \quad x \in \partial D. \quad (48)$$

We assume the following.

(NA2) For any $M > 0$ the restrictions to $[0, \infty) \times \bar{D} \times [0, M]$ of the function f and its derivatives up to order two belong to $C^{1-,1-,1-}([0, \infty) \times \bar{D} \times [0, M])$.

(NA3) There are $P > 0$ and a continuous function $m: [P, \infty) \rightarrow (0, \infty)$ such that $f(t, x, u) \leq -m(u)$ for any $t \geq 0$, any $x \in \bar{D}$ and any $u \geq P$.

By the theory in [7], for each $t_0 \geq 0$ and each $u_0 \in X^+$ there is a (classical) solution $u(\cdot; t_0, u_0)$ of (47)+(48), defined on $[t_0, \infty)$, with initial condition $u(t_0; t_0, u_0)(x) = u_0(x)$, such that $u(t; t_0, u_0) \in X$ for all $t \geq t_0$. By the comparison principle, there holds $u(t; t_0, u_0) \in X^+$ for all $t \geq t_0$.

Definition 6.1. Equation (47)+(48) is said to be *forward uniformly persistent* if there is $\eta > 0$ such that for any $u_0 \in X^+ \setminus \{0\}$ there is $\tau(u_0) \geq 0$ with the property that

$$u(t; t_0, u_0) \geq \eta \varphi_{\text{princ}}$$

for all $t_0 \geq 0$ and all $t \geq \tau(u_0) + t_0$.

Note that $u \equiv 0$ is the solution of (47)+(48).

Consider the linearization of (47)+(48) along 0,

$$\begin{cases} v_t = \Delta v + f_0(t, x)v, & x \in D, \\ \mathcal{B}v = 0, & x \in \partial D, \end{cases} \quad (49)$$

where $f_0(t, x) = f(t, x, 0)$. We also have that for each $t_0 \geq 0$ and each $v_0 \in X$ there is a (classical) solution $v(\cdot; t_0, v_0)$ of (49), defined on $[t_0, \infty)$, with initial condition $v(t_0; t_0, v_0)(x) = v_0(x)$, such that $v(t; t_0, v_0) \in X$ for all $t \geq t_0$.

It follows from (NA1) and (NA2) that the assumptions (SM1) through (SM3) are satisfied for (49), with $a = (\delta_{ij}, 0, 0, f_0, 0)$. Consequently, the theory presented in Sect. 4.2 applies.

Let $[\lambda_{\min}, \lambda_{\max}]$ stand for the principal spectrum interval of (49). We then have

Theorem 6.1. *If $\lambda_{\min} > 0$ then (47)+(48) is forward uniformly persistent.*

For any function $g: \mathbb{R} \times \bar{D} \times [0, \infty) \rightarrow \mathbb{R}$ and any $t \in \mathbb{R}$ we write $g \cdot t(\tau, x, u) := g(\tau + t, x, u)$, $\tau \in \mathbb{R}$, $x \in \bar{D}$, $u \geq 0$.

We extend the function f to a function $\bar{f}: \mathbb{R} \times \bar{D} \times [0, \infty) \rightarrow \mathbb{R}$ by putting $\bar{f}(t, x, u) := f(0, x, u)$ for $t < 0$, $x \in \bar{D}$ and $u \geq 0$.

Put

$$Z := \text{cl} \{ \bar{f} \cdot t : t \in \mathbb{R} \} \quad (50)$$

with the open-compact topology, where the closure is taken in the open-compact topology. By the Ascoli–Arzelà theorem, the set Z is a compact metrizable space. Further, if $g \in Z$ and $t \in \mathbb{R}$ then $g \cdot t =: \zeta_t g \in Z$. Hence $(Z, \{\zeta_t\}_{t \in \mathbb{R}})$ is a compact flow.

Put

$$Z_0 := \bigcap_{s \geq 0} \text{cl} \{ \bar{f} \cdot t : t \in [s, \infty) \}. \quad (51)$$

Z_0 , as the ω -limit set of a forward orbit in the compact flow $(Z, \{\zeta_t\}_{t \in \mathbb{R}})$, is nonempty, compact, connected and invariant.

Put

$$\tilde{Z}_0 := \{ \bar{f} \cdot t : t \geq 0 \} \cup Z_0 = \text{cl} \{ \bar{f} \cdot t : t \geq 0 \}. \quad (52)$$

The set \tilde{Z}_0 is a closed, hence compact, subset of Z . Further, it is *forward invariant*: for any $g \in \tilde{Z}_0$ and any $t \geq 0$ there holds $g \cdot t \in \tilde{Z}_0$.

For any $g \in \tilde{Z}_0$, consider the following semilinear second order parabolic equation of Kolmogorov type,

$$\begin{cases} u_t = \Delta u + g(t, x, u)u, & t > 0, x \in D, \\ \mathcal{B}u = 0, & t > 0, x \in \partial D. \end{cases} \quad (53)$$

By the theory in [7], the following holds.

Proposition 6.2. *For each $u_0 \in X^+$ and each $g \in \tilde{Z}_0$ there exists a unique solution $u(\cdot; u_0, g)$ of (53), defined on $[0, \infty)$, satisfying the initial condition $u(0; u_0, g) = u_0$, such that $u(t; u_0, g) \in X^+$ for all $t \geq 0$. That solution is classical. Further, the mapping*

$$[[0, \infty) \times X^+ \times \tilde{Z}_0 \ni (t, u_0, g) \mapsto u(t; u_0, g) \in X]$$

is continuous.

Observe that $u(\cdot + t_0; t_0, u_0) = u(\cdot; u_0, \tilde{f}_0 \cdot t_0)$ for $t_0 \geq 0$.

Let Y_0 and \tilde{Y}_0 be defined as follows,

$$Y_0 := \{g_0 : \exists g \in Z_0 \text{ such that } g_0(t, x) = g(t, x, 0), t \in \mathbb{R}, x \in \bar{D}\},$$

and

$$\tilde{Y}_0 := \{g_0 : \exists g \in \tilde{Z}_0 \text{ such that } g_0(t, x) = g(t, x, 0), t \in \mathbb{R}, x \in \bar{D}\}.$$

The sets Y_0 and \tilde{Y}_0 are considered endowed with the open-compact topology. As the images of the compact sets Z_0 and \tilde{Z}_0 , respectively, under restriction, they are compact.

For $t_0 \in \mathbb{R}$ and $g_0 \in \tilde{Y}_0$ consider

$$\begin{cases} v_t = \Delta v + g_0(t, x)v, & t > t_0, x \in D, \\ \mathcal{B}v = 0, & t > t_0, x \in \partial D. \end{cases} \quad (54)$$

By the theory in [7], for any $v_0 \in X$, $t_0 \in \mathbb{R}$ and $g_0 \in \tilde{Y}_0$, (54) has a unique (classical) solution $v(t; t_0, v_0, g_0)$, defined on $[t_0, \infty)$, with $v(t_0; t_0, v_0, g_0) = v_0$, such that $v(t; t_0, v_0, g_0) \in X$ for all $t \geq t_0$.

Observe that for any $g \in \tilde{Z}_0$, $u \equiv 0$ is the solution of (53) and (54) with $g_0(t, x) = g(t, x, 0)$ is the linearization of (53) along $u \equiv 0$. Put $U_{g_0}(t, t_0)v_0 := v(t; t_0, v_0, g_0)$. If $g_0 = \tilde{f}_0 \cdot t_0$ and $t_0 \geq 0$, we write $U_{g_0}(t, t_0)$ as $U(t, t_0)$.

Lemma 6.3. *For each $t > 0$ there holds*

$$\frac{\|u(t; \rho u_0, g) - \rho U_{g_0}(t, 0)u_0\|_X}{\rho} \rightarrow 0 \quad \text{as } \rho \rightarrow 0^+$$

uniformly in $g \in \tilde{Z}_0$ and $u_0 \in X^+$ with $\|u_0\|_X = 1$, where $g_0(t, x) = g(t, x, 0)$.

Proof. It follows from [18, Theorem 7.1.5]. □

Lemma 6.4. *Assume that $\lambda_{\min} > 0$. Then there is $T > 0$ such that*

$$U_{g_0}(T, 0)\phi_{\text{princ}} \gg 2\phi_{\text{princ}} \quad \text{for all } g_0 \in Y_0.$$

Proof. Let $[\tilde{\lambda}_{\min}, \tilde{\lambda}_{\max}]$ be the principal spectrum of (54) over Y_0 . By Theorem 4.3, $\lambda_{\min} = \tilde{\lambda}_{\min}$ and hence $\tilde{\lambda}_{\min} > 0$. The lemma then follows from [18, Lemma 7.1.16]. □

Proof of Theorem 6.1. Let $T > 0$ be as in Lemma 6.4. As the mapping $[Y_0 \ni g_0 \mapsto U_{g_0}(T, 0)\varphi_{\text{princ}} \in X]$ is continuous and Y_0 is compact, the set $\{U_{g_0}(T, 0)\varphi_{\text{princ}} - 2\varphi_{\text{princ}} : g_0 \in Y_0\}$ is compact, too. Further, this set is, by Lemma 6.4, contained in the open set X^{++} . Therefore $\varepsilon_0 := \inf\{\|(U_{g_0}(T, 0)\varphi_{\text{princ}} - 2\varphi_{\text{princ}}) - v\|_X : g_0 \in Y_0, v \in \partial X^+\}$ is positive. By linearity,

$$\inf\{\|(rU_{g_0}(T, 0)\varphi_{\text{princ}} - 2r\varphi_{\text{princ}}) - v\|_X : g_0 \in Y_0, v \in \partial X^+\} = r\varepsilon_0 \quad (55)$$

for any $r > 0$.

It follows from Lemma 6.3 that there is $r_0 > 0$ such that

$$\|u(T + t; t, r\varphi_{\text{princ}}) - rU(T + t, t)\varphi_{\text{princ}}\|_X \leq \frac{r\varepsilon_0}{3}$$

for all $t \geq 0$ and all $r \in (0, r_0]$.

We claim that there is $T_1 \geq 0$ such that for each $t \geq T_1$ one can find $g \in Z_0$ such that $\|U(T + t, t)\varphi_{\text{princ}} - U_{g_0}(T, 0)\varphi_{\text{princ}}\|_X < \varepsilon_0/3$. Indeed, for each $g \in Z_0$ there is $\delta = \delta(g) > 0$ such that for any $h \in Z$, if $d(g, h) < \delta$ then $\|U_{h_0}(T, 0)\varphi_{\text{princ}} - U_{g_0}(T, 0)\varphi_{\text{princ}}\|_X < \varepsilon_0/3$, where $d(\cdot, \cdot)$ stands for the metric in Z . Since Z_0 is compact, there are finitely many $g^{(1)}, \dots, g^{(n)} \in Z_0$ such that the union of the open balls (in Z) with center $g^{(k)}$ and radius $\delta(g^{(k)})$, $k = 1, \dots, n$, covers Z_0 . Denote this union by B . It suffices now to find $T_1 \geq 0$ such that $\tilde{f} \cdot t \in B$ for all $t \geq T_1$, and the existence of such T_1 follows from the fact that Z_0 is, by definition, the ω -limit set (in the compact flow $(Z, \{\zeta_t\})$) of \tilde{f} .

Fix for the moment $t \geq T_1$, and let $g \in Z_0$ be such that $\|U(T + t, t)\varphi_{\text{princ}} - U_{g_0}(T, 0)\varphi_{\text{princ}}\|_X < \varepsilon_0/3$. We estimate

$$\begin{aligned} & \| (u(T + t; t, r\varphi_{\text{princ}}) - 2r\varphi_{\text{princ}}) - (rU_{g_0}(T, 0)\varphi_{\text{princ}} - 2r\varphi_{\text{princ}}) \|_X \\ &= \| u(T + t; t, r\varphi_{\text{princ}}) - rU_{g_0}(T, 0)\varphi_{\text{princ}} \|_X \\ &\leq \| u(T + t; t, r\varphi_{\text{princ}}) - rU(T + t, t)\varphi_{\text{princ}} \|_X \\ &\quad + \| rU(T + t, t)\varphi_{\text{princ}} - rU_{g_0}(T, 0)\varphi_{\text{princ}} \|_X \\ &< \frac{r\varepsilon_0}{3} + \frac{r\varepsilon_0}{3} \end{aligned}$$

for any $r \in (0, r_0]$. It follows from (55) that $u(t + T; t, r\varphi_{\text{princ}}) - 2r\varphi_{\text{princ}} \in X^{++}$, that is, $u(t + T; t, r\varphi_{\text{princ}}) \gg 2r\varphi_{\text{princ}}$, for any $r \in (0, r_0]$ and any $t \geq T_1$.

Fix a nonzero $u_0 \in X^+$. By the comparison principle for parabolic equations, $u(t; t_0, u_0) \gg 0$, that is, $u(t; t_0, u_0)$ belongs to the open subset X^{++} of X , for any $t > t_0$. Since $[T_1, T_1 + T] \times \{u_0\} \times \tilde{Z}_0$ is compact, it follows from Proposition 6.2 that the set $\{u(t; u_0, g) : t \in [T_1 + 1, T_1 + T + 1], g \in \tilde{Z}_0\} (\subset X^{++})$ is compact. Consequently, the set $\{u(t + t_0; t_0, u_0) : t_0 \geq 0, t \in [T_1 + 1, T_1 + T + 1]\}$ has compact closure contained in X^{++} . By arguments as in the proof of [18, Theorem 7.1.6], there is $\tilde{r} > 0$ such that $u(t + t_0; t_0, u_0) \geq \tilde{r}\varphi_{\text{princ}}$ for all $t_0 \geq 0$ and $t \in [T_1 + 1, T_1 + T + 1]$.

Assume $\tilde{r} \geq r_0$. Then for each $t \in [T_1 + 1, T_1 + T + 1]$ and each $t_0 \geq 0$ we have $u(t + T + t_0; t_0, u_0) = u(t + T + t_0; t + t_0, u(t + t_0; t_0, u_0)) \gg u(t + T + t_0; t + t_0, r_0 \varphi_{\text{princ}}) \gg 2r_0 \varphi_{\text{princ}}$. By induction, we have $u(t + nT + t_0; t_0, u_0) \gg 2r_0 \varphi_{\text{princ}}$ for all $n = 1, 2, \dots$. Therefore we can take $\tau(u_0) = T_1 + T + 1$.

Assume $\tilde{r} < r_0$. Then for each $t \in [T_1 + 1, T_1 + T + 1]$ and each $t_0 \geq 0$ such that $u(t + t_0; t_0, u_0) \geq r \varphi_{\text{princ}}$ for some $r < r_0$ we have $u(t + T + t_0; t_0, u_0) = u(t + T + t_0; t + t_0, u(t + t_0; t_0, u_0)) \geq u(t + T + t_0; t + t_0, r \varphi_{\text{princ}}) \gg 2r \varphi_{\text{princ}}$. Repeating this procedure sufficiently many times we obtain that $u(t + nT + t_0; t_0, u_0) \gg 2^n r \varphi_{\text{princ}}$ as long as $2^{n-1} r \leq r_0$. After some calculation we conclude that we can take $\tau(u_0) = (\lfloor \frac{\ln r_0 - \ln \tilde{r}}{\ln 2} \rfloor + 2)T + T_1 + 1$.

In both cases, $\eta = 2r_0$. \square

We finish the section by giving a sufficient condition for the assumptions in Theorem 6.1 to hold.

A function $\hat{f}_0 \in C(\bar{D})$ is called a *time-averaged function* of f_0 if there are subsequences $(s_n)_{n=1}^\infty$ and $(t_n)_{n=1}^\infty$, with $0 < s_n < t_n$ for all $n = 1, 2, \dots$, $\lim_{n \rightarrow \infty} s_n = \infty$, $\lim_{n \rightarrow \infty} (t_n - s_n) \rightarrow \infty$, such that

$$\hat{f}_0(x) = \lim_{n \rightarrow \infty} \frac{1}{t_n - s_n} \int_{s_n}^{t_n} f(t, x, 0) dt$$

uniformly for $x \in \bar{D}$.

Let $\hat{Y} := \{\hat{f}_0 : \hat{f}_0 \text{ is a time-averaged function of } f_0\}$. For a given $\hat{f}_0 \in \hat{Y}$, denote by $\lambda_{\text{princ}}(\hat{f}_0)$ the principal eigenvalue of

$$\begin{cases} \Delta u + \hat{f}_0(x)u = \lambda u, & x \in D, \\ \mathcal{B}u = 0, & x \in \partial D. \end{cases} \quad (56)$$

Theorem 6.5. *If $\lambda_{\text{princ}}(\hat{f}_0) > 0$ for any $\hat{f}_0 \in \hat{Y}$, then (47)+(48) is forward uniformly persistent.*

Proof. Observe that the standing assumptions in Sect. 5.2 hold for (49). By Theorem 5.3(1), there is $\hat{f}_0 \in \hat{Y}$ such that $\lambda_{\min} \geq \lambda_{\text{princ}}(\hat{f}_0)$. An application of Theorem 6.1 concludes the proof. \square

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