Chapter 2
Existence results for pullback attractors

In this chapter we develop the existence theory for pullback attractors in a way that recovers well known results for the global attractors of autonomous systems as a particular case (see, for example, Babin and Vishik 1992; Chepyzhov and Vishik 2002; Cholewa and Dlotko 2000; Chueshov 1999; Hale 1988; Ladyzhenskaya 1991; Robinson 2001; Temam 1988).

We give a number of existence results of different 'flavours': they all require some boundedness and compactness properties of the process, but the way that these are combined varies, and which theorem is more suitable will depend on the application. We give a brief summary here of the corresponding autonomous statements, which are somewhat simpler. Much of the work in this chapter is in finding the appropriate non-autonomous generalisations, which often require some uniformity assumptions that are not immediately obvious.

We say that $T(\cdot)$ is a bounded semigroup if $\bigcup_{t\geq 0} T(t)B$ is bounded for every bounded B, and that D is an attracting set for $T(\cdot)$ if $\lim_{t\to\infty} \mathrm{dist}(T(t)B, D) = 0$ for all bounded subsets B of X.

Theorem 2.1. *Let $T(\cdot)$ be a bounded semigroup on a Banach space X. The following statements are equivalent:*

(a) $T(\cdot)$ has a global attractor $\mathscr{A}$;
(b) $T(\cdot)$ has a compact attracting set K;
(c) $T(\cdot)$ has a bounded attracting set and is asymptotically compact;
(d) $T(\cdot)$ is asymptotically compact and there is a bounded set that attracts points.

A sufficient condition for the validity of the statements $(a)-(d)$ is that

(e) $T(\cdot)$ has a bounded attracting set and is flattening.

If X is uniformly convex, then (e) is equivalent to each of the statements $(a)-(d)$.

The characterisation in (b) is proved in Theorem 2.12. While the most elegant, the existence of a compact attracting set is usually hard to check, and the result is often used in a weakened form: there is a global attractor if there is a compact absorbing set, i.e. a compact set K such that for any bounded B

A.N. Carvalho et al., *Attractors for infinite-dimensional non-autonomous dynamical systems*, Applied Mathematical Sciences 182, DOI 10.1007/978-1-4614-4581-4_2,

$$T(t)B \subset K \qquad \text{for all} \qquad t \geq t(B)$$

for some $t(B)$. In this formulation, Theorem 2.12 is easily applicable to ordinary differential equations (we treat two such examples after the statement of the theorem), and in certain relatively straightforward partial differential equation (PDE) applications (e.g. the two-dimensional Navier–Stokes equations in the space L^2; see Chap. 11).

However, the requirement of the existence of a compact absorbing set is very strong, and there are simple examples that have global attractors but no compact absorbing set (e.g. the equation $\dot{x} = -x$ in any infinite-dimensional space has $\{0\}$ as the global attractor). For this reason, the characterisation in (c), whose non-autonomous counterpart can be found in Theorem 2.23, is probably the most generally applicable. A semigroup is *asymptotically compact* if for every sequence $t_k \to \infty$ and $\{x_k\} \in B$, with B bounded, $T(t_k)x_k$ has a convergent subsequence. For example, we use the equivalent non-autonomous result in Chap. 15 to treat a damped wave equation; it can also be used for equations on unbounded domains for which the semigroup is not compact.

The 'point dissipativity' of (d), i.e. the requirement to attract only individual trajectories, is particularly suited for the analysis of autonomous gradient semigroups, for which this type of dissipativity property is almost automatic. We treat such systems, which play a major role in many of our subsequent considerations, in Sect. 2.5.1.

The 'flattening' property in (e) is one way to ensure asymptotic compactness. This property requires that for every bounded set $B \subset X$ and every $\varepsilon > 0$ there exists a $T(B,\varepsilon)$ and a finite-dimensional subspace X_ε of X such that

$$\bigcup_{t \geq T} P_\varepsilon T(t)B \text{ is bounded}$$

and

$$\|(I - P_\varepsilon)\left(\bigcup_{t \geq T} T(t)B\right)\|_X < \varepsilon,$$

where $P_\varepsilon : X \to X_\varepsilon$. The nice thing about this formulation, whose non-autonomous statement is given in Theorem 2.27, is that one need only make estimates in the phase space X. We use this idea to prove the existence of a pullback attractor for the 2D Navier–Stokes equation in H^1 in Chap. 11.

We build our attractors from omega-limit sets, which we now introduce.

2.1 Omega-limit sets

We start by generalising the notion of an ω-limit set to deal with processes, choosing to define our non-autonomous limit sets using the pullback procedure. Eventually we will build our pullback attractor as a union of ω-limit sets.

Throughout this section, $S(\cdot,\cdot)$ is a process on a metric space (X,d).

Definition 2.2. The pullback ω-limit set at time t of a subset B of X is defined by

$$\omega(B,t) := \bigcap_{\sigma \le t} \overline{\bigcup_{s \le \sigma} S(t,s)B}$$

or, equivalently,

$$\omega(B,t) = \Big\{ y \in X : \text{there are sequences } \{s_k\} \le t,\ s_k \to -\infty \text{ as } k \to \infty, \\ \text{and } \{x_k\} \text{ in } B,\ \text{such that } y = \lim_{k\to\infty} S(t,s_k)x_k \Big\}. \tag{2.1}$$

Note that we have used here, and will use throughout the book, the shorthand notation 'a sequence $\{x_k\} \in X$' for 'a sequence $\{x_k\}_{k=1}^{\infty}$ with $x_k \in X$ for all $k \in \mathbb{N}$'; and for sequences of real numbers we will often write '$\{s_k\} \le t$' to mean '$\{s_k\}_{k=1}^{\infty}$ with $s_k \in \mathbb{R}$ and $s_k \le t$ for all $k \in \mathbb{N}$'. Similarly, we will write 'a sequence $\{t_k\} \to \infty$' to mean 'a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \in \mathbb{R}$ and $t_k \to \infty$ as $k \to \infty$'.

Clearly, if $T(\cdot)$ is a semigroup and $S_T(\cdot,\cdot)$ is the corresponding process, then $\omega(B,t)$ is independent of t and coincides with the definition of the ω-limit set for semigroups (Hale 1988; Temam 1988):

$$\omega(B) = \bigcap_{s \ge 0} \overline{\bigcup_{r \ge s} T(r)B}.$$

The following lemma will be used many times throughout the book.

Lemma 2.3. *Suppose that K is a compact subset of X and that $\{x_n\} \in X$ is a sequence with* $\mathrm{dist}(x_n,K) \to 0$ *as $n \to \infty$. Then $\{x_n\}$ has a convergent subsequence whose limit lies in K.*

Proof. Given $k \in \mathbb{N}$, take x_{n_k} such that $\mathrm{dist}(x_{n_k},K) < 1/k$. Then, there exists $y_k \in K$ with $d(x_{n_k},y_k) < 1/k$. Now, by the compactness of K, there exists a subsequence (which we relabel) y_k such that $\lim_{k\to\infty} y_k = y_0 \in K$. The result now follows since $d(x_{n_k},y_0) \le d(x_{n_k},y_k) + d(y_k,y_0)$. $\square$

We now want to find conditions under which $\omega(B,t)$ is non-empty and invariant and pullback attracts B at time t. We can deal quickly with the question of invariance:

Lemma 2.4. *Let $S(\cdot,\cdot)$ be a process in a metric space X.*

(i) For any $B \subset X$, $\omega(B,s)$ is positively invariant: $S(t,s)\omega(B,s) \subseteq \omega(B,t)$, $t \ge s$.
(ii) If $\omega(B,s)$ is compact and pullback attracts B at time s, then $S(t,s)\omega(B,s) = \omega(B,t)$ for all $t \ge s$.
(iii) If $\omega(B,t)$ is compact and pullback attracts C at time t, where C is a connected set that contains B, then $\omega(B,t)$ is connected.

Proof. (i) If $\omega(B,t) = \varnothing$, then there is nothing to show. If $\omega(B,s) \neq \varnothing$, then from the continuity of $S(t,s)$ and from (2.1) one immediately sees that $S(t,s)\omega(B,s) \subseteq \omega(B,t)$.

(ii) If $\omega(B,s)$ is compact and pullback attracts B, then $\omega(B,t) \subseteq S(t,s)\omega(B,s)$. Indeed, for $x \in \omega(B,t)$ there are sequences $\{\sigma_k\} \leq t$ with $\sigma_k \to -\infty$, and $\{x_k\} \in B$ such that $S(t,\sigma_k)x_k \to x$ as $k \to \infty$. Since $\sigma_k \to -\infty$, there exists a $k_0 \in \mathbb{N}$ such that $\sigma_k \leq s$ for all $k \geq k_0$. Hence $S(t,s)S(s,\sigma_k)x_k = S(t,\sigma_k)x_k \to x$ for $k \geq k_0$. Since $\omega(B,s)$ is compact and pullback attracts B at time s, $\mathrm{dist}(S(s,\sigma_k)x_k, \omega(B,s)) \to 0$ as $k \to \infty$. It is then easy to see that $\{S(s,\sigma_k)x_k\}$ has a subsequence that converges to some $y \in \omega(B,s)$. It follows from the continuity of $S(t,s)$ that $S(t,s)y = x$. Hence $\omega(B,t) = S(t,s)\omega(B,s)$.

(iii) Finally, we prove the assertion about the connectedness of $\omega(B,t)$. Suppose that $\omega(B,t)$ is disconnected; then $\omega(B,t)$ is the disjoint union of two non-empty compact sets ω_1, ω_2 (which are therefore separated by a positive distance 2δ). Since $\omega(B,t)$ attracts C and $B \subseteq C$, it follows that $\omega(B,t) = \omega(C,t)$, and there exists $s_0 < 0$ such that $S(t,s)C \subset \mathscr{O}_\delta(\omega(C,t))$ for all $s \leq s_0$. From the connectedness of C, there exists $i \in \{1,2\}$ such that $S(t,s)C \bigcap \mathscr{O}_\delta(\omega_i) = \varnothing$ for all $s \leq s_0$ [for any $c \in C$, $\{S(t,s)c : s \leq s_0\}$ may not intersect $\mathscr{O}_\delta(\omega_1)$ and $\mathscr{O}_\delta(\omega_2)$ without leaving both]. This contradicts the fact that both ω_1 and ω_2 are non-empty. $\square$

We note that the proof of connectedness in part (iii) implies that the pullback attractor must be connected if it exists.

Corollary 2.5. *If $A(t)$ is compact and pullback attracts C at time t, where C is a connected set that contains $A(t)$, then $A(t)$ is connected. In particular, if X is a Banach space, or a metric space in which balls are connected, then the pullback attractor is connected if it exits.*

Lemma 2.4 can be recast in the language of semigroups (cf. Lemma 3.1.1 in Hale 1988):

Corollary 2.6. *Let $T(\cdot)$ be a semigroup in a metric space X.*

(i) For any $B \subset X$, $T(t)\omega(B) \subseteq \omega(B)$.
(ii) If $\omega(B)$ is compact and attracts B, then $T(t)\omega(B) = \omega(B)$ for all $t \geq 0$.
(iii) If $\omega(B)$ is compact and attracts C, where C is a connected set that contains B, then $\omega(B)$ is connected.

We now look for conditions under which we can guarantee that $\omega(B,t)$ pullback attracts B. Our first result shows that $\omega(B,t)$ pullback attracts B whenever B is pullback attracted at time t by any compact set.

Note that, from Lemma 2.4 (ii), $S(t,s)\omega(B,s) = \omega(B,t)$ for all $t \geq s$ whenever $\omega(B,s)$ is compact and pullback attracts B. For completeness we include this invariance property in the statements of Lemmas 2.7 and 2.10, but this comes 'for free' from Lemma 2.4.

Lemma 2.7. *Let $S(\cdot,\cdot)$ be a process in a metric space X. Suppose that B is a non-empty bounded subset of X that is pullback attracted by some compact set K at time* *t. Then $\omega(B,t)$ is non-empty and compact and pullback attracts B at time t, and $S(\tau,t)\omega(B,t) = \omega(B,\tau)$ for all $\tau \geq t$.*

Proof. First observe that for any sequence $\{x_n\} \in B$ and any sequence $\{s_n\} \to -\infty$, it follows from the fact that K attracts B that $\mathrm{dist}(S(t,s_n)x_n, K) \to 0$. Lemma 2.3 now implies that $\{S(t,s_n)x_n\}$ has a convergent subsequence, and by definition the limit of this subsequence must be an element of $\omega(B,t)$, which shows that $\omega(B,t)$ is non-empty.

We prove that $\omega(B,t)$ pullback attracts B at time t by contradiction. Assume that there exists an $\varepsilon > 0$, a sequence $\{s_n\} \to -\infty$, and a sequence $\{x_n\} \in B$ such that

$$\mathrm{dist}(S(t,s_n)x_n, \omega(B,t)) > \varepsilon \qquad \text{for all} \quad n \in \mathbb{N}. \tag{2.2}$$

But we have just shown that there must be a subsequence of $\{S(t,s_n)x_n\}$ that converges to an element of $\omega(B,t)$, contradicting (2.2).

Finally, $\omega(B,t)$ is compact since $\omega(B,t) \subset K$ and $\omega(B,t)$ is closed (from its definition). $\square$

This result will be of most interest when there is a compact family $K(\cdot)$ such that $K(t)$ pullback attracts B for each $t \in \mathbb{R}$. The following concept is useful in applications to obtain the pullback attraction for ω-limit sets (and hence the existence of pullback attractors) without having to find such a compact pullback attracting family $K(\cdot)$ explicitly.

Definition 2.8. A process $S(\cdot,\cdot)$ in a metric space X is said to be *pullback asymptotically compact* if, for each $t \in \mathbb{R}$, each sequence $\{s_k\} \leq t$ with $s_k \to -\infty$ as $k \to \infty$, and each bounded sequence $\{x_k\} \in X$ the sequence $\{S(t,s_k)x_k\}$ has a convergent subsequence.

If $T(\cdot)$ is a semigroup, then the corresponding process $S_T(\cdot,\cdot)$ is pullback asymptotically compact if and only if for each bounded sequence $\{x_k\} \in X$ and sequence $\{t_k\} \geq 0$ with $t_k \to +\infty$ as $k \to \infty$, the sequence $\{T(t_k)x_k\}$ has a convergent subsequence. In this case, the semigroup $T(\cdot)$ is said to be[1] *asymptotically compact*.

Armed with Lemma 2.7, it is simple to show that a process with a family of compact pullback attracting sets is asymptotically compact, so in particular any process with a pullback attractor *must* be pullback asymptotically compact.

Lemma 2.9. *If $S(\cdot,\cdot)$ has a family of compact pullback attracting sets $K(\cdot)$, then it is pullback asymptotically compact.*

Proof. Take sequences $\{s_k\} \leq t$ with $s_k \to -\infty$ and $\{x_k\} \in X$ contained in a bounded set B; then, since $\mathrm{dist}(S(t,s)B, K(t)) \to 0$ as $s \to -\infty$ and $K(t)$ is compact, Lemma 2.3 guarantees that $\{S(t,s_k)x_k\}$ has a convergent subsequence. $\square$

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[1]Note that this is the classical definition of asymptotic compactness for a semigroup (e.g. Ladyzhenskaya 1991; Temam 1988), which is stronger (it implies boundedness) than the one in Hale (1988) or Raugel (2002) in which one also must assume that $\{T(t_k)x_k\}$ is bounded.

Rather than investigate the converse directly, we pursue conditions ensuring that $\omega(B,t)$ pullback attracts B at time t, as in Lemma 2.7.

Lemma 2.10. *Let $S(\cdot,\cdot)$ be a pullback asymptotically compact process and suppose that B is a non-empty bounded subset of X. Then, for each $t \in \mathbb{R}$, $\omega(B,t)$ is non-empty and compact and pullback attracts B at time t and $S(\tau,t)\omega(B,t) = \omega(B,\tau)$ for all $\tau \geq t$.*

Proof. Note first that there exists a time s_0 such that

$$\overline{\cup_{s \leq s_0} S(t,s)B} \text{ is bounded.}$$

If not, there would exist a sequence $\{s_k\} \to -\infty$ and a sequence $\{x_k\} \in B$ such that $\{S(t,s_k)x_k\}$ is unbounded, which would contradict the asymptotic compactness. Now for any sequences $\{x_k\} \in B$ and $\{s_k\} \leq s_0$, with $s_k \to -\infty$ as $k \to \infty$, it follows from the fact that $S(\cdot,\cdot)$ is pullback asymptotically compact that there exists a subsequence of $\{S(t,s_k)x_k\}$ that converges to some $y \in X$. Then $y \in \omega(B,t)$ and $\omega(B,t)$ is non-empty. That $\omega(B,t)$ pullback attracts B follows exactly as in the proof of Lemma 2.7.

To finish, we show that $\omega(B,t)$ is compact. Given a sequence $\{y_k\} \in \omega(B,t)$, there are $x_k \in B$ and $\{s_k\} \leq \min(s_0,-k)$, such that $d(S(t,s_k)x_k,y_k) \leq \frac{1}{k}$. Since $\{S(t,s_k)x_k\}$ has a subsequence that converges to an element y of $\omega(B,t)$, it follows that $\{y_k\}$ has a subsequence that converges to $y \in \omega(B,t)$, and hence $\omega(B,t)$ is compact. □

The autonomous version of this result is pleasingly simple.

Corollary 2.11. *If $T(\cdot)$ is an asymptotically compact semigroup and B is a non-empty bounded subset of X, then $\omega(B)$ is non-empty, compact, and invariant and attracts B.*

2.2 First result: from the existence of a compact attracting set

Our first result on the existence of pullback attractors is a generalisation of the analogous one for autonomous dynamical systems (Temam 1988; Hale 1988; Babin and Vishik 1992; Robinson 2001); the closest result to the form here is given by Crauel (2001) (see also results in Hale 1988). It shows that the existence of a pullback attractor is equivalent to the existence of a family of compact pullback attracting sets: given such a family one can obtain the additional invariance property via a suitable construction in terms of ω-limit sets.

Theorem 2.12. *If $S(\cdot,\cdot)$ is a process in a metric space X, then the following statements are equivalent:*

- *$S(\cdot,\cdot)$ has a pullback attractor $\mathscr{A}(\cdot)$.*
- *There exists a family of compact sets $K(\cdot)$ that pullback attracts bounded subsets of X under $S(\cdot,\cdot)$.*

In either case

$$\mathscr{A}(t) = \overline{\bigcup\{\omega(B,t):\ B\subset X,\ B\ \textit{bounded}\}}, \tag{2.3}$$

and $\mathscr{A}(\cdot)$ is minimal in the sense that, if there exists another family of closed bounded sets $\hat{\mathscr{A}}(\cdot)$ that pullback attracts bounded subsets of X under $S(\cdot,\cdot)$, then $\mathscr{A}(t)\subseteq\hat{\mathscr{A}}(t)$ for all $t\in\mathbb{R}$.

Proof. If $S(\cdot,\cdot)$ has a pullback attractor $\mathscr{A}(\cdot)$, then each $\mathscr{A}(t)$ is compact and pullback attracts bounded subsets of X at time t.

To prove the converse, we proceed as follows. First note that, as an immediate consequence of the characterisation in (2.1), $\omega(B,t)\subseteq K(t)$, for all $B\subset X$ bounded and all $t\in\mathbb{R}$. It follows from Lemma 2.7 that $\omega(B,t)$ attracts B, and then, from Lemma 2.4 (ii), that $\omega(B,t)$ is invariant. Thus, if we define $\mathscr{A}(t)$ by (2.3), then we produce a compact set that pullback attracts all bounded subsets of X.

The invariance of $\mathscr{A}(\cdot)$ follows from the invariance of each ω-limit set $\omega(B,\cdot)$. Indeed, given $x_0\in\mathscr{A}(s)$, there exist $x_n\in\omega(B_n,s)$ with $x_n\to x_0$ as $n\to+\infty$. Then $S(t,s)x_n = y_n\in\omega(B_n,t)$ and, by the continuity of $S(t,s)$, $S(t,s)x_n = y_n\to S(t,s)x_0$, which implies that $S(t,s)x_0\in\mathscr{A}(t)$, and so $S(t,s)\mathscr{A}(s)\subseteq\mathscr{A}(t)$. Now, choose some $y_0\in\mathscr{A}(t)$. Then there exist $y_n\in\omega(B_n,t)$ with $y_n\to y_0$ as $n\to+\infty$. But then, again by the invariance of the family $\omega(B_n,\cdot)$, there exist $x_n\in\omega(B_n,s)$ with $S(t,s)x_n = y_n$. But since $x_n\in\omega(B_n,s)\subseteq\mathscr{A}(s)$, and $\mathscr{A}(s)$ is compact, there is a subsequence x_{n_j} that converges to some $x_0\in\mathscr{A}(s)$, for which $S(t,s)x_0 = \lim_{j\to\infty}S(t,s_{n_j})x_{n_j} = \lim_{j\to\infty}y_{n_j} = y_0$. It follows that $S(t,s)\mathscr{A}(s)\supseteq\mathscr{A}(t)$, and so $\mathscr{A}(\cdot)$ is invariant.

The minimality property follows simply from the observation that if $\hat{\mathscr{A}}(t)$ is closed and bounded and pullback attracts bounded sets at time t, then $\omega(B,t)\subseteq\hat{\mathscr{A}}(t)$ for all bounded subsets B of X, and hence $\mathscr{A}(t)\subseteq\hat{\mathscr{A}}(t)$. □

The following corollary for semigroups allows for a simpler characterisation of the global attractor than is available in the non-autonomous case.

Corollary 2.13. *Let $T(\cdot)$ be a semigroup in a metric space X. Then $T(\cdot)$ has a global attractor $\mathscr{A}$ if and only if there exists a compact set K that attracts bounded subsets of X under $T(\cdot)$, and in this case $\mathscr{A}=\omega(K)$.*

Proof. It is an immediate corollary of Theorem 2.12 that

$$\mathscr{A} = \overline{\bigcup\{\omega(B):\ B\subset X,\ B\ \text{bounded}\}}$$

is the global attractor for $T(\cdot)$. It is immediate from this that $\mathscr{A}\supseteq\omega(K)$, while, since K attracts bounded subsets of X, we must have $\omega(B)\subseteq\omega(K)$ for all bounded subsets B of X, which completes the proof. □

We note that in many applications one can prove something stronger than the existence of a compact pullback attracting set (the main hypothesis of Theorem 2.12), namely the existence of a compact pullback absorbing set.

Definition 2.14. A set $B \subset X$ *pullback absorbs bounded sets at time* $t \in \mathbb{R}$ if, for each bounded subset D of X, there exists $T = T(t,D) \leq t$ such that

$$S(t,s)D \subseteq B, \text{ for all } s \leq T.$$

A family $B(\cdot)$ pullback absorbs bounded sets if $B(t)$ pullback absorbs bounded sets at time t, for each $t \in \mathbb{R}$.

If a set pullback absorbs bounded sets at time t, then clearly it pullback attracts bounded sets at time t.

In the context of ordinary differential equations (ODEs), for which the phase space is finite-dimensional, the existence of a bounded absorbing set is equivalent to the existence of a compact absorbing set, and Theorem 2.12 is relatively straightforward to apply. In the following section we use this result to study an example of a saddle-node bifurcation in a non-autonomous scalar ODE. In Chap. 9 we apply the result to investigate the behaviour of a non-autonomous coupled Lotka–Volterra system. We use Theorem 2.12 to prove the existence of a pullback attracting set, which we are then able to reduce to a single point, thereby identifying a dynamically significant global solution of the system; in both cases the fact that the system is order preserving plays a key role. We also apply Theorem 2.12 to the two-dimensional Navier–Stokes equations in Sect. 11.3.

We continue to develop the abstract theory of existence results for pullback attractors in Sect. 2.3.

2.2.1 Example: a saddle-node bifurcation

As an example we consider, after Langa et al. (2002), a non-autonomous version of the simple ODE

$$\dot{x} = a - bx^2, \qquad b > 0,$$

which models a saddle-node bifurcation. For $a < 0$ all trajectories tend to $-\infty$; for $a = 0$ positive solutions tend to zero and negative solutions blow up to $-\infty$ in a finite time; and for $a > 0$ there is a unique attracting solution $\sqrt{a/b}$ for compact subsets in $(-\sqrt{a/b}, +\infty)$, while $-\sqrt{a/b}$ is unstable. We denote the solution operator for this autonomous equation by $T_{a,b}(t)$.

Here we consider

$$\dot{x} = a - b(t)x^2, \qquad x(s) = x_0, \tag{2.4}$$

where $a \in \mathbb{R}$ and $b : \mathbb{R} \to \mathbb{R}$ is continuous with $b(t) \geq 0$ for all $t \in \mathbb{R}$ and

$$\int_{-\infty}^{0} b(t)\,\mathrm{d}t = \int_{0}^{\infty} b(t)\,\mathrm{d}t = +\infty. \tag{2.5}$$

We will see that this non-autonomous equation behaves in a similar way to its autonomous counterpart. Since we will analyse the equation by considering a process on a closed subset of $\mathbb{R}$, even in this remarkably simple case we make use of the possibility of defining pullback attractors in metric spaces as well as linear spaces.

First, note that if $a < 0$, then $\dot{x} \leq a$, and $S(t,s)x_0 \to -\infty$ for every $x_0 \in \mathbb{R}$, both as $t \to +\infty$ ('forwards') and as $s \to -\infty$ ('pullback'); while if $a = 0$, then the equation can be solved explicitly to yield

$$S(t,s)x_s = \frac{x_s}{1 + x_s \int_s^t b(r)\,\mathrm{d}r}.$$

Using (2.5) it follows that if $x_s > 0$, then $S(t,s)x_s \to 0$ as $t \to +\infty$ or $s \to -\infty$, whereas if $x_s < 0$, then the solution 'blows up' to $-\infty$ in a finite time (either forwards or pullback).

The behaviour for $a > 0$ is more interesting. We assume that $b(t)$ is bounded above and below, i.e. that there exists b_0 and b_1 with $0 < b_0 \leq b_1$ such that

$$b_0 \leq b(t) \leq b_1.$$

The equation is then simple to analyse since the solution of (2.4) is bounded above and below by those of the autonomous equations

$$\dot{x} = a - b_0 x^2 \quad \text{and} \quad \dot{x} = a - b_1 x^2,$$

respectively [both with $x(s) = x_0$], i.e.

$$T_{a,b_1}(t-s)x_0 \leq S(t,s)x_0 \leq T_{a,b_0}(t-s)x_0.$$

Note that the interval $[-\sqrt{a/b_1}, +\infty)$ is positively invariant for the three systems (the non-autonomous equation and the 'bounding' autonomous equations).

We consider the process $S(\cdot,\cdot)$ restricted to $[-\sqrt{a/b_1}, +\infty)$, which is a complete metric space when we use the usual distance on $\mathbb{R}$. For any bounded set $B \subset [-\sqrt{a/b_1}, +\infty)$ there exists a $\tau_0(B)$ such that

$$S(t,s)B \subset \left[\frac{1}{2}\sqrt{a/b_1}, 2\sqrt{a/b_0}\right] \qquad t - s \geq \tau_0(B); \tag{2.6}$$

in particular, there is a closed bounded (and hence compact) pullback absorbing set. We can now apply Theorem 2.12 to deduce the existence of a pullback attractor $\mathcal{A}(t)$ for (2.4) in $[-\sqrt{a/b_1}, +\infty)$. Since the attractor is a connected subset of $\mathbb{R}$ (Corollary 2.5), it must in fact be an interval,

$$\mathcal{A}(t) = [a_-(t), a_+(t)] \subset I_a = [\sqrt{a/b_1}, \sqrt{a/b_0}];$$

since the phase space is one-dimensional, the process is order preserving, and so $a_{\pm}(\cdot)$ are global solutions of (2.4). We now show that in fact $a_-(t) = a_+(t)$ for all t. To do this, consider the difference $z(t) = a_+(t) - a_-(t)$, which satisfies the equation

$$\dot{z} = -b(t)(a_+^2(t) - a_-^2(t)) = -b(t)[a_+(t) + a_-(t)]z \leq -\frac{\sqrt{a/b_1}}{2} b(t)z.$$

Integrating we obtain

$$z(t) \leq z(s) \exp\left(-\frac{\sqrt{a/b_1}}{2} \int_s^t b(r)\,\mathrm{d}r \right);$$

letting $s \to -\infty$ and using (2.5) shows that $z(t) = 0$, i.e. that $a_+(t) = a_-(t)$. So the pullback attractor consists of a single (positive) global solution $a(t)$. Note that in this example the global solution $a(t)$ is also forwards attracting; using (2.6) any solution $x(\cdot)$ is eventually bounded below by $\frac{1}{2}\sqrt{a/b_1}$, and the preceding argument can be repeated [replacing $a_-(\cdot)$ and $a_+(\cdot)$ by $x(\cdot)$ and $a(\cdot)$, appropriately ordered] to show that $|x(t) - a(t)| \to 0$ as $t \to +\infty$.

The paper by Langa et al. (2002) considers the less straightforward situation in which $0 < b(t) \leq b$ and $b(t) \to 0$, while preserving the integral condition (2.5). In this case the pullback attractor is still a pullback attracting positive global solution, but this global solution is now unbounded (it tends to $+\infty$ as $|t| \to \pm\infty$), and there is also a negative global solution that, while important for the dynamics, is not contained in the pullback attractor (nor is its unstable set), cf. Lemma 1.16.

2.3 Second result: from the existence of a bounded attracting set

In the proof of Theorem 2.12 we appealed to Lemma 2.7 to guarantee attracting properties of ω-limit sets. But we have already seen that there is a parallel result to Lemma 2.7, namely Lemma 2.10, that uses the notion of asymptotic compactness rather than assuming the existence of a compact attracting family. It is no surprise, therefore, that we can replace one assumption with the other in our attractor theorem (Theorem 2.12) to give a more easily applicable result.

Before this, we give a sufficient condition for a process to be pullback asymptotically compact that can often be verified in applications.

Definition 2.15. A process $S(\cdot,\cdot)$ is said to be *pullback bounded* if for each bounded set B and every $t \in \mathbb{R}$ the 'pullback orbit' of B at time $t \in \mathbb{R}$,

$$\gamma_p(B,t) := \bigcup_{s \leq t} S(t,s)B,$$

is bounded.

Note that if $T(\cdot)$ is a semigroup, then the corresponding process $S_T(\cdot,\cdot)$ is pullback bounded if and only if $\gamma^+(B) = \bigcup_{t \geq 0} T(t)B$ is bounded for each bounded subset B of X. In this case, we say that the semigroup $T(\cdot)$ is bounded.

Definition 2.16. A process $S(\cdot,\cdot)$ is called *pullback eventually compact* if it is pullback bounded (Definition 2.15) and there exists a $\tau \geq 0$ such that, if B is a bounded subset of X and $t \in \mathbb{R}$, then $\overline{S(t,t-\tau)B}$ is compact.

Lemma 2.17. *Let $S(\cdot,\cdot)$ be a process in a metric space X. If $S(\cdot,\cdot)$ is pullback eventually compact, then $S(\cdot,\cdot)$ is pullback asymptotically compact.*

Proof. Let $\{x_j\} \in X$ be a bounded sequence, and $\{s_j\} \leq t$ such that $s_j \to -\infty$. If $B = \gamma_p(\{x_j\},t-\tau)$, then B is bounded, and therefore $S(t,t-\tau)B$ is relatively compact and contains $\{S(t,s_j)x_j\}$. It follows that $\{S(t,s_j)x_j\}$ is relatively compact. □

A semigroup $T(\cdot)$ is *eventually compact* if it is bounded and there exists a $t_0 > 0$ such that $\overline{T(t_0)B}$ is compact for each bounded subset B of X.

Corollary 2.18. *Let $T(\cdot)$ be an eventually compact semigroup in a metric space X. Then $T(\cdot)$ is asymptotically compact.*

We now add an assumption of 'dissipativity' to asymptotic compactness in order to regain attracting properties of our ω-limit sets and to recover the pullback attractor once more.

Definition 2.19. We say that a process $S(\cdot,\cdot)$ is *pullback bounded dissipative* if there exists a family $B(\cdot)$ of bounded sets such that $B(t)$ pullback attracts bounded sets at time t, for each $t \in \mathbb{R}$.

If $T(\cdot)$ is a semigroup and $S_T(\cdot,\cdot)$ the corresponding process, then $S_T(\cdot,\cdot)$ is pullback bounded dissipative if and only if $T(\cdot)$ is bounded dissipative, i.e. there exists a bounded set B that attracts all bounded subsets of X under $T(\cdot)$.

Theorem 2.20. *If $S(\cdot,\cdot)$ is pullback asymptotically compact, then for each $t \in \mathbb{R}$ the set $\mathscr{A}(t)$ given by (2.3) is closed and invariant and pullback attracts bounded subsets of X at time t. Furthermore, the family $\mathscr{A}(\cdot)$ is minimal among families $C(\cdot)$ such that for each $t \in \mathbb{R}$ the set $C(t)$ is closed and pullback attracts bounded subsets of X at time t. If in addition $S(\cdot,\cdot)$ is pullback bounded dissipative, then $\mathscr{A}(t)$ is also bounded for each $t \in \mathbb{R}$.*

Proof. Observe that the hypotheses of Lemma 2.10 are satisfied, and so $\omega(B,t)$ is non-empty, compact, and invariant and pullback attracts B at time t for any bounded subset B of X. Hence, if $\mathscr{A}(t)$ is defined by (2.3), i.e.

$$\mathscr{A}(t) = \overline{\bigcup \{\omega(B,t) : \; B \subset X, \; B \; \text{ bounded } \}},$$

$\mathscr{A}(\cdot)$ is closed and invariant and pullback attracts bounded subsets of X. If $C(t)$ is closed and pullback attracts bounded sets at time t, it is clear that $\omega(B,t) \subseteq C(t)$ for each bounded subset B of X, and consequently $\mathscr{A}(t) \subseteq C(t)$. Now, if $S(\cdot,\cdot)$ is pullback bounded dissipative, for each $t \in \mathbb{R}$ there is a bounded set $B(t)$ that pullback bounded attracts bounded subsets of X. Hence $\omega(D,t) \subset \overline{B(t)}$ for each bounded subset D of X and $\mathscr{A}(t) \subset \overline{B(t)}$, which shows that $\mathscr{A}(t)$ is bounded. □

Note that this result *does not give any compactness* of the set $\mathscr{A}(t)$. This is only a restriction in an infinite-dimensional setting, as $\mathscr{A}(t)$ is actually bounded and closed. However, in the semigroup case we do not have such a restriction, and the corresponding result appears much stronger.

Corollary 2.21. *If $T(\cdot)$ is bounded dissipative and asymptotically compact, then it has a global attractor $\mathscr{A}$.*

Proof. To show that $\mathscr{A}$ is compact, take $\{x_n\} \in \mathscr{A}$. Since $\mathscr{A}$ is invariant, $x_n = T(n)y_n$ with $y_n \in \mathscr{A}$. Since $\{y_n\}$ is bounded, it follows from the asymptotic compactness of $T(\cdot)$ that $\{T(n)y_n\} = \{x_n\}$ has a convergent subsequence. □

Obtaining the equivalent result for processes requires a stronger hypothesis that imposes some uniformity in the 'dissipativity' of $S(\cdot,\cdot)$.

Definition 2.22. We say that a process $S(\cdot,\cdot)$ is *strongly pullback bounded dissipative* if for each $t \in \mathbb{R}$ there is a bounded subset $B(t)$ of X that pullback attracts bounded subsets of X at time τ for each $\tau \leq t$; that is, given a bounded subset D of X and $\tau \leq t$, $\lim_{s\to-\infty} \operatorname{dist}(S(\tau,s)D, B(t)) = 0$.

Note that the family $B(\cdot)$ given in this definition does not need to have a bounded union. Nevertheless, we can choose it in such a way that, for each $t \in \mathbb{R}$, $\bigcup_{s\leq t} B(s)$ is bounded.

The following theorem gives a sufficient condition for the existence of a compact pullback attractor $\mathscr{A}(\cdot)$ that is bounded in the past, i.e.

$$\bigcup_{s\leq t} \mathscr{A}(s)$$

is bounded for each $t \in \mathbb{R}$. Note that it is desirable that the pullback attractor belong to this class of sets since it would then be contained in the class of sets that it itself attracts (see Sect. 2.6 for more on such self-consistent basins of attraction).

Theorem 2.23. *If a process $S(\cdot,\cdot)$ is strongly pullback bounded dissipative and pullback asymptotically compact and $B(\cdot)$ is a family of bounded subsets of X such that, for each $t \in \mathbb{R}$, $B(t)$ pullback attracts bounded subsets of X at time τ for each*

$\tau \le t$, then $S(\cdot,\cdot)$ has a compact pullback attractor $\mathscr{A}(\cdot)$ such that $\mathscr{A}(t) = \omega(\overline{B}(t),t)$ and $\bigcup_{s\le t} \mathscr{A}(s)$ is bounded for each $t \in \mathbb{R}$.

Proof. We only need to check that $\mathscr{A}(t)$ as defined by (2.3) is compact. For each fixed $\tau \le t$, since $B(t)$ pullback attracts all bounded sets at time τ, it follows that $\omega(D,\tau) \subseteq \overline{B}(t)$ for every bounded $D \subset X$. Since $\omega(D,\cdot)$ is invariant,

$$\omega(D,t) = S(t,\tau)\omega(D,\tau) \subseteq S(t,\tau)\overline{B}(t) \qquad \text{for all} \quad \tau \le t.$$

It follows that

$$\omega(D,t) \subseteq \bigcap_{\sigma\le t} \overline{\bigcup_{\tau\le\sigma} S(t,\tau)B(t)} = \omega(\overline{B}(t),t).$$

Since this holds for every bounded set D, it follows that $\mathscr{A}(t) \subset \omega(\overline{B}(t),t)$, and consequently $\mathscr{A}(t)$ is compact. Since clearly $\omega(\overline{B}(t),t) \subset \mathscr{A}(t)$, it in fact follows that $\mathscr{A}(t) = \omega(\overline{B}(t),t)$.

We have already shown that $\mathscr{A}(\tau) \subseteq \overline{B}(t)$ for all $\tau \le t$, so $\mathscr{A}(\cdot)$ is bounded in the past, as claimed. $\square$

Note that if a process has a pullback attractor that is bounded in the past, then it must be strongly pullback bounded dissipative [setting $B(t) = \cup_{s\le t}\mathscr{A}(s)$]; and we have already seen (Lemma 2.9) that any process with a pullback attractor must be pullback asymptotically compact. So the conditions in Theorem 2.23 are in fact necessary and sufficient for the existence of a pullback attractor that is bounded in the past.

This method will be applied in Chap. 15 to the dissipative damped wave equation

$$u_{tt} + \beta(t)u_t = \Delta u + f(u).$$

2.4 Third result: from the pullback flattening property

We now turn to another approach, due to Ma et al. (2002) for autonomous systems and extended by Y. Wang et al. (2006) to the non-autonomous case, which makes a different kind of compactness assumption that is easier to check in some applications. Referred to as 'Condition (C)' by Ma et al., the term 'flattening' was coined by Kloeden and Langa (2007) in a paper that extended the autonomous theory to treat random dynamical systems. Note that the results here are restricted to Banach spaces (and are at their most natural in Hilbert spaces).

Definition 2.24. A process $S(\cdot,\cdot)$ on a Banach space X is said to be *pullback flattening* if, given $t \in \mathbb{R}$, for every bounded set B in X and $\varepsilon > 0$ there exists a $T_0 = T_0(B,\varepsilon,t) \in \mathbb{R}$ and a finite-dimensional subspace X_ε of X along with a mapping $P_\varepsilon : X \to X_\varepsilon$ such that

$$\bigcup_{s\le T_0} P_\varepsilon\, S(t,s)B \ \text{ is bounded}$$

and

$$\left\| (I-P_\varepsilon)\left(\bigcup_{s\le T_0} S(t,s)B\right)\right\|_X < \varepsilon, \tag{2.7}$$

where (2.7) is understood in the sense that $\|(I-P_\varepsilon)S(t,s)x_0\|_X < \varepsilon$ for all $x_0 \in B$ and $s \le T_0$.

A semigroup $T(\cdot)$ is *flattening* if the corresponding process $S_T(\cdot,\cdot)$ is pullback flattening (of course, the autonomous definition was in fact introduced first; see Ma et al. 2002).

We now show that the pullback flattening property implies that $S(\cdot,\cdot)$ is pullback asymptotically compact, which is an essential ingredient for the existence of a pullback attractor (Theorems 2.20 and 2.23). We will also show that if X is a uniformly convex Banach space, then the converse to this implication also holds. In particular, this shows that in any uniformly convex Banach space, any process with a pullback attractor must be pullback flattening.

Recall that a Banach space X is uniformly convex if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that, given $x, y \in X$,

$$\|x\|_X, \|y\|_X \le 1, \qquad \|x-y\| > \varepsilon \qquad \Rightarrow \qquad \frac{\|x+y\|}{2} < 1-\delta;$$

for instance, Hilbert spaces, L^p spaces for $p \in (1,\infty)$, and Sobolev spaces $W^{s,p}$ for $p \in (1,\infty)$ are uniformly convex; see, for example, Brézis (1983), Sect. III.7.

The key property of such spaces that we will use is that if U is a finite-dimensional subspace, then for every $x \in X$ there exists a unique closest point in U, which can be used to define a mapping $P : X \to U$ (of course, in a Hilbert space this is simply the orthogonal projection onto U, which is linear). Given $x \in X$, that such a point exists is clear; its uniqueness follows since if $\hat{a} \neq a'$ and

$$\|x-\hat{a}\| = \|x-a'\| = d := \inf_{u\in U} \|x-u\|,$$

then $d^{-1}\|\hat{a}-a'\| = \|d^{-1}(x-\hat{a}) - d^{-1}(x-a')\| > 0$, whence

$$\frac{d^{-1}\|(x-\hat{a})+(x-a')\|}{2} = d^{-1}\left\|x - \frac{\hat{a}+a'}{2}\right\| < 1,$$

i.e.

$$\left\|x - \frac{\hat{a}+a'}{2}\right\| < d,$$

which contradicts the definition of d.

Note that for this canonical projector (which is not necessarily linear in general), since $x = Px + (x-Px)$ and $\|x-Px\| \le \|x\|$ (as $0 \in U$), it follows that

$$\|Px\| \le 2\|x\| \tag{2.8}$$

(if X is a Hilbert space, then P is linear and $\|Px\| \le \|x\|$).

Theorem 2.25. *Let $S(\cdot,\cdot)$ be a process on a Banach space X. If $S(\cdot,\cdot)$ is pullback flattening, then it is pullback asymptotically compact, and for every bounded $B \subset X$,*

$$\bigcup_{s \le T_0} S(t,s)B$$

is bounded for some $T_0 = T_0(B,t)$. Conversely, if X is uniformly convex and $S(\cdot,\cdot)$ is pullback asymptotically compact, then it is pullback flattening.

Proof. Let B be a bounded subset of X, and fix $t \in \mathbb{R}$. The boundedness property follows trivially from the definition of the pullback flattening property.

To show that $S(\cdot,\cdot)$ is pullback asymptotically compact, let $\{s_n\} \le t$ be a sequence with $s_n \to -\infty$, and $\{x_n\} \in B$. Since $S(\cdot,\cdot)$ is pullback flattening, for each $k \in \mathbb{N}$ there exists a finite-dimensional subspace X_k of X, a bounded projector $P_k : X \to X_k$, and a T_k such that

$$\bigcup_{s \le T_k} P_k\, S(t,s)B \text{ is bounded}$$

and

$$\left\| (I - P_k) \left(\bigcup_{s \le T_k} S(t,s)B \right) \right\|_X < 1/k.$$

It follows that there exists an n_k such that for all $n \ge n_k$,

$$\{P_k\, S(t,s_n)x_n\} \text{ is a bounded subset of } X_k \qquad \text{and} \qquad \|(I-P_k)S(t,s_n)x_n\|_X < 1/k.$$

Since the set $\{S(t,s_n)x_n\}$ is bounded and for any $k \in \mathbb{N}$ can be covered by a finite number of balls of radius $\frac{2}{k}$, its closure is compact and the required pullback asymptotic compactness follows.

Now we suppose that X is a uniformly convex Banach space and show that asymptotic compactness implies the pullback flattening property.

Let B be a bounded set in X. From Lemma 2.10, $\omega(B,t)$ is non-empty, compact, and invariant and pullback attracts B at time t. Since $\omega(B,t)$ is compact, there exists an $\ell_\varepsilon \in \mathbb{N}$, and $x_1, \ldots, x_{\ell_\varepsilon}$ in $\omega(B,t)$ such that

$$\omega(B,t) \subset \bigcup_{i=1}^{\ell_\varepsilon} B_X\left(x_i, \frac{\varepsilon}{4}\right),$$

where $B_X(x,r)$ is the ball in X centred at x and of radius r. From the fact that $\omega(B,t)$ pullback attracts B, given $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that

$$\operatorname{dist}(\overline{\bigcup_{s \le t-\tau} S(t,s)B}, \omega(B,t)) < \frac{\varepsilon}{4} \qquad \text{for all} \quad \tau \ge n_\varepsilon,$$

from which

$$\overline{\bigcup_{s\le t-n_\varepsilon} S(t,s)B} \subset \bigcup_{i=1}^{\ell_\varepsilon} B_X\left(x_i,\frac{\varepsilon}{2}\right). \tag{2.9}$$

Now let $X_\varepsilon := \operatorname{span}\{x_1,x_2,\dots,x_{\ell_\varepsilon}\}$, and let $P_\varepsilon : X \to X_\varepsilon$ be the map onto the closest point in X_ε. It follows from (2.9) and the fact that $\|P_\varepsilon x\| \le 2\|x\|$ [see (2.8)] that

$$P_\varepsilon\left(\bigcup_{s\le t-\tau_\varepsilon} S(t,s)B\right) \text{ is a bounded subset of } X,$$

whereas, since $x_i \in X_\varepsilon$ and $\|x - P_\varepsilon x\| = \operatorname{dist}(x,X_\varepsilon)$ by definition,

$$\left\|(I-P_\varepsilon)\left(\bigcup_{s\le t-\tau_\varepsilon} S(t,s)B\right)\right\| \le \frac{\varepsilon}{2} < \varepsilon,$$

i.e. $S(\cdot,\cdot)$ is pullback flattening. □

Corollary 2.26. *Assume that $T(\cdot)$ is a semigroup in a uniformly convex Banach space X. Then $T(\cdot)$ is flattening if and only if $T(\cdot)$ is bounded and asymptotically compact.*

Now we state the main result of this section whose proof follows immediately from Theorems 2.23 and 2.25.

Theorem 2.27. *Suppose that a process $S(\cdot,\cdot)$ is pullback flattening and strongly pullback bounded dissipative. Then it has a pullback attractor $\mathscr{A}(\cdot)$. Moreover, if X is a uniformly convex Banach space and $S(\cdot,\cdot)$ has a pullback attractor $\mathscr{A}(\cdot)$, then $S(\cdot,\cdot)$ is pullback flattening.*

Corollary 2.28. *Suppose that a semigroup $T(\cdot)$ is flattening and bounded dissipative. Then it has a global attractor $\mathscr{A}$. Moreover, if X is a uniformly convex Banach space and $T(\cdot)$ has a global attractor $\mathscr{A}$, then $T(\cdot)$ is flattening.*

We use this approach in Sect. 11.4 to show that the 2D Navier–Stokes equations have a pullback attractor in the phase space H^1.

2.5 Pullback point dissipativity

In the previous results we proved the existence of compact sets that (pullback) attract all bounded subsets of X by combining assumptions of bounded dissipativity with asymptotic compactness assumptions on the process $S(\cdot,\cdot)$. The aim of this section is to parallel the results from the autonomous theory that only make the assumptions of point dissipativity and asymptotic compactness but nevertheless deduce results about the attraction of bounded subsets (Hale 1988; Raugel 2002).

The development of a non-autonomous version of these results rounds off our generalisations of the autonomous theory. It is perhaps the case, however, that 'point dissipativity plus asymptotic compactness' is a more useful criterion in autonomous than in non-autonomous systems; we consider one particularly important autonomous application, gradient semigroups, in Sect. 2.5.1.

We begin by defining a new notion of dissipativity; while essentially we ask that 'pullback orbits are bounded', we also require some uniformity, restoring the autonomous feature that behaviour depends on the time elapsed rather than on both the initial and final times.

Definition 2.29. Let $S(\cdot,\cdot)$ be a process in a metric space X. We say that a bounded set B *uniformly strongly pullback attracts* points of X at time t if

$$\lim_{\tau\to\infty}\left[\sup_{s\le t}\operatorname{dist}(S(s,s-\tau)x,B(t))\right]=0.$$

If there exists a bounded family $B(\cdot)$ such that $B(t)$ uniformly strongly pullback attracts points of X at time t for each $t\in\mathbb{R}$, then we say that $S(\cdot,\cdot)$ is *uniformly strongly pullback point dissipative*.

We make the corresponding definition of *uniformly strongly pullback compact dissipative*, replacing x by a compact set K throughout Definition 2.29 and of *uniformly strongly pullback bounded dissipative*, replacing x by a bounded set B.

Note that if $T(\cdot)$ is a semigroup and $S_T(\cdot,\cdot)$ is the corresponding process, then $S_T(\cdot,\cdot)$ is uniformly strongly pullback point dissipative if and only if $T(\cdot)$ is point dissipative in the sense of Hale (1988).

Definition 2.30. We say that a process $S(\cdot,\cdot)$ in X is *strongly pullback bounded* if, for each $t\in\mathbb{R}$ and bounded subset B of X, $\bigcup_{\tau\le t}\gamma_p(B,\tau)$ is bounded, i.e.

$$\bigcup_{\tau\le t}\bigcup_{s\le\tau}S(\tau,s)B$$

is bounded.

If $T(\cdot)$ is a semigroup and $S_T(\cdot,\cdot)$ the corresponding process, then $S_T(\cdot,\cdot)$ is strongly pullback bounded if and only if $S_T(\cdot,\cdot)$ is pullback bounded if and only if $T(\cdot)$ is a bounded semigroup.

As already remarked, we want to find conditions that ensure that a (uniformly strongly pullback) point dissipative process is in fact strongly pullback bounded dissipative. The first step is to find conditions under which we can strengthen point dissipativity to compact dissipativity. The key property here turns out to be (essentially) equicontinuity of the family of operators $\{S(t,s):\ t-s=\tau\}$, i.e. those elements of the process that correspond to the same elapsed time.

Lemma 2.31. *Let $S(\cdot,\cdot)$ be uniformly strongly pullback point dissipative, pullback asymptotically compact, and strongly pullback bounded. Assume also that for each*

$t \in \mathbb{R}$ and $s > 0$, the family $\{S_{\tau,\tau-s} : \tau \leq t\}$ is equicontinuous at each $x \in X$. Then $S(\cdot,\cdot)$ is uniformly strongly pullback compact dissipative.

Proof. Fix $t \in \mathbb{R}$ and let $B(t)$ be a bounded subset of X that strongly pullback attracts points of X at time t: in particular, given any $x \in X$, there exists an $n_x \in \mathbb{N}$ such that for all $s \geq n_x$

$$\mathrm{dist}(S(r,r-s)x, B(t)) < 1/2 \qquad \text{for all} \quad r \leq t.$$

Now one can use the equicontinuity hypothesis to guarantee the existence of an $\varepsilon_x > 0$ such that for all $s \geq n_x$,

$$\mathrm{dist}(S(r,r-s)B_{\varepsilon_x}(x), B(t)) < 1 \qquad \text{for all} \quad r \leq t.$$

Given any compact set $K \subset X$, there is a $p \in \mathbb{N}$ and $\{x_1,\ldots,x_p\} \in K$ such that $K \subset \cup_{i=1}^{p} B_{\varepsilon_{x_i}}(x_i)$. It follows that with $n_K = \max_i n_{x_i}$, for all $s \geq n_K$,

$$S(r,r-s)K \subset \bigcup_{\tau \leq t} \gamma_p(B(t),\tau) = C(t) \qquad \text{for all} \quad r \leq t,$$

where $B_1(t) := \{x \in X : d(x,y) \leq 1 \text{ for some } y \in B(t)\}$ is clearly a bounded subset of X. Since $S(\cdot,\cdot)$ is strongly pullback bounded, $C(t)$ is bounded and $C(t) \supset C(s)$ for all $s \leq t$. $\square$

To proceed from compact dissipativity to bounded dissipativity, we need to strengthen our notion of asymptotic compactness.

Definition 2.32. We say that a process $S(\cdot,\cdot)$ is *strongly pullback asymptotically compact* if, for each $t \in \mathbb{R}$, each bounded sequence $\{x_k\} \in X$, and any sequences $\{s_k\}, \{\tau_k\}$ with $s_k \leq \tau_k \leq t$ and $\tau_k - s_k \to \infty$ as $k \to \infty$, the sequence $\{S(\tau_k, s_k)x_k\}$ is relatively compact.

If $T(\cdot)$ is a semigroup, then the corresponding process $S_T(\cdot,\cdot)$ is strongly pullback asymptotically compact if and only if $S_T(\cdot,\cdot)$ is pullback asymptotically compact if and only if $T(\cdot)$ is asymptotically compact.

We are now in a position to boost compact dissipativity to bounded dissipativity, using this strong definition of asymptotic compactness.

Theorem 2.33. *If a process $S(\cdot,\cdot)$ is uniformly strongly pullback compact dissipative and strongly pullback asymptotically compact, then $S(\cdot,\cdot)$ is strongly pullback bounded dissipative.*

Proof. Since $S(\cdot,\cdot)$ is strongly pullback compact dissipative, there exists a closed bounded set $B(t)$ that strongly pullback attracts compact subsets of X at time t.

First we prove that, for each bounded subset D of X, $\omega(D,\tau) \subseteq B(t)$ for each $\tau \leq t$. Indeed, if $y \in \omega(D,\tau)$, then there is a sequence $\{s_k\} \leq \tau$ with $s_k \to -\infty$ as $k \to \infty$ and a sequence $\{x_k\} \subset D$ such that $S(\tau,s_k)x_k \to y$ as $k \to \infty$. Taking a sequence $\{r_k\}$ with $\tau \geq r_k \geq s_k$ and $\min\{\tau - r_k, r_k - s_k\} \to \infty$ as $k \to \infty$ and using the fact that $S(\cdot,\cdot)$

is strongly pullback asymptotically compact, one can find a subsequence (which we relabel) and a $z \in X$ such that $z_k := S(r_k, s_k)x_k \to z$ as $k \to \infty$.

Choose $\varepsilon > 0$. From the compactness of the set $K = \{z_k : k \in \mathbb{N}\} \cup \{z\}$, there is an $n_K \in \mathbb{N}$ such that $\mathrm{dist}(S(\tau, r_k)K, B(t)) < \varepsilon$ whenever $\tau - r_k \geq n_K$. Thus, for all suitably large k,

$$S(\tau, s_k)x_k = S(\tau, r_k)[S(r_k, s_k)x_k] \subset S(\tau, r_k)K,$$

and so

$$\mathrm{dist}(S(\tau, s_k)x_k, B(t)) < \varepsilon$$

for all k sufficiently large. This implies that $\omega(D, \tau) \subseteq B(t)$ for each $\tau \leq t$.

Since $\omega(D, \tau)$ pullback attracts D at time τ, it follows that $B(t)$ pullback attracts bounded subsets of X at time τ for each $\tau \leq t$; i.e. $S(\cdot, \cdot)$ is strongly pullback bounded dissipative. □

We can now deduce the main result of this section.

Theorem 2.34. *Let $S(\cdot, \cdot)$ be a process with the property that, for each $t \in \mathbb{R}$ and $\tau > 0$, $\{S(s, s - \tau) : s \leq t\}$ is equicontinuous at x for each $x \in X$. If $S(\cdot, \cdot)$ is uniformly strongly pullback point dissipative, strongly pullback bounded, and strongly pullback asymptotically compact, then $S(\cdot, \cdot)$ is strongly pullback bounded dissipative. Consequently, $S(\cdot, \cdot)$ has a pullback attractor $\mathscr{A}(\cdot)$ that is bounded in the past.*

Corollary 2.35. *If $T(\cdot)$ is a bounded semigroup that is point dissipative and asymptotically compact, then it has a global attractor.*

A sufficient condition for a semigroup to be asymptotically compact, involving a splitting into two components, one compact and one that vanishes asymptotically, is given in Lemma 3.2.6 of Hale (1988) and Theorem 1.1 of Temam (1988). We now prove an analogous result for processes, giving a sufficient condition for a process to be strongly pullback asymptotically compact (Definition 2.32). From here to the end of this section we assume that X is a Banach space with norm $\|\cdot\|$.

We start with a preliminary definition.

Definition 2.36. A family of continuous maps $\{U(t, s) : t \geq s\}$ (which need not be a process) is called *strongly compact* if for each time t and each bounded $B \subset X$ there exists a $T_B \geq 0$ and a compact set $K \subset X$ such that $U(\tau, s)B \subset K$ for all $s \leq \tau \leq t$ with $\tau - s \geq T_B$.

Theorem 2.37. *Let $S(\cdot, \cdot)$ be a strongly pullback bounded process such that*

$$S(t, s) = T(t, s) + U(t, s),$$

where $U(\cdot,\cdot)$ is strongly compact and there exists a function $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ with $k(\cdot,r)$ non-increasing for each $r > 0$ and $k(\sigma,r) \to 0$ as $\sigma \to \infty$, such that for all $s \le t$ and $x \in X$ with $\|x\| \le r$,

$$\|T(t,s)x\| \le k(t-s,r).$$

Then the process $S(\cdot,\cdot)$ is strongly pullback asymptotically compact.

Proof. Given a bounded sequence $\{x_n\} \subset B(0,r)$, and sequences $\{s_k\}$ and $\{\tau_k\}$ with $s_k \le \tau_k \le t$ and $\tau_k - s_k \to \infty$ as $k \to \infty$; since $U(\cdot,\cdot)$ is strongly compact there exists a compact set $K = K_{B(0,r)}$ such that $U(\tau_k, s_k)x_k \in K$ for all k sufficiently large. By hypothesis $\|T(\tau_k, s_k)x_k\| \le k(\tau_k - s_k, r)$, which tends to zero as $k \to \infty$. The strong pullback asymptotic compactness of $S(\cdot,\cdot)$ now follows from Lemma 2.3. □

2.5.1 An abstract application: gradient semigroups

As we have already remarked, the existence of a global attractor for a semigroup implies that the system is bounded dissipative and asymptotically compact. Thus, the existence result that requires point dissipativity, asymptotic compactness, and boundedness of the semigroup provides alternative, rather than weaker, hypotheses (cf. Raugel 2002). However, it may be that these are easier to check in certain applications. Indeed, this method is particularly well suited to autonomous gradient systems, for which point dissipativity is almost automatic. We give an abstract treatment of such systems here, deriving some results about the structure of the attractor to which we will return later.

We say that $x^* \in X$ is an equilibrium point for the semigroup $T(\cdot)$ if it is a fixed point for the map $T(t)$ for each $t \ge 0$; that is, $T(t)x^* = x^*$ for each $t \ge 0$. We denote by $\mathscr{E}$ the set of equilibrium points for $T(\cdot)$.

Definition 2.38. A semigroup $T(\cdot)$ is said to be *gradient* if there is a continuous function $V : X \to \mathbb{R}$, a *Lyapunov function*, with the following properties:

(i) $t \mapsto V(T(t)x)$ is non-increasing for each $x \in X$; and
(ii) If x is such that $V(T(t)x) = V(x)$ for all $t \ge 0$, then $x \in \mathscr{E}$.

The simplest example of a gradient system is the ODE

$$\dot{x} = -\nabla V(x), \qquad x \in \mathbb{R}^n, \tag{2.10}$$

where $V : \mathbb{R}^n \to \mathbb{R}$ is C^1. Then if $x(t)$ is a solution of (2.10), it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t} V(x(t)) = -|\nabla V(x(t))|^2; \tag{2.11}$$

it is clear that $V(x(t))$ is non-increasing and that if the left-hand side of (2.11) is zero, then so is $\nabla V(x(t))$, and hence [from (2.10)] $x(t)$ must be constant.

A less trivial example is the scalar reaction–diffusion equation on a bounded domain $\Omega \subset \mathbb{R}^n$,

$$u_t - \Delta u = f(u), \qquad u|_{\partial\Omega} = 0. \tag{2.12}$$

If one imposes appropriate (dimension-dependent) growth conditions on f, then this equation generates a semigroup on $H_0^1(\Omega)$ and the functional

$$V(u) = \frac{1}{2}\|\nabla u\|_{L^2}^2 - \int_\Omega F(u(x))\,\mathrm{d}x, \qquad \text{with} \qquad F(s) = \int_0^s f(r)\,\mathrm{d}r,$$

is continuous from H_0^1 into $\mathbb{R}$ (see Chap. 12) . To see that $V(u(t))$ is non-increasing along trajectories, multiply (2.12) by u_t and integrate over Ω to obtain

$$\|u_t\|_{L^2}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 = \int f(u)u_t = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega F(u(x,t))\,\mathrm{d}x,$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}V(u(t)) = -\|u_t\|_{L^2}^2.$$

It is also clear from this equality that if $V(u(t))$ is constant, then $u_t = 0$, and so $u(t)$ is also constant.

In a gradient system, the ω-limit set of every (point) initial condition must be a subset of the set of equilibria. Under certain assumptions the same is true of the 'backwards' α-limit sets, which we define as follows.

Definition 2.39. Given $x \in X$, suppose that there exists a backwards-bounded solution $\phi : \mathbb{R} \to X$ such that $\phi(0) = x$. Then the *α-limit set of x along ϕ*, $\alpha_\phi(x)$, is given by

$$\alpha_\phi(x) = \{y \in X : \text{there is a sequence } \{t_n\},\ t_n \to -\infty \text{ as } n \to \infty, \text{ with } \lim_{n\to\infty} \phi(t_n) = y\}.$$

We can now prove that global solutions of gradient semigroups are asymptotic both forwards and backwards to the set of equilibria, in the following sense.

Lemma 2.40. *If $T(\cdot)$ is a gradient semigroup, then $\omega(x)$ is a subset of $\mathscr{E}$ for each $x \in X$. If $x \in X$ and there is a backwards-bounded solution $\phi : \mathbb{R} \to X$ with $\phi(0) = x$, then $\alpha_\phi(x)$ is a subset of $\mathscr{E}$.*

Proof. If $\omega(x) = \varnothing$, then the result is trivial. If $\omega(x) \neq \varnothing$ and $y \in \omega(x)$, then there is a sequence $\{t_n\} \geq 0$ with $t_n \to \infty$ as $n \to \infty$ such that $T(t_n)x \to y$ as $n \to \infty$. From the continuity of V,

$$V(T(t+t_n)x) = V(T(t)[T(t_n)x]) \to V(T(t)y) \qquad \text{as} \quad n \to \infty$$

for each $t \geq 0$. Since V is non-increasing along solutions,

$$V(T(t_n)x) \geq V(T(t+t_n)x) \geq V(y)$$

for each $t \geq 0$ and $n \in \mathbb{N}$. Now one can take the limit as $n \to \infty$; since $T(t_n)x \to y$ and V is continuous,

$$V(y) \geq \lim_{n\to\infty} V(T(t+t_n)x) = V(T(t)y) \geq V(y),$$

i.e. $V(T(t)y) = V(y)$ for each $t \geq 0$. Property (ii) in Definition 2.38 now ensures that $y \in \mathscr{E}$.

Since the case $\alpha_\phi(x) = \varnothing$ is trivial, we assume that $\alpha_\phi(x)$ is non-empty and show first that V is constant on $\alpha_\phi(x)$. Since V is continuous, $V(\phi(t))$ is non-increasing, and for some sequence $\{t_n\} \to -\infty$ the sequence $\{V(\phi(t_n)\}$ is convergent, it follows that $V_\phi := \lim_{t\to-\infty} V(\phi(t))$ exists.

It is also true that $\alpha_\phi(x)$ is positively invariant since, if $y \in \alpha_\phi(x)$ is given by $y = \lim_{n\to\infty} \phi(t_n)$, then $T(t)y = \lim_{n\to\infty} T(t)\phi(t_n)$ and

$$\lim_{n\to\infty} T(t)\phi(t_n) = \lim_{n\to\infty} \phi(t+t_n)$$

and is therefore another element of $\alpha_\phi(x)$.

It follows that if $y \in \alpha_\phi(x)$, then $V(T(t)y) = V_\phi$ for all $t \geq 0$, and hence (using property (ii) of Definition 2.38 again) $y \in \mathscr{E}$. ☐

It is tempting to interpret this lemma as guaranteeing that all solutions converge towards $\mathscr{E}$, but this presupposes that $\omega(x)$ attracts x. As we saw in Sect. 2.1, we need some further conditions on $T(\cdot)$ to guarantee this. Since these also ensure that $T(\cdot)$ has a global attractor, we combine these results in the following theorem.

Theorem 2.41. *Assume that $T(\cdot)$ is a gradient semigroup that is bounded and asymptotically compact and has a bounded set of equilibria $\mathscr{E}$. Then $\omega(x)$ attracts x for every $x \in X$, and consequently $T(\cdot)$ has a global attractor $\mathscr{A}$. Furthermore, if $\mathscr{E}$ consists of isolated points, then for each $x \in X$ there exists an $e \in \mathscr{E}$ such that*

$$\lim_{t\to\infty} T(t)x = e. \tag{2.13}$$

Recall that $T(\cdot)$ is bounded if for any bounded set B, $\gamma^+(B)$ (the forward orbit of B) is bounded (see the remark following Definition 2.15).

Proof. For each $x \in X$, $\gamma^+(x)$ is bounded by assumption. Since $T(\cdot)$ is asymptotically compact, it follows from Corollary 2.11 that $\omega(x)$ attracts x. Since $\omega(x) \subset \mathscr{E}$ for every $x \in X$ and $\mathscr{E}$ is bounded, $T(\cdot)$ is point dissipative. The semigroup $T(\cdot)$ therefore satisfies the assumptions of Corollary 2.35, and so has a global attractor.

Now, since $\omega(x)$ attracts x, we can appeal to Corollary 2.6 to guarantee that $\omega(x)$ is connected. Since $\omega(x)$ is a subset of $\mathscr{E}$, if the points of $\mathscr{E}$ are isolated, then $\omega(x)$ must be a singleton, from which (2.13) follows. ☐

We can completely describe the structure of an attractor in a gradient system: it is the unstable set of the set of equilibria. We have already defined the unstable set in the case of a non-autonomous system (Definition 1.15), but we recall the definition here and specialise to the autonomous case. If E is an invariant set, then the unstable set of E, $W^{\mathrm{u}}(E)$, is given by

$$\begin{aligned} W^{\mathrm{u}}(E) := \{y \in X : \text{ there is a global solution } \phi : \mathbb{R} \to X \\ \text{such that} \quad \phi(0) = y \quad \text{and} \quad \lim_{t\to-\infty} \operatorname{dist}(\phi(t), E) = 0\}. \end{aligned} \tag{2.14}$$

Note that if E is a single point e (an equilibrium), then the convergence condition in (2.14) is simply $\lim_{t\to-\infty} \phi(t) = e$.

This class of gradient systems is essentially the only class of autonomous systems for which we have a detailed knowledge of the makeup of the attractor (although we will manage to extend this in Chap. 5). We will need the following simple topological lemma.

Lemma 2.42. *If X_t is compact and connected and $X_t \subseteq X_s$ for $t \geq s$, then $\mathbb{X} = \cap_{t\geq 0} X_t$ is connected.*

Proof. First we show that $\operatorname{dist}(X_t, \mathbb{X}) \to 0$ as $t \to \infty$. If not, then there exists an $\varepsilon > 0$, $t_n \to \infty$, and $x_n \in X_{t_n}$ such that

$$\operatorname{dist}(x_n, \mathbb{X}) > \varepsilon.$$

Since X_{t_1} is compact and $x_n \in X_{t_1}$ for every n, it follows that $x_n \to x_0$, where $\operatorname{dist}(x_0, \mathbb{X}) > \varepsilon$. But since $x_n \to x_0$, $x_0 \in \mathbb{X}$, a contradiction.

Now, if $\mathbb{X}$ is not connected, then there exist open sets O_1, O_2 such that $O_1 \cap O_2 = \varnothing$, $\mathbb{X} \cap O_i \neq \varnothing$, and $\mathbb{X} \subset O_1 \cup O_2$. In particular, there exists a $\delta > 0$ such that $d(x_1, x_2) \geq \delta$ if $x_1 \in O_1$, $x_2 \in O_2$. If t_0 is sufficiently large so that

$$\operatorname{dist}(X_t, \mathbb{X}) < \delta/3$$

then it follows that X_t is disconnected, or that $X_t \cap O_i = \varnothing$ for $i = 1$ or $i = 2$. Both of these contradict the original assumptions, so $\mathbb{X}$ must be connected. □

Theorem 2.43. *If $T(\cdot)$ is a gradient semigroup with a global attractor $\mathscr{A}$ and a set of equilibria $\mathscr{E}$, then $\mathscr{A} = W^{\mathrm{u}}(\mathscr{E})$. In particular, if $\mathscr{E} = \{e_1^*, \ldots, e_n^*\}$ is finite, then*

$$\mathscr{A} = \bigcup_{i=1}^{n} W^{\mathrm{u}}(e_i^*). \tag{2.15}$$

Proof. If $x \in \mathscr{A}$, then there is a global solution $\phi : \mathbb{R} \to X$ through x. Since $\phi(\mathbb{R}) \subset \mathscr{A}$ and $\mathscr{A}$ is compact, $\alpha_\phi(x) \subset \mathscr{E}$ is not empty and $\lim_{t\to-\infty} \operatorname{dist}(\phi(t), \alpha_\phi) = 0$. It follows that $\mathscr{A} \subseteq W^{\mathrm{u}}(\mathscr{E})$. To prove equality, note that if $x \in W^{\mathrm{u}}(\mathscr{E})$, then there is a global solution $\phi : \mathbb{R} \to X$ through x and $\mathrm{dist}(\phi(t), \mathscr{E}) \to 0$ as $t \to \pm\infty$. It follows that $\phi(\mathbb{R})$ is bounded, and since it is also invariant, it follows that $\phi(\mathbb{R}) \subset \mathscr{A}$ and $x = \phi(0) \in \mathscr{A}$.

To show that (2.15) holds when $\mathscr{E}$ is finite, it suffices to show that $\alpha_\phi(x)$ is connected, since then $\alpha_\phi(x)$ is a singleton, and it follows that through every $x \in \mathscr{A}$ there is a global solution $\phi : \mathbb{R} \to X$ with $\phi(t) \to e$ as $t \to -\infty$ for some $e \in \mathscr{E}$.

To show that $\alpha_\phi(x)$ is connected, observe that

$$\alpha_\phi(x) = \bigcap_{t\le 0} \overline{\bigcup_{s\le t} \phi(s)}.$$

So $\alpha_\phi(x)$ is the intersection of a nested sequence of compact connected sets and, hence (by Lemma 2.42), connected. □

2.5.2 Example: the Chafee–Infante equation

A canonical autonomous example in which we have a very precise idea of the structure of the global attractor is the Chafee–Infante equation (Chafee and Infante 1974) on a one-dimensional domain $[0, \pi]$,

$$u_t - u_{xx} = \lambda u - u^3, \qquad u(0,t) = u(\pi,t) = 0. \tag{2.16}$$

Since this equation is a particular case of (2.12), it is a gradient system, so we can understand much of the structure of its attractor if we understand its set of equilibria.

Any equilibrium u of (2.16) must be a solution of the elliptic equation

$$-u_{xx} = \lambda u - u^3, \qquad u(0,t) = u(\pi,t), \tag{2.17}$$

and we are able to investigate solutions of this equation by considering solutions of the two-dimensional ODE

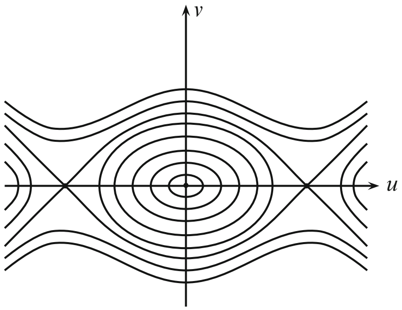
$$\begin{aligned} u_x &= v \\ v_x &= -\lambda u + u^3. \end{aligned} \tag{2.18}$$

We can analyse this using phase plane ideas, treating x as the time variable. Note first that

$$E(u,v) := \frac{v^2}{2} + \lambda u^2 - \frac{u^4}{4}$$

is constant along any solution. The phase portrait for (2.18) is shown in Fig. 2.1: trajectories are closed orbits while $0 \le E(u,v) < \lambda^2/2$.

Fig. 2.1 Phase portrait for (2.18)



To find a solution of (2.18) that satisfies the correct boundary conditions for (2.17), we need to find trajectories that start with $u = 0$ (i.e. on the v-axis) at $x = 0$ and return to the v-axis when $x = \pi$. For a given value of E the velocity in the u coordinate is

$$u_x = v = \sqrt{2E - \lambda u^2 + \frac{u^4}{2}};$$

a solution with this value of E starts with $u = 0$, $v = \sqrt{2E}$, and moves around clockwise until it strikes the u axis at $u = u_0$, where

$$E = \lambda \frac{u_0^2}{2} - \frac{u_0^4}{4}.$$

The 'time' $x(E)$ it has taken to reach this point is given by

$$x(E) = \int_0^{u_0} \frac{1}{\sqrt{2E - \lambda u^2 + \frac{u^4}{2}}}\, \mathrm{d}u.$$

Properties of the fixed points of (2.18) follow from the following properties of the integral $x(E)$:

(i) As $E \to \lambda^2/4$, $x(E) \to \infty$;
(ii) As $E \to 0^+$, $x(E) \to \frac{\pi}{2\sqrt{\lambda}}$; and
(iii) $x(E)$ is a strictly increasing function of E.

In particular, it follows that for $E \in (0, \frac{\lambda^2}{4})$

$$\frac{\pi}{2\sqrt{\lambda}} < x(E) < \infty.$$

To obtain a solution of (2.17) from a solution of (2.18), we need a solution with $2nx(E) = \pi$ for some integer n; we circle around the origin $n/2$ times, ending up

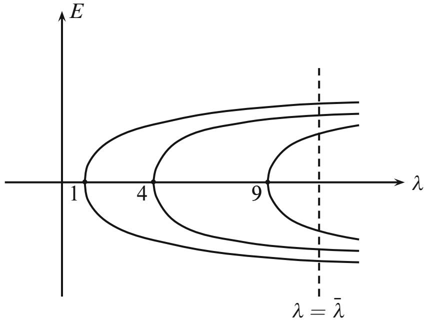


Fig. 2.2 Bifurcation of equilibria as λ is increased. For example, at $\lambda = \bar{\lambda}$ there are seven equilibria

back on the v-axis. To find the number of fixed points of our equation (2.16), we therefore must find the number of distinct values of E for which $2nx(E) = \pi$.

First, if $\lambda < 1$, then $\pi/\sqrt{\lambda} > \pi$, and so the only solution that fulfils our criteria is the origin. This corresponds to the equilibrium $u \equiv 0$; we label this ϕ_0.

If $1 < \lambda < 2^2$, then the values of $x(E)$ are bounded below by $\pi/4$ but include the value $\pi/2$; so there are two new fixed points, corresponding to orbits that perform a half loop, and we call these $\phi_1^{\pm}$. Similarly, if $2^2 < \lambda < 3^2$, then we also have orbits that loop around $3/2$ times since one of the orbits has $x(E) = \pi/6$; we call these $\phi_2^{\pm}$ and have five equilibria in total.

Continuing in this way provides us with a full description of the fixed points of the system, illustrated schematically in Fig. 2.2.

Theorem 2.44. *If $n^2 < \lambda \leq (n+1)^2$, then there are $2n+1$ equilibria of the Chafee–Infante equation (2.16); $\phi_0^{\pm}$ and n pairs $\phi_j^{\pm}$, $j = 1, \ldots, n$. The function $\phi_j^{\pm}$ has j zeros in $(0, \pi)$.*

2.6 Pullback attractors with more general basins of attraction

We have concentrated so far on sets that pullback attract fixed bounded subsets of X. A consequence of this is that unless the pullback attractor is bounded in the past, it does not lie in the class of sets that it is required to attract. This prevents one deducing the uniqueness of pullback attractors if the requirement of minimality (not needed in the autonomous case) is dropped.

Moreover, if we only know that a process is pullback dissipative and asymptotically compact, then we can conclude only that a minimal *closed* attractor exists (Theorem 2.20), a distinction that is important in infinite-dimensional phase spaces. To guarantee the compactness of the pullback attractor, we have had to impose strong pullback dissipativity, which also implies that the pullback attractor must be bounded in the past (Theorem 2.23). But the pullback attractor can be compact without being bounded in the past, as one can see from Theorem 2.12, or, with more generality, when the pullback attractor is associated to a random differential equation (Sect. 1.7).

We have already remarked that pullback attraction of fixed bounded sets implies the pullback attraction of time-dependent families that are bounded in the past. But in fact it is common in applications that there is a pullback attractor that attracts more general time-dependent families, and in this section we develop a theory that allows for these more general basins of attraction. It would have been possible to develop all the preceding theory in this more general setting, but the greater generality did not seem to merit the resulting complication of the presentation. Nevertheless, within this framework we can prove the uniqueness of attractors and their compactness from the appropriate definitions of pullback dissipative and pullback asymptotic compact processes (Theorem 2.50).

In what follows we will consider the collection $\mathscr{M}$ consisting of all time-dependent families of non-empty subsets of X,

$$D(\cdot) = \{D(t) : \ D(t) \subset X, \ D(t) \neq \varnothing\}_{t\in\mathbb{R}}.$$

If $D(\cdot)$ and $D'(\cdot)$ are elements of $\mathscr{M}$, then we write $D'(\cdot) \subseteq D(\cdot)$ to mean that $D'(t) \subseteq D(t)$ for all $t \in \mathbb{R}$.

Definition 2.45. A subset $\mathscr{D}$ of $\mathscr{M}$ is called *inclusion closed* if whenever $D(\cdot) \in \mathscr{D}$ and $D'(\cdot) \in \mathscr{M}$ is such that $D'(\cdot) \subseteq D(\cdot)$, then $D'(\cdot) \in \mathscr{D}$. We call such a collection a *universe* (of sets).

Note that because of the requirement that $\mathscr{D}$ must be inclusion closed, the collection of all constant families $D(\cdot)$ where $D(t) = D$ for all $t \in \mathbb{R}$ is not an allowable universe of sets. Instead, the minimal universe that includes these sets (‘the bounded universe’ $\mathscr{D}_{\mathrm{B}}$) consists of all time-dependent families $D(\cdot)$ such that for some bounded set D, $D(t) \subset D$ for every $t \in \mathbb{R}$.

Another simple example is the collection of all families $D(\cdot)$ such that for some $\lambda > 0$

$$\sup_{x\in D(t)} \|x\|\mathrm{e}^{\lambda t} \to 0 \qquad \text{as} \quad t \to -\infty.$$

In this direction a particularly useful universe is the collection of *tempered sets* $\mathscr{T}$ consisting of families $D(\cdot)$ such that

$$t \mapsto \sup_{x\in D(t)} \|x\|$$

grows subexponentially as $t \to -\infty$ (see, among others, Flandoli and Schmalfuß 1996; Kloeden and Langa 2007; Marín-Rubio and Real 2009; Y. Wang et al. 2006).

We now present a number of definitions that parallel those of Sect. 1.4 but that are now referred to a given universe $\mathscr{D}$ rather than only fixed bounded sets.

Definition 2.46. Let $\mathscr{D}$ be a universe of sets. A family of compact sets $\mathscr{A}_{\mathscr{D}}(\cdot)$ is said to be the pullback $\mathscr{D}$-attractor for the process $S(\cdot,\cdot)$ if

(i) $\mathscr{A}_{\mathscr{D}}(\cdot)$ is invariant;
(ii) $\mathscr{A}_{\mathscr{D}}(\cdot)$ pullback attracts every $D(\cdot) \in \mathscr{D}$,

$$\lim_{s\to-\infty} \operatorname{dist}(S(t,s)D(s), \mathscr{A}_{\mathscr{D}}(t)) = 0 \quad \text{for all} \quad t \in \mathbb{R};$$

and
(iii) $\mathscr{A}_{\mathscr{D}}(\cdot)$ is minimal: if there is another family of closed sets $C(\cdot)$, satisfying property (ii), then $\mathscr{A}_{\mathscr{D}}(t) \subseteq C(t)$ for all $t \in \mathbb{R}$.

Different universes provide different 'basins of attraction' and will give rise to different attractors, reflecting different aspects of the dynamics. Indeed, consider the example

$$\dot{x} = f(t,x) \qquad x(\tau) = x_0 \in \mathbb{R},$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is the function

$$f(t,x) = \begin{cases} -x, & x \in [-\mathrm{e}^{-t}, \mathrm{e}^{-t}] \\ -x - x(x - \mathrm{e}^{-t})\mathrm{e}^{t}, & \mathrm{e}^{-t} \le |x| \le 2\mathrm{e}^{-t} \\ -2x, & |x| \ge 2\mathrm{e}^{-t}. \end{cases}$$

If $\mathscr{D}$ contains $D(\cdot) = \{[-\mathrm{e}^{-t}, \mathrm{e}^{-t}] : \, t \in \mathbb{R}\}$, then the pullback $\mathscr{D}$-attractor $\mathscr{A}_{\mathscr{D}}(\cdot)$ will satisfy

$$[-\mathrm{e}^{-t}, \mathrm{e}^{-t}] \subset \mathscr{A}_{\mathscr{D}}(t) \subset [-2\mathrm{e}^{-t}, 2\mathrm{e}^{-t}].$$

On the other hand, if we only wish to attract bounded sets, then the pullback attractor will be $\mathscr{A}(\cdot)$, with $\mathscr{A}(t) = \{0\}$, for all $t \in \mathbb{R}$.

Within this framework it is also natural to try to find the largest possible universe $\mathscr{D}$ for which a pullback $\mathscr{D}$-attractor exists. If we return to the simple scalar equation

$$\dot{x} = -\alpha x + f(t)$$

(this was (1.10) above), recall that the explicit solution is

$$x(t) = \mathrm{e}^{-\alpha(t-s)}x(s) + \int_s^t \mathrm{e}^{-\alpha(t-r)} f(r)\,\mathrm{d}r.$$

We remarked (essentially) that if

$$\int_{-\infty}^0 \mathrm{e}^{\alpha r} f(r)\,\mathrm{d}r$$

converges, then

$$x^*(t) = \int_{-\infty}^t \mathrm{e}^{-\alpha(t-r)} f(r)\,\mathrm{d}r$$

pullback attracts bounded sets of initial conditions. But it is clear that one can in fact let $x(s)$ grow as $s \to -\infty$, provided that $\mathrm{e}^{\alpha s}x(s) \to 0$ as $s \to -\infty$, so one could take

$$\{x(\cdot) :\ \mathrm{e}^{\alpha s}x(s) \to 0 \qquad \text{as} \qquad s \to -\infty\}$$

as the universe $\mathscr{D}$. [One can find similar results in more involved PDE examples: see García-Luengo et al. (2012a,b) and Łukaszewicz 2010, among others.]

Conditions for the existence of $\mathscr{D}$-attractors closely parallel those for pullback attractors for bounded sets: the following definitions are unsurprising.

Definition 2.47. Let X be a metric space, $S(\cdot,\cdot)$ a process on X, and $\mathscr{D}$ a universe of sets in X. Given $D(\cdot) \in \mathscr{D}$, the pullback ω-limit of $D(\cdot)$ is defined as

$$\omega(D(\cdot),t) := \bigcap_{T \le t} \overline{\bigcup_{s \le T} S(t,s)D(s)}.$$

Definition 2.48. Let X be a metric space, $S(\cdot,\cdot)$ a process on X, and $\mathscr{D}$ a universe of sets in X. The process $S(\cdot,\cdot)$ is said to be *pullback $\mathscr{D}$-asymptotically compact* if, for each $D(\cdot) \in \mathscr{D}$ and $t \in \mathbb{R}$, for all sequences $\{s_k\} \le t$ with $s_k \to -\infty$ as $k \to \infty$ and $\{x_k\} \in X$ with $x_k \in D(s_k)$ for all $k \in \mathbb{N}$, then $\{S(t,s_k)x_k\}$ has a convergent subsequence.

The previous existence theorems for pullback attractors can now be written with respect to a universe $\mathscr{D}$. We give two general results that are the analogues of Theorems 2.20 and 2.23 (see Caraballo et al. 2006a).

For the existence theorem we will need the following generalisation of Lemma 2.10; we omit the proof since it follows line by line that of Lemma 2.10 with minimal changes.

Lemma 2.49. *Let $\mathscr{D}$ be a universe of sets and $S(\cdot,\cdot)$ a pullback $\mathscr{D}$-asymptotically compact process. Then for any $B(\cdot) \in \mathscr{D}$, $\omega(B(\cdot),t)$ is non-empty, compact, and invariant and pullback attracts $B(\cdot)$ at time t.*

The following theorem contains Theorem 2.23 as a particular case if we take $\mathscr{D}$ to be the bounded universe $\mathscr{D}_{\mathrm{B}}$ (the minimal universe containing all time-independent bounded sets).

Theorem 2.50. *Let $\mathscr{D}$ be a universe in X and $S(\cdot,\cdot)$ a process. Suppose that $S(\cdot,\cdot)$ is pullback $\mathscr{D}$-asymptotically compact and that there exists a $B(\cdot) \in \mathscr{D}$ that pullback attracts all families in $\mathscr{D}$. Then $\omega(B(\cdot)) \in \mathscr{D}$ is the unique pullback $\mathscr{D}$-attractor for $S(\cdot,\cdot)$ and is also the maximal invariant family in $\mathscr{D}$.*

Proof. First we show that $\omega(B(\cdot),t)$ pullback attracts every $D(\cdot) \in \mathscr{D}$ at time t. Indeed, it is immediate from the definition of $\omega(D(\cdot),t)$ and the fact that $B(t)$ pullback attracts $D(\cdot)$ at time t that $\omega(D(\cdot),t) \subseteq B(t)$. It then follows from the invariance of $\omega(D(\cdot),\cdot)$ and the fact that $\omega(B(\cdot),t)$ pullback attracts $B(\cdot)$ at time t that $\omega(D(\cdot),t) \subseteq \omega(B(\cdot),t)$.

The minimality is a straightforward consequence of the fact that $\omega(B(\cdot),t) \subset B(t)$ (which implies in particular that $\omega(B(\cdot),\cdot) \in \mathscr{D}$) and the invariance of the family $\omega(B(\cdot),\cdot)$. The maximality follows since for any invariant family $C(\cdot) \in \mathscr{D}$,

$$\lim_{s\to-\infty} \operatorname{dist}(S(t,s)C(s),\mathscr{A}(t)) = \operatorname{dist}(C(t),\mathscr{A}(t)) = 0,$$

so that $C(t) \subseteq \mathscr{A}(t)$ for all $t \in \mathbb{R}$, and we also obtain the uniqueness of the family $\mathscr{A}(\cdot)$ in the basin $\mathscr{D}$. □

If we suppose that $\mathscr{D}_{\mathrm{B}} \subset \mathscr{D}$ (so that any bounded set $D \subset X$ is included in the universe $\mathscr{D}$), then there is a straightforward relationship between the pullback attractor for bounded sets, $\mathscr{A}(\cdot)$, and the pullback $\mathscr{D}$-attractor, $\mathscr{A}_{\mathscr{D}}(\cdot)$. Indeed, thanks to the minimality of $\mathscr{A}(\cdot)$,

$$\mathscr{A}(t) \subseteq \mathscr{A}_{\mathscr{D}}(t) \qquad \text{for all} \quad t \in \mathbb{R}.$$

On the other hand, suppose that $\mathscr{A}_{\mathscr{D}}(\cdot)$ is bounded in the past. In this case, $\mathscr{A}_{\mathscr{D}}(\cdot)$ is attracted to $\mathscr{A}(\cdot)$, and since it is invariant, it follows that

$$\mathscr{A}_{\mathscr{D}}(t) \subseteq \mathscr{A}(t) \qquad \text{for all} \quad t \in \mathbb{R}.$$

We therefore have the following result (Marín-Rubio and Real 2009):

Lemma 2.51. *Suppose that $\mathscr{D}$ is a universe that contains every bounded subset of X and that $\mathscr{A}_{\mathscr{D}}(\cdot)$ is bounded in the past. Then $\mathscr{A}_{\mathscr{D}}(\cdot) = \mathscr{A}(\cdot)$, where $\mathscr{A}(\cdot)$ is the pullback attractor for bounded subsets of X.*

Notes

The development of this theory can be found in many papers. The first results on the existence of pullback attractors can be found in the appendix of the book by Vishik (1992) and in the paper by Chepyzhov and Vishik (1994), there termed ‘kernel sections’, and in the paper on random attractors by Crauel et al. (1997). The last of these provides the first general results on pullback ω-limit sets and the existence of attractors along the lines of our Theorem 2.12.

The existence result for asymptotically compact processes (Theorem 2.23) comes from Caraballo et al. (2006a) (written there for a general basin of attraction) and was generalised further in Caraballo et al. (2010a). An interesting application of this theoretical result is to treat non-autonomous PDEs in unbounded domains, as in Caraballo et al. (2006a), Marín-Rubio and Real (2007), and B. Wang (2009), or to the case of non-autonomous hyperbolic PDEs, as in Caraballo et al. (2010b) and Y. Wang (2008). Theorems 2.25 and 2.27 come from Y. Wang et al. (2006), where the flattening property (a coinage of Kloeden and Langa 2007) is called ‘condition (C)’, following the terminology introduced in the paper of Ma et al. (2002), which treated the autonomous case. The results in Sect. 2.5 related to pullback point dissipativity all come from Caraballo et al. (2010a,b).

Some general works on bifurcations for non-autonomous differential equations using pullback attraction have appeared in Kloeden and Siegmund (2005), Rasmussen (2007c), and Langa et al. (2002, 2006).

The results of our Sect. 2.6 on basins of attraction are based on the work of Marín-Rubio and Real (2009). Referring attraction to a universe of sets is a natural technique when dealing with attractors for random dynamical systems, and it was used by Crauel and Flandoli (1994) and Flandoli and Schmalfuß (1996); in this context, a particularly useful universe is the collection of *tempered sets* $\mathscr{T}$ consisting of families $\{D(\omega)\}_{\omega\in\Omega}$ such that

$$t \mapsto \sup_{x\in D(\theta_t\omega)} \|x\|$$

grows subexponentially as $t \to -\infty$ (see Arnold 1998; Flandoli and Schmalfuß 1996; Liu 2007, among others). An alternative approach, in which the phase space is allowed to depend as time, is adopted by Di Plinio et al. (2011, 2012).

Finally, other time-dependent subsets of the phase space that are pullback attracting have been also introduced in the literature. For instance, exponential attractors (Langa et al. 2010a; Efendiev et al. 2005; Czaja and Efendiev 2011; Carvalho and Sonner 2012) and inertial manifolds (Koksch and Siegmund 2002; Z. Wang et al. 1998) have been generalised to treat non-autonomous problems. The theory has also been extended to treat multivalued processes, i.e. those coming from non-autonomous differential equations that are set-valued or for which the uniqueness of solutions is unknown (Caraballo et al. 2003; Caraballo et al. 2005).

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Attractors for infinite-dimensional non-autonomous dynamical systems
Carvalho, A.N. de; Langa Rosado, J.A.; Robinson, J.
2013, XXXVI, 412 p., Hardcover
ISBN: 978-1-4614-4580-7