

Preface

Since the beginning of general relativity, Lorentzian geometry has provided the language and background for such a theory, as well as a mathematical arena where prospective extensions of Riemannian techniques could be sounded out. However, the variety and depth of its developments have consolidated Lorentzian geometry as a branch of differential geometry which is interesting by itself, as it provides applications to many parts of mathematical physics and yields an appealing framework where many mathematical techniques merge: geometric analysis, functional analysis, partial differential equations, Lie groups, and so on.

Ten years ago, a biennial series of meetings focused on Lorentzian Geometry was born in the town of Benalmádena (Spain). In the sixth edition, celebrated in Granada, September 2012, around 120 researchers of 18 countries gathered to discuss on the new trends of this geometry. In fact, the progress along this decade has attracted a renewed interest for many researchers: long-standing open problems have been solved, outstanding Lorentzian spaces and groups have been classified, new applications to mathematical relativity and high energy physics have been found, and further connections with other geometries have been developed.

In this volume, a sampler of the recent progress in Lorentzian geometry is presented. Topics such as geodesics, constant mean curvature submanifolds, trapped surfaces, gravitational collapse, classifications of manifolds with relevant symmetries, connections with Finsler geometry, and applications to mathematical physics are included. The contributions to this volume give a general perspective on these topics and provide new substantial results in some of them.

Let us give a very short overview of the contents.

The first five contributions constitute a block devoted to several problems on notable surfaces (maximal, constant mean curvature, umbilical, trapped) in Lorentzian manifolds. They are studied from different viewpoints, which include connections with other classical parts of differential geometry and mathematical relativity. More precisely, Fujimori, Kawakami, Kokubu, Rossmann, Umehara, and Yamada introduce and develop an original notion of extended hyperbolic metric (i.e., a hyperbolic metric with a certain kind of singularities on a Riemann surface). Surfaces endowed with such metrics will be related to surfaces of constant mean

curvature one in de Sitter space \mathbb{S}_1^3 . This relation is developed specifically for catenoids in \mathbb{S}_1^3 , and the classification of such catenoids will provide a classification of the corresponding moduli space of hyperbolic metrics. Albuje and Alías revisit both the classical Calabi–Bernstein theorem (i.e., the only entire maximal graphs in Lorentz-Minkowski space \mathbb{L}^3 are the space-like planes) and quite a few of its extensions. Very recent generalizations to a product spacetime $M^2 \times \mathbb{L}^1$ are specially considered. In particular, a local approach based on a parabolicity criterium is introduced so that a new proof to Calabi–Bernstein result is achieved. Senovilla focuses on umbilical space-like 2-surfaces in a Lorentzian manifold of dimension four. He introduces the notion of *ortho-umbilicity* and provides an original criterion to characterize total umbilicity in terms of the commutativity of two independent Weingarten operators. Some consequences are analyzed, and extensions to arbitrary signatures and higher dimensions are also discussed. Mars focuses on marginally outer trapped surfaces (MOTS), which play an important role in gravitational theory as indicators of strong gravitational fields and, eventually, of black hole boundaries (*event horizons*). They share some of the properties of minimal hypersurfaces, in particular, the existence of a useful notion of *stability*. The implications of stability on the topology of MOTS, its interplay with spacetime symmetries, and, then, the stability of Killing horizons are carefully analyzed. As a further development, Jaramillo analyzes the existence of a set of inequalities involving the area, angular momentum, and charges of stably outermost marginally trapped surfaces in a generic spacetime under natural hypotheses. These inequalities provide lower bounds for the area of spatial sections that offers quasi-local models of black hole horizons. As in Mars contribution, the extension to a Lorentzian setting of tools employed in minimal surfaces in Riemannian contexts is emphasized.

The next three contributions are devoted to properties of geodesics and gravitational collapse. Caponio makes a thorough analysis of the notion of convexity for a hypersurface in a semi-Riemannian manifold. Classical Bishop’s Riemannian result stating that infinitesimal convexity is equivalent to local convexity is reviewed, and its failure for the Lorentzian case is remarked. The analogous problem for the Finsler case has been solved very recently, and the author shows that the techniques for this more general case are also applicable to the semi-Riemannian problem. Applications to geodesic connectivity and further questions are also discussed. Variational methods and techniques of global analysis in manifolds are used by Bartolo, Candela, and Flores in order to study geodesics in spacetimes. After the successful results in the last two decades about the existence of connecting geodesics in causally well-behaved spacetimes, the authors focus on Gödel-type spacetimes. The results on this case are reviewed, and further improvements are obtained. Giambó and Magli analyze the geometry of isotropic fluids under gravitational collapse. Under general assumptions defining the fluid model, the null geodesics and causal structure, as well as the possible formation of horizons and nature of singularities, are discussed, with special attention to the case of bariotropic fluids obeying a linear equation of state.

The next block of three contributions is related to the recently developed connection between the class of (conformally) standard stationary spacetimes and the class of Finsler manifolds of Randers type. Javaloyes gives a general overview of such a stationary-to-Randers correspondence. This includes relations already developed at the levels of light-like geodesics, causality or causal boundaries on the stationary side, with different Finslerian elements (geodesics, convexity/completeness, and Busemann boundaries, respectively), as well as further prospective relations on topics such as isometry groups and curvature. In this framework, Matveev solves a natural question on arbitrary Finsler manifolds. More specifically, he characterizes when a Finsler metric F can be made complete by using a trivial projective change ($F \rightarrow F + df$). This question is inspired in a result that can be proved for the class of Randers metrics as a direct consequence of the stationary-to-Randers correspondence. Flores and Herrera contribution has several aims. Firstly, they review both the new redefinition of the classical *causal boundary* of a spacetime and the tools for its computation. These include, on the one hand, Penrose's conformal boundary and, on the other, connections with several boundaries in differential geometry (Cauchy, Gromov, Busemann), which have been developed for Finsler manifolds recently. Secondly, by using such tools, the causal boundary of the stationary part of Kerr spacetime is computed explicitly here.

The last four contributions study different aspects of symmetries of Lorentzian manifolds. Lichtenfelz, Piccione, and Zeghibs contribution provides a critical survey on some topics about the isometry group of a Lorentzian manifold. They revisit carefully the subtleties to endow this group with a Lie group structure. Then, recent results on (compact or not) Lie groups acting on a compact Lorentzian manifold are reviewed. Honda and Tsukada progress towards the local classification of conformally flat homogeneous Lorentzian manifolds. Such a complete classification is available in the Riemannian setting, but in the Lorentzian one has been obtained only in dimension three. As the Schouten tensor determines the curvature in this case, the authors focus on its algebraic structure and obtain the classification for all cases with nontrivial Jordan form, except when a triple root of the minimal polynomial exists. Díaz-Ramos gives an updated review on polar and hyperpolar actions on symmetric spaces, including the discussion of the differences between the compact and the noncompact cases. The study is carried out at the Riemannian level, and the Lorentzian one appears as a prospective challenge. Finally, Gilkey and Nikčević, after surveying some known results in geometric realizability (including the semi-Riemannian and para-Hermitian settings), provide a new result on Kähler and para-Kähler Weyl structures. Specifically, a decomposition of the corresponding space of curvature tensors (which stresses the differences between dimension 4 and higher) is obtained. Then, every (para-)Kähler algebraic structure is shown to be geometrically realized by a (para-)Kähler manifold.

Summing up, these contributions, as a whole, provide a progress and an updated guide for many of the most interesting topics in present-day research on Lorentzian geometry. Thus, we would like to acknowledge the careful work of all the contributors, as well as of the anonymous referees. We would also like to thank

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