Chapter 2
Basic Continuum Kinematics

The theme of this chapter was stated with exuberance and in an idealistic deterministic extreme by Marquis Pierre-Simon de Laplace (1759–1827): "Thus, we must consider the present state of the universe as the effect of its previous state and as the cause of those states to follow. An intelligent being which, for a given point in time, knows all the forces acting upon the universe and the positions of the objects of which it is composed, supplied with facilities large enough to submit these data to numerical analysis, would include in the same formula the movements of the largest bodies of the universe and those of the lightest atom. Nothing would be uncertain for it, and the past and future would be known to it."[1]

2.1 The Deformable Material Model, the Continuum

In the deformable material model all types of motion are permitted, but the deformational motions are usually the major concern. Consider the image O of an object in Euclidean space. The object is in a configuration $O(0)$ at $t = 0$ and in a configuration $O(t)$ at time t (Fig. 2.1). The mathematical representation of the motion of a three-dimensional deformable continuum gives a complete history of the motion of each point P on the object $O(0)$, $P \subset O(0)$; in words, $P \subset O(0)$ means all points P contained in ($\subset$) the image of the object, O, at $t = 0$. In order to identify each point P in the object $O(0)$ and to follow the movement of that point in subsequent configurations of the object $O(t)$, each point on an object is given a reference location in a particular coordinate system, called the reference coordinate system. The selection of the reference configuration is the choice of the modeler; here the reference configuration is taken as the configuration of the object at time $t = 0$. To distinguish between the reference location of a point on an object and a location of the same point at a later time, the terminology of "particle" and "place"

[1] Translated by John H. Van Drie (http://www.johnvandrie.com).

S.C. Cowin, *Continuum Mechanics of Anisotropic Materials*,
DOI 10.1007/978-1-4614-5025-2_2,

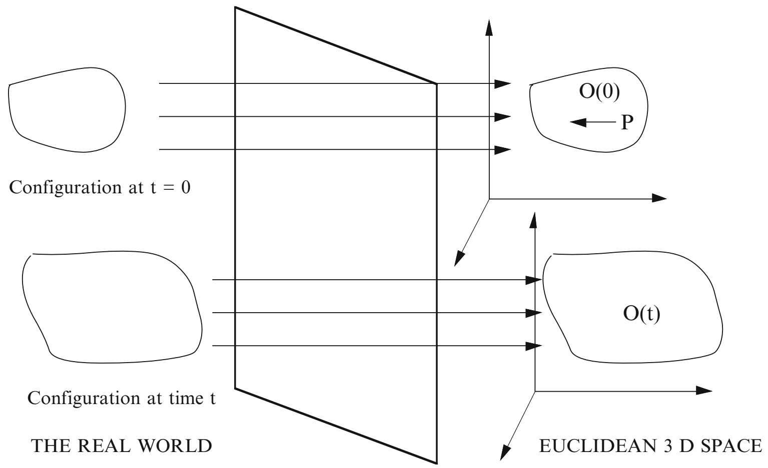

Fig. 2.1 Representation of the motion of an object in Euclidean 2D space

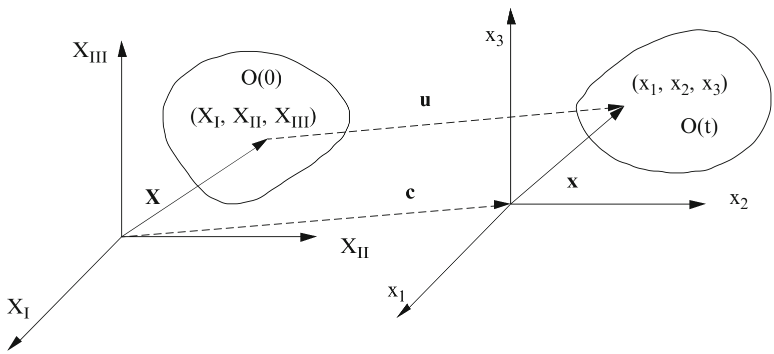

Fig. 2.2 Details of the representation in Euclidean 2D space

of a particle is introduced. Each point $P \subset O(0)$ in the continuum model of the object is labeled by its position in the reference configuration (Fig. 2.2). This procedure assigns a location to each point in the object and such points are called particles. A position vector of a point in a given coordinate system is a vector from the origin of coordinates to that point. In this case the reference configuration is a three-dimensional Cartesian coordinate system with base vectors $\mathbf{e}_\alpha$, $\alpha = \mathrm{I, II, III}$, and coordinates X_α; the position of the particle is described by the vector

$$\mathbf{X} = X_\alpha \mathbf{e}_\alpha.$$

As a simplifying convention, instead of saying the vector $\mathbf{X}$ describes the position of a particle, we define it to be the particle. Thus the notation $\mathbf{X}$ has replaced the notation P and one can speak of all $\mathbf{X} \subset O(0)$ as a complete representation of the object in the reference configuration.

If the motion of one particle $\mathbf{X}$ of an object can be represented, then the motion of all the particles of the object, $\mathbf{X} \subset O(0)$, can be represented. A second coordinate system with axes x_i, $i = 1, 2, 3$, and base vectors $\mathbf{e}_i$, $i = 1, 2, 3$, is introduced to represent the present position of the object $O(t)$; this also represents the present positions of the particles. The triplet (x_1, x_2, x_3), denoted in the shorthand direct notation by $\mathbf{x}$, represents the place at time t of the particle $\mathbf{X}$. The motion of the particle $\mathbf{X}$ is then given by

$$x_1 = \chi_1(X_{\mathrm{I}}, X_{\mathrm{II}}, X_{\mathrm{III}}, t), \quad x_2 = \chi_2(X_{\mathrm{I}}, X_{\mathrm{II}}, X_{\mathrm{III}}, t), \quad x_3 = \chi_3 = (X_{\mathrm{I}}, X_{\mathrm{II}}, X_{\mathrm{III}}, t) \tag{2.1}$$

which is a set of three scalar-valued functions whose arguments are the particle $\mathbf{X}$ and time t and whose values are the components of the place $\mathbf{x}$ at time t of the particle $\mathbf{X}$. Since $\mathbf{X}$ can be any particle in the object, $\mathbf{X} \subset O(0)$, the motion (2.1) describes the motion of the entire object $\mathbf{x} \subset O(t)$ and (2.1) is thus referred to as the *motion* of the object O. In the direct shorthand or vector notation (2.1) is written

$$\mathbf{x} = \chi(\mathbf{X},\ t) \quad \text{for all} \quad \mathbf{X} \subset O(0). \tag{2.2}$$

This is called the *material description of motion* because the material particles $\mathbf{X}$ are the independent variables. Generally the form of the motion, (2.1) or (2.2), is unknown in a problem, and the prime kinematic assumption for all continuum models is that such a description of the motion of an object is possible.

However, if the motion is known, then all the kinematic variables of interest concerning the motion of the object can be calculated from it; this includes velocities, accelerations, displacements, strains, rates of deformation, etc. The present, past, and future configurations of the object are all known. The philosophical concept embedded in the representation (2.2) of a motion is that of determinism. The determinism of the eighteenth century in physical theory was modified by humbler notions of "uncertainty" in the nineteenth century and by the discovery of extreme sensitivity to starting or initial conditions known by the misnomer "chaos" in the twentieth century. The quote of the Marquis Pierre-Simon de Laplace (1759–1827) at the beginning of the chapter captures the idea of determinism underlying the representation (2.2).

A translational rigid object motion is a special case of (2.2) represented by,

$$\mathbf{x} = \mathbf{X} + \mathbf{h}(t) \quad \text{for all} \quad \mathbf{X} \subset O(0), \tag{2.3}$$

where $\mathbf{h}(t)$ is a time-dependent vector. A rotational rigid object motion is a special case of (2.2) represented by

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X}, \quad \mathbf{Q}(t)\mathbf{Q}(t)^{\mathrm{T}} = \mathbf{1} \quad \text{for all} \quad \mathbf{X} \subset O(0), \tag{2.4}$$

where $\mathbf{Q}(t)$ is a time-dependent orthogonal transformation. It follows that a general rigid object motion is a special case of (2.2) represented by

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X} + \mathbf{h}(t),\ \mathbf{Q}(t)\mathbf{Q}(t)^{\mathrm{T}} = \mathbf{1} \quad \text{for all} \quad \mathbf{X} \subset O(0). \tag{2.5}$$

A motion of the form (2.2) is said to be a planar motion if the particles always remain in the same plane. In this case (2.2) becomes

$$x_1 = \chi_1(X_{\mathrm{I}}, X_{\mathrm{II}}, t), \quad x_2 = \chi_2(X_{\mathrm{I}}, X_{\mathrm{II}}, t), x_3 = X_{\mathrm{III}}. \tag{2.6}$$

Another subset of the motion is a deformation of an object from one configuration to another, say from the configuration at $t = 0$ to the configuration at $t = t^*$. In this case the motion (2.2) becomes a deformation

$$\mathbf{x} = \Psi(\mathbf{X}) \quad \text{for all} \quad \mathbf{X} \subset O(0), \tag{2.7}$$

where

$$\Psi(\mathbf{X}) = \chi(\mathbf{X},\ t^*) \quad \text{for all} \quad \mathbf{X} \subset O(0). \tag{2.8}$$

A 3D motion picture or 3D video of the motion of an object may be represented by a subset of the motion (2.2) because a discrete number of images (frames) per second are employed,

$$\mathbf{x} = \chi(\mathbf{X},\ n/\zeta) \quad \text{for all} \quad \mathbf{X} \subset O(0),\ n = 0,\ 1,\ 2, \ldots, \tag{2.9}$$

where ζ is the number of images (frames) per second.

Example 2.1.1
Consider the special case of a planar motion given by

$$x_1 = A(t)X_{\mathrm{I}} + C(t)X_{\mathrm{II}} + E(t), \quad x_2 = D(t)X_{\mathrm{I}} + B(t)X_{\mathrm{II}} + F(t), \quad x_3 = X_{\mathrm{III}}, \tag{2.10}$$

where $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, $F(t)$ are arbitrary functions of time. Further specialize this motion by the selections

$$A(t) = 1 + t, \quad C(t) = t, \quad E(t) = 3t, \quad B(t) = 1 + t, \quad D(t) = t, \quad F(t) = 2t, \tag{2.11}$$

for $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$. With these selections the motion becomes

$$x_1 = (1 + t)X_{\mathrm{I}} + tX_{\mathrm{II}} + 3t, \quad x_2 = tX_{\mathrm{I}} + (1 + t)X_{\mathrm{II}} + 2t, \quad x_3 = X_{\mathrm{III}}. \tag{2.12}$$

The problem is to find the positions of the unit square whose corners are at the material points $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (0, 0)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (1, 0)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (1, 1)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (0, 1)$ at times $t = 1$ and $t = 2$.

Solution: For convenience let the spatial (x_1, x_2, x_3) and material $(X_{\mathrm{I}}, X_{\mathrm{II}}, X_{\mathrm{III}})$ coordinate systems coincide and then consider the effect of the motion (2.12) on the

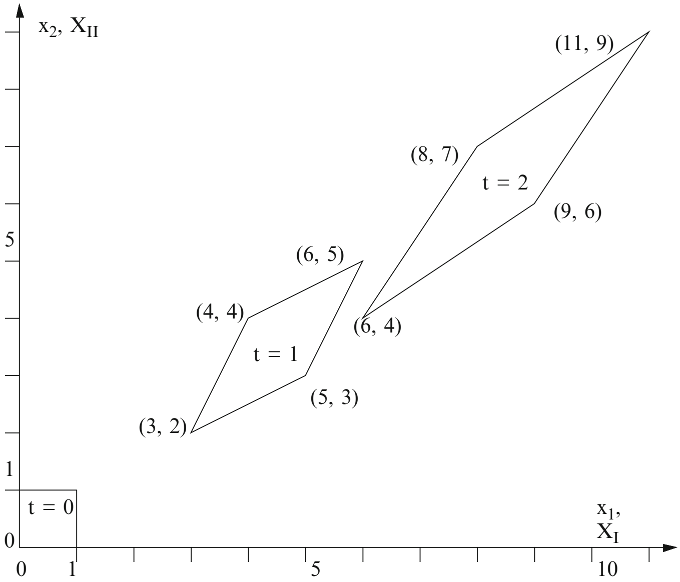

Fig. 2.3 The movement of a square at $t = 0$ due to the motion (2.12)

unit square whose corners are at the material points $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (0, 0)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (1, 0)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (1, 1)$, $(X_{\mathrm{I}}, X_{\mathrm{II}}) = (0, 1)$. At $t = 0$ the motion (2.12) specifies that $x_1 = X_{\mathrm{I}}$, $x_2 = X_{\mathrm{II}}$, and $x_3 = X_{\mathrm{III}}$ so that $t = 0$ has been taken as the reference configuration. The square at $t = 0$ is illustrated in Fig. 2.3. At $t = 1$ the motion (2.12) specifies the places $\mathbf{x}$ of the particles $\mathbf{X}$ as follows:

$$x_1 = 2X_{\mathrm{I}} + X_{\mathrm{II}} + 3, \quad x_2 = X_{\mathrm{I}} + 2X_{\mathrm{II}} + 2, \quad x_3 = X_{\mathrm{III}}.$$

Thus the particles at the four corners of the unit square have the following new places $\mathbf{x}$ at $t = 1$:

$$(3,\ 2) = \chi(0,\ 0),\ (5,\ 3) = \chi(1,\ 0),\ (6,\ 5) = \chi(1,\ 1),\ (4,\ 4) = \chi(0,\ 1).$$

A sketch of the deformed and translated unit square at $t = 1$ is shown in Fig. 2.3. At $t = 2$ the motion (2.12) specifies the places $\mathbf{x}$ of the particles $\mathbf{X}$ as follows:

$$x_1 = 3X_{\mathrm{I}} + 2X_{\mathrm{II}} + 6, \quad x_2 = 2X_{\mathrm{I}} + 3X_{\mathrm{II}} + 4, \quad x_3 = X_{\mathrm{III}}.$$

Thus the particles at the four corners of the unit square have the following new places at $t = 2$:

$$(6, 4) = \chi(0, 0), (9, 6) = \chi(1, 0), (11, 9) = \chi(1, 1), (8, 7) = \chi(0, 1).$$

A sketch of the deformed and translated unit square at $t = 2$ is shown in Fig. 2.3.

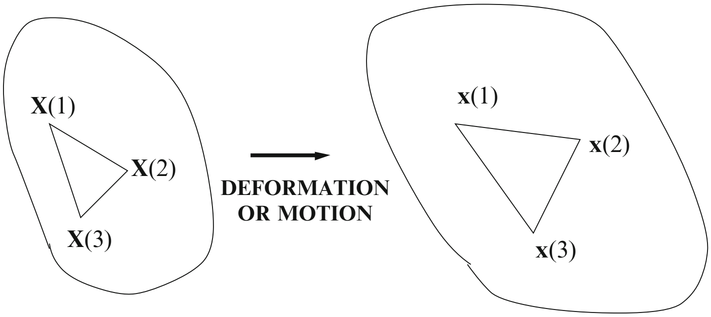

Fig. 2.4 The experimental measurement of a planar homogeneous motion. The reference frame is the laboratory reference frame. The three initial positions ($\mathbf{X}^{(1)}$, $\mathbf{X}^{(2)}$, $\mathbf{X}^{(3)}$) of the markers are indicated as well as their positions ($\mathbf{x}^{(1)}$, $\mathbf{x}^{(2)}$, $\mathbf{x}^{(3)}$) at time t. In many experiments the markers are attached to a specimen of soft tissue that is undergoing a planar homogeneous motion in order to quantify the motion

Example 2.1.2
An experimental technique in widespread use in the measurement of the planar homogeneous motion of a deformable object is to place three markers (dots or beads) in triangular pattern (so that the markers are not collinear) on the deformable object before a motion. The initial locations of the three markers are recorded relative to a fixed laboratory frame of reference as $(X_{\mathrm{I}}^{(1)}, X_{\mathrm{II}}^{(1)})$, $(X_{\mathrm{I}}^{(2)}, X_{\mathrm{II}}^{(2)})$, and $(X_{\mathrm{I}}^{(3)}, X_{\mathrm{II}}^{(3)})$, Fig. 2.4. If the process is automated a camera is used to follow the motion of the three markers with time and to digitize the data in real time. The instantaneous locations of the three markers at a time t is recorded relative to a fixed laboratory frame of reference as $(x_1^{(1)}(t), x_2^{(1)}(t))$, $(x_1^{(2)}(t), x_2^{(2)}(t))$ and $(x_1^{(3)}(t), x_2^{(3)}(t))$, Fig. 2.4. From these data the experimentalist calculates the time-dependent coefficients $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ of the homogeneous planar motion (2.10). Determine the formulas used in the calculation of the time-dependent coefficients $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ from the data $(X_{\mathrm{I}}^{(1)}, X_{\mathrm{II}}^{(1)})$, $(X_{\mathrm{I}}^{(2)}, X_{\mathrm{II}}^{(2)})$, $(X_{\mathrm{I}}^{(3)}, X_{\mathrm{II}}^{(3)})$, $(x_1^{(1)}(t), x_2^{(1)}(t))$, $(x_1^{(2)}(t), x_2^{(2)}(t))$, and $(x_1^{(3)}(t), x_2^{(3)}(t))$.

Solution: The data on the motion of each marker provide two equations that may be used for the determination of the time-dependent coefficients. Since there are three markers, a total of six equations is obtained. Three markers are used because it is known that six equations will be needed to solve the linear system of equations for the six unknowns, $A(t), B(t), C(t), D(t), E(t)$, and $F(t)$. Using the notation for the data and the representation of the homogeneous planar motion (2.10), these six equations are as follows:

$$
\begin{aligned}
x_1^{(1)}(t) &= A(t)X_{\mathrm{I}}^{(1)} + C(t)X_{\mathrm{II}}^{(1)} + E(t), \quad & x_2^{(1)}(t) &= D(t)X_{\mathrm{I}}^{(1)} + B(t)X_{\mathrm{II}}^{(1)} + F(t),\\
x_1^{(2)}(t) &= A(t)X_{\mathrm{I}}^{(2)} + C(t)X_{\mathrm{II}}^{(2)} + E(t), \quad & x_2^{(2)}(t) &= D(t)X_{\mathrm{I}}^{(2)} + B(t)X_{\mathrm{II}}^{(2)} + F(t),\\
x_1^{(3)}(t) &= A(t)X_{\mathrm{I}}^{(3)} + C(t)X_{\mathrm{II}}^{(3)} + E(t), \quad & x_2^{(3)}(t) &= D(t)X_{\mathrm{I}}^{(3)} + B(t)X_{\mathrm{II}}^{(3)} + F(t).
\end{aligned}
$$

The solution to these six equations is

$$A(t) = \frac{X_{\mathrm{II}}^{(1)} x_1^{(2)}(t) - X_{\mathrm{II}}^{(1)} x_1^{(3)}(t) - X_{\mathrm{II}}^{(2)} x_1^{(1)}(t) + X_{\mathrm{II}}^{(2)} x_1^{(3)}(t) + X_{\mathrm{II}}^{(3)} x_1^{(1)}(t) - X_{\mathrm{II}}^{(3)} x_1^{(2)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}},$$

$$B(t) = \frac{X_{\mathrm{I}}^{(2)} x_2^{(1)}(t) - X_{\mathrm{I}}^{(3)} x_2^{(1)}(t) - X_{\mathrm{I}}^{(1)} x_2^{(2)}(t) + X_{\mathrm{I}}^{(3)} x_2^{(2)}(t) + X_{\mathrm{I}}^{(1)} x_2^{(3)}(t) - X_{\mathrm{I}}^{(2)} x_2^{(3)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}},$$

$$C(t) = \frac{-X_{\mathrm{I}}^{(1)} x_1^{(2)}(t) + X_{\mathrm{I}}^{(2)} x_1^{(1)}(t) + X_{\mathrm{I}}^{(1)} x_1^{(3)}(t) - X_{\mathrm{I}}^{(2)} x_1^{(3)}(t) - X_{\mathrm{I}}^{(3)} x_1^{(1)}(t) + X_{\mathrm{I}}^{(3)} x_1^{(2)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}},$$

$$D(t) = \frac{X_{\mathrm{II}}^{(1)} x_2^{(2)}(t) - X_{\mathrm{II}}^{(2)} x_2^{(1)}(t) - X_{\mathrm{II}}^{(1)} x_2^{(3)}(t) + X_{\mathrm{II}}^{(2)} x_2^{(3)}(t) + X_{\mathrm{II}}^{(3)} x_2^{(1)}(t) - X_{\mathrm{II}}^{(3)} x_2^{(2)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}},$$

$$\begin{aligned} E(t) = {} & \frac{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} x_1^{(3)}(t) - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} x_1^{(2)}(t) - X_{\mathrm{I}}^{(1)} X_{\mathrm{II}}^{(2)} x_1^{(3)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}} \\ & + \frac{X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} x_1^{(1)}(t) + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} x_1^{(2)}(t) - X_{\mathrm{I}}^{(2)} X_{\mathrm{II}}^{(3)} x_1^{(1)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}}, \end{aligned}$$

$$\begin{aligned} F(t) = {} & \frac{-X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} x_2^{(2)}(t) + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} x_2^{(1)}(t) + X_{\mathrm{I}}^{(2)} X_{\mathrm{II}}^{(1)} x_2^{(3)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}} \\ & + \frac{-X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} x_2^{(3)}(t) - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)} x_2^{(1)}(t) + X_{\mathrm{I}}^{(1)} X_{\mathrm{II}}^{(3)} x_2^{(2)}(t)}{X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(2)} - X_{\mathrm{II}}^{(1)} X_{\mathrm{I}}^{(3)} - X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(1)} + X_{\mathrm{II}}^{(2)} X_{\mathrm{I}}^{(3)} + X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(1)} - X_{\mathrm{II}}^{(3)} X_{\mathrm{I}}^{(2)}}. \end{aligned}$$

Example 2.1.3

Consider again the experimental technique described in Example 2.1.2, but in this case a deformation rather than a motion, Fig. 2.4. Suppose that the initial locations of the markers are recorded relative to the fixed laboratory frame of reference as $(X_{\mathrm{I}}^{(1)}, X_{\mathrm{II}}^{(1)}) = (0, 0)$, $(X_{\mathrm{I}}^{(2)}, X_{\mathrm{II}}^{(2)}) = (1, 0)$, and $(X_{\mathrm{I}}^{(3)}, X_{\mathrm{II}}^{(3)}) = (0, 1)$. The deformed locations of the three markers relative to the same fixed laboratory frame of reference are $(x_1^{(1)}, x_2^{(1)}) = (1, 2)$, $(x_1^{(2)}, x_2^{(2)}) = (2, 3.25)$, and $(x_1^{(3)}, x_2^{(3)}) = (2.5, 3.75)$. From these data the constant coefficients A, B, C, D, E, and F of the homogeneous planar deformation (2.10) are determined.

Solution: The solution for the motion coefficients $A(t), B(t), C(t), D(t), E(t)$, and $F(t)$ obtained in Example 2.1.2 may be used in the solution to this problem. One simply assigns the time-dependent positions in the formulas for $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ to be fixed rather than time dependent by setting $(x_1^{(1)}(t), x_2^{(1)}(t)) = (x_1^{(1)}, x_2^{(1)})$, $(x_1^{(2)}(t), x_2^{(2)}(t)) = (x_1^{(2)}, x_2^{(2)})$, and $(x_1^{(3)}(t), x_2^{(3)}(t)) = (x_1^{(3)}, x_2^{(3)})$. The coefficients are no longer functions of time so they are denoted by A, B, C, D, E, and F. They are evaluated by substituting the initial and final locations of the set of particles, $(X_{\mathrm{I}}^{(1)}, X_{\mathrm{II}}^{(1)}) = (0, 0), (X_{\mathrm{I}}^{(2)}, X_{\mathrm{II}}^{(2)}) = (1, 0), (X_{\mathrm{I}}^{(3)}, X_{\mathrm{II}}^{(3)}) = (0, 1)$, and $(x_1^{(1)}, x_2^{(1)}) = (1, 2), (x_1^{(2)}, x_2^{(2)}) = (2, 3.25), (x_1^{(3)}, x_2^{(3)}) = (2.5, 3.75)$, respectively, into the last set of equations in Example 2.1.2. The values obtained are $A = 1$, $B = 1.75$, $C = 1.5$, $D = 1.25$, $E = 1$, and $F = 2$ and they are obtained by substituting the values for the relevant points given in the statement of the problem above into the last set of equations in Example 2.1.2. The planar homogeneous deformation then has the representation

$$x_1 = 2X_{\mathrm{I}} + 1.5X_{\mathrm{II}} + 1, \quad x_2 = 1.25X_{\mathrm{I}} + 1.75X_{\mathrm{II}} + 2, \quad x_3 = X_{\mathrm{III}},$$

which is a particular case of (2.10) To double check this calculation one can check to see if each marker is mapped correctly from its initial position to its final position.

There are two coordinate systems with respect to which a gradient may be taken, either the spatial coordinate system $\mathbf{x}$, (x_1, x_2, x_3), or the reference material coordinate system $\mathbf{X}$, $(X_{\mathrm{I}}, X_{\mathrm{II}}, X_{\mathrm{III}})$. To distinguish between gradients with respect to these two systems, the usual gradient symbol ∇ will be used to indicate a gradient with respect to the spatial coordinate system $\mathbf{x}$, and the gradient symbol $\nabla_{\mathbf{O}}$ with a subscripted boldface $\mathbf{O}$ will indicate a gradient with respect to the material coordinate system $\mathbf{X}$. The (material) deformation gradient tensor $\mathbf{F}$ is defined by

$$\mathbf{F} = [\nabla_{\mathbf{O}} \otimes \chi(\mathbf{X}, t)]^{\mathrm{T}} \quad \text{for all } \mathbf{X} \subset O(0). \tag{2.13}$$

The (spatial) inverse deformation gradient tensor $\mathbf{F}^{-1}$ is defined by

$$\mathbf{F}^{-1} = [\nabla \otimes \chi^{-1}(\mathbf{x}, t)]^{\mathrm{T}} \quad \text{for all } \mathbf{x} \subset O(t), \tag{2.14}$$

where

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t) \quad \text{for all } \mathbf{x} \subset O(t) \tag{2.15}$$

is the inverse of the motion (2.2). The components of $\mathbf{F}$ and $\mathbf{F}^{-1}$ are

$$\mathbf{F} = \left[\frac{\partial x_i}{\partial X_\alpha}\right] = \begin{bmatrix} \frac{\partial x_1}{\partial X_{\mathrm{I}}} & \frac{\partial x_1}{\partial X_{\mathrm{II}}} & \frac{\partial x_1}{\partial X_{\mathrm{III}}} \\ \frac{\partial x_2}{\partial X_{\mathrm{I}}} & \frac{\partial x_2}{\partial X_{\mathrm{II}}} & \frac{\partial x_2}{\partial X_{\mathrm{III}}} \\ \frac{\partial x_3}{\partial X_{\mathrm{I}}} & \frac{\partial x_3}{\partial X_{\mathrm{II}}} & \frac{\partial x_3}{\partial X_{\mathrm{III}}} \end{bmatrix} \text{ and } \mathbf{F}^{-1} = \left[\frac{\partial X_\alpha}{\partial x_i}\right] = \begin{bmatrix} \frac{\partial X_{\mathrm{I}}}{\partial x_1} & \frac{\partial X_{\mathrm{I}}}{\partial x_2} & \frac{\partial X_{\mathrm{I}}}{\partial x_3} \\ \frac{\partial X_{\mathrm{II}}}{\partial x_1} & \frac{\partial X_{\mathrm{II}}}{\partial x_2} & \frac{\partial X_{\mathrm{II}}}{\partial x_3} \\ \frac{\partial X_{\mathrm{III}}}{\partial x_1} & \frac{\partial X_{\mathrm{III}}}{\partial x_2} & \frac{\partial X_{\mathrm{III}}}{\partial x_3} \end{bmatrix} \tag{2.16}$$

respectively. Using the chain rule for partial derivatives it is easy to verify that $\mathbf{F}^{-1}$ is indeed the inverse of $\mathbf{F}$,

$$\mathbf{F}\mathbf{F}^{-1} = \mathbf{F}^{-1}\mathbf{F} = \mathbf{1}. \tag{2.17}$$

Recall that any motion can be decomposed into a sum of a translational, rotational, and deformational motion. The deformation gradient tensors remove the translational motion as may be easily seen because the translational motion is a separate function of time (cf., e.g., 2.2) that must be independent of the particle $\mathbf{X}$. Thus only the rotational motion and the deformational motion determine $\mathbf{F}$. If $\mathbf{F} = \mathbf{1}$ there are no rotational or deformational motions. If $\mathbf{F} = \mathbf{Q}(t)$, $\mathbf{Q}(t)\mathbf{Q}(t)^{\mathrm{T}} = \mathbf{1}$, it follows from (2.4) that the motion is purely rotational and there is no deformational motion. The *deformation gradient* $\mathbf{F}$ is so named because it is a measure of the deformational motion as long as $\mathbf{F} \neq \mathbf{Q}(t)$. If $\mathbf{F} = \mathbf{Q}(t)$, then the motion is rotational and we replace $\mathbf{F}$ by $\mathbf{Q}(t)$.

The determinant of the tensor of deformation gradients, J, is the Jacobian of the transformation from $\mathbf{x}$ to $\mathbf{X}$, thus

$$J \equiv \mathrm{Det}\,\mathbf{F} = 1/\mathrm{Det}\,\mathbf{F}^{-1}, \tag{2.18}$$

where it is required that

$$0 < J < \infty \tag{2.19}$$

so that a finite continuum volume always remains a finite continuum volume.

If $\mathbf{c}$ represents the position vector of the origin of the coordinate system used for the configuration at time t relative to the origin of the coordinate system used for the configuration at $t = 0$, then the displacement vector $\mathbf{u}$ of the particle $\mathbf{X}$ is given by (Fig. 2.2),

$$\mathbf{u} = \mathbf{x} - \mathbf{X} + \mathbf{c}. \tag{2.20}$$

The displacement vectors $\mathbf{u}$ for all the particles $\mathbf{X} \subset O(0)$ are given by

$$\mathbf{u}(\mathbf{X}, t) = \chi(\mathbf{X}, t) - \mathbf{X} + \mathbf{c}(t), \quad \mathbf{X} \subset O(0), \tag{2.21}$$

or by

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{x} - \chi^{-1}(\mathbf{x}, t) + \mathbf{c}(t), \quad \mathbf{x} \subset O(t). \tag{2.22}$$

Two gradients of the displacement field $\mathbf{u}$ may then be calculated, one with respect to the spatial coordinate system $\mathbf{x}$ denoted by the usual gradient symbol ∇ and one with respect to the material coordinate system $\mathbf{X}$ denoted by the gradient symbol $\nabla_{\mathbf{O}}$, thus

$$[\nabla_{\mathbf{O}} \otimes \mathbf{u}(\mathbf{X}, t)]^{\mathrm{T}} = \mathbf{F}(\mathbf{X}, t) - \mathbf{1} \quad \text{and} \quad [\nabla \otimes \mathbf{u}(\mathbf{x}, t)]^{\mathrm{T}} = \mathbf{1} - \mathbf{F}^{-1}(\mathbf{x}, t), \tag{2.23}$$

when (2.12) and (2.14) are employed. Often the base vectors of the coordinate systems $\mathbf{X}$ and $\mathbf{x}$ are taken to coincide, in which case the position vector $\mathbf{c}$ of the origin of the $\mathbf{x}$ system relative to the $\mathbf{X}$ system is zero. The selection of the coordinate system is always the prerogative of the modeler and such selections are usually made to simplify the analysis of the resulting problem.

Example 2.1.4
Compute the deformation gradient and the inverse deformation gradient for the motion given by (2.12). Then compute the Jacobian of the motion and both the spatial and material gradients of the displacement vector.

Solution: The deformation gradients and the inverse deformation gradients for this motion are obtained from (2.16) to (2.12), thus

$$\mathbf{F} = \begin{bmatrix} 1+t & t & 0 \\ t & 1+t & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{F}^{-1} = \frac{1}{1+2t}\begin{bmatrix} 1+t & -t & 0 \\ -t & 1+t & 0 \\ 0 & 0 & 1+2t \end{bmatrix},$$

a result that can be verified using $\mathbf{F}\mathbf{F}^{-1} = \mathbf{1}$ or $\mathbf{F}^{-1}\mathbf{F} = \mathbf{1}$. It is then easy to show that $J = 1 + 2t$. It also follows from (2.22) that

$$[\nabla_{\mathbf{O}} \otimes \mathbf{u}]^{\mathrm{T}} = t\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [\nabla \otimes \mathbf{u}]^{\mathrm{T}} = \frac{t}{1+2t}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Problems

2.1.1. Sketch the shape and position of the unit square with corners at (0, 0), (1, 0), (1, 1), and (0, 1) subjected to the motion in (2.10) for the seven special cases, (a) through (g) below. The shape and position are to be sketched for each of the indicated values of t.

(a) Translation. $A(t) = 1, B(t) = 1, C(t) = 0, D(t) = 0, E(t) = 2t, F(t) = 2t$ and values of $t = 0, 1, 2$.
(b) Uniaxial extension. $A(t) = 1 + t, B(t) = 1, C(t) = 0, D(t) = 0, E(t) = 0, F(t) = 0$ and values of $t = 0, 1, 2, 3$.
(c) Biaxial extension. $A(t) = 1 + t$, $B(t) = 1 + 2t$, $C(t) = 0$, $D(t) = 0$, $E(t) = 0$, $F(t) = 0$ and values of $t = 0, 1, 2$.
(d) Simple shearing (R). $A(t) = 1$, $B(t) = 1$, $C(t) = t$, $D(t) = 0$, $E(t) = 0$, $F(t) = 0$ and values of $t = 0, 1, 2$.
(e) Simple shearing (U). $A(t) = 1$, $B(t) = 1$, $C(t) = 0$, $D(t) = t$, $E(t) = 0$, $F(t) = 0$ and values of $t = 0, 1, 2$.
(f) Rigid Rotation (cw). $A(t) = \cos(\pi t/2)$, $B(t) = \cos(\pi t/2)$, $C(t) = \sin(\pi t/2)$, $D(t) = -\sin(\pi t/2)$, $E(t) = 0$, $F(t) = 0$ and values of $t = 0, 1, 2, 3, 4$.

(g) Rigid rotation (ccw). $A(t) = \cos(\pi t/2)$, $B(t) = \cos(\pi t/2)$, $C(t) = -\sin(\pi t/2)$, $D(t) = \sin(\pi t/2)$, $E(t) = 0$, $F(t) = 0$ and values of $t = 0, 1, 2, 3, 4$.

2.1.2. Sketch the shape and position of the square with corners at $(-1, -1)$, $(1, -1)$, $(1, 1)$, and $(-1, 1)$ at times $t = 0, 1, 2, 3, 4$. The square is subjected to the motion in (2.10) with the values of $A(t)$, $B(t)$, $C(t)$, $D(t)$, $E(t)$, and $F(t)$ being those given in 2.1(f), the rigid rotation (clockwise) motion.

2.1.3. For the six motions of the form (2.10) given in Problem 2.1.1, namely 2.1.1 (a) through 2.1.1(f), compute the deformation gradient tensor $\mathbf{F}$, its Jacobian J, and its inverse $\mathbf{F}^{-1}$. Discuss briefly the significance of each of the tensors computed. In particular, explain the form or value of the deformation gradient tensor $\mathbf{F}$ in terms of the motion.

2.1.4. Using the planar homogeneous deformation (2.10), with the values of A, B, C, D, E, and F calculated in Example 2.1.2, show that deformation (2.10) predicts the final positions of the three markers when the initial marker locations $(X_{\mathrm{I}}^{(1)}, X_{\mathrm{II}}^{(1)}) = (0, 0)$, $(X_{\mathrm{I}}^{(2)}, X_{\mathrm{II}}^{(2)}) = (1, 0)$, and $(X_{\mathrm{I}}^{(3)}, X_{\mathrm{II}}^{(3)}) = (0, 1)$ are substituted into it.

2.1.5. In Example 2.1.2 an experimental technique in widespread use in the measurement of the planar homogeneous motion of the deformable object was described and a system of equations was set and solved to determine the time-dependent parameters appearing in the equations describing the planar homogeneous motion. Consider the same problem when the problem is not planar, but three-dimensional. How many markers are necessary in three dimensions and how must the markers be arranged so that they provide the information necessary to determine the time-dependent parameters appearing in the equations describing the three-dimensional homogeneous motion? Explain the process.

2.2 Rates of Change and the Spatial Representation of Motion

The velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ of a particle $\mathbf{X}$ are defined by

$$\mathbf{v} = \dot{\mathbf{x}} = \left.\frac{\partial \chi}{\partial t}\right|_{\mathbf{X}\text{ fixed}}, \quad \mathbf{a} = \ddot{\mathbf{x}} = \left.\frac{\partial^2 \chi}{\partial t^2}\right|_{\mathbf{X}\text{ fixed}}, \tag{2.24}$$

where $\mathbf{X}$ is held fixed because it is the velocity or acceleration of that particular particle that is being determined. The *spatial description of motion* (as opposed to the *material description of motion* represented by (2.2)) is obtained by solving (2.2) for $\mathbf{X}$,

$$\mathbf{X} = \chi^{-1}(\mathbf{x},\ t) \quad \text{for all } \mathbf{X} \subset O(0) \tag{2.25}$$

and substituting the result into the first of the expressions (2.24) for the velocity; thus $\mathbf{v} = \dot{\mathbf{x}} = \dot{\chi}(\mathbf{X}, t)$ becomes

$$\mathbf{v} = \dot{\mathbf{x}} = \dot{\chi}(\chi^{-1}(\mathbf{x}, t), t) = \mathbf{v}(\mathbf{x}, t) \tag{2.26a}$$

or

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\chi_1{}^{-1}(\mathbf{x}, t), \chi_2{}^{-1}(\mathbf{x}, t), \chi_3{}^{-1}(\mathbf{x}, t), t), \tag{2.26b}$$

which emphasizes that the time dependence of the spatial representation of velocity is both explicit and implicit. This representation of the velocity with the places $\mathbf{x}$ as independent variables is called the *spatial representation of motion*. A quantity is said to be in the spatial representation if its independent variables are the places $\mathbf{x}$ and not the particles $\mathbf{X}$. In the material representation the independent variables are the particles $\mathbf{X}$; compare the material description of motion, (2.2), with (2.26). The *material time derivative* is the time derivative following the material particle $\mathbf{X}$; it is denoted by a superposed dot or D/Dt and it is defined as the partial derivative with respect to time with $\mathbf{X}$ held constant. The material time derivative is easy to calculate in the material representation. It is more complicated to calculate in the spatial representation. To determine the acceleration in the spatial representation we must calculate the material time rate of the spatial representation of velocity (2.26). The notation D/Dt introduced above is illustrated using the definitions of (2.24):

$$\mathbf{a} = \left.\frac{\partial^2 \chi}{\partial t^2}\right|_{\mathbf{X}\text{ fixed}} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{X}\text{ fixed}} \equiv \frac{D\mathbf{v}}{Dt}. \tag{2.27}$$

A formula for $D\mathbf{v}/Dt$ is obtained by observing the explicit and implicit time dependence of the spatial representation of velocity (2.26b) and noting that the time derivative associated with the implicit dependencies may be obtained using the chain rule, thus

$$\frac{D\mathbf{v}}{Dt} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \left.\frac{\partial \mathbf{v}}{\partial x_i}\frac{\partial x_i}{\partial t}\right|_{\mathbf{X}\text{ fixed}} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \frac{\partial \mathbf{v}}{\partial x_i} v_i, \tag{2.28}$$

a result that may be written more simply as

$$\frac{D\mathbf{v}}{Dt} = \left.\frac{\partial \mathbf{v}}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \mathbf{v} \cdot \nabla \mathbf{v}. \tag{2.29}$$

The time rate computed by holding the places $\mathbf{x}$ fixed is called the local time rate. In general, the material time rate is related to the local time rate by the following operator expression that follows from (2.29),

$$\frac{D}{Dt} = \left.\frac{\partial}{\partial t}\right|_{\mathbf{x}\text{ fixed}} + \mathbf{v} \cdot \nabla, \tag{2.30}$$

where D/Dt is the material time rate of change, $\partial/\partial t$ is the local time rate of change and $\mathbf{v}\cdot\nabla$ determines the convective change of the quantity.

The second order tensor formed by taking the spatial gradient of the velocity field $\mathbf{v} = \mathbf{v}(\mathbf{x}, t)$ is called the tensor of velocity gradients and is denoted by $\mathbf{L}$, thus

$$\mathbf{L} = [\nabla \otimes \mathbf{v}]^{\mathrm{T}} = \left[\frac{\partial v_i}{\partial x_j}\right] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}. \tag{2.31}$$

$\mathbf{L}$ is decomposed into a symmetric part $\mathbf{D}$ called *the rate-of-deformation tensor*, and a skew symmetric part $\mathbf{W}$ called the *spin tensor*, thus

$$\mathbf{L} = \mathbf{D} + \mathbf{W}, \quad \mathbf{D} = (1/2)(\mathbf{L} + \mathbf{L}^{\mathrm{T}}), \quad \mathbf{W} = (1/2)(\mathbf{L} - \mathbf{L}^{\mathrm{T}}). \tag{2.32}$$

The three nonzero components of $\mathbf{W}$ can be formed into an axial vector (1/2) ($\nabla \times \mathbf{v}$) which represents the local rotational motion and is called the angular velocity or one-half the vorticity.

The rate-of-deformation tensor $\mathbf{D}$ defined by the second of (2.32) has the component representation

$$\mathbf{D} = \frac{1}{2}\left[\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j}\right] = \frac{1}{2}\begin{bmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1} & \frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2} & 2\frac{\partial v_3}{\partial x_3} \end{bmatrix}. \tag{2.33}$$

The components of $\mathbf{D}$ along the diagonal are called normal rates of deformation and the components off the diagonal are called shear rates of deformation. The normal rates of deformation, D_{11}, D_{22}, and D_{33}, are measures of instantaneous time rate of change of the material filament instantaneously coincident with the 1, 2, and 3 axes, respectively, and the shear rates of deformation, D_{23}, D_{13}, and D_{12}, are equal to one-half the time rate of decrease in an originally right angle between material

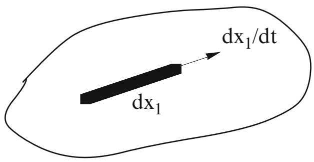

Fig. 2.5 An illustration for the geometric interpretation of the D_{11} component of the rate-of-deformation tensor **D**. A vector of infinitesimal length representing the present position of an infinitesimal material filament coinciding with the x_1 at time t is denoted by dx_1. The instantaneous time rate of change of the material filament instantaneously coincident with dx_1 is $dv_1 = D_{11}dx_1$. The expression $dv_1 = D_{11}dx_1$ shows dv_1 as a linear function of dx_1 at any point **x** and time t. Thus the geometric interpretation of $D_{11} = (d\dot{x}_1/dx_1)$ is that it is the instantaneous time rate of change of dx_1 at time t relative to dx_1 at time t

filaments instantaneously situate upon the 2 and 3 axes, the 1 and 3 axes, and the 1 and 2 axes, respectively.

The rate-of-deformation tensor **D** represents instantaneous rates of change, that is to say how much a quantity is changing compared to its present size. Let dx_1 be a vector of infinitesimal length representing the present position of an infinitesimal material filament coinciding with the x_1 at time t, Fig. 2.5. The instantaneous time rate of change of the material filament instantaneously coincident with dx_1 is $dv_1 = D_{11}dx_1$, a result that follows from the entry in the first column and first row of (2.33). The expression $dv_1 = D_{11}dx_1$ shows dv_1 as a linear function of dx_1 at any point **x** and time t. Thus the geometric interpretation of $D_{11} = (d\dot{x}_1/dx_1)$ is that it is the instantaneous time rate of change of dx_1 at time t relative to dx_1 at time t. Similar geometric interpretations exist for D_{22} and D_{33}.

The geometric interpretation of the normal rate of shearing components D_{11}, D_{22}, and D_{33} is easily extended to obtain a geometric interpretation of the trace of **D** which is also the divergence of the velocity, tr $\mathbf{D} = \nabla\cdot\mathbf{v}$. If dv represents an element of volume in the spatial coordinate system, $dv = dx_1dx_2dx_3$ (Fig. 2.6), the material time rate of change of dv can be computed using the type of formula developed in the previous paragraph; $d\dot{x}_1 = D_{11}dx_1$, $d\dot{x}_2 = D_{22}dx_2$, and $d\dot{x}_3 = D_{33}dx_3$, thus

$$d\dot{v} = \frac{D}{Dt}(dx_1dx_2dx_3) = (D_{11} + D_{22} + D_{33})dx_1dx_2dx_3 = (\text{tr }\mathbf{D})dv \tag{2.34}$$

or, noting from the definition of **D** that

$$\text{tr}\mathbf{D} = D_{11} + D_{22} + D_{33} = \nabla\cdot\mathbf{v} = \text{div }\mathbf{v} \tag{2.35}$$

it follows that

$$\text{tr}\mathbf{D} = \nabla\cdot\mathbf{v} = \frac{d\dot{v}}{dv}. \tag{2.36}$$

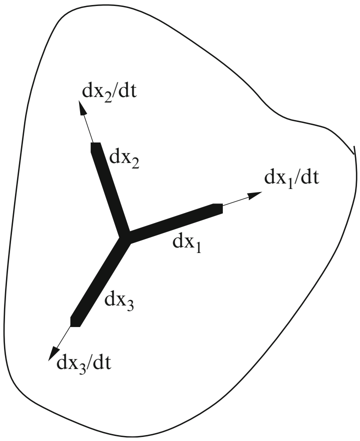

Fig. 2.6 Illustration for the geometric interpretation of the trace of the rate-of-deformation tensor **D** as the instantaneous time rate of change of volume. The material time rate of change of an element of volume in the spatial coordinate system, $dv = dx_1 dx_2 dx_3$, is shown to be $d\dot{v} = \mathrm{tr}\mathbf{D}dv = \nabla \cdot \mathbf{u}dv$, thus the $\nabla \cdot \mathbf{v}$ or tr**D** has the geometric interpretation as the instantaneous time rate of change of material volume

Thus the $\nabla \cdot \mathbf{v}$ or tr**D** have the geometric interpretation as the instantaneous time rate of change of material volume. Another way of viewing this result is to say that the divergence of the velocity field is the time rate of change of a material volume relative to how large it is at the instant (2.35).

The off-diagonal components of the rate-of-deformation tensor, for example D_{12}, represent rates of shearing. D_{12} is equal to one-half the time rate of decrease in an originally right angle between the filaments d**x**(1) and d**x**(2), Fig. 2.7. To see this, note that the dot product of material filaments d**x**(1) and d**x**(2) axes may be written as

$$\mathbf{dx}(1) \cdot \mathbf{dx}(2) = |\mathbf{dx}(1)||\mathbf{dx}(2)| \cos \theta_{12},$$

where θ_{12} is the angle between the two filaments. In the calculation of the material time derivative of the dot product above, $\mathbf{dx}(1) \cdot \mathbf{dx}(2)$, we will employ the formula $dv_i = d\dot{x}_i = L_{ij} dx_j$ that follows from (2.31). The material time derivative of both sides of the equation above is then computed;

$$\begin{aligned}\frac{D}{Dt}(\mathbf{dx}(1) \cdot \mathbf{dx}(2)) &= \mathbf{d\dot{x}}(1) \cdot \mathbf{dx}(2) + \mathbf{dx}(1) \cdot \mathbf{d\dot{x}}(2) \\ &= L_{ij} dx_j(1) dx_2(2) + dx_i(1) L_{ij} dx_j(2) = 2D_{ij} dx_i(1) dx_j(2) \\ &= |\mathbf{d\dot{x}}(1)||\mathbf{dx}(2)| \cos \theta_{12} + |\mathbf{dx}(1)||\mathbf{d\dot{x}}(2)| \cos \theta_{12} \\ &\quad - \dot{\theta}_{12} |\mathbf{dx}(1)||\mathbf{dx}(2)| \sin \theta_{12}\end{aligned}$$

and, since we are interested in the instant that $\theta_{12} = \pi/2$, it follows that

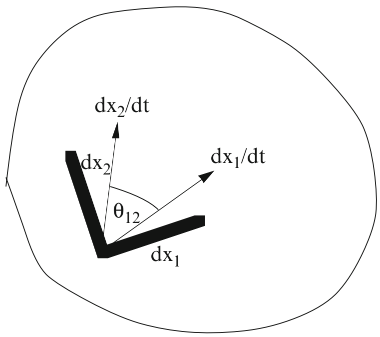

Fig. 2.7 An illustration for the geometric interpretation of the rate of shearing strain component D_{12} of the rate-of-deformation tensor **D**. The *heavy black lines* represent two material filaments of infinitesimal length that are instantaneously perpendicular. The *thin black lines* represent the same two material filaments in the next instant. The instantaneous time rate of change of the angle between the two material filaments, the rate at which the two filaments are coming together or separating is $\dot{\theta}_{12}$. The geometric interpretation of D_{12} is that it is one half the instantaneous time of decrease in an originally right angle between dx_1 and dx_2, $D_{12} = -\dot{\theta}_{12}/2$

$$2D_{ij}dx_i(1)dx_j(2) = -\dot{\theta}_{12}|\mathbf{dx}(1)|\,|\mathbf{dx}(2)|.$$

Finally, if we take $\mathbf{dx}(1) = dx_1\mathbf{e}_1$ and $\mathbf{dx}(2) = dx_2\mathbf{e}_2$ it may be concluded that

$$D_{12} = -\frac{\dot{\theta}_{12}}{2},$$

confirming that D_{12} is equal to one-half the material time rate of decrease in an originally right angle between dx_1 and dx_2. The geometric interpretations of D_{13} and D_{23} are similar. These geometric interpretations of the components of **D** as the instantaneous time rate of change of filaments, angles, and volume are the rationale for calling **D** the *rate of deformation* tensor.

Example 2.2.1
Calculate the velocity and acceleration in the material representation of the motion (2.12) of Example 2.1.1, then determine the spatial representation. Verify that the acceleration computed in the spatial representation is the same as the acceleration computed in the material representation. Calculate the tensor of velocity gradients **L**, the rate of deformation tensor **D**, and the spin tensor **W** for this motion.

Solution: The velocity and acceleration for this motion are given by (2.24) as

$$\dot{x}_1 = X_{\mathrm{I}} + X_{\mathrm{II}} + 3, \quad \dot{x}_2 = X_{\mathrm{I}} + X_{\mathrm{II}} + 2, \quad \dot{x}_3 = 0, \quad \ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0.$$

In order to find the spatial representation for this motion we must invert the system of equations (2.12) representing the motion, thus

$$X_{\mathrm{I}} = \frac{1}{1+2t}\{(1+t)x_1 - tx_2 - t^2 - 3t\},$$
$$X_{\mathrm{II}} = \frac{1}{1+2t}\{-tx_1 + (1+t)x_2 + t^2 - 2t\}, \quad X_{\mathrm{III}} = x_3$$

then, substituting these expressions into the previous equations for the velocities, the spatial representation of this motion is obtained:

$$v_1 = \frac{1}{1+2t}\{x_1 + x_2 + 3 + t\}, \quad v_2 = \frac{1}{1+2t}\{x_1 + x_2 + 2 - t\}, \quad v_3 = 0.$$

It is known from the first calculation in this example that this motion is one of zero acceleration, $\ddot{x}_1 = \ddot{x}_2 = \ddot{x}_3 = 0$. This may be verified by calculating the acceleration of the spatial representation of the motion above using the material time derivative (2.29), thus

$$\begin{aligned} a_1 &= \frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial x_1} + v_2\frac{\partial v_1}{\partial x_2} + v_3\frac{\partial v_1}{\partial x_3} \\ &= \frac{-2}{(1+2t)^2}\{x_1 + x_2 + 3 + t\} + \frac{1}{1+2t} + \frac{1}{(1+2t)^2}\{2(x_1+x_2)+5\} = 0, \end{aligned}$$

$$\begin{aligned} a_2 &= \frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial x_1} + v_2\frac{\partial v_2}{\partial x_2} + v_3\frac{\partial v_2}{\partial x_3} \\ &= \frac{-2}{(1+2t)^2}\{x_1 + x_2 + 2 - t\} - \frac{1}{1+2t} + \frac{1}{(1+2t)^2}\{2(x_1+x_2)+5\} = 0. \end{aligned}$$

The tensor of velocity gradients $\mathbf{L}$ for the motion (2.12) is obtained by substituting the spatial representation for the motion obtained above into (2.31); thus

$$\mathbf{L} = \frac{1}{1+2t}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The rate-of-deformation tensor $\mathbf{D}$ for this motion is equal to $\mathbf{L}$. The spin tensor $\mathbf{W}$ is zero for this motion.

Problems

2.2.1. For the first six motions of the form (2.10) given in Problem 2.1.1, namely 2.1.1(a) through 2.1.1(f), determine the velocity and acceleration in the material (Lagrangian) representation, the velocity and acceleration in the spatial (Eulerian) representation, and the three tensors $\mathbf{L}$, $\mathbf{D}$, and $\mathbf{W}$. Discuss briefly how these algebraic calculations relate to the geometry of the motion.

2.2.2. The motion of a continuum is given by: $x_1 = X_{\mathrm{I}} + X_{\mathrm{II}}t + X_{\mathrm{III}}t^2$, $x_2 = X_{\mathrm{II}} + X_{\mathrm{III}}t + X_{\mathrm{I}}t^2$, $x_3 = X_{\mathrm{III}} + X_{\mathrm{I}}t + X_{\mathrm{II}}t^2$.

(a) Find the inversion of this motion.
(b) Determine the velocity and the acceleration in the material representation.
(c) Find the velocity in the spatial (Eulerian) representation for this motion.
(d) Find the three tensors **L**, **D**, and **W** for this motion.
(e) Find the tensor of deformation gradients **F** for this motion.

2.2.3. The motion of a continuum is given by:

$$x_1 = X_{\mathrm{I}} + X_{\mathrm{II}} \sin(\pi t), \quad x_2 = X_{\mathrm{II}} - X_{\mathrm{I}} \sin(\pi t), \quad x_3 = X_{\mathrm{III}}.$$

(a) Determine the deformation gradient **F** of this deformation.
(b) Determine the instantaneous configuration image of the set of points $(X_{\mathrm{I}})^2 + (X_{\mathrm{II}})^2 = 1$ in the reference configuration.
(c) Describe the geometry of the set of points $(X_{\mathrm{I}})^2 + (X_{\mathrm{II}})^2 = 1$ in the reference configuration and describe what happens to this set of points in the motion of the continuum as time t increases.

2.3 Infinitesimal Motions

The term infinitesimal motion is used to describe the case when the deformation, including rotation, is small. This does not mean that the displacement vector is small; one can have large displacements but small strain infinitesimal motions. Large displacements associated with small strain infinitesimal motions occur in very thin long rods. The criterion for infinitesimal motion is that the square of the gradients of displacement be small compared to the gradients of displacement themselves. Thus, for infinitesimal motions, the squares and products of the nine quantities

$$\frac{\partial u_1}{\partial x_1}, \frac{\partial u_1}{\partial x_2}, \frac{\partial u_1}{\partial x_3}, \frac{\partial u_2}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_2}{\partial x_3}, \frac{\partial u_3}{\partial x_1}, \frac{\partial u_3}{\partial x_2}, \frac{\partial u_3}{\partial x_3} \tag{2.37}$$

must be small compared to their own values. This means, for example, $\{\partial u_2/\partial x_1\}^2$ is required to be much smaller than $\partial u_2/\partial x_1$; each such square and product of these nine quantities is so small that it may be neglected compared to the quantity itself. Using this criterion of smallness, representations of the kinematics variables for infinitesimal motions will be developed in this section.

If the motion is infinitesimal the deformation gradient tensor **F** must not deviate significantly from the unit tensor **1**, the magnitude of the deviation being restricted by the criterion on the deformation gradients stated in the previous paragraph.

The deformation gradient $\mathbf{F}$ may be expressed, using (2.22), in terms of $[\nabla_{\mathbf{O}} \otimes \mathbf{u}]^{\mathrm{T}}$, which is a matrix of components $[\partial u_i/\partial X_\alpha]$, as

$$\mathbf{F} = \mathbf{1} + [\nabla_{\mathbf{O}} \otimes \mathbf{u}]^{\mathrm{T}}. \tag{2.38}$$

Since

$$\nabla_{\mathbf{O}} = \mathbf{F}^{\mathrm{T}} \cdot \nabla \quad \text{or} \quad \frac{\partial}{\partial X_\alpha} = \frac{\partial}{\partial x_j}\frac{\partial x_j}{\partial X_\alpha} = \frac{\partial}{\partial x_j} F_{j\alpha} \tag{2.39}$$

it follows from (2.38) and a result obtained in Appendix A, namely that the transpose of a product of matrices is equal to the product of the transposed matrices in reverse order, $[\mathbf{AB}]^{\mathrm{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$, that

$$\mathbf{F} = \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot \mathbf{F}. \tag{2.40}$$

This result may be used as a recursion formula for $\mathbf{F}$. In that role this formula for F can be substituted into itself once,

$$\mathbf{F} = \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot \mathbf{F} \tag{2.41}$$

and then again and again,

$$\mathbf{F} = \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot [\nabla \otimes \mathbf{u}]^{\mathrm{T}} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot [\nabla \otimes \mathbf{u}]^{\mathrm{T}} \cdot [\nabla \otimes \mathbf{u}]^{\mathrm{T}} + \text{h.o.t.}, \tag{2.42}$$

where h.o.t. stands for "higher order terms." If the terms of second order according to the criterion (2.37) are neglected, the $\mathbf{F}$ is approximated by

$$\mathbf{F} \approx \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}}. \tag{2.43}$$

From (2.22) it is known that

$$\mathbf{F}^{-1} = \mathbf{1} - [\nabla \otimes \mathbf{u}]^{\mathrm{T}}, \tag{2.44}$$

a formula that is accurate in the approximation because $\mathbf{F}\cdot\mathbf{F}^{-1} = \mathbf{1}$ when terms of second order are neglected.

Two important conclusions may be made from this result. First, for infinitesimal motions the difference between the use of material and spatial coordinates is insignificant, thus $\mathbf{X}$ and $\mathbf{x}$ are equivalent as are the gradient operators $\nabla_{\mathbf{O}}$ and ∇. Concerning these operators note from (2.39) that

$$\nabla_{\mathbf{O}} \otimes \mathbf{u} = \mathbf{F}^{\mathrm{T}} \cdot [\nabla \otimes \mathbf{u}] \tag{2.45}$$

and, substituting for $\mathbf{F}$ using (2.42),

$$\nabla_{\mathbf{O}} \otimes \mathbf{u} = \nabla \otimes \mathbf{u} + [\nabla \otimes \mathbf{u}] \cdot [\nabla \otimes \mathbf{u}]$$

thus, to neglect terms of second order,

$$\nabla_{\mathbf{O}} \otimes \mathbf{u} \approx \nabla \otimes \mathbf{u}. \tag{2.46}$$

For infinitesimal motions, the movement of boundaries due to motion is neglected because the small movement is equivalent to the difference in the use of material and spatial coordinates, which is insignificant. Therefore in all the following considerations of infinitesimal motions the coordinates $\mathbf{x}$ will be used without reference to their material or spatial character, because the result is correct independent of their character. The second important conclusion is that, for infinitesimal motions, $\mathbf{F}$ has the representation

$$\mathbf{F} = \mathbf{1} + [\nabla \otimes \mathbf{u}(\mathbf{x}, t)]^{\mathrm{T}}. \tag{2.47}$$

In the special case when the infinitesimal motion is a rigid object rotation, $\mathbf{F} = \mathbf{Q}$ and $\mathbf{Q} = \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}}$. The requirement that $\mathbf{Q}$ be orthogonal, $\mathbf{Q}^{\mathrm{T}} \cdot \mathbf{Q} = \mathbf{Q} \cdot \mathbf{Q}^{\mathrm{T}} = \mathbf{1}$, $\mathbf{Q} \cdot \mathbf{Q}^{\mathrm{T}} = (\mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}}) \cdot (\mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}})^{\mathrm{T}} = \mathbf{1} + [\nabla \otimes \mathbf{u}]^{\mathrm{T}} + (\nabla \otimes \mathbf{u}) + (\nabla \otimes \mathbf{u})^{\mathrm{T}} \cdot (\nabla \otimes \mathbf{u}) = \mathbf{1}$, means that

$$(\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u} = 0, \tag{2.48}$$

since $(\nabla \otimes \mathbf{u})^{\mathrm{T}} \cdot (\nabla \otimes \mathbf{u})$ represents terms of the second order terms that are neglected. Defining the symmetric and skew symmetric parts of $\nabla \otimes \mathbf{u}$ as $\mathbf{E}$ and $\mathbf{Y}$,

$$\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u}), \quad \mathbf{Y} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} - \nabla \otimes \mathbf{u}), \tag{2.49}$$

it is seen from (2.48) that $\mathbf{E}$ must be zero when the infinitesimal motion is a rigid object rotation. It may also be seen that the orthogonal rotation $\mathbf{Q}$ characterizing the infinitesimal rigid object rotation is given by

$$\mathbf{Q} = \mathbf{1} + \mathbf{Y}, \tag{2.50}$$

where $\mathbf{Y}$, defined by (2.49), is skew symmetric, $\mathbf{Y} = -\mathbf{Y}^{\mathrm{T}}$, and $\mathbf{Y}\mathbf{Y}^{\mathrm{T}}$ is a second order term, since it is a square of the coefficients (2.37) which are neglected compared to the values of $\mathbf{Y}$.

Returning to the total infinitesimal motion, the definitions (2.49) of $\mathbf{E}$ and $\mathbf{Y}$ may be used to rewrite (2.42) as

$$\mathbf{F} = \mathbf{1} + \mathbf{E} + \mathbf{Y}. \tag{2.51}$$

It has been established that $\mathbf{F}$ represents only the rotational and deformational motion because the translational portion of the motion, being independent of the coordinates, was removed by spatial or material differentiation. It has also been noted that the special case of $\mathbf{F} = \mathbf{1}$ corresponds to no rotational and no deformational motion. Further it has been shown (2.50) that $\mathbf{Y}$ is associated with pure rigid object rotation. This means that $\mathbf{E}$ must be the tensor representing the deformation. This is indeed the case, as will be shown below. $\mathbf{E}$ is called the *infinitesimal strain tensor* and $\mathbf{Y}$ is called the *infinitesimal rotation tensor*. The representation (2.51) for the tensor of deformation gradients then demonstrates that, for infinitesimal motions, $\mathbf{F} - \mathbf{1}$ may be decomposed into the sum of two terms, $\mathbf{E}$ and $\mathbf{Y}$, which represent the deformational and rigid rotational characteristics of the infinitesimal motion, respectively.

The strain tensor $\mathbf{E}$, defined by the first of (2.49), has the component representation

$$\mathbf{E} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right] = \frac{1}{2}\begin{bmatrix} 2\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} & 2\frac{\partial u_2}{\partial x_2} & \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} & \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} & 2\frac{\partial u_3}{\partial x_3} \end{bmatrix}. \tag{2.52}$$

The components of $\mathbf{E}$ along the diagonal are called normal strains and the components off the diagonal are called shear strains. The normal strains, E_{11}, E_{22}, and E_{33}, are measures of change in length per unit length along the 1, 2 and 3 axes, respectively, and the shear strains, E_{23}, E_{13}, and E_{12}, are one-half of the changes in the angle between the 2 and 3 axes, the 1 and 3 axes and the 1 and 2 axes, respectively.

The geometric interpretation of the components of the strain tensor $\mathbf{E}$ stated in the previous paragraph will be analytically developed here. Let dx_1 be a vector of infinitesimal length representing the present position of an infinitesimal material filament coinciding with the x_1 at time t. The displacement of this material filament instantaneously coincident with dx_1 is $du_1 = E_{11}dx_1$, a result that follows from the entry in the first column and first row of (2.52). The expression $du_1 = E_{11}dx_1$ is the change in length of dx_1 as a consequence of the strain as illustrated in Fig. 2.8. Thus the geometric interpretation of $E_{11} = (du_1/dx_1)$ is that it is the change in length per unit length of dx_1. Similar geometric interpretations exist for E_{22} and E_{33}.

The geometric interpretation of the normal strain components E_{11}, E_{22} and E_{33} is easily extended to obtain a geometric interpretation of the trace of the small strain tensor tr $\mathbf{E}$, or equivalently the divergence of the displacement field $\nabla\cdot\mathbf{u}$, tr $\mathbf{E} = \nabla\cdot\mathbf{u}$. If $dv_o = dx_1dx_2dx_3$ represents an undeformed element of volume (Fig. 2.9), the deformed volume is given by $dv = (dx_1 + du_1)(dx_2 + du_2)(dx_3 + du_3)$. Using $du_1 = E_{11}dx_1$, $du_2 = E_{22}dx_2$ and $du_3 = E_{33}dx_3$, the deformed volume is given by $dv = (1 + E_{11})(1 + E_{22})(1 + E_{33})dv_o$. Expanding $dv = (1 + E_{11})(1 + E_{22})(1 + E_{33})dv_o$ and recognizing that the squares of displacement gradients (2.37)

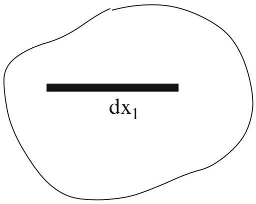

Fig. 2.8 An illustration for the geometric interpretation of the normal strain component E_{11}. The *left* and *right* illustrations of this figure represent the undeformed and deformed configurations, respectively. The *heavy black line* represents the same material filament in the two configurations. E_{11} is equal to the change in length per unit length of the filament between the two configurations. The original length is dx_1 and the change in length due to the deformation is du_1, thus $E_{11} = du_1/dx_1$

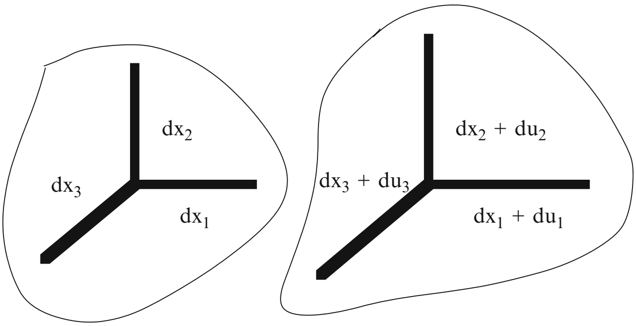

Fig. 2.9 An illustration for the geometric interpretation of the trace of the strain tensor, tr $\mathbf{E}$, or the divergence of the displacement field, $\nabla \cdot \mathbf{u}$, tr$\mathbf{E} = \nabla \cdot \mathbf{u}$. The *left* and *right* illustrations of this figure represent the undeformed and deformed configurations, respectively. The *heavy black lines* represent the same material filaments in the two configurations. The volume element in the undeformed configuration is $dv_o = dx_1 dx_2 dx_3$ and the deformed volume is given by $dv = (dx_1 + du_1)(dx_2 + du_2)(dx_3 + du_3)$. It may be shown (see text) that $dv = (1 + \text{tr}\mathbf{E})dv_o$. Thus the tr$\mathbf{E}$ $= \nabla \cdot \mathbf{u}$ represents the change in volume per unit volume, $(dv - dv_o)/dv_o$

may be neglected, it follows that $dv = (1 + \text{tr}\mathbf{E})dv_o$. Thus the tr$\mathbf{E}$ represents the change in volume per unit volume, $(dv - dv_o)/dv_o$.

The off-diagonal components of the strain tensor, for example E_{12}, represent the shearing strains. E_{12} is equal to one-half the change in angle that was originally a right angle between the x_1 and x_2 axes. To construct this geometric result algebraically, the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are considered, see Fig. 2.10. After deformation these vectors are $\mathbf{F}\mathbf{e}_1$ and $\mathbf{F}\mathbf{e}_2$, respectively, or, since $\mathbf{F} = \mathbf{1} + \mathbf{E}$, the deformed vectors are given by $\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1$ and $\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2$, respectively. The dot product of the vectors $\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1$ and $\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2$ is $(\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1)\cdot(\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2) = \mathbf{e}_1\cdot\mathbf{e}_2 + \mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_2 + \mathbf{e}_2\cdot\mathbf{E}\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_2$, but since the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal, $\mathbf{e}_1\cdot\mathbf{e}_2 = 0$, and also since $\mathbf{E}\mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_2$ is a higher order term because it contains the squares of the displacement gradients (2.37), this expression reduces to $(\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1)\cdot(\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2) = \mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_2 + \mathbf{e}_2\cdot\mathbf{E}\mathbf{e}_1$. This result is further reduced by noting that $\mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_2 = \mathbf{e}_2\cdot\mathbf{E}\mathbf{e}_1 = E_{12}$,

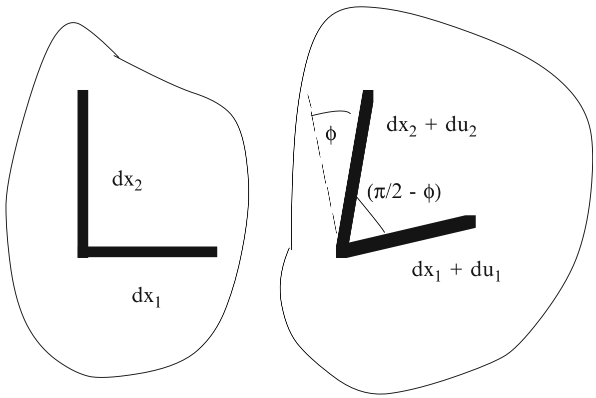

Fig. 2.10 Illustration for the geometric interpretation of the shearing strain E_{12}. The *left* and *right* illustrations of this figure represent the undeformed and deformed configurations, respectively. The *heavy black lines* represent the same material filaments in the two configurations. E_{12} is equal to one-half the change in angle that was originally a right angle between the x_1 and x_2 axes, $\phi/2$ in this figure

thus it follows that $(\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1)\cdot(\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2) = 2E_{12}$. Recalling the formula (A61) for the dot product of two vectors, say **u** and **v**, as equal to the magnitude of the first times the magnitude of the second times the cosine of the angle (say ζ) between them, $\mathbf{u} \cdot \mathbf{v} = u_i v_i = |\mathbf{u}| \cdot |\mathbf{v}| \cos\zeta$, it follows that $2E_{12} = |\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1|\,|\mathbf{e}_2 + \mathbf{E}\mathbf{e}_2| \cos(\pi/2 - \phi)$, where the angle $(\pi/2 - \phi)$ is illustrated in Fig. 2.10. The magnitude of $|\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1|$ is the square root of $(\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1)\cdot(\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1) = \mathbf{e}_1\cdot\mathbf{e}_1 + \mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_1 + \mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_1 + \mathbf{E}\mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_1$, but since $\mathbf{e}_1\cdot\mathbf{e}_1 = 1$ and $\mathbf{E}\mathbf{e}_1\cdot\mathbf{E}\mathbf{e}_1$ is a higher order term, this reduces to the square root of $1 + 2E_{11}$, by a parallel of the arguments used above to obtain the formula for $2E_{12}$. At this point a classical approximation is used. This approximation is that $1 + \varepsilon \approx \sqrt{(1 + 2\varepsilon)}$ if summands of the order ε^2 may be neglected; the proof of this approximation follows easily if one squares it. Then, since the square of E_{11} is a higher order term, the square root of $1 + 2E_{11}$ is given by $1 + E_{11}$, thus $2E_{12} = (1 + E_{11})(1 + E_{22}) \cos(\pi/2 - \phi)$ or $2E_{12} = (1 + E_{11})(1 + E_{22})\sin\phi$ or expanding; $2E_{12} = \sin\phi + (E_{11} + E_{22})\sin\phi + E_{11}E_{22}\sin\phi$. Finally, since the angle ϕ is small, $\sin\phi$ is small as are E_{11} and E_{22}, thus the neglect of higher order terms gives $2E_{12} = \phi$, and the interpretation of E_{12} as one-half the change in an angle that was originally a right angle between the x_1 and x_2 axes (Fig. 2.10). These geometric interpretations of the components of **E** as the change in the length of filaments, the change in angles and the change in volume deformation between the undeformed and the deformed configurations are the rationale for calling **E** the *strain* tensor.

Example 2.3.1
The deformation gradient and the inverse deformation gradient for the motion given by (2.12) were computed in Example 2.1.4. Determine the restriction on the motion given by (2.12) so the motion is infinitesimal. Find the strain tensor **E** and the rotation tensor **Y** for the infinitesimal motion.

Solution: Comparison of the expressions $\nabla_{\mathbf{O}} \otimes \mathbf{u}(\mathbf{X}, t)$ and $\nabla \otimes \mathbf{u}(\mathbf{x}, t)$ obtained in Example 2.1.4 shows that these two expressions coincide only for very small times t, only if t^2 is much less than t. In this case $\nabla_{\mathbf{O}} \otimes \mathbf{u}(\mathbf{X}, t) = \nabla \otimes \mathbf{u}(\mathbf{x}, t)$ and

$$\nabla \otimes \mathbf{u} = t \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From this expression for $\nabla \otimes \mathbf{u}$ and (2.49), the rotation tensor is determined to be $\mathbf{Y} = \mathbf{0}$, and the strain tensor $\mathbf{E}$ is given by

$$\mathbf{E} = t \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

as long as t is small.

Problem

2.3.1. For the motions of the form (2.10) given in Problem 2.1.1, namely 2.1.1(a) through 2.1.1(g), determine the conditions under which the motion remains infinitesimal and compute the infinitesimal strain and rotation tensors, $\mathbf{E}$ and $\mathbf{Y}$. Discuss briefly the significance of each of the seven strain tensors computed. In particular, explain the form or value of the strain tensor in terms of the motion.

2.4 The Strain Conditions of Compatibility

Calculating the strain tensor $\mathbf{E}$ given the displacement field $\mathbf{u}$ is a relatively simple matter; one just substitutes the displacement field $\mathbf{u}$ into the formula (2.49) for the strain displacement relations, $\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u})$. Situations occur in which it is desired to calculate the displacement field $\mathbf{u}$ given the strain tensor $\mathbf{E}$. This inverse problem is more difficult because the strain displacement relations, $\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u})$, become a system of first order partial differential equations for the displacement field $\mathbf{u}$. Given the significance of the displacement field $\mathbf{u}$ in an object we generally want to insure that the displacement field $\mathbf{u}$ is continuous and single valued. There are real situations in which the displacement field $\mathbf{u}$ might be discontinuous and multiple valued, but these situations will be treated as special cases. In general it is desired that the integral of the strain–displacement relations, the displacement field $\mathbf{u}$, is continuous and single valued. The conditions of compatibility insure this. The conditions of compatibility are equations that the strain tensor must satisfy so that when the strain–displacement relations are integrated, the resulting displacement field $\mathbf{u}$, is continuous and single valued. The conditions of compatibility may be written in the direct notation as

$$\nabla \times \mathbf{E} \times \nabla = 0 \tag{2.53}$$

or in the index notation as

$$e_{ijk} e_{pmn} \frac{\partial^2 E_{jm}}{\partial x_k \partial x_n} = 0 \tag{2.54}$$

or in scalar form as the following six equations:

$$\begin{aligned}
&\frac{\partial^2 E_{11}}{\partial x_2 \partial x_3} = \frac{\partial}{\partial x_1}\left\{-\frac{\partial E_{23}}{\partial x_1} + \frac{\partial E_{31}}{\partial x_2} + \frac{\partial E_{12}}{\partial x_3}\right\}, \quad 2\frac{\partial^2 E_{12}}{\partial x_1 \partial x_2} = \frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_1^2}, \\
&\frac{\partial^2 E_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left\{-\frac{\partial E_{31}}{\partial x_2} + \frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1}\right\}, \quad 2\frac{\partial^2 E_{23}}{\partial x_2 \partial x_3} = \frac{\partial^2 E_{22}}{\partial x_3^2} + \frac{\partial^2 E_{33}}{\partial x_2^2}, \\
&\frac{\partial^2 E_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left\{-\frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1} + \frac{\partial E_{31}}{\partial x_2}\right\}, \quad 2\frac{\partial^2 E_{31}}{\partial x_3 \partial x_1} = \frac{\partial^2 E_{33}}{\partial x_1^2} + \frac{\partial^2 E_{11}}{\partial x_3^2}.
\end{aligned} \tag{2.55}$$

Equations (2.53) and (2.54) are symmetric second rank tensors in three dimensions and therefore have the six components given by (2.55). It follows that each of the six scalar equations (2.55) must be satisfied in order to insure compatibility. The conditions (2.53) are a direct consequence of the definition of strain, that is to say that $\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u}) = (1/2)(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u})$ implies that $\nabla \times \mathbf{E} \times \nabla = 0$. To see that this is true, consider the result of operating on $\mathbf{E} = (1/2)(\mathbf{u} \otimes \nabla + \nabla \otimes \mathbf{u})$ from the left by $\nabla \times$ and from the right by $\times \nabla$; one obtains the expression

$$2(\nabla \times \mathbf{E} \times \nabla) = \nabla \times \mathbf{u} \otimes \nabla \times \nabla + \nabla \times \nabla \otimes \mathbf{u} \times \nabla. \tag{2.56}$$

The operator $\nabla \times \nabla$, which occurs in both terms on the right hand side of (2.56) is called the "curl grad"; the curl of the gradient applied to a function f is zero, $\nabla \times \nabla f = 0$. In the indicial notation this is easy to see, $\nabla \times \nabla f = e_{ijk}(\partial f/\partial x_j \partial x_k)$ $\mathbf{e}_i = 0$, because of the symmetry of the indices on the partial derivatives and skew-symmetry in the components of the alternator (see Appendix A.8). Both terms on the right hand side of (2.56) contain the operator curl grad, $\nabla \times \nabla$, applied to a function, hence $\nabla \times \mathbf{E} \times \nabla = 0$. It may also be shown that the reverse is true, namely that $\nabla \times \mathbf{E} \times \nabla = 0$ implies that $\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u})$. Thus $\mathbf{E} = (1/2)((\nabla \otimes \mathbf{u})^{\mathrm{T}} + \nabla \otimes \mathbf{u})$ is a necessary and sufficient condition that $\nabla \times \mathbf{E} \times \nabla = 0$.

In order to both prove and motivate this result consider the two integration paths from the point P^{o} to the point P' in an object (Fig. 2.11). If the result of the integration from the point P^{o} to the point P' is to be the same along all paths chosen between these two points, then the value of the integral around any closed path in the object must be zero. This means that the integrand of the integral must be an exact differential (see Appendix A.15 Exact differentials). Recall the theorem at the

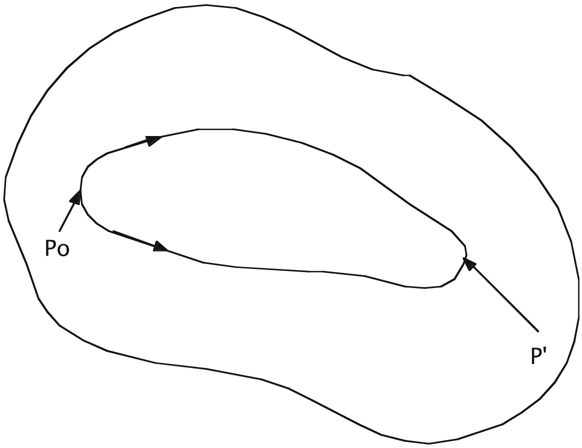

Fig. 2.11 Illustration of two integration paths from the point P^o to the point P' in an object. If the result of the integration from the point P^o to the point P' is to be the same along all paths chosen between these two points, then the value of the integral around any closed path in the object must be zero. This means that the integrand of the integral must be an exact differential

start of most texts on ordinary differential equations concerning exact differentials: If $M(x, y)$ and $N(x, y)$ are continuous functions and have continuous partial derivatives in a region of the x–y plane, then the expression $M(x, y)dx + N(x, y)dy$ is an exact differential if and only if $\partial M/\partial y = \partial N/\partial x$ throughout the region. This theorem will be applied to prove that the compatibility relations $\nabla \times \mathbf{E} \times \nabla = 0$ are both necessary and sufficient conditions for the continuous and single-valued nature of the displacement field obtained by integration from the strain–displacement relations. If the displacement vector is known at the point P^o then integration of d$\mathbf{u}$ from the point P^o to the point P' (Fig. 2.11) will determine $\mathbf{u}(\mathbf{x}')$, thus,

$$\mathbf{u}(\mathbf{x}') = \mathbf{u}^o + \int_{P^o}^{P'} \mathrm{d}\mathbf{u} = \mathbf{u}^o + \int_{P^o}^{P'} (\nabla \otimes \mathbf{u})^T \cdot \mathrm{d}\mathbf{x}. \tag{2.57}$$

Recall from (2.43) and (2.51) that

$$(\nabla \otimes \mathbf{u})^T = \mathbf{E} + \mathbf{Y} \tag{2.58}$$

it follows that

$$\mathbf{u}(\mathbf{x}') = \mathbf{u}^o + \int_{P^o}^{P'} \mathbf{E} \cdot \mathrm{d}\mathbf{x} + \int_{P^o}^{P'} \mathbf{Y} \cdot \mathrm{d}\mathbf{x}. \tag{2.59}$$

The last integral in the previous result may be rewritten as

$$\int_{P^o}^{P'} \mathbf{Y} \cdot \mathrm{d}\mathbf{x} = \int_{P^o}^{P'} \mathbf{Y} \cdot \mathrm{d}(\mathbf{x} - \mathbf{x}') \tag{2.60}$$

and integrated by parts, thus

$$\int_{P^o}^{P'} \mathbf{Y} \cdot \mathrm{d}\mathbf{x} = -\mathbf{Y}^o \cdot (\mathbf{x}^o - \mathbf{x}') + \int_{P^o}^{P'} \mathrm{d}\mathbf{x} \cdot \nabla \otimes \mathbf{Y} \cdot (\mathbf{x} - \mathbf{x}'). \tag{2.61}$$

Placing the result (2.61) into (2.59) it follows that

$$\mathbf{u}(\mathbf{x}') = \mathbf{u}^o - \mathbf{Y}^o \cdot (\mathbf{x}^o - \mathbf{x}') + \int_{P^o}^{P'} \mathrm{d}\mathbf{x} \cdot [\mathbf{E} + \nabla \otimes \mathbf{Y} \cdot (\mathbf{x} - \mathbf{x}')] \tag{2.62}$$

or, in the indicial notation,

$$u_i(\mathbf{x}') = u_i^o - Y_{ik}^o(x_k^o - x_k') + \int_{P^o}^{P'} \left[E_{im} - \frac{\partial Y_{ik}}{\partial x_m}(x_k - x_k') \right] \mathrm{d}x_m. \tag{2.63}$$

The relationship between the derivatives of the rotation and strain tensors,

$$\frac{\partial Y_{ik}}{\partial x_m} = \frac{\partial E_{im}}{\partial x_k} - \frac{\partial E_{mk}}{\partial x_i}, \tag{2.64}$$

may easily be verified by substituting the formulas (2.49) relating **E** and **Y** to the displacement gradients. When the relationship (2.64) is substituted into (2.63) it becomes

$$u_i(\mathbf{x}') = u_i^o - Y_{ik}^o(x_k^o - x_k') + \int_{P^o}^{P'} R_{im} \, \mathrm{d}x_m, \tag{2.65}$$

where

$$R_{im} = E_{im} - \left\{ \frac{\partial E_{im}}{\partial x_k} - \frac{\partial E_{mk}}{\partial x_i} \right\} (x_k - x_k'). \tag{2.66}$$

The condition that the integrand in the integral in (2.65) be an exact differential is then expressed as the condition

$$\frac{\partial R_{im}}{\partial x_k} = \frac{\partial R_{ik}}{\partial x_m}. \tag{2.67}$$

When (2.67) is substituted into (2.66), the result

$$0 = \left\{ \frac{\partial^2 E_{mq}}{\partial x_i \partial x_k} + \frac{\partial^2 E_{ik}}{\partial x_q \partial x_m} - \frac{\partial^2 E_{kq}}{\partial x_i \partial x_m} - \frac{\partial^2 E_{im}}{\partial x_q \partial x_k} \right\} (x_q - x'_q) \tag{2.68}$$

is satisfied only when the compatibility conditions (2.54), or equivalently (2.54) or (2.55) or $\nabla \times \mathbf{E} \times \nabla = 0$, hold. Thus $\nabla \times \mathbf{E} \times \nabla = 0$ is a necessary and sufficient condition that the integration of the strain–displacement relations will yield a single-valued and continuous displacement field.

Problems

2.4.1. For the motions of the form (2.10) given in Problem 2.1.1, namely 2.1.1(a) through 2.1.1(g), determine if the infinitesimal strain tensors, **E**, calculated in 2.3.1 satisfy the conditions of compatibility.

2.4.2. Is the following strain state possible for an object in which the displacement field must be continuous and single valued? Justify your answer analytically.

$$\mathbf{e} = c \begin{bmatrix} x_3(x_1^2 + x_2^2) & x_1 x_2 x_3 & 0 \\ x_1 x_2 x_3 & x_3 x_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.4.3. Demonstrate the validity of the formula (2.64) by substituting the formulas relating **E** and **Y** to the displacement gradients (2.49) into (2.64) and show that an identity is obtained. This is more easily done in the indicial notation.

2.4.4. Verify that substitution of the formula (2.67) into (2.66) leads to the result (2.68). This is much more easily done in the indicial notation.

http://www.springer.com/978-1-4614-5024-5

Continuum Mechanics of Anisotropic Materials
Cowin, S.C.
2013, XIII, 425 p., Hardcover
ISBN: 978-1-4614-5024-5