Chapter 1
Uniformities and Topologies

In this chapter I define uniform structures and uniformly continuous mappings in the language of pseudometrics. I derive their basic properties and their relationship to topologies.

1.1 Uniform Structures and Mappings

There are several equivalent ways to define uniform spaces. I find the definition based on pseudometrics most convenient, because it simplifies the construction of uniformly continuous functions.

When Δ_0 and Δ_1 are two pseudometrics on a non-empty set S, consider the pointwise maximum $\Delta_0 \vee \Delta_1$ of Δ_0 and Δ_1:

$$(\Delta_0 \vee \Delta_1)(x,y) := \Delta_0(x,y) \vee \Delta_1(x,y) \text{ for } x,y \in S.$$

Clearly $\Delta_0 \vee \Delta_1$ is also a pseudometric.

When $\mathscr{P}$ is a set of pseudometrics on a set S and Δ is a pseudometric on S, write $\Delta \ll \mathscr{P}$ iff

$$\forall \varepsilon > 0\ \exists \Delta_\varepsilon \in \mathscr{P}\ \exists \theta > 0\ \forall x,y \in S\ \ [\, \Delta_\varepsilon(x,y) < \theta\ \Rightarrow\ \Delta(x,y) < \varepsilon \,].$$

Definition 1.1. A *uniform structure* (or a *uniformity*, for short) on a non-empty set S is a set $\mathscr{U}$ of pseudometrics on S with these properties:

(U1) If $\Delta_0, \Delta_1 \in \mathscr{U}$, then $\Delta_0 \vee \Delta_1$.
(U2) If Δ is a pseudometric on S such that $\Delta \ll \mathscr{U}$, then $\Delta \in \mathscr{U}$.
(U3) If $x,y \in S$, $x \neq y$, then there is $\Delta \in \mathscr{U}$ such that $\Delta(x,y) > 0$.

A *uniform space* X is a non-empty set S (*set of points of* X) together with a uniformity on S. Let $\mathsf{UP}(X)$ denote the uniformity of X (the set of pseudometrics), and let $\mathsf{UP_b}(X)$ denote the set of bounded pseudometrics in $\mathsf{UP}(X)$. ■

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Some authors define uniform spaces using only conditions equivalent to (U1) and (U2); in their terminology, uniform spaces as defined here would be called *Hausdorff* (or *separated*) uniform space.

From now on, the set of points of any uniform space X is written simply as X in expressions such as $|X|$, $x \in X$ and $Y \subseteq X$, and "the set X" means the set of points of X. Note however that extra care is needed for expressions that include equality. When X and Y are uniform spaces, $X = Y$ means that X and Y are equal as uniform spaces, not merely as sets or some other objects with some structure "forgotten".

Bounded pseudometrics are in some respects better behaved than general (unbounded) ones. Fortunately it is often possible to replace a general pseudometric by a bounded one. If Δ is a pseudometric, then $1 \wedge \Delta$ is also a pseudometric, and $\Delta \in \mathsf{UP}(X)$ if and only if $1 \wedge \Delta \in \mathsf{UP}(X)$.

Every uniform space defines a topology on its set of points, as follows: If $x \in X$ and Δ is a pseudometric on X, write

$$\odot[x, \Delta] := \{y \in X \mid \Delta(x, y) < 1\}.$$

Theorem 1.2. *If X is a uniform space, then the collection of sets $\odot[x, \Delta]$, where $\Delta \in \mathsf{UP}(X)$ and $x \in X$, is a base of a Hausdorff topology on the set of points of X.*

Proof. To prove that the collection is a base of a topology, take any $x_0, x_1 \in X$, $\Delta_0, \Delta_1 \in \mathsf{UP}(X)$ and $x \in \odot[x_0, \Delta_0] \cap \odot[x_1, \Delta_1]$. Then

$$\odot[x, \Delta/r] \subseteq \odot[x_0, \Delta_0] \cap \odot[x_1, \Delta_1]$$

where $r := \min(1 - \Delta(x, x_0), 1 - \Delta(x, x_1)) > 0$ and $\Delta := \Delta_0 \vee \Delta_1$. The pseudometric Δ/r belongs to $\mathsf{UP}(X)$ because $\Delta/r \ll \{\Delta\}$ and $\Delta \in \mathsf{UP}(X)$.

In view of property (U3) in Definition 1.1, the resulting topology is Hausdorff.

□

For any uniform space X, the topology defined in Theorem 1.2 is called simply *the topology* of X. Thus every uniform space X is also a topological space, which gives meaning to open and closed subsets of X, continuous mappings and homeomorphisms to and from X, and expressions such as $\mathsf{C}(X)$ and $\mathsf{C_b}(X)$. By Corollary 1.23, the topology of every uniform space is completely regular.

The following description of the closure operation in the topology of X is often useful:

Theorem 1.3. *For every uniform space X, $x \in X$ and $A \subseteq X$, the following are equivalent:*

(i) $x \in \overline{A}$ *(x is in the closure of A).*
(ii) $\Delta(x, A) = 0$ *for every* $\Delta \in \mathsf{UP}(X)$.
(iii) $\Delta(x, A) < 1$ *for every* $\Delta \in \mathsf{UP}(X)$.
(iv) $\Delta(x, A) = 0$ *for every* $\Delta \in \mathsf{UP_b}(X)$.
(v) $\Delta(x, A) < 1$ *for every* $\Delta \in \mathsf{UP_b}(X)$.

Proof. Evidently (ii)⇒(iii)⇒(v) and (ii)⇒(iv)⇒(v), and (i)⇔(iii) from the definition of the topology of X.

For every $\Delta \in \mathsf{UP}(X)$ and $r > 0$, the pseudometric $1 \wedge r\Delta$ is in $\mathsf{UP}_\mathsf{b}(X)$. That proves (v)⇒(ii). □

For a mapping $\varphi\colon X \to Y$ and a pseudometric Δ on Y, the pseudometric $\overleftarrow{\varphi}\Delta$ on X is defined by $\overleftarrow{\varphi}\Delta(x,y) := \Delta(\varphi(x),\varphi(y))$ for $x,y \in X$. For a set Φ of mappings from X to Y and a bounded pseudometric Δ on Y, the pseudometric $\overleftarrow{\Phi}\Delta$ on X is defined by

$$\overleftarrow{\Phi}\Delta(x,y) := \sup\,\{\,\overleftarrow{\varphi}\Delta(x,y) \mid \varphi \in \Phi\,\}\,,\ x,y \in X.$$

Definition 1.4. When X and Y are uniform spaces, a mapping $\varphi\colon X \to Y$ is *uniformly continuous* iff $\overleftarrow{\varphi}\Delta \in \mathsf{UP}(X)$ for every $\Delta \in \mathsf{UP}(Y)$. A mapping $\varphi\colon X \to Y$ is a *uniform isomorphism* iff it is bijective and both φ and its inverse are uniformly continuous.

A set Φ of mappings from X to Y is *uniformly equicontinuous* iff $\overleftarrow{\Phi}\Delta \in \mathsf{UP}(X)$ for every $\Delta \in \mathsf{UP}_\mathsf{b}(Y)$. ■

Clearly $\varphi\colon X \to Y$ is uniformly continuous if and only if $\overleftarrow{\varphi}\Delta \in \mathsf{UP}(X)$ for every bounded pseudometric $\Delta \in \mathsf{UP}(Y)$. Thus φ is uniformly continuous if and only if the singleton set $\{\varphi\}$ is uniformly equicontinuous.

Theorem 1.5. *Let X and Y be two uniform spaces. Every uniformly continuous mapping from X to Y is continuous.*

Proof. Let $\varphi\colon X \to Y$ be a uniformly continuous mapping. To prove that φ is continuous, it is enough to show that the set $\varphi^{-1}(\odot[y,\Delta])$ is open in X for every $\Delta \in \mathsf{UP}(Y)$ and $y \in Y$.

Take any $x \in \varphi^{-1}(\odot[y,\Delta])$. Then $\varphi(x) \in \odot[y,\Delta]$; hence there exists $r > 0$ such that $\odot[\varphi(x),\Delta/r] \subseteq \odot[y,\Delta]$. The pseudometric $\overleftarrow{\varphi}\Delta$ is in $\mathsf{UP}(X)$ because φ is uniformly continuous, and

$$\varphi(\odot[x,\overleftarrow{\varphi}\Delta/r]) \subseteq \odot[\varphi(x),\Delta/r] \subseteq \odot[y,\Delta],$$

hence $\odot[x,\overleftarrow{\varphi}\Delta/r] \subseteq \varphi^{-1}(\odot[y,\Delta])$. Thus $\varphi^{-1}(\odot[y,\Delta])$ is an open set. □

When X is a uniform space and T is a topological space, say that X or the uniformity of X is *compatible with T* or with the topology of T iff X and T have the same set of points and the topology of X is the topology of T.

Typically many uniformities on a set are compatible with the same topology, and in that sense the passage from a uniform space to its topological space “forgets” some structure. However, no structure is forgotten when the space is compact:

Theorem 1.6. *Every continuous mapping from a compact uniform space to any uniform space is uniformly continuous.*

Proof. Let X be a compact uniform space, and $\varphi\colon X \to Y$ a continuous mapping from X to a uniform space Y.

Take any $\Delta \in \mathsf{UP}(Y)$ and $\varepsilon > 0$. I claim there is $\Delta_\varepsilon \in \mathsf{UP}(X)$ such that

$$\forall\, x,y \in X \;\; [\, \Delta_\varepsilon(x,y) < 1/2 \;\Rightarrow\; \overleftarrow{\varphi}\Delta(x,y) < \varepsilon \,].$$

Since φ is continuous, the set $\varphi^{-1}(\odot[\varphi(x), 2\Delta/\varepsilon])$ is open in X for every $x \in X$. Hence there are pseudometrics $\Delta_x \in \mathsf{UP}(X)$, $x \in X$, such that

$$\odot[x, \Delta_x] \subseteq \varphi^{-1}(\odot[\varphi(x), 2\Delta/\varepsilon])$$

for every $x \in X$. By compactness, there is a finite set $D \subseteq X$ such that the sets $\odot[z, 2\Delta_z]$, $z \in D$, cover X.

The pseudometric $\Delta_\varepsilon := \max_{z\in D} \Delta_z$ is in $\mathsf{UP}(X)$. If $x,y \in X$, $\Delta_\varepsilon(x,y) < 1/2$, find $z \in D$ such that $\Delta_z(z,x) < 1/2$. Then

$$\Delta_z(z,y) \le \Delta_z(z,x) + \Delta_\varepsilon(x,y) < 1\,,$$

which means that $x,y \in \odot[z, \Delta_z] \subseteq \varphi^{-1}(\odot[\varphi(z), 2\Delta/\varepsilon])$, and

$$\overleftarrow{\varphi}\Delta(x,y) = \Delta(\varphi(x), \varphi(y)) \le \Delta(\varphi(x), \varphi(z)) + \Delta(\varphi(z), \varphi(y)) < \varepsilon.$$

That proves the claim, and it follows that $\overleftarrow{\varphi}\Delta \in \mathsf{UP}(X)$. □

Corollary 1.7. *Every homeomorphism of compact uniform spaces is a uniform isomorphism.* □

The set of uniform structures on a fixed set of points is partially ordered by inclusion. When X and Y are two uniform spaces on the same set of points, say that the uniformity of X is *finer than* the uniformity of Y and that the uniformity of Y is *coarser than* the uniformity of X iff $\mathsf{UP}(X) \supseteq \mathsf{UP}(Y)$; that is, iff the identity mapping from X to Y is uniformly continuous. Say that X is finer or coarser than Y iff the same holds for the uniformities of X and Y.

The *discrete uniformity* on a set S is the set of all pseudometrics on S. It is the finest of all uniformities on S. A uniform space is *uniformly discrete* iff its uniformity is the discrete uniformity.

1.2 Cardinal Reflections, Compactness and Completeness

When Δ is a pseudometric on a set S and $Y \subseteq S$, say that Y is Δ-*uniformly discrete* iff there exists $\varepsilon > 0$ such that $\Delta(x,y) \ge \varepsilon$ for all $x,y \in Y$, $x \ne y$. When X is a uniform space and $Y \subseteq X$, say that Y is *uniformly discrete in* X iff it is Δ-uniformly discrete for some $\Delta \in \mathsf{UP}(X)$. Evidently a uniform space is uniformly discrete if and only if it is uniformly discrete in itself.

Theorem 1.8. *Let κ be an infinite cardinal and S a non-empty set. The following two properties of a pseudometric Δ on S are equivalent:*

(i) *If $Y \subseteq S$ is Δ-uniformly discrete in S, then $|Y| < \kappa$.*

(ii) *For every $\varepsilon > 0$, there exists $Y \subseteq S$ such that $|Y| < \kappa$ and $\Delta(x,Y) < \varepsilon$ for every $x \in S$.*

Proof. To prove (i)⇒(ii), suppose that (i) holds, and take any $\varepsilon > 0$. Consider the family of all sets $Y \subseteq S$ such that $\Delta(x,y) \geq \varepsilon$ whenever $x,y \in Y$, $x \neq y$. By Zorn's Lemma, this family contains a maximal set Y_0. We have $\Delta(x,Y_0) < \varepsilon$ for every $x \in S$ because Y_0 is maximal, and $|Y_0| < \kappa$ by (i).

To prove (ii)⇒(i), suppose that (ii) holds, and take any Δ-uniformly discrete set $Y \subseteq S$. There is some $\varepsilon > 0$ such that $\Delta(x,y) \geq \varepsilon$ for all $x,y \in Y$, $x \neq y$. By (ii), there exists $Y' \subset S$ such that $|Y'| < \kappa$ and $\Delta(x,Y') < \varepsilon/2$ for every $x \in S$. Thus there is a mapping $\varphi \colon Y \to Y'$ such that $\Delta(x,\varphi(x)) < \varepsilon/2$ for every $x \in Y$, and φ is injective because $\Delta(x,y) \geq \varepsilon$ for $x,y \in Y$, $x \neq y$. It follows that $|Y| \leq |Y'| < \kappa$. □

Corollary 1.9. *Let κ be an infinite cardinal. The following three properties of a uniform space X are equivalent:*

(i) *If $Y \subseteq X$ is uniformly discrete in X, then $|Y| < \kappa$.*

(ii) *For every $\Delta \in \mathsf{UP}(X)$ and $\varepsilon > 0$, there exists $Y \subseteq X$ such that $|Y| < \kappa$ and $\Delta(x,Y) < \varepsilon$ for every $x \in X$.*

(iii) *For every $\Delta \in \mathsf{UP}(X)$, there exists $Y \subseteq X$ such that $|Y| < \kappa$ and $\Delta(x,Y) < 1$ for every $x \in X$.* □

Lemma 1.10. *Let κ be an infinite cardinal, let $\{\Delta_0, \Delta_1, \ldots, \Delta_j\}$ be a non-empty finite set of pseudometrics on a non-empty set S and let Δ be a pseudometric on S such that $\Delta \ll \Delta_0 \vee \Delta_1 \vee \cdots \vee \Delta_j$. If the pseudometrics Δ_i, $i = 0,1,\ldots,j$, have property (ii) in Theorem 1.8, then so does Δ.*

Proof. Write $\Delta' := \Delta_0 \vee \Delta_1 \vee \cdots \vee \Delta_j$. To prove that Δ has property (ii) in Theorem 1.8, take any $\varepsilon > 0$. There is $\theta > 0$ such that if $\Delta'(x,y) < \theta$, then $\Delta(x,y) < \varepsilon$. There are sets $Y_i \subseteq S$, $i = 0,1,\ldots,j$, such that $|Y_i| < \kappa$ and $\Delta'(x,Y_i) < \theta$ for every $x \in S$. The set $Y := \bigcup_i Y_i$ satisfies $|Y| < \kappa$ and $\Delta'(x,Y) < \theta$, and hence also $\Delta(x,Y) < \varepsilon$, for every $x \in X$. □

Lemma 1.11. *Let X be any uniform space and κ an infinite cardinal. Then*

$$\mathscr{U} := \{\, \Delta \in \mathsf{UP}(X) \mid \textit{if } Y \subseteq X \textit{ is } \Delta\textit{-uniformly discrete in } X \textit{ then } |Y| < \kappa \,\}$$

is a uniform structure that is compatible with the topology of X.

Proof. By Lemma 1.10, $\mathscr{U}$ has properties (U1) and (U2) in Definition 1.1.

If $\Delta \in \mathsf{UP}(X)$ and $x_0 \in X$, then the pseudometric Δ' defined by

$$\Delta'(x,y) := 1 \wedge |\Delta(x,x_0) - \Delta(y,x_0)|$$

for $x,y \in X$ is in $\mathscr{U}$ (in fact, every Δ'-uniformly discrete set in X is finite), and $\odot[x_0, \Delta/\varepsilon] = \odot[x_0, \Delta'/\varepsilon]$ whenever $0 < \varepsilon < 1$. It follows that $\mathscr{U}$ has property (U3) because $\mathsf{UP}(X)$ does, and the topology of $\mathscr{U}$ is the same as that of $\mathsf{UP}(X)$. □

Definition 1.12. Let X be any uniform space and α any ordinal. The *α-th cardinal reflection* of X, denoted $\mathsf{p}_\alpha X$, is the uniform space with the same points as X and the uniformity $\mathscr{U}$ defined in Lemma 1.11 with $\kappa = \aleph_\alpha$. ■

By Lemma 1.11, X and $\mathsf{p}_\alpha X$ have the same topology. Clearly a uniform space X satisfies the conditions in Corollary 1.9 with $\kappa = \aleph_\alpha$ if and only if $X = \mathsf{p}_\alpha X$. (As was noted above, the equality $X = \mathsf{p}_\alpha X$ means that X and $\mathsf{p}_\alpha X$ are equal as uniform spaces, not merely as sets or topological spaces.)

Among the cardinal reflections, p_0 and p_1 will be of most interest in the sequel. The reflection p_0 captures the concept of a precompact (sometimes called totally bounded) space.

Definition 1.13. A uniform space X is *precompact* iff it satisfies the conditions in Corollary 1.9 with $\kappa = \aleph_0$. The 0-th cardinal reflection $\mathsf{p}_0 X$ is called the *precompact reflection* of X and written simply as $\mathsf{p}X$. ■

Compact and precompact spaces are closely related, as the next theorem shows. First note that every convergent net in a uniform space is Cauchy, in the sense of the following definition.

Definition 1.14. A net $\{x_\gamma\}_\gamma$ in a uniform space X is *Cauchy* iff for every $\Delta \in \mathsf{UP}(X)$ the estimate $\Delta(x_\beta, x_\gamma) < 1$ holds for almost all β, γ. The space X is *complete* iff every Cauchy net in X converges in X. ■

Theorem 1.15. *A uniform space is compact if and only if it is precompact and complete.*

Proof. Let X be a uniform space whose topology is compact. To prove X is precompact, take any $\Delta \in \mathsf{UP}(X)$. The collection of open sets $\odot[x, \Delta]$, $x \in X$, covers X. Being compact, X is covered by a finite subcollection. Thus X has property (iii) in Corollary 1.9 with $\kappa = \aleph_0$.

Next observe that if a Cauchy net has a cluster point x, then it converges to x. Since X is compact, every net in X has a cluster point, and therefore every Cauchy net in X converges.

To prove the converse, let X be precompact and complete. From property (iii) in Corollary 1.9 with $\kappa = \aleph_0$, we get that every universal net in X is Cauchy. Thus every universal net in X converges, which means that X is compact. □

1.3 Metric Spaces and Real Functions

When X is a metric space and Δ is its metric,

$$\mathscr{U}(\Delta) := \{\Delta' \mid \Delta' \text{ is a pseudometric on } X \text{ and } \Delta' \ll \{\Delta\}\}$$

is a uniformity on the set X. In fact, it is the uniformity induced by Δ, as defined further on in Sect. 2.1. Say that a uniform space is *metrizable by* Δ iff its uniformity is $\mathscr{U}(\Delta)$, and *metrizable* iff it is metrizable by some metric Δ.

Many metrics Δ on a given set may determine the same uniform structure $\mathscr{U}(\Delta)$, so that the passage from a metric space to the uniform space "forgets" some structure. Nevertheless, the passage preserves uniform concepts. Indeed, let X_i with metrics Δ_i, $i = 0,1$, be two metric spaces and let φ be a mapping from X_0 to X_1. Then φ is uniformly continuous with respect to the uniform structures $\mathscr{U}(\Delta_0)$ and $\mathscr{U}(\Delta_1)$ if and only if

$$\forall \varepsilon > 0\, \exists \theta > 0\, \forall x,y \in X_0\ \left[\, \Delta_0(x,y) < \theta \ \Rightarrow\ \Delta_1(\varphi(x),\varphi(y)) < \varepsilon \,\right].$$

Thus uniform continuity with respect to $\mathscr{U}(\Delta_0)$ and $\mathscr{U}(\Delta_1)$ is equivalent to the usual notion of uniform continuity for metric spaces. Clearly the topology of a metric Δ is the same as the topology of the uniformity $\mathscr{U}(\Delta)$. Similarly, by the following lemma, the completeness as usually defined for a metric space is the same as the completeness of the corresponding uniform space.

Lemma 1.16. *A metrizable uniform space X is complete if and only if every Cauchy sequence in X converges.*

Proof. The condition is obviously necessary. To prove its necessity, let X be metrizable by a metric Δ, assume that every Cauchy sequence in X converges and let $\{x_\gamma\}_{\gamma \in \Gamma}$ be a Cauchy net in X. There are $\gamma(j) \in \Gamma$ for $j \in \omega$ such that $\gamma(0) \le \gamma(1) \le \cdots$ and $\Delta(x_\beta, x_\gamma) < 1/(j+1)$ for all $\beta, \gamma \ge \gamma(j)$. The sequence $\{x_{\gamma(j)}\}_j$ is Cauchy, and its limit is also the limit of the net $\{x_\gamma\}_\gamma$. $\square$

These observations show that there is no danger of confusion or ambiguity when we adopt the view that every metric space *is* a uniform space. The meaning of expressions such as $\mathsf{UP}(X)$, $\mathsf{p}_\alpha X$ and $Y \Subset X$ (defined further on) for a metric space X is obtained by treating X as a uniform space.

$\mathbb{R}$ is a metric space with the usual metric $\Delta_{\mathbb{R}}(x,y) := |x-y|$, $x,y \in \mathbb{R}$. In the sequel, the uniformity induced by $\Delta_{\mathbb{R}}$ is called simply *the uniformity* of $\mathbb{R}$, and $\mathbb{R}$ is always considered as a uniform space with this uniformity.

For a uniform space X, let $\mathsf{U}(X)$ denote the space of uniformly continuous real-valued functions on X, and let $\mathsf{U_b}(X)$ denote the space of bounded functions in $\mathsf{U}(X)$.

Lemma 1.17. *Let X be a uniform space and let f be a real-valued function on X. If $\{f_\gamma\}_\gamma$ is a net of functions $f_\gamma \in \mathsf{U}(X)$ such that* $\lim_\gamma \|f_\gamma - f\|_X = 0$, *then* $f \in \mathsf{U}(X)$.

Proof. I shall prove that $\overleftarrow{f}\Delta_{\mathbb{R}} \in \mathsf{UP}(X)$. Take any $\varepsilon > 0$. There exists γ such that $\|f_\gamma - f\|_X < \varepsilon$. Since $f_\gamma \in \mathsf{U}(X)$, there are $\Delta' \in \mathsf{UP}(X)$ and $\theta > 0$ such that

$$\forall x,y \in X\ \left[\, \Delta'(x,y) < \theta \ \Rightarrow\ |f_\gamma(x) - f_\gamma(y)| < \varepsilon \,\right].$$

For $x,y \in X$, we have

$$\overleftarrow{f}\Delta_{\mathbb{R}}(x,y) = |f(x)-f(y)| \le |f(x)-f_\gamma(x)| + |f_\gamma(x)-f_\gamma(y)| + |f_\gamma(y)-f(y)| < 3\varepsilon,$$

which shows that $\overleftarrow{f}\Delta_{\mathbb{R}} \ll \mathsf{UP}(X)$ and therefore $\overleftarrow{f}\Delta_{\mathbb{R}} \in \mathsf{UP}(X)$. □

Theorem 1.18. *Let X be any uniform space.*

1. $\mathsf{U}(X)$ *is a Riesz subspace of* $\mathbb{R}^X$.
2. *If* $f \in \mathsf{U}(X)$ *and* $\inf\{|f(x)| \mid x \in X\} > 0$, *then* $1/f \in \mathsf{U_b}(X)$.
3. $\mathsf{U_b}(X)$ *is a Riesz subspace of* $\mathbb{R}^X$ *and a Banach lattice with the norm* $\|\cdot\|_X$.
4. *If* $f_0, f_1 \in \mathsf{U_b}(X)$, *then* $f_0 f_1 \in \mathsf{U_b}(X)$.

Proof. 1. Take any $f_0, f_1 \in \mathsf{U}(X)$, and let $f := f_0 \vee f_1$, $g := f_0 + f_1$. Then

$$\begin{aligned}\overleftarrow{f}\Delta_{\mathbb{R}} &\le \overleftarrow{f_0}\Delta_{\mathbb{R}} \vee \overleftarrow{f_1}\Delta_{\mathbb{R}},\\ \overleftarrow{g}\Delta_{\mathbb{R}} &\le \overleftarrow{f_0}\Delta_{\mathbb{R}} + \overleftarrow{f_1}\Delta_{\mathbb{R}} \le 2(\overleftarrow{f_0}\Delta_{\mathbb{R}} \vee \overleftarrow{f_1}\Delta_{\mathbb{R}}),\end{aligned}$$

hence $f, g \in \mathsf{U}(X)$. A similar but simpler argument shows that $rf_0 \in \mathsf{U}(X)$ for every $r \in \mathbb{R}$.

2. Take any $f \in \mathsf{U}(X)$ such that $r := \inf\{|f(x)| \mid x \in X\} > 0$. Then $g := 1/f$ is in $\mathsf{U}(X)$ because

$$\overleftarrow{g}\Delta_{\mathbb{R}} \le (1/r^2)\overleftarrow{f}\Delta_{\mathbb{R}}$$

and clearly $\|g\|_X \le 1/r$.

3. Since $\mathsf{U_b}(X) = \mathsf{U}(X) \cap \ell_\infty(X)$ and $\ell_\infty(X)$ is a Riesz subspace of $\mathbb{R}^X$, $\mathsf{U_b}(X)$ is a Riesz subspace of $\mathbb{R}^X$ by Part 1. By Lemma 1.17, $\mathsf{U_b}(X)$ is complete in the norm $\|\cdot\|_X$.

4. Take any $f_0, f_1 \in \mathsf{U_b}(X)$, and let $f := f_0 f_1$. Then

$$\overleftarrow{f}\Delta_{\mathbb{R}} \le \|f_1\|_X \overleftarrow{f_0}\Delta_{\mathbb{R}} + \|f_0\|_X \overleftarrow{f_1}\Delta_{\mathbb{R}} \le (\|f_0\|_X + \|f_1\|_X)(\overleftarrow{f_0}\Delta_{\mathbb{R}} \vee \overleftarrow{f_1}\Delta_{\mathbb{R}})$$

which shows that $f \in \mathsf{U_b}(X)$. □

Definition 1.19. Let X be any uniform space. A set of real-valued functions on X is said to be $\mathsf{UE}(X)$, or simply UE when X is understood, iff it is uniformly equicontinuous and bounded at every point of X. A set of real-valued functions on X is said to be $\mathsf{UEB}(X)$, or simply UEB when X is understood, iff it is uniformly equicontinuous and $\|\cdot\|_X$ bounded. ■

Lemma 1.20. *Let X be a uniform space.*

1. *A set* $\mathscr{F} \subseteq \mathbb{R}^X$ *is* $\mathsf{UE}(X)$ *if and only if there are a pseudometric* $\Delta \in \mathsf{UP}(X)$ *and a* Δ-1-*Lipschitz function* $h\colon X \to \mathbb{R}^+$ *such that* $\mathscr{F} \subseteq \mathsf{BLip}(\Delta, h)$.
2. *A set* $\mathscr{F} \subseteq \mathbb{R}^X$ *is* $\mathsf{UEB}(X)$ *if and only if there are a pseudometric* $\Delta \in \mathsf{UP}(X)$ *and* $r \in \mathbb{R}^+$ *such that* $\mathscr{F} \subseteq r\mathsf{BLip_b}(\Delta)$.

Proof. If $\mathscr{F} \subseteq \mathbb{R}^X$ is uniformly equicontinuous and

$$\Delta(x,y) := \sup_{f \in \mathscr{F}} |f(x) - f(y)|,\ x,y \in X,$$

$$h(x) := \sup_{f \in \mathscr{F}} |f(x)|,\ x \in X,$$

then $\Delta \in \mathsf{UP}(X)$, h is Δ-1-Lipschitz and $\mathscr{F} \subseteq \mathsf{BLip}(\Delta, h)$. □

1.4 Uniformizable Topological Spaces

The results in this section show that a topological space has a compatible uniformity if and only if it is completely regular. Moreover, for every completely regular topology there exists the finest compatible uniformity.

Theorem 1.21. *Let X be a uniform space and $x \in X$.*

1. *For every $\Delta \in \mathsf{UP}(X)$, the function $\backslash_y \Delta(x,y)$ is uniformly continuous on X.*
2. *If $V \subseteq X$ is an open set and $x \in V$, then there exists $f \in \mathsf{U_b}(X)$ such that $0 \leq f \leq 1$, $f(x) = 0$ and f is 1 on $X \setminus V$.*

Proof. 1. Let $\varphi := \backslash_y \Delta(x,y)$. Then $\overleftarrow{\varphi}\Delta_{\mathbb{R}} \in \mathsf{UP}(X)$ because $\overleftarrow{\varphi}\Delta_{\mathbb{R}} \leq \Delta$.
2. V is a union of basic open sets. Thus $x \in \odot[x_0, \Delta] \subseteq V$ for some $x_0 \in X$ and $\Delta \in \mathsf{UP}(X)$, and $\odot[x, \Delta/r] \subseteq V$ where $r := 1 - \Delta(x, x_0) > 0$. The function f defined by

$$f(y) := 1 \wedge \frac{1}{r}\Delta(y,x)$$

is as required. □

Corollary 1.22. *The topology of any uniform space X is the coarsest topology for which every function in $\mathsf{U_b}(X)$ is continuous.* □

Corollary 1.23. *The topology of every uniform space is completely regular.* □

Conversely, every completely regular topology is the topology of a uniform space:

Theorem 1.24. *Let T be a completely regular topological space and let $\mathscr{U}$ be the set of all pseudometrics Δ on T such that for each $x \in T$ the function $\backslash_y \Delta(x,y)$ is continuous on T. Then $\mathscr{U}$ is a uniformity compatible with T.*

Proof. Clearly $\mathscr{U}$ has properties (U1) and (U2) in Definition 1.1.

By the definition of $\mathscr{U}$, every set $\odot[x, \Delta]$, where $x \in X$ and $\Delta \in \mathscr{U}$, is open in the topology of T. On the other hand, if V is an open subset of T and $x_0 \in V$, there is a function $f \in \mathsf{C_b}(T)$ such that $f(x_0) = 0$ and f is 1 on $T \setminus V$. Define the pseudometric $\Delta \in \mathscr{U}$ by $\Delta(x,y) := |f(x) - f(y)|$, $x, y \in T$. Then $\odot[x_0, \Delta] \subseteq V$. Hence $\mathscr{U}$ has property (U3) and is compatible with T. □

Definition 1.25. When T is completely regular topological space, the *fine uniformity of* T is the uniformity $\mathscr{U}$ in Theorem 1.24. The *fine uniform space of* T, denoted $\mathsf{F}T$, is the set T with the fine uniformity of T.

Since every uniform space is a completely regular topological space, this also defines the fine uniform space $\mathsf{F}X$ for every uniform space X. A uniform space X is *fine* iff $X = \mathsf{F}X$. ■

Theorem 1.26. *Let X be a uniform space.*

1. *If Y is a uniform space and φ is a continuous mapping from X to Y, then φ is uniformly continuous from $\mathsf{F}X$ to Y.*
2. *Every uniformity on the set X that is compatible with the topology of X is coarser than $\mathsf{F}X$.*

Proof. To prove Part 1, take any continuous mapping $\varphi\colon X \to Y$ and any $\Delta \in \mathsf{UP}(Y)$. For every $x \in X$ the function $\backslash_y \Delta(\varphi(x), y)$ is continuous on Y; hence the function $\backslash_z \overleftarrow{\varphi}\Delta(x,z) = \backslash_z \Delta(\varphi(x), \varphi(z))$ is continuous on X. Thus $\overleftarrow{\varphi}\Delta \in \mathsf{UP}(\mathsf{F}X)$ by the definition of $\mathsf{F}X$.

Part 2 is a special case of Part 1 for the identity mapping φ on the set X. □

By Theorem 1.26, the fine uniformity of a completely regular space T is the finest uniformity compatible with the topology of T. Two uniform spaces X and Y on the same set of points are compatible with same topology if and only if $\mathsf{F}X = \mathsf{F}Y$.

Exercise 1.27. Show that $\mathsf{U}(\mathsf{F}X) = \mathsf{C}(X)$ and $\mathsf{U_b}(\mathsf{F}X) = \mathsf{C_b}(X)$ for every uniform space X. ■

1.5 Notes for Chap. 1

In his memoir published in 1937, Weil [179] defined uniform spaces and proved their basic properties. Since then, the theory has grown far beyond the brief account here. The foundations of uniform spaces are presented in most books on general topology, such as Bourbaki [12], Čech [24], Császár [31] and Engelking [47]. Isbell's monograph [100] is a classic of the field, covering a number of advanced topics. Bentley et al. [6] trace the history of uniform concepts and include pointers to major developments in uniform spaces up to the early 1990s.

A uniform structure may be defined in several ways, all equivalent to Weil's original definition. The three most popular approaches are those using entourages, uniform covers and pseudometrics. Definition 1.1 is an instance of the latter, adapted from Choquet [25] and Gillman and Jerison [81, 15.3].

Cardinal reflections in Sect. 1.2 are taken from Isbell [100], where $\mathsf{p}_\alpha X$ is written as $X_{\aleph_\alpha}$ and $\mathsf{p}_1 X$ is also written as eX.

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